Ancient Indian Trigonometry

Greek

trigōnon	triangle
metron	measure

Ancient Indian Mathematics

India

Āryabhata (Indian astronomer, 5th century AD)

↓

Wrote Arya bhatay

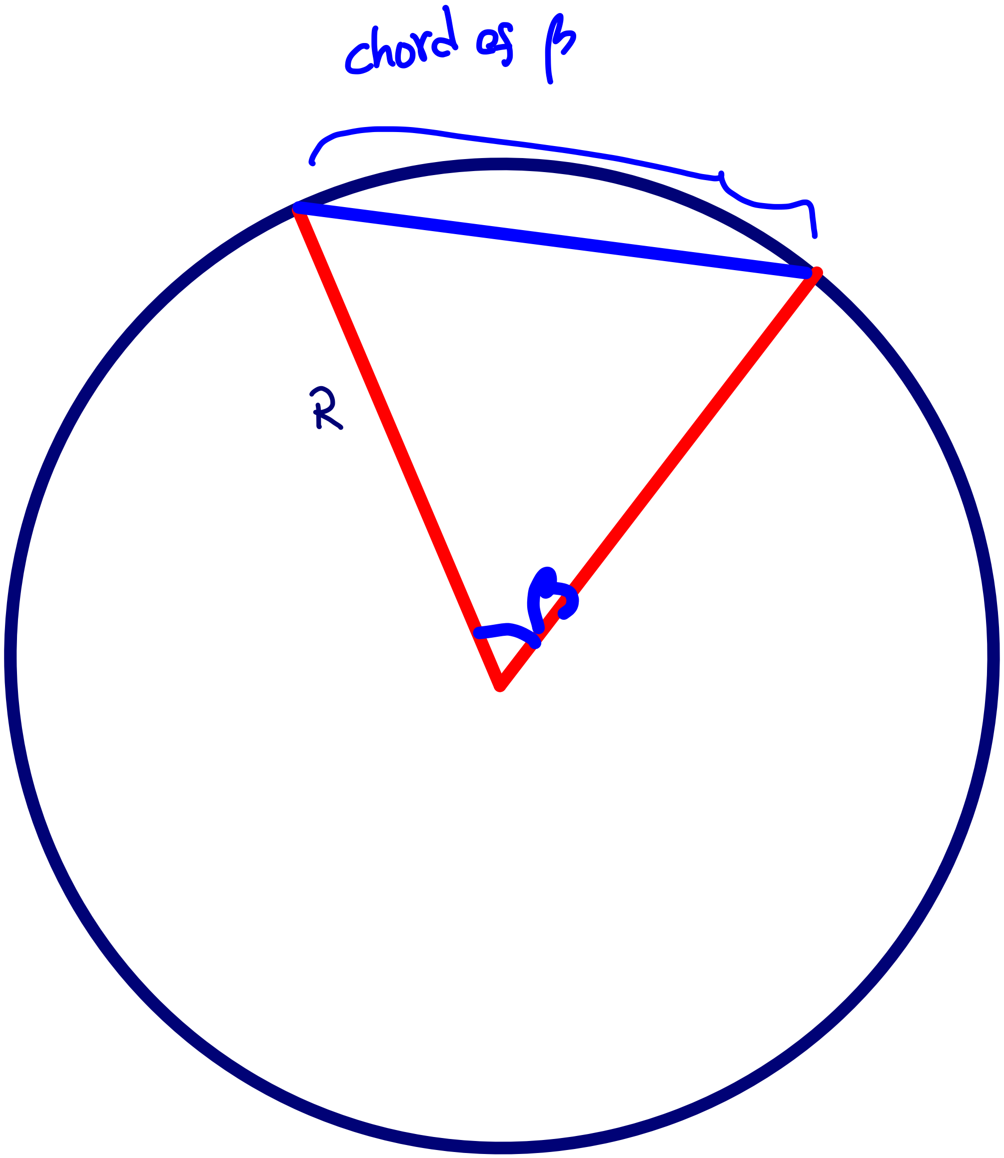
घ्लकि किग्र हक्य धकि किच स्ग झश ङ्व क्ल प्त फ छ कलार्धज्याः ॥

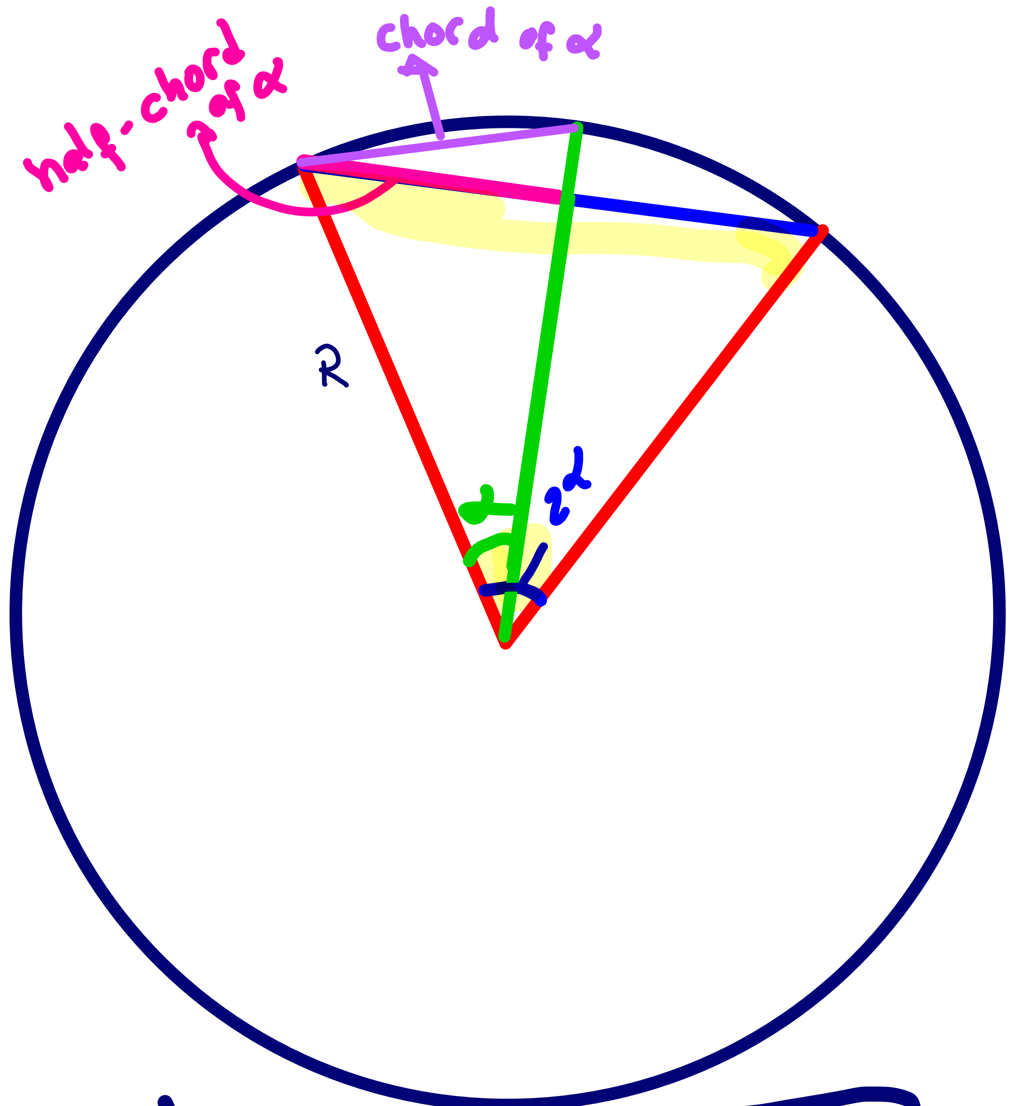
Source: Sines in terse verse
by Roddam Narasimha

https://www.nature.com/articles/414851a3

See also Prof Phillips column:

http://www.math.stonybrook.edu/~tony/whatsnew/
jun02/06-2002-media.html#sanscriti





β central angle
(that is, vertex coincides with center of the circle

$\boxed{\text{Half-chord of } \alpha \text{ is half the chord of } 2\alpha = \tfrac{1}{2}\,\text{chord}(2\alpha)}$

Review

$\underset{\text{radians}}{\alpha} \longrightarrow \underset{\text{in degrees}}{\dfrac{\alpha \cdot 360}{2\pi}} \longrightarrow \underset{\text{in minutes}}{\alpha \cdot \dfrac{360 \cdot 60}{2\pi}}$

$\dfrac{360 \cdot 60}{2\pi} \sim 3438$

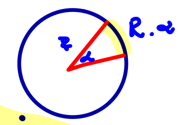
$\pi \sim 3.141$

α angle

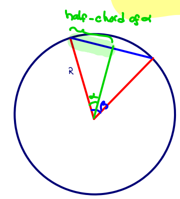
$\underset{\text{in minutes}}{\alpha} \longrightarrow \alpha \cdot \dfrac{2\pi}{360 \cdot 60}$ in rad.

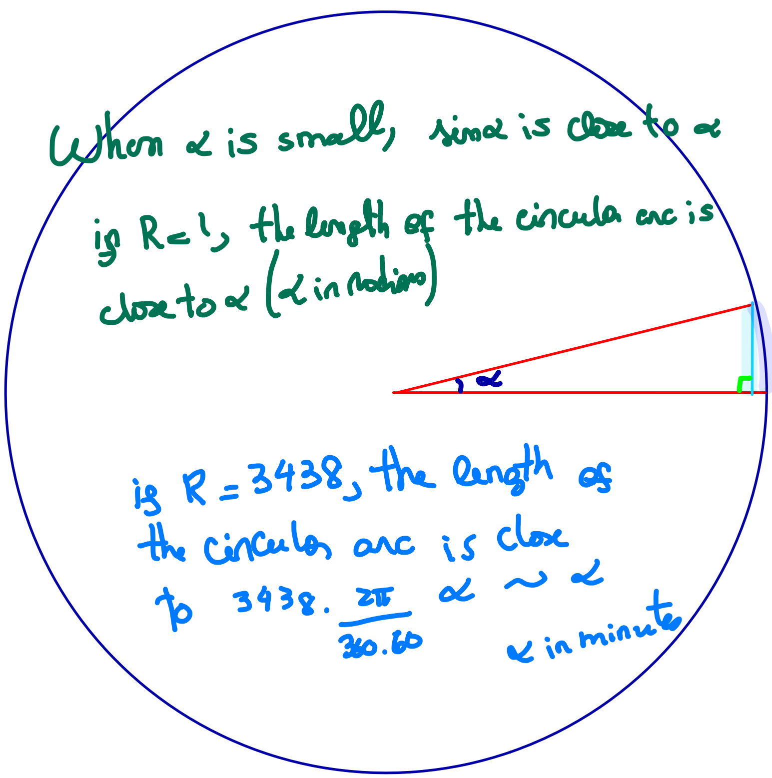
On a circle of radius R, the length of a circular arc of angle α (in radians) is $R \cdot \underset{\text{in rad.}}{\alpha}$

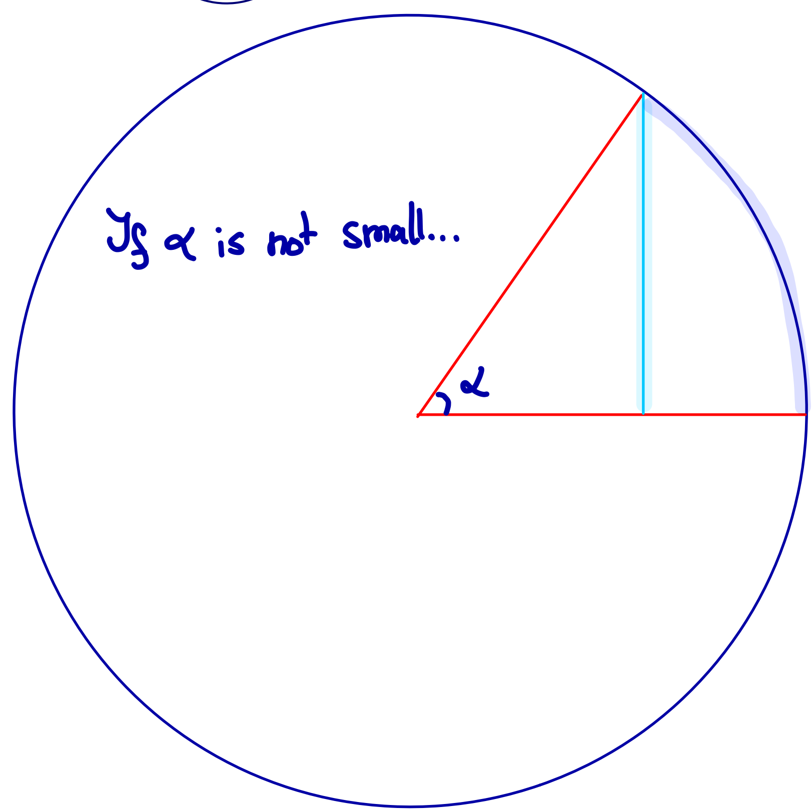
$\sim R \dfrac{2\pi}{360 \cdot 60} \cdot \alpha$ if α in minute



The half chord of an angle α is $\frac{1}{2}\,\text{chord}(2\alpha) = R \sin\alpha$







Indian mathematicians used half the chord of twice the angle. $\frac{1}{2}\text{chord}(2\alpha)$

Aryabhata wrote "jya" (for "ardha-jya", half-chord)

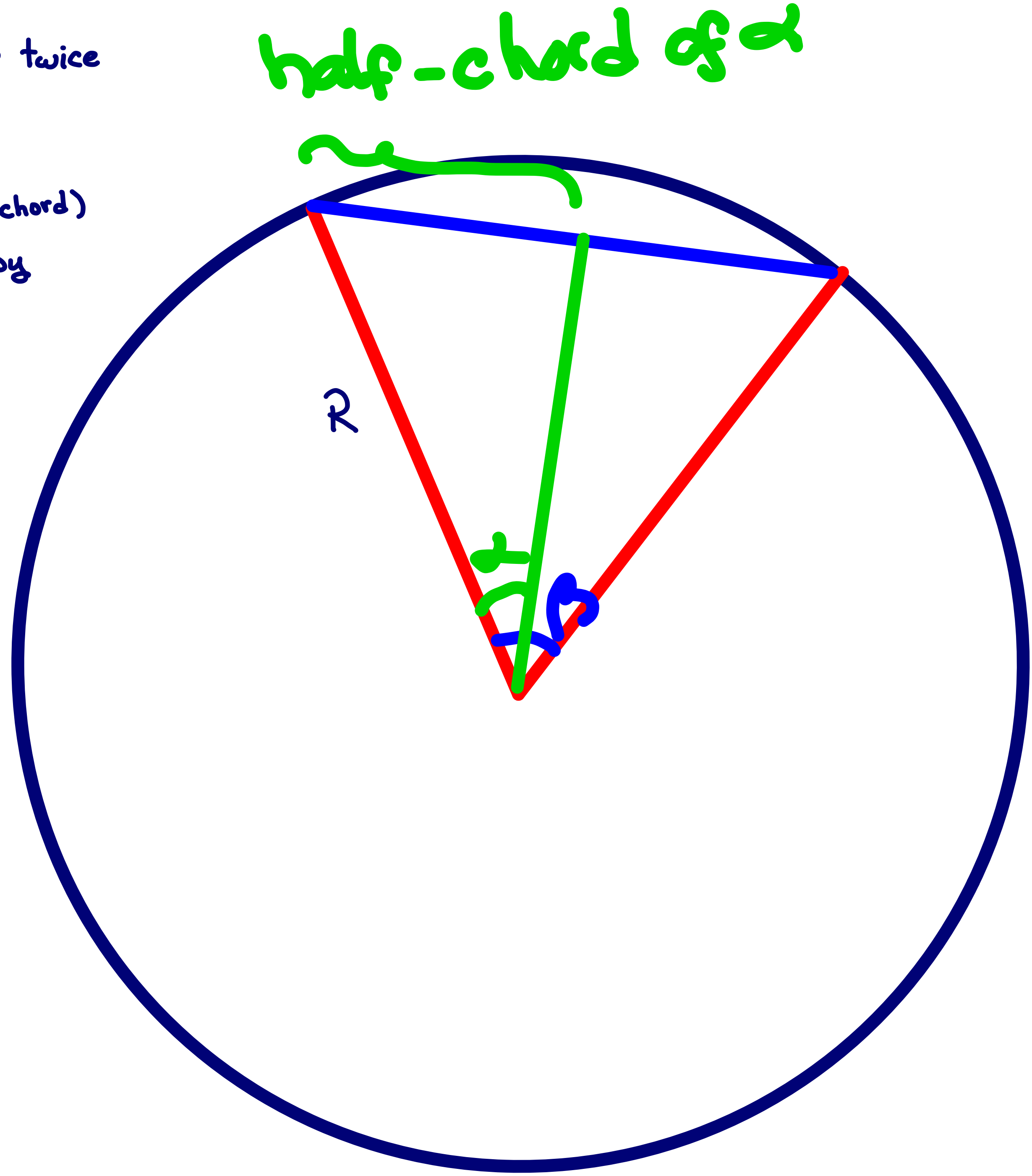
This was phonetically translated to "jiba" by Arab mathematicians, and written "jb".

The closest real Arab word was "jaib", which means "bay." or "breast"

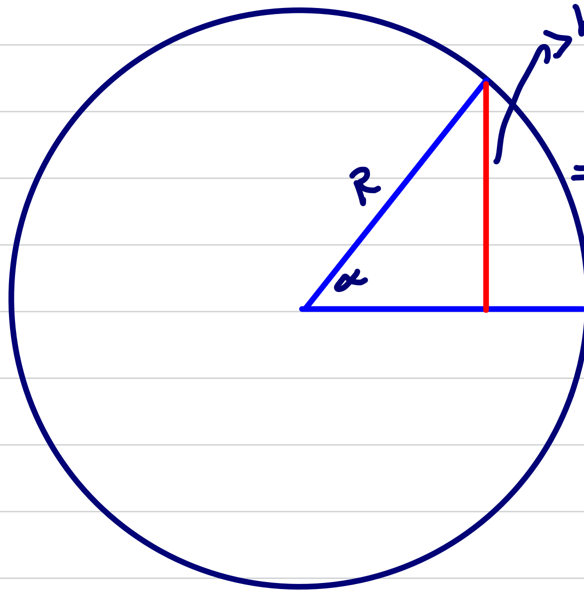
The Latin word for "bay" is...

sinus

$$\sin\alpha = \frac{\frac{1}{2}\text{chord}(2\alpha)}{R}$$



Aryabhata Table of half-chords



half-chord of α $= R\sin\alpha = \frac{1}{2}\,\text{chord}(2\alpha)$

For us, $\sin\alpha$ is the ratio, $\dfrac{\text{half chord of } \alpha}{R}$

For Indian mathematicians, it was the length of a segment, $R.\sin\alpha$

half chord
- $R.\sin\alpha$
- $R\,\alpha$

For small angles, (expressed in minutes) $R\sin\alpha$ is approx $R.\dfrac{\pi}{180\cdot 60}\cdot\alpha$ (Note α in radians is $\dfrac{\pi}{180\cdot 60}\cdot\alpha_{min}$)

If one wants to make $R\sin\alpha$ close to α for small values of α, one needs $R = \dfrac{180\cdot 60}{\pi}$

Now, Aryabhatta wrote that 3.1416 is a good approximation of π. $R = \dfrac{180\cdot 60}{3.1416} = 3437'\,44''\ldots$

rounded up to 3438.

$$S_{n+1} - S_n = S_1 - \frac{1}{225}(S_1 + \cdots + S_n)$$

$$\sin(2\alpha) = 1 - 2\sin^2\alpha$$

makhi-bhakhi-phakhi-dhakhi-ṇakhi-ñakhi-ṅakhi-hasjha-skaki-kiṣga-śghaki-kighva,
ghlaki-kigra-hakya-dhaki-kica-sga-jhaśa-ṅva-kla-pta-pha-cha-kalārdhajyāḥ.

THE SANSKRIT ALPHABET

संस्कृतवर्णमाला SAÑSKṚTA-VARṆA-MĀLĀ

vowels (*svara*)

simple – short & long:

अ A	आ Ā	इ I	ई Ī	उ U	ऊ Ū	ऋ Ṛ	ॠ Ṝ
guttural		*palatal*		*labial*		*cerebral (retroflex)*	

diphthongs – long:				*anusvāra*	*visarga*		
ए E	ऐ AI	ओ O	औ AU	अं AṀ	अः AḤ	ऌ Ḷ	ॡ Ḹ
						dental	

consonants (*vyañjana*)

mutes or stops (*sparsha*)

class – location	*hard (non-voiced)* *simple*	*aspirate*	*soft (voiced)* *simple*	*aspirate*	*nasal*
gutturals – throat	क KA	ख KHA	ग GA	घ GHA	ङ ṄA
palatals – middle of mouth	च ĆA	छ ĆHA	ज JA	झ JHA	ञ ÑA
cerebrals – roof of mouth	ट ṬA	ठ ṬHA	ड ḌA	ढ ḌHA	ण ṆA
dentals – teeth	त TA	थ THA	द DA	ध DHA	न NA
labials – lips	प PA	फ PHA	ब BA	भ BHA	म MA

semi-vowels (*antastha*) – soft

य YA	र RA	ल LA	व VA
palatal	*cerebral*	*dental*	*labio-dental*

sibilants – *hard* & pure aspirate – *soft* (*ūṣhman*)

श ŚA	ष ṢA	स SA	ह HA
palatal	*cerebral*	*dental*	*guttural*

special conjunct consonants क्ष KṢA त्र TRA ज्ञ JÑA

घ्लकि किग्र हक्य धकि किच स्ग झश ङ्व क्ल प्त फ छ कलार्धज्याः ॥

Worksheet 3

makhi-bhakhi-phakhi-dhakhi-ṇakhi-ñakhi-ṅakhi-hasjha-skaki-kiṣga-śghaki-kighva, ghlaki-kigra-hakya-dhaki-kica-sga-jhaśa-ṅva-kla-pta-pha-cha-kalārdhajyāḥ.

1	Word from verse				Number from verse
2					
3	makhi	khi=kha x i=200	ma=25		225
4	bhakhi	khi=kha x i=200	bha=24		224
5	phakhi				
6	dhakhi	khi=kha x i=200	dha=19		219
7	.nakhi	khi=kha x i=200	.na=15		215
8	~nakhi				
9	"nakhi	khi=kha x i=200	"na=5		205
10	hasjha	ha=100	sa=90	jha=9	199
11	skaki	ki=100	sa=90	ka=1	191
12	ki.sga	ki=100	.sa=80	ga=3	183
13	"sghaki				
14	kighva	ki=100	va=60	gha=4	164
15	ghlaki	ki=100	gha=4	la=50	154
16	kigra				
17	hakya	ha=100	ya=30	ka=1	131
18	dhaki	dha=19	ki=ka x i=100		119
19	kica				
20	sga	sa=90 g	a=3		93
21	"sjha	"sa=70	jha=9		79
22	"nva	"na=5	va=60		65
23	kla	ka=1	la=50		51
24	pta	pa=21	ta=16		37
25	pha				
26	cha	cha=7			7

Classified consonants

ka	1
kha	2
ga	3
gha	4
“na	5
ca	6
cha	7
ja	8
jha	9
~na	10
.ta	11
.tha	12
.da	13
.dha	14
.na	15
ta	16
tha	17
da	18
dha	19
na	20
pa	21
pha	22
ba	23
aha	24
ma	25

Unclassified consonants

ya	30
ra	40
la	50
va	60
“sa	70
.sa	80
sa	90
ha	100

Vowels

a	1
i	100
u	100^2
.r	100^3
.l	100^4
e	100^5
ai	100^6
o	100^7
au	100^8

Word from verse	Number from verse
	0
makhi $s_1 = d_1$	225
bhakhi $= d_2$	224
phakhi	222
dhakhi	219
.nakhi	215
~nakhi	210
"nakhi	205
hasjha	199
skaki	191
ki.sga	183
"sghaki	174
kighva	164
ghlaki	154
kigra	143
hakya	131
dhaki	119
kica	106
sga	93
"sjha	79
"nva	65
kla	51
pta	37
pha	22
cha	7

घ्लकि किग्र हक्य धकि किच स्ग झश ङ्व क्ल प्त फ छ कलार्धज्याः ॥

makhi-bhakhi-phakhi-dhakhi-ṇakhi-ñakhi-ṅakhi-hasjha-skaki-kiṣga-śghaki-kighva, ghlaki-kigra-hakya-dhaki-kica-sga-jhaśa-ṅva-kla-pta-pha-cha-kalārdhajyāḥ.

$\alpha = 225'$

Stanza I, 10. The twenty-four sine [differences] reckoned in minutes of arc are 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, 7.

2α

3α

HW Problem 1

Find each of the levels.

(Example :

0 - number from verse.)

Find 1, 2, 3, 4, 5

Words from verse	0	1	2	3	4	5
	0	0	0	0.00000	0.00000	0.0000000
makhi	225	225	225	0.06545	0.06540	-0.0000419
bhakhi	224	450	449	0.13060	0.13053	-0.0000730
phakhi	222	675	671	0.19517	0.19509	-0.0000813
dhakhi	219	900	890	0.25887	0.25882	-0.0000524
.nakhi	215	1125	1105	0.32141	0.32144	0.0000317
~nakhi	210	1350	1315	0.38249	0.38268	0.0001936
"nakhi	205	1575	1520	0.44212	0.44229	0.0001712
hasjha	199	1800	1719	0.50000	0.50000	0.0000000
skaki	191	2025	1910	0.55556	0.55557	0.0000147
ki.sga	183	2250	2093	0.60878	0.60876	-0.0000227
"sghaki	174	2475	2267	0.65939	0.65935	-0.0000492
kighva	164	2700	2431	0.70710	0.70711	0.0000096
ghlaki	154	2925	2585	0.75189	0.75184	-0.0000508
kigra	143	3150	2728	0.79348	0.79335	-0.0001312
hakya	131	3375	2859	0.83159	0.83147	-0.0001185
dhaki	119	3600	2978	0.86620	0.86603	-0.0001759
kica	106	3825	3084	0.89703	0.89687	-0.0001604
sga	93	4050	3177	0.92408	0.92388	-0.0002042
"sjha	79	4275	3256	0.94706	0.94693	-0.0001321
"nva	65	4500	3321	0.96597	0.96593	-0.0000428
kla	51	4725	3372	0.98080	0.98079	-0.0000175
pta	37	4950	3409	0.99156	0.99144	-0.0001200
pha	22	5175	3431	0.99796	0.99786	-0.0001050
cha	7	5400	3438	1.00000	1.00000	0.0000000

Three interpretations of

प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम् ।
तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि ॥ १२ ॥

[2] K. S. SHUKLA: Āryabhaṭīya of Āryabhaṭa, Critically Edited with Introduction, English Translation, Commentary and Indexes, Indian National Science Academy, New Delhi 1976.

The above translation is based on Prabhākara's interpretation of the text. The same interpretation is given by the commentators Someśvara, Sūryadeva (b. 1191 A. D.), Yallaya (1480 A. D.) and Raghunātha-rāja (1597 A.D.). It is interesting to note that this interpre-

1

The first Rsine divided by itself and then diminished by the quotient gives the second Rsine-difference. The same first Rsine diminished by the quotients obtained by dividing each of the preceding Rsines by the first Rsine gives the remaining Rsine-differences.

Datta and Singh, following the commentator Parameśvara (1431 A.D.), have translated the text as follows :

"The first Rsine divided by itself and then diminished by the quotient will give the second difference. For computing any other difference, (the sum of) all the preceding differences is divided by the first Rsine and the quotient is subtracted from the preceding difference. Thus, all the remaining differences (can be calculated)."[1]

2

The commentator Nīlakaṇṭha (c. 1500 A.D.) interprets the text as follows :

"The first Rsine divided by itself and then diminished by the quotient gives the second Rsine-difference. To obtain any other Rsine-difference, divide the preceding Rsine by the first Rsine and multiply the quotient by the difference between the first and second Rsine-differences and subtract the resulting product from the preceding Rsine-difference."

3

$S_n = R\sin(n\alpha)$ $\alpha = 225'$ $n = 1, 2, \dots 24$

$R = 3438$

$d_n = S_n - S_{n-1}$ $n \geq 2$

1

The above translation is based on Prabhākara's interpretation of the text. The same interpretation is given by the commentators Someśvara, Sūryadeva (b. 1191 A. D.), Yallaya (1480 A. D.) and Raghunātha-rāja (1597 A.D.). It is interesting to note that this interpre-

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2

"The first Rsine divided by itself and then diminished by the quotient will give the second difference. For computing any other difference, (the sum of) all the preceding differences is divided by the first Rsine and the quotient is subtracted from the preceding difference. Thus, all the remaining differences (can be calculated)."[1]

$$d_2 = S_1 - \frac{S_1}{S_1}$$

$$d_{n+1} = S_1 - \frac{S_1 + S_2 + \dots + S_n}{S_1} \quad n \geq 2$$

The commentator Nīlakaṇṭha (c. 1500 A.D.) interprets the text as follows :

$R.\sin(n\alpha)$
$n=1$

3

"The first Rsine divided by itself and then diminished by the quotient gives the second Rsine-difference. To obtain any other Rsine-difference, divide the preceding Rsine by the first Rsine and multiply the quotient by the difference between the first and second Rsine-differences and subtract the resulting product from the preceding Rsine-difference."

d_2

$S_1 = 225$

$$d_2 = S_1 - \frac{S_1}{S_1}$$

$$d_{n+1} = d_n - \frac{S_n}{S_1}(d_1 - d_2)$$

(Note $d_1 - d_2 = 2R\sin\alpha\,(1 - \cos\alpha)$)

Apparently, Aryabhata did not use his own rule! It is conjectured that he copied the values, probably from Ptolomy.

HW Problem 2

$S_i = R\sin(i\alpha)$

$d_n = S_n - S_{(n-1)}$

$d_1 = \alpha$

Prove that $d_{n+1} = \frac{S_n}{S_1} \cdot 4\sin^2(\alpha/2)$, $n \geq 3$

QED RD

Extra credit

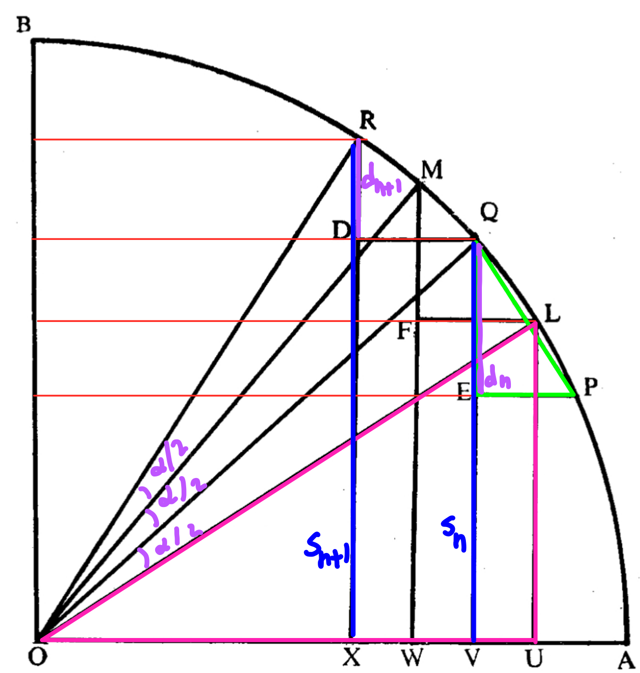
With the notation of HW problem 3, prove that

$$d_n - d_{n+1} = \frac{S_n}{S_1}(d_1 - d_2)$$

Hint

ΔQEP and ΔOUL are similar

ΔRDQ and ΔOWM are similar



HW Problem 3

Why did Aryabha use a circle of radius 3438?

Philosophy is written in that great book which ever lies before our eyes – I mean the universe – but we cannot understand it if we do not first learn the language and grasp the symbols, in which it is written.

This book is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth.

Galileo Galilei

Galileo

Square root extraction

Multiplication and square root extraction. This next excerpt from f.70r shows the multiplication $1111\frac{1}{30} \times 500$ and the extraction of the square root of that product. The multiplication is done as above, with $500 \times \frac{1}{30}$ reduced to $50 \times \frac{1}{3}$ and approximated as 17, giving 555517 as the product.

The square root algorithm shows a fundamental improvement of the *a danda*-like method explained by Cataneo (and by Fibonacci and Pacioli before him). The earlier authors understood that the odd and even-placed digits of the radicand had to be treated differently, and that each *pair* of radicand digits contributes a single digit to the root. Nevertheless in the operation the radicand digits were brought ("given") down one by one. This resulted in a convoluted explanation like those shown above in italics in Cataneo's root extraction. To analyze the difference anachronistically, let us look at the spot in Cataneo's operation where he has just brought down the 5. At that moment the current root is 23, and the next digit of the radicand is 6. Cataneo has *Now you need to find a number which multiplied by the double of the root you have found, i.e. the double of* 23 *which is* 46, *that product can be subtracted from* 185 *and from the remainder joined with the* 6, *the next digit of* 54756, *can be subtracted the product of that number [with itself] with remainder not larger than the double of the root you will have found.* Suppose we call "that number" x. Cataneo's sentence translates to the three inequalities

$$0 \leq 185 - 46x$$

$$0 \leq ((185 - 46x) \cdot 10 + 6) - x^2$$

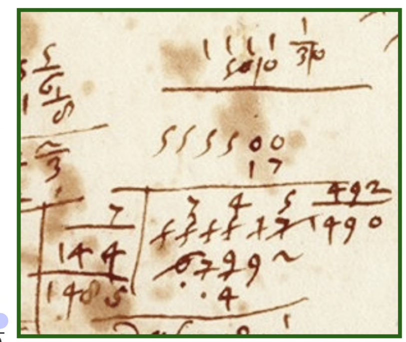
$$((185 - 46x) \cdot 10 + 6) - x^2 \leq 2 \cdot (230 + x)$$

which must all be satisfied by x. But if the 5 and the 6 are brought down *together* the problem simplifies to

$$0 \leq 1856 - (460 + x)x$$

$$x \text{ is the largest such number.}$$

I don't know if Galileo was taught this improvement or figured it out himself. It appears, slightly mangled, in the work of Georg von Peuerbach (1423-1461). In Galileo's operation the digits are brought down two by two to form a new partial radicand, and the next digit of the root will be the largest single digit which, when appended to double the current root, and then used to multiply that adjunction, is smaller than that partial radicand. This is the way square root extraction was still taught in elementary school halfway through the 20th century. Galileo continues the condensed notation from long division (multiplications and subtractions preformed mentally, and only essential digits recorded) resulting in a *galera*-like picture. If the operation is written as below with the digits suppressed by Galileo reinserted small, it could have been the work of a 20th-century schoolchild. The schoolchild might have continued the calculation by inserting a decimal point and bringing down pairs of zeros to get $745.33012\ldots$; Galileo uses the "remainder over twice the root" fractional approximation: $\frac{492}{1490}$ which in decimals would be $.330201\ldots$.



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$(a+b)^2 = a^2 + b^2 + 2ab$

7	. 7 4 5
	5 5 5 5 1 7 $\frac{492}{1490}$
. 7	4 9
$(7 \times 2 \times 10 + 4) \times 4$	6 5 5 2×745
74 1 4 4 ×4	5 7 6
$(74 \times 2 \times 10 + 5) \times 5$	7 9 1 7
1 4 8 5 ×5	7 4 2 5
	4 9 2

$745 + \frac{492}{1490} \quad (2 \cdot 745 = 1490)$

FEATURE COLUMN Monthly essays on mathematical topics

Galileo's Arithmetic

The manuscripts allow us very unusual, if not unique, access to the private calculations of a great scientist; it is as if we could look over his shoulder and watch him at work...

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Stony Brook University
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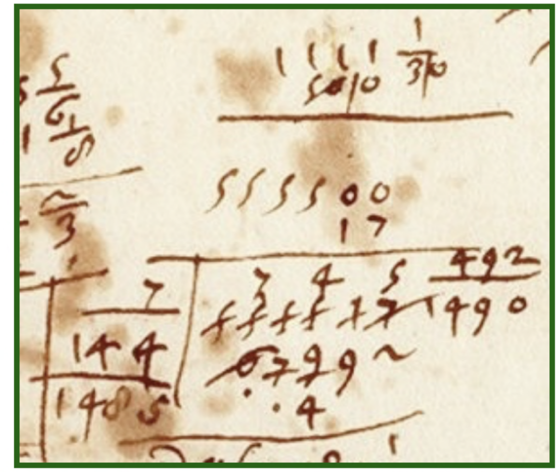
7

74

$$\begin{array}{c|cccccc|c}
 & 7 & & 4 & & 5 & & \\
 & 5 & 5 & 5 & 5 & 1 & 7 & \frac{492}{1490} \\
7 & 4 & 9 & & & & & \\
(7\times 2\times 10+4)\times 4 & & 6 & 5 & 5 & & & 2\times 745 \\
1\;4\;4\;\times 4 & & 5 & 7 & 6 & & & \\
(74\times 2\times 10+5)\times 5 & & & 7 & 9 & 1 & 7 & \\
1\;4\;8\;5\;\times 5 & & & 7 & 4 & 2 & 5 & \\
 & & & & 4 & 9 & 2 &
\end{array}$$



$$555517 = 745^2 + 492 \rightarrow \text{remainder}$$

$$\sqrt{555517} \sim 745 + \frac{492}{1490}$$

HW Problem 4 Compute $\sqrt{438749}$ as Galileo.

Extra credit: Galileo approximated the value of $\sqrt{555517}$ by $745 + \frac{492}{1490}$.

Explain why this value $\left(745 + \frac{492}{1490}\right)$ or $745 + \frac{492}{1491}$ are good approximations