Linear Algebra

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individual linear combination $a_1 v_1 + \cdots + a_n v_n$ and $\mathrm{span}(v_1, \ldots, v_n)$.

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For example,
"$v_1, \ldots, v_n$ are linearly dependent if
$a_1v_1 + \cdots + a_nv_n = 0 \implies$ at least one of a_i does not equal 0."

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The graders consider introducing negative credits for a clear demonstration of illiteracy.

Midterm Problem 3

Prove or give a counterexample:
If v_1, v_2, v_3, v_4 is a basis of vector space V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U, v_4 \notin U$, then v_1, v_2 is a basis of U.

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Counterexample: $U = \operatorname{span}(v_1, v_2, v_3 + v_4) \neq \operatorname{span}(v_1, v_2)$.

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Counterexample: $U = \operatorname{span}(v_1, v_2, v_3 + v_4) \neq \operatorname{span}(v_1, v_2)$.
$v_3 + v_4 \notin \operatorname{span}(v_1, v_2)$,
because otherwise $v_3 + v_4 = av_1 + bv_2$
which would contradict to linear independence of v_1, v_2, v_3, v_4. ■

Midterm Problem 4

Let $w_1, \ldots, w_m$ be a linearly independent list of vectors in a vector space V and $u \in V$. What values can the dimension of $\operatorname{span}(w_1 + u, \ldots, w_m + u)$ take?

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Vectors $w_1 - w_m, \ldots, w_{m-1} - w_m$ are linearly independent, because a non-trivial linear relation $x_1(w_1 - w_m) + \cdots + x_{m-1}(w_{m-1} - w_m) = 0$ would imply linear dependence of $w_1, \ldots, w_m$.

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If $u = -w_m$, then $\operatorname{span}(w_1 + u, \dots, w_m + u) = \operatorname{span}(w_1 - w_m, \dots, w_{m-1} - w_m)$
and $\dim \operatorname{span}(w_1 + u, \dots, w_m + u) = m - 1$. ∎

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Solution: x.

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Theorem. Each finite-dimensional vector space V is isomorphic to $\mathbb{F}^{\dim V}$.

Linear map vs. its values on basis

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Proof. Existense.

Consider linear maps $T_v : \mathbb{F}^n \to V$ and $T_w : \mathbb{F}^n \to W$,
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Then $\mathbb{F}^n \xrightarrow{T_v} V \xrightarrow{T} W$ maps $e_j \mapsto v_j \mapsto w_j$. Hence $T_v \circ T = T_w$
and $T = T_v^{-1} \circ T_w$. ■

Reformulation. Any map $\{v_1, \dots, v_n\} \to W$ from a basis of V to a vector space is extended uniquely to a linear map $V \to W$.

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Conclusion: any linear map $\mathbb{F}^p \to \mathbb{F}^q$ is multiplication by a $q \times p$-matrix.

Matrices

3.30 **Definition.** Let q and p denote positive integers.
A p-by-q matrix A is a rectangular array of elements of F with q rows and p columns:

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The kth column of $\mathcal{M}(T)$ is formed of the coordinates of the kth basis vector v_k.

The matrix of composition

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If $T : U \to V$ and $S : V \to W$ are linear maps, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

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&= B_{1,k}(A_{1,1}w_1 + A_{2,1}w_2 + \cdots + A_{m,1}w_m) \\
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&+ \ldots \\
&+ B_{n,k}(A_{1,n}w_1 + A_{2,n}w_2 + \cdots + A_{m,n}w_m)
\end{aligned}
$$

The matrix of composition

3.43 The matrix of the composition of linear maps

If $T : U \to V$ and $S : V \to W$ are linear maps, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

Proof. Let u, v, w be bases of U, V, W, respectively, and $\mathcal{M}(S) = A$, $\mathcal{M}(T) = B$.

$$
\begin{aligned}
(ST)u_k = S(B_{1,k}v_1 + B_{2,k}v_2 + \cdots + B_{n,k}v_n) &= B_{1,k}Sv_1 + B_{2,k}Sv_2 + \cdots + B_{n,k}Sv_n \\
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$$= (w_1 \dots w_m) \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} \begin{pmatrix} B_{1,k} \\ \vdots \\ B_{n,k} \end{pmatrix}$$

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$$

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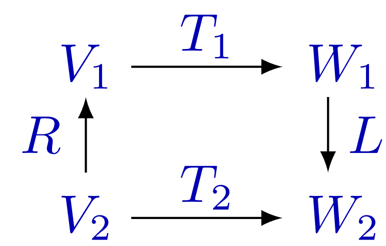
$$

Hence $(STu_1 \dots STu_p) = (w_1 \dots w_m) \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} \begin{pmatrix} B_{1,1} & \dots & B_{1,p} \\ \vdots & & \vdots \\ B_{n,1} & \dots & B_{n,p} \end{pmatrix}$

Right-left equivalence of linear maps

Right-left equivalence of linear maps

Linear maps $T_1 : V_1 \to W_1$, $T_2 : V_2 \to W_2$ are called **isomorphic** if there exist isomorphisms $R : V_2 \to V_1$ and $L : W_1 \to W_2$ such that $T_2 = L \circ T_1 \circ R$.



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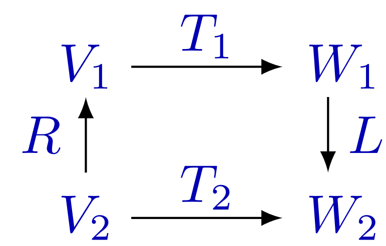

Another name: **right-left equivalent** or **R-L-equivalent**.

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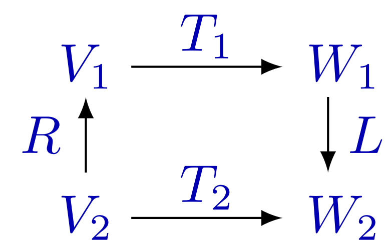

R-L-equivalence is an equivalence.

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$$\begin{array}{ccc} V_1 & \xrightarrow{T_1} & W_1 \\ {\scriptstyle R}\uparrow & & \downarrow{\scriptstyle L} \\ V_2 & \xrightarrow{T_2} & W_2 \end{array}$$

R-L-equivalence is an equivalence.

Proof. Reflexivity: $T = L \circ T \circ R$ for $R = \mathrm{id}_V$, $L = \mathrm{id}_W$ and any $T : V \to W$.

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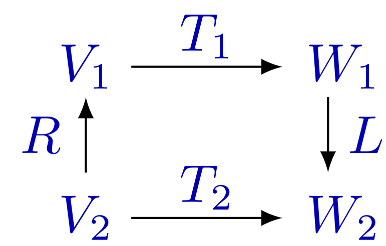

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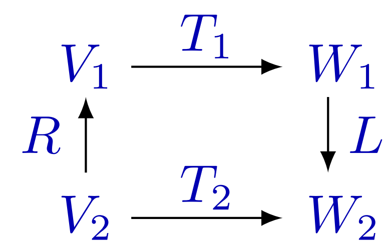

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Proof. **Reflexivity:** $T = L \circ T \circ R$ for $R = \mathrm{id}_V$, $L = \mathrm{id}_W$ and any $T : V \to W$.
Symmetry: If $T_2 = L \circ T_1 \circ R$, then $L^{-1} \circ T_2 \circ R^{-1} = T_1$.
Transitivity: If $T_2 = L_1 \circ T_1 \circ R_1$ and $T_3 = L_2 \circ T_2 \circ R_2$, then

$$T_3 = L_2 \circ T_2 \circ R_2 = L_2 \circ L_1 \circ T_1 \circ R_1 \circ R_2 .$$

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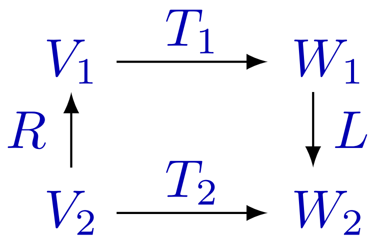

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R-L-equivalent maps $T_1 : V_1 \to W_1$, $T_2 : V_1 \to W_2$ have

- isomorphic domains V_1 and V_2 ,
- isomorphic target spaces W_1 and W_2 ,
- isomorphic null spaces $\mathrm{null}\, T_1$ and $\mathrm{null}\, T_2$ and
- isomorphic ranges $\mathrm{range}\, T_1$ and $\mathrm{range}\, T_2$.

Classification of linear maps

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3.22 **Fundamental Theorem of Linear Maps.**
Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$.
Then $\operatorname{range} T$ is finite-dimensional and $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$.

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The proof above provides classification of linear maps between finite-dimensional vector spaces up to R-L equivalences.

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Extend $Tu_1, \dots, Tu_p$ to a basis $Tu_1, \dots, Tu_p, w_1, \dots, w_r$ of W.
This basis and the bases constructed above define isomorphisms $R : \mathbb{F}^p \oplus \mathbb{F}^q \to V$ and $L : \mathbb{F}^q \oplus \mathbb{F}^r \to W$ such that $L^{-1} \circ T \circ R : \mathbb{F}^p \oplus \mathbb{F}^q \to \mathbb{F}^q \oplus \mathbb{F}^r$ is 0 on $\mathbb{F}^p$ and maps identically $\mathbb{F}^q \to \mathbb{F}^q$.