MAT 322 Final exam practice problems

Analysis in several dimensions

May , 2017

1. **Implicit function theorem: a manifold defined by a set of equations, dimension of this manifold**
   (a) Prove that the equation $x^2 + y^4 + z^6 = 3$ defines a smooth manifold M in R^3.
   (b) Find the dimension of M.
   (c) Find the equations of the tangent space of M at the point $(1, 1, 1)$. What is the dimension of the tangent space?

2. **Implicit function theorem: a manifold defined by a set of equations, dimension of this manifold** A manifold defined by more than one equation.

3. **Manifolds and their tangent spaces: as a graph of a function, parametrized, as set of solutions of equations.** Consider the subset of R^3 defined by $4x^2 + y^2 + z^2 = 1$.
   (a) Find a parametrization of M.
   (b) Can M be describe as the graph of a function?
   (c) Find a tangent space of M at the point $(0, 1/\sqrt{2}, 1/\sqrt{2})$, using the definition of M as a solution of equations.
   (d) Find a tangent space of M at the point $(0, 1/\sqrt{2}, 1/\sqrt{2})$, using the parametrization of M.
   (e) Find a tangent space of M at the point $(0, 1/\sqrt{2}, 1/\sqrt{2})$, using the definition of M as the graph of a function.

4. **integral by definition, dyadic cubes.**
   (a) Compute the upper and lower sums $U_1(f)$ and $L_1(f)$ where $f(x, y) = x^2 + y$ if $0 \leq x, y \leq 1$, and $f(x, y) = 0$ otheriwse.
   (b) Can you determine without computing the integral $\int f(x, y)|dxdy|$, whether the inequalities $L_1(f) \leq \int f(x, y)|dxdy| \leq U_1(f)$ hold?

5. **Measure zero - Volume**
   (a) Find, if possible, a set of measure 0 and volume not defined or not 0.
   (b) Find, if possible, a set of volume 0 and measure not defined or not 0.

6. **When a function is integrable** Let Q be the subset of R^2 defined by $|x| + |y| \leq 1$. Prove that the function $f(x, y) = sin(x^2 + cos(y))I_Q$ is integrable.

7. **1-Form fields : Integrate by definition** Consider the 1-form in R^2, $\sigma = \frac{y}{x^2+y^2}dx$. Following the procedure we did in class, find the integral of σ against the arc of the circle of radius 2 centered at the origin, running from $(1, 0)$ to $(-1, 0)$, in the clockwise direction. (You just need to set up the appropriate sequence of partitions of the curve, write down the sum for each partition and express the result as a limit. You do not need to compute the limit.)

8. **Manifold orientation** Find an orientation of the surface S in R^3 defined by the equation $x^2+y^4+z^6 = 1$.

9. **Form fields : integrate over parametrized domains and over manifolds. Generalized Stokes Theorem. "nice" subsets of Manifolds : orientation of boundary** Denote by S the portion of the sphere of radius 2 in R^3, that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy plane. Orient S by a vector field defined by the upwards pointing normal
   (a) Find the integral $\int_{\partial S} \omega$, where $\omega = xzdx + yzdy + xydz$.
   (b) Find the integral $\int_S d\omega$..
   (c) These two integrals should be the same, why?

10. **Forms: definition, elementary forms, exterior differential** Consider $\omega = xyzdy$.
   (a) Compute $d\omega$.
   (b) What do you obtain when you apply $d\omega$ to a point in R^3.
   (c) What do you obtain when you apply ω to a point in R^3.

11. **Form fields : integrate over parametrized domains** Set up each of the following integrals of form fields over parametrized domains as an ordinary multiple integrals.
   (a) $\int_{[\gamma(I)]} y^2dy + x^2dz$ where $\gamma(t) = (t^3, t^2 + 1, t^2 - 1)$ and $I = [0, s]$
   (b) $\int_{[\gamma(U)]} sin(y)^2 dx \wedge dz$ where $\gamma(u, v) = (u^2 - v, uv, v^4)$ and $U = [0, s] \times [0, r]$