Problem set Week 11

The *Farey sequence of order* n, denoted by $\mathcal{F}_b$ is the set

$$\{\frac{a}{b} \text{ such that } (a,b)=1 \text{ and } 0 \leq a \leq b\}$$

ordered from smallest to largest element.

For example, the Farey sequence of order 2 and 3 are $0/1, 1/2, 1/1$ and $0/1, 1/3, 1/2, 2/3, 1/1$ respectively.

Define Farey addition of fractions by $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$.

1. Write down the Farey sequences of order 6 and 7, $\mathcal{F}_6$ and $\mathcal{F}_7$.

2. The fraction $\frac{a}{b}$, $0 < a < b$, $(a,b)=1$ appears first in $\mathcal{F}_b$. Show that if $\frac{a}{b} = \langle a_0, a_1, \ldots, a_n \rangle$ with $a_n > 1$ then its neighbors in $\mathcal{F}_b$ are C_{n-1} and $C_{n-1,-1+a_n}$ where C_{n-1} is the n-th convergent of $\frac{a}{b} =$ and $C_{n-1,x} = \frac{xp_{n-1}+p_{n-2}}{xq_{n-1}+q_{n-2}}$.

3. Show that the Farey addition of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$ is between $\frac{a}{b}$ and $\frac{c}{d}$. Moreover, the first term that appears (in some $\mathcal{F}_k$) between $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a+c}{b+d}$.

4. Prove that if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive terms in $\mathcal{F}_n$, and $\frac{a}{b} < \frac{c}{d}$ then $c.b - a.d = 1$. (Hint: use mathematical induction).

5. Show all of the Farey fractions so obtained in the above constructions are reduced (namely, numerator and denominator are relatively prime)

6. Determine the number of fractions in the Farey sequence of order n. (Hint: The Euler Φ function might be helpful).

7. (bonus) Determine the sum of the fractions in the Farey sequence of order n. (Hint: This is half of the previous value).

8. (bonus) For each pair of relative prime positive integers p and q construct a circle of diameter $\frac{1}{q^2}$ and resting on the x-axis at $\frac{p}{q}$. These circles are the *Ford circles* (In the Figure you can see an example of the Ford circles for the Farey sequence). Show that no two of these circles intersect at more than one point and two such circles are tangent if and only if the corresponding resting point fractions are neighbors in the Farey sequence of order n for some n.

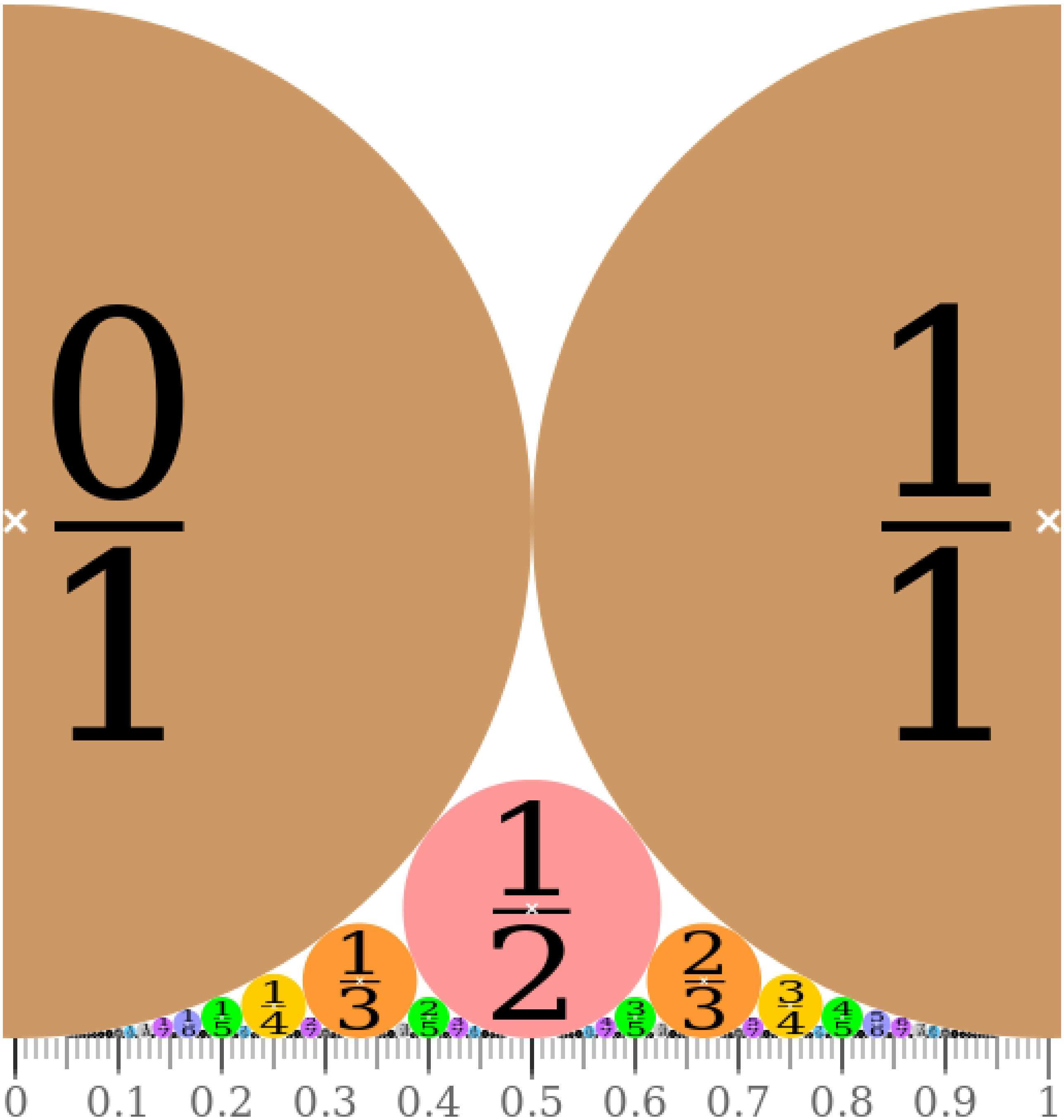

Figure 1: Ford circles (image from Wikipedia)