A proof of Pilgrim’s conjecture

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A Thurston map is a pair \((f, P_f)\) where \(f : S^2 \to S^2\) is an orientation-preserving branched self-cover of \(S^2\) of degree \(d_f \geq 2\) and \(P_f\) is a finite forward invariant set that contains all critical values of \(f\).
A *Thurston map* is a pair \((f, P_f)\) where \(f : \mathbb{S}^2 \rightarrow \mathbb{S}^2\) is an orientation-preserving branched self-cover of \(\mathbb{S}^2\) of degree \(d_f \geq 2\) and \(P_f\) is a finite forward invariant set that contains all critical values of \(f\).

In particular, the branched cover \(f\) must be postcritically finite.
Two Thurston maps \( f \) and \( g \) are combinatorially equivalent if and only if there exist two homeomorphisms \( h_1, h_2 : \mathbb{S}^2 \to \mathbb{S}^2 \) such that the diagram commutes, \( h_1 |_{P_f} = h_2 |_{P_f} \), and \( h_1 \) and \( h_2 \) are homotopic relative to \( P_f \).
Theorem (Thurston’s Theorem)

A postcritically finite branched cover $f : \mathbb{S}^2 \to \mathbb{S}^2$ with hyperbolic orbifold is either Thurston-equivalent to a rational map $g$ (which is then necessarily unique up to conjugation by a Möbius transformation), or $f$ has a Thurston obstruction.
Some further notations

- $\mathcal{T}_f$ is the Teichmüller space modeled on the marked surface $(\mathbb{S}^2, P_f)$
- $\mathcal{M}_f$ is the corresponding moduli space
- Recall that $\mathcal{T}_f$ can be defined as the quotient of the space of all diffeomorphisms from $(\mathbb{S}^2, P_f)$ to the Riemann sphere. We write $\tau = \langle h \rangle$ if a point $\tau$ is represented by a homeomorphism $h$
- $Q(\mathbb{P}, h(P_f))$ is the cotangent space at a point $\tau = \langle h \rangle$ in the Teichmüller space $T_f$ which is canonically isomorphic to the space of all integrable meromorphic quadratic differentials on the marked Riemann surface corresponding to $\tau$
Maps between Teichmüller spaces

**Pullback map**

Suppose we have a (unbranched) covering map $h: A \to B$ between finite type surfaces $A$ and $B$. Then we can define $h^*: \mathcal{T}(B) \to \mathcal{T}(A)$ that acts by pulling back complex structures from $B$ to $A$. 

**Projection map**

Suppose we have an inclusion map $i: A \to B$ between finite type surfaces $A$ and $B$. This happens exactly when $A$ can be obtained from $B$ by removing finitely many points. Then we can define the forgetful projection $i^*: \mathcal{T}(A) \to \mathcal{T}(B)$ which "forgets" the positions of extra punctures.
Maps between Teichmüller spaces

### Pullback map

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### Projection map

Suppose we have an inclusion map $i: A \rightarrow B$ between finite type surfaces $A$ and $B$. This happens exactly when $A$ can be obtained from $B$ by removing finitely many points. Then we can define the forgetful projection $i_*: \mathcal{T}(A) \rightarrow \mathcal{T}(B)$ which “forgets” the positions of extra punctures.
In our setting we have the unbranched cover $f: \mathbb{S}^2 \setminus f^{-1}(P_f) \to \mathbb{S}^2 \setminus P_f$ and the identity injection $\text{id}: \mathbb{S}^2 \setminus f^{-1}(P_f) \to \mathbb{S}^2 \setminus P_f$ since $f^{-1}(P_f) \supset P_f$. Denote $\sigma_f = \text{id} \circ f^*: \mathcal{T}_f \to \mathcal{T}_f$. 

Another definition of Thurston's iteration $(\mathbb{S}^2, P_f) \to (P_f, \tau_1)$ $(\mathbb{S}^2, P_f) \to (P_f, \tau_\tau)$ - $h_1 f\tau_1 f\tau$. 

Thurston's iteration
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In our setting we have the unbranched cover
\[ f : \mathbb{S}^2 \setminus f^{-1}(P_f) \to \mathbb{S}^2 \setminus P_f \]
and the identity injection
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since \( f^{-1}(P_f) \supset P_f \). Denote
\[ \sigma_f = \text{id}_* \circ f^* : \mathcal{T}_f \to \mathcal{T}_f. \]

Another definition of Thurston’s iteration

\[
\begin{align*}
(\mathbb{S}^2, P_f) & \xrightarrow{h_1} (\mathbb{P}, h_1(P_f)) \\
\downarrow f & \quad \quad \downarrow f_{\tau} \\
(\mathbb{S}^2, P_f) & \xrightarrow{h_\tau} (\mathbb{P}, h_\tau(P_f))
\end{align*}
\]
Fixed Points of $\sigma_f$

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(S^2, P_f) & \xrightarrow{h_\tau} (\mathbb{P}, h_\tau(P_f))
\end{align*}
\]

Lemma

A Thurston map $f$ is equivalent to a rational function if and only if $\sigma_f$ has a fixed point.
Lemma (Douady, Hubbard)

There exists an intermediate cover $\mathcal{M}'_f$ of $\mathcal{M}_f$ (so that $\mathcal{T}_f \xrightarrow{\pi_1} \mathcal{M}'_f \xrightarrow{\pi_2} \mathcal{M}_f$ are covers and $\pi_2 \circ \pi_1 = \pi$) such that

i. $\pi_2$ is finite,

ii. $\mathcal{T}_f \xrightarrow{\pi_1} \mathcal{M}'_f \xrightarrow{\pi_2} \mathcal{M}_f$ commutes for some map $\tilde{\sigma}_f: \mathcal{M}'_f \to \mathcal{M}_f$,

iii. If $\pi_1(\tau_1) = \pi_1(\tau_2)$ then $f_{\tau_1} = f_{\tau_2}$ up to pre- and post-composition by Moebius transformations.
Co-derivative of $\sigma_f$

**Definition**

The push-forward operator is locally defined by the formula

$$g_*q|_U = \sum_i g_i^* q,$$

where $g_i$ are all inverse branches of $g$. 

Lemma $\sigma_f$ is a holomorphic self-map of $T_f$ and the co-derivative of $\sigma_f$ satisfies

$$(d\sigma_f(\tau))^* = (f\tau)^*$$

where $(f\tau)^*$ is the push-forward operator on quadratic differentials.
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**Definition**

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**Lemma**

$\sigma_f$ is a holomorphic self-map of $T_f$ and the co-derivative of $\sigma_f$ satisfies $(d\sigma_f(\tau))^\ast = (f_\tau)_\ast$ where $(f_\tau)_\ast$ is the push-forward operator on quadratic differentials.
Metrics on Teichmüller space

Metric definitions

For a meromorphic integrable quadratic differential on $\mathbb{P}$ we define

- its Teichmüller norm

$$||q||_T = 2 \int_{\mathbb{P}} |q|$$

and

- its Weil-Petersson norm

$$||q||_{WP} = \left( \int_{\mathbb{P}} \rho^{-2} |q|^2 \right)^{1/2}$$
Estimates on the norm of $d\sigma_f^*$

**Lemma**

$\|(d\sigma_f)^*\|_T \leq 1.$
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\| (d\sigma_f)^* \|_T \leq 1.
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**Proof.**

\[
\int_U |g^* q| = \int_U \left| \sum_i g_i^* q \right| \leq \sum_i \int_U |g_i^* q| = \sum_i \int_{U_i} |q|
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**Corollary (almost)**

*There exists at most one fixed point of $\sigma_f$, hence the uniqueness in Thurston’s theorem follows.*
Lemma

\[ \| (d \sigma_f)^* \|_{WP} \leq \sqrt{d}. \]
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Proof.

$$\int_{U} \frac{|g_\ast q|^2}{\rho^2} = \int_{U} \frac{|\sum_i g_i^\ast q|^2}{\rho^2} \leq d \sum_i \int_{U} \frac{|g_i^\ast q|^2}{\rho^2} =$$

$$= d \sum_i \int_{U} \frac{|q|^2}{g^\ast \rho^2} \leq d \int_{g^{-1}(U)} \frac{|q|^2}{\rho_1^2},$$
Lemma
\[ \|(d\sigma_f)^*\|_{WP} \leq \sqrt{d}. \]

Proof.
\[
\int_U \frac{|g^* q|^2}{\rho^2} = \int_U \left\| \sum_i g_i^* q \right\|^2 \leq d \sum_i \int_U \frac{|g_i^* q|^2}{\rho^2} = \\
= d \sum_i \int_{U_i} \frac{|q|^2}{g^* \rho^2} \leq d \int_{g^{-1}(U)} \frac{|q|^2}{\rho_1^2},
\]

Corollary
\( \sigma_f \) is Lipschitz with respect to the WP-metric.
Yet some more definitions:

- a closed curve $\gamma$ is *essential* if every component of $S^2 \setminus \gamma$ contains at least two points of $P_f$
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- \( f^{-1}(\Gamma) \) is the multicurve of all essential mutually non-homotopic preimages of curves in \( \Gamma \).
Invariant multicurves

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- A multicurve $\Gamma$ is *invariant* if $f^{-1}(\Gamma) \subseteq \Gamma$.
- A multicurve $\Gamma$ is *completely invariant* if $f^{-1}(\Gamma) = \Gamma$. 

The augmented Teichmüller space $\mathcal{T}_f$ - the set of all marked noded stable Riemann surfaces of the same type

- The augmented Teichmüller space $\mathcal{T}_f$ is a stratified space with strata $S_\Gamma$ corresponding to different multicurves $\Gamma$ on $(\mathbb{S}^2, P_f)$. In particular, $\mathcal{T}_f = S_\emptyset$. 

Structure of the boundary of the augmented Teichmüller space
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- $S_\Gamma$ is the product of Teichmüller spaces of these components.

- Within each stratum one can define its own natural Teichmüller and Weil-Petersson metrics.
Augmented Teichmüller space

Lemma

The quotient $\overline{M}_f$ of the augmented Teichmüller space by the action of the pure mapping class group is compact.
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Recall the diagram
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Recall the diagram

\[
\begin{array}{ccc}
\overline{T}_f & \xrightarrow{\sigma_f} & \overline{T}_f \\
\pi & & \pi \\
\pi_1 & & \pi_2 \\
\pi & & \pi \\
\overline{\mathcal{M}}_f & & \overline{\mathcal{M}}_f \\
\overline{\mathcal{M}}'_f & & \\
\overline{\mathcal{M}}_f & & \overline{\mathcal{M}}_f \\
\end{array}
\]
Theorem (Masur)

The augmented Teichmüller space $\tilde{T}_f$ is homeomorphic to the WP-completion of the Teichmüller space.
Theorem (Masur)

The augmented Teichmüller space $\overline{T}_f$ is homeomorphic to the WP-completion of the Teichmüller space.

Corollary

$\sigma_f$ extends continuously to $\overline{T}_f$. 
Definition of $\sigma_f$ on the boundary

We represent points in $\overline{T}_f$ not only by homeomorphisms but also by continuous maps from $(S^2, P_f)$ to a noded Riemann surface that are allowed to send a whole simple closed curve to a node. Consider such an $h$ representing some point in $\overline{T}_f$. 
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We complete this diagram as before

$$
\begin{array}{ccc}
(\mathbb{S}^2, P_f) & \xrightarrow{h_1} & (R_1, P_f) \\
\downarrow f & & \downarrow \{f^C_i\} \\
(\mathbb{S}^2, P_f) & \xrightarrow{h} & (R, P_f) \\
\end{array}
$$
Action of $\sigma_f$ on $\mathcal{T}_f$

**Theorem**

The map $\sigma_f$ as defined above is continuous on $\mathcal{T}_f$.

**Remark.**

Note that by definition $\sigma_f$ maps any stratum $S_\Gamma$ into the stratum $S_{f^{-1}}(\Gamma)$, therefore invariant boundary strata are in one-to-one correspondence with completely invariant multicurves.
Denote by $\mathcal{C}$ the set of all homotopy classes of essential simple closed curve. Define Thurston linear operator $M: \mathbb{R}^\mathcal{C} \to \mathbb{R}^\mathcal{C}$ by setting

$$M(\gamma) = \sum_{f(\gamma_i) = \gamma, \deg f|_{\gamma_i}} \frac{1}{\deg f|_{\gamma_i}} \gamma_i.$$ 

Every multicurve $\Gamma$ has its associated Thurston matrix $M_\Gamma$ which is the restriction of $M$ to $\mathbb{R}^\Gamma$. 

Definition
Thurston matrix and obstructions

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Definition

Since all entries of $M_\Gamma$ are non-negative real, the leading eigenvalue $\lambda_\Gamma$ of $M_\Gamma$ is also real and non-negative. A multicurve $\Gamma$ is a Thurston obstruction if $\lambda_\Gamma \geq 1$. 
**Definition**

We call $\Gamma$ *simple* if there exists a leading eigenvector of $M_\Gamma$ with positive coordinates. Each multicurve has a simple sub-multicurve with the same leading eigenvalue.
Dynamics near boundary strata

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**Definition**
An invariant stratum $S$ of $\overline{T}_f$ is *weakly attracting* if there exists a nested decreasing sequence of neighborhoods $U_n$ such that $\sigma_f(U_n) \subset U_n$ and $\bigcap U_n = S$. 
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**Definition**
An invariant stratum $S$ of $\overline{T}_f$ is *weakly repelling* if for any compact set $K \subset S$ there exists a neighborhood $K \subset U$ such that every point of $U \cap T_f$ escapes from $U$ after finitely many iterations.
Lemma

If \( \Gamma = \{ \gamma_1, \gamma_2, \ldots, \gamma_m \} \) is a completely invariant simple multicurve and \( \lambda_{\Gamma} \geq 1 \), then \( S_{\Gamma} \) is weakly attracting. Otherwise it is weakly repelling.
Sketch of the proof of Thurston’s theorem

Pick any starting point \( \tau \in \mathcal{T}_f \) and consider \( \tau_n = \sigma_f^n(\tau) \). Take an accumulation point in \( \overline{\mathcal{M}_f} \) of the projection of \( \tau_n \) to the moduli space on the stratum of smallest possible dimension. For simplicity we assume that \( \tau_n \) accumulates on some \( \tau_0 \in \mathcal{S}_\Gamma \).
Pick any starting point $\tau \in \mathcal{T}_f$ and consider $\tau_n = \sigma_f^n(\tau)$. Take an accumulation point in $\overline{M}_f$ of the projection of $\tau_n$ to the moduli space on the stratum of smallest possible dimension. For simplicity we assume that $\tau_n$ accumulates on some $\tau_0 \in S_\Gamma$.

- If $\Gamma = \emptyset$ then $\tau_0$ is a fixed point of $\sigma_f$. 
Sketch of the proof of Thurston’s theorem

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- If $\Gamma = \emptyset$ then $\tau_0$ is a fixed point of $\sigma_f$.
- If $\Gamma \neq \emptyset$ then $\Gamma$ must be a Thurston obstruction. Otherwise, $\mathcal{S}_\Gamma$ is weakly repelling and therefore $\tau_n$ can not have an accumulation point there.
Pilgrim’s theorems

Definition

The *canonical* obstruction $\Gamma_f$ is the set of all homotopy classes of curves $\gamma$ that satisfy $l(\gamma, \sigma^n_f(\tau)) \to 0$ for all $\tau \in \mathcal{T}_f$. 

Theorem (Canonical Obstruction Theorem)

If for a Thurston map with hyperbolic orbifold its canonical obstruction is empty then it is Thurston equivalent to a rational function. If the canonical obstruction is not empty then it is a Thurston obstruction.

Theorem

For any point $\tau \in \mathcal{T}_f$ there exists a bound $L = L(\tau, f) > 0$ such that for any essential simple closed curve $\gamma \not\in \Gamma_f$ the inequality $l(\gamma, \sigma^n_f(\tau)) \geq L$ holds for all $n$. 
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**Theorem (Canonical Obstruction Theorem)**

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**Theorem**

For any point $\tau \in \mathcal{T}_f$ there exists a bound $L = L(\tau, f) > 0$ such that for any essential simple closed curve $\gamma \notin \Gamma_f$ the inequality $l(\gamma, \sigma^n_f(\tau)) \geq L$ holds for all $n$. — *the accumulation set of* $\{\pi(\tau_n)\}$ *is precompact in* $S_{[\Gamma_f]}$
Recall that the action on any invariant stratum is given by pullbacks of complex structures by a collection of maps $\sigma_{f_C}$ for all components $C$ of any surface in the stratum. Combinatorics of the process is very simple: we have a map from a finite set into itself, every component is pre-periodic. The whole action, therefore, can be characterized by studying cycles of components. For each cycle $Y$ there are three cases, the composition $f^Y$ of all coverings in the cycle is either of the following:

- a homeomorphism,
- a Thurston map with a parabolic orbifold,
- a Thurston map with a hyperbolic orbifold.
Theorem

If a cycle $Y$ of components a topological surface corresponding to the stratum $S_{\Gamma_f}$ has hyperbolic orbifold then $f^Y$ is not obstructed and, hence, equivalent to a rational map.
Pilgrim’s conjecture

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**Idea of the proof**

We may assume that our component is mapped to itself. Take an accumulation point $\tau_0 \in S_{\Gamma_f}$ of $\sigma^n_f(\tau)$. Let $\tau'$ be the coordinate corresponding to $Y$. Let $\Gamma_Y$ be the canonical obstruction for $f^Y$. 
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**Idea of the proof**

We may assume that our component is mapped to itself. Take an accumulation point $\tau_0 \in S_{\Gamma_f}$ of $\sigma_f^n(\tau)$. Let $\tau'$ be the coordinate corresponding to $Y$. Let $\Gamma_Y$ be the canonical obstruction for $f^Y$. Then the accumulation set of $\sigma_f^n(\tau')$ must be a subset of the closure of $S_{\Gamma_f} \cup \Gamma_Y$. On the other hand, it is clearly a subset of the accumulation set of $\sigma_f^n(\tau) \subset S_{\Gamma_f}$. This means $\Gamma_Y$ must be empty.
Generalized Pilgrim’s conjecture

**Theorem**

If a cycle $Y$ of components a topological surface corresponding to the stratum $S_{\Gamma_f}$ is a Thurston map then the canonical obstruction of $f^Y$ is empty.
Generalized Pilgrim’s conjecture

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*If a cycle $Y$ of components a topological surface corresponding to the stratum $S_{\Gamma_f}$ is a Thurston map then the canonical obstruction of $f^Y$ is empty.*

This, with some extra effort, leads to a complete topological description of canonical obstructions.