Discovered sol'n only told about it to del Fiore

Del Fiore challenged Tartaglia. Tartaglia found sol'n of $x^3+ax=b$ during the challenge

Cardano begged Tartaglia for the sol'n promising silence, and told it to his student, Ludovico Ferrari

Tartaglia told the sol'n of $x^3+ax=b$ in a poem.

Cardano discovered the sol'n of the general cubic
Ferrari found a sol of the quartic, Cdegree 4) using the sol'n of the cubic

Cardano and Ferrari read the sol' of the cubic in del Ferro unpublished papers, and Cardano included it in his book "Ars Magne"

In short, mathematics only exists in a living community of mathematicians that spreads understanding and breaths life into ideas both old and new.

Bill Thurston

The question of who is the first person to ever set foot on some square meter of land is really secondary.

Revolutionary change does matter, but revolutions are few, and they are not self-sustaining --- they depend very heavily on the community of mathematicians.

Quando chel cubo con le coſe appreſſo
Se agguaglia à qualche numero diſcreto
Troman dui altri differenti in eſſo.
Dapoi terrai queſto per conſueto
Che'l lor produtto ſempre ſia eguale
Al terzo cubo delle coſe neto,
El reſiduo poi ſuo generale
Delli lor lati cubi ben ſottratti
Varra la tua coſa principale.
In el ſecondo de cotesti atti
Quando che'l cubo reſtaſſe lui ſolo
Tu oſſerverai queſt'altri contratti,
Del numer farai due tal part'à uolo
Che l'una in l'altra ſi produca ſchietto
El terzo cubo delle coſe in ſtolo
Delle qual poi, per commun precetto
Torrai li lati cubi inſieme gionti
Et cotal ſomma ſara il tuo concetto.
El terzo poi de queſti noſtri conti
Se ſolue col ſecondo ſe ben guardi
Che per natura ſon quaſi congionti.
Queſti trouai, & non con paßi tardi
Nel mille cinquecente, quatro e trenta
Con fondamenti ben ſald'e gagliardi
Nella citta dal mar'intorno centa.

When the cube and the things together
Are equal to some discrete number
Find two other numbers differing in this one.

Then you will keep this as a habit
That their product should always be equal
Exactly to the cube of a third of the things.

The remainder then as a general rule
Of their cube roots subtracted
Will be equal to your principal thing.

In the second of these acts,
When the cube remains alone
You will observe these other agreements:

You will at once divide the number into two parts
So that the one times the other produces clearly
The cube of a third of the things exactly.

Then of these two parts, as a habitual rule,
You will take the cube roots added together,
And this sum will be your thought.

The third of these calculations of ours
Is solved with the second if you take good care,
As in their nature they are almost matched.

These things I found, and not with sluggish steps,
In the year one thousand five hundred, four and thirty
With foundations strong and sturdy

In the city girdled by the sea.

Solve $x^3+6x=20$

$x^3+ax=b$

Recall $\begin{cases} u-v=b \\ u\cdot v=(a/3)^3 \end{cases}$

$\sqrt[3]{u}-\sqrt[3]{v}=x$

$\begin{cases} u-v=b \\ u\cdot v=\left(\frac{a}{3}\right)^3 \end{cases}$

$u-\frac{(a/3)^3}{u}=b$

$u^2-bu-(a/3)^3=0$

$\Delta=b^2+4(a/3)^3$

$u=\frac{1}{2}\left(b+\sqrt{b^2+4(\frac{a}{3})^3}\right)=\frac{b}{2}+\sqrt{\left(\frac{b}{2}\right)^2+\left(\frac{a}{3}\right)^3}$

$v=-\frac{b}{2}+\sqrt{\left(\frac{b}{2}\right)^2+\left(\frac{a}{3}\right)^3}$

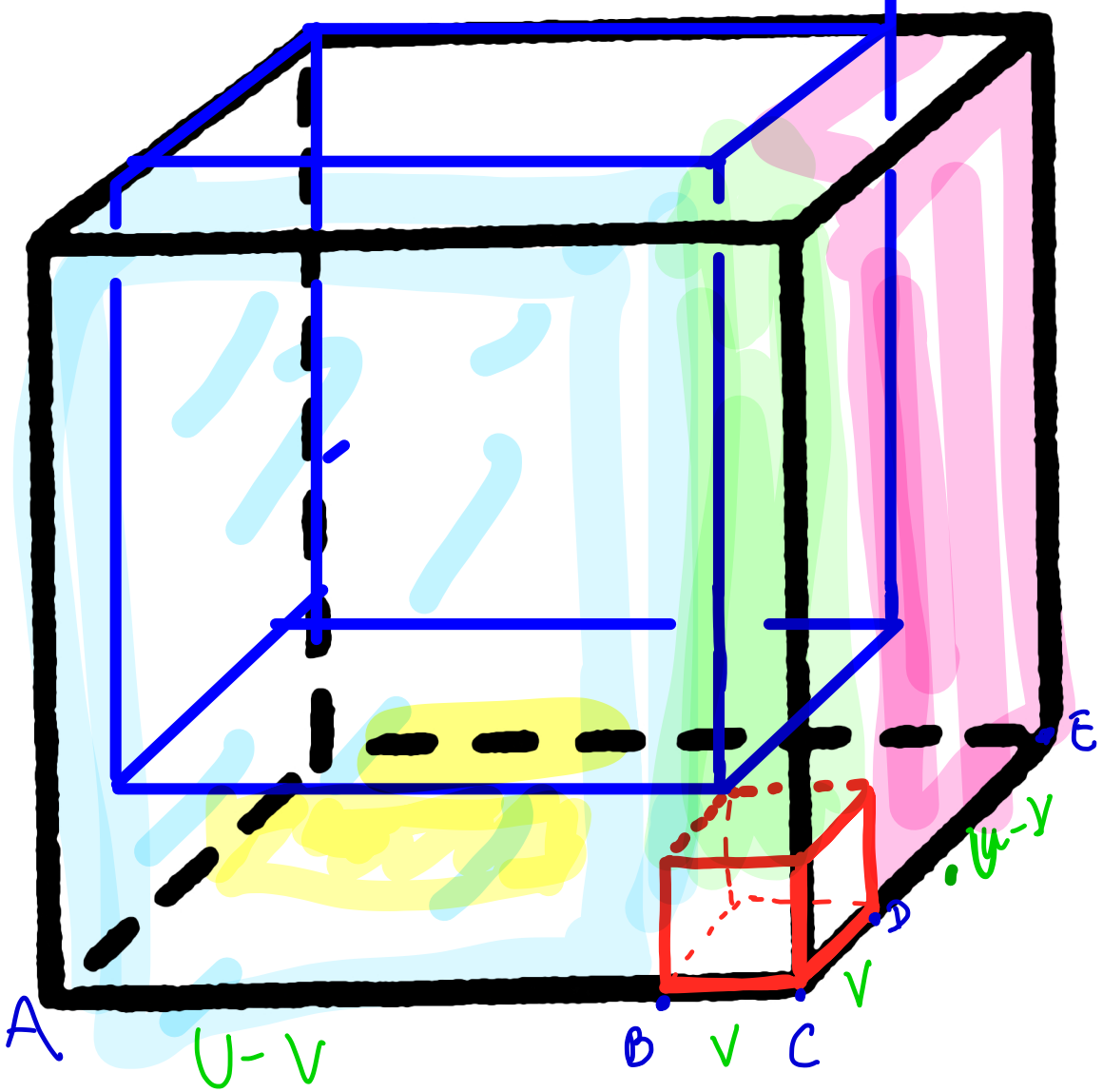
$\sqrt{3}$ is the only pos. number a such that $a^2=3$

(Solve $x^2=3 \rightsquigarrow \pm\sqrt{3}$

$x^3+6x=20$

$x=\sqrt[3]{10+\sqrt{108}}+\sqrt[3]{-10+\sqrt{108}}=2$

How did Tartaglia found his solution? A possible argument.



$U = \sqrt[3]{u}$

$V = \sqrt[3]{v}$

$x = U - V$

$x^3 + ax = b$

$\begin{cases} u - v = b \\ u \cdot v = \left(\frac{a}{3}\right)^3 \end{cases}$

$x = \sqrt[3]{u} - \sqrt[3]{v}$

$U^3 = V^3 + (U-V)^3 + (U-V)^2 \cdot V + V^2 (U-V) + (U-V) UV$

$U^3 - V^3 = x^3 + (U-V) \cdot V (U - V + V + 2U)$

$b = x^3 + x \, 3UV = x^3 + ax$

Descartes: La Geometry (appendix of the Discourse of Method)

- Consistent application of the well developed algebra of the 1600's to the geometrical analysis of the ancients
- He also rejected the request of homogeneity; so x, xy, x^2... were considered as line segments.
  - no worry about dimensionality


GEOMETRIA,
à
RENATO DES CARTES
Anno 1637 Gallicè edita: postea autem
Una cum NOTIS
FLORIMONDI DE BEAVNE,
In Curia Blesensi Consiliarii Regii, Gallicè conscriptis in
Latinum sermonem versis, & Commentariis illustratis,
Operâ atque studio
FRANGISCI à SCHOOTEN,
in Acad. Lugd. Batava Mathesews Professoris.
Quorum omnium Catalogum pagina versa exhibet.
AMSTELODAMI,
Apud Ludovicum & Danielem Elzevirios,
CIↃ IↃC XXX.

- exponents on a variable to indicate power
- x, y, z for unkown quantity
- a, b, c... for known quantities
- He would write $\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$

At the end of XVI Century, algebra begun to look more like today.

Viete worked at the french court as a cryptographer

Perhaps this job inspired him to use letter to stand for numbers

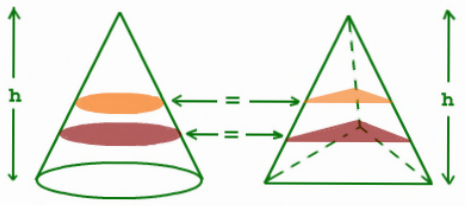
A, E, I, O, U unknow quantities

consonants → know quantity

He would write $ax^2+bx=c$ as

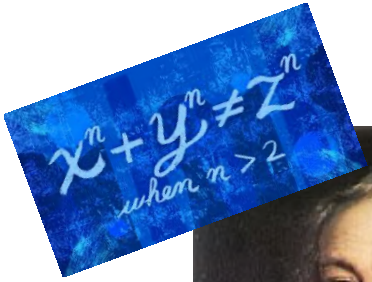
B in A squared + C plane in A eq. D solid

Cavalieri



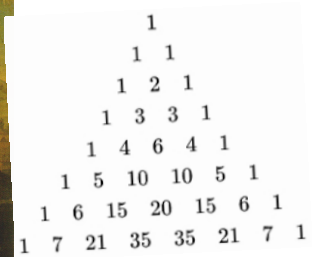
Descartes

Cartesian



Fermat

Pascal



Pierre de Fermat ~1600

French lawyer
Came closer to analytic geometry
Wrote a paper on geometry in Viète's notation

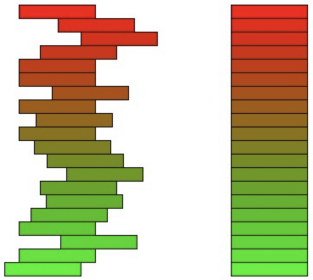
- Fermat introduced "Questions about numbers"

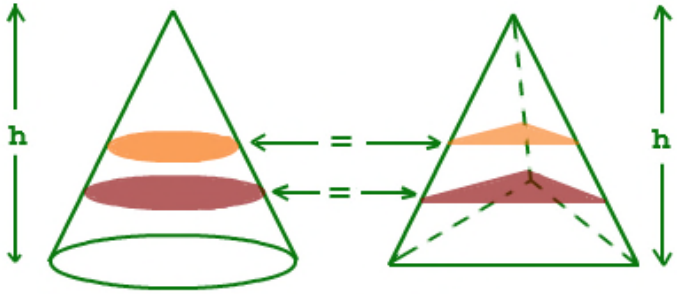
or - can a square be equal to a cube plus one? 3, 2
ie $\exists x, y$ integ $x^2 = y^3 + 1$

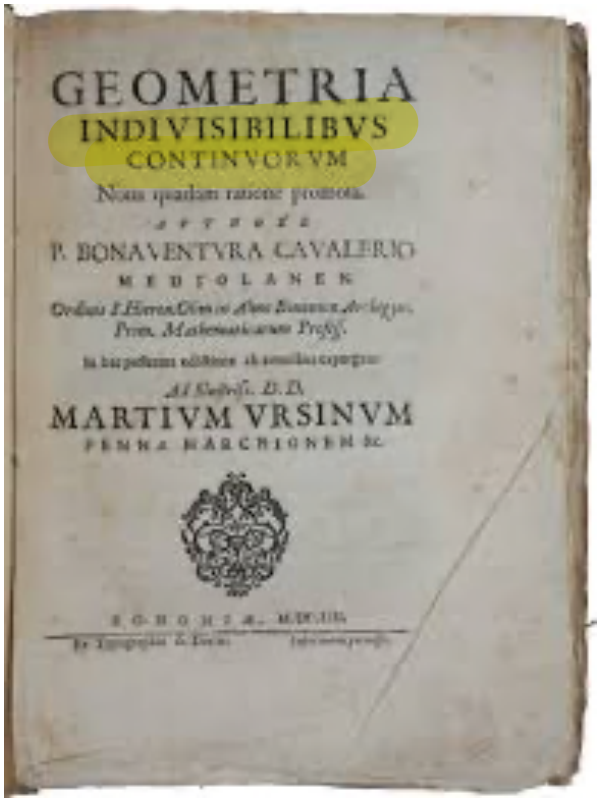
Wrote proofs in letters
to friends.
Mathematicians were not ready
for questions about what cannot be done

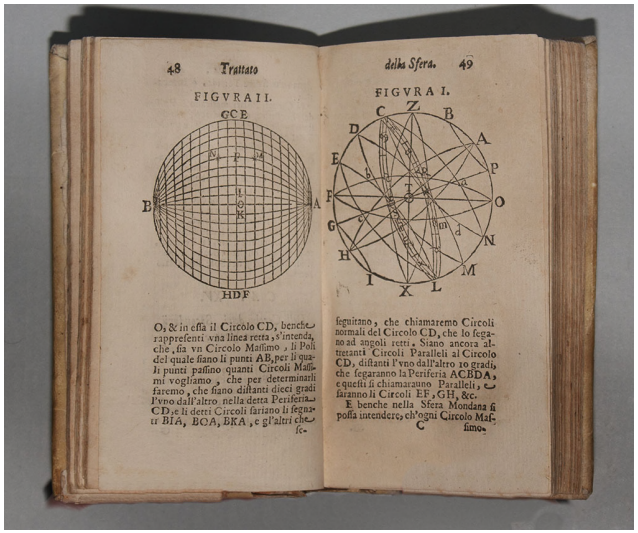
Will

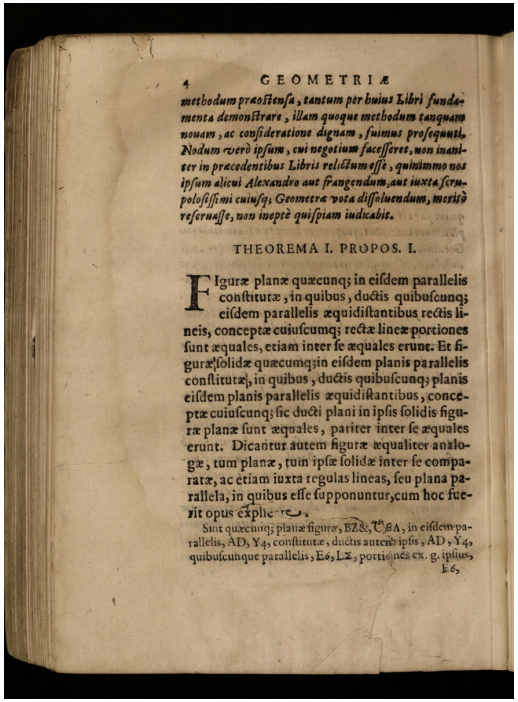
Cavalieri ~ 1600
Idea of "indivisibles"

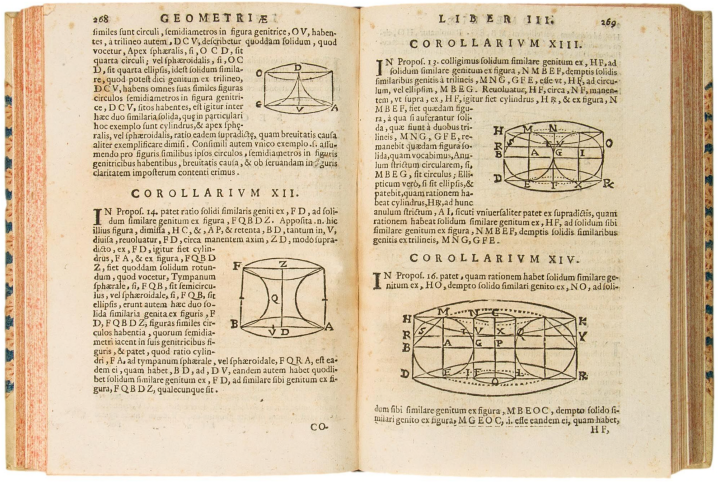




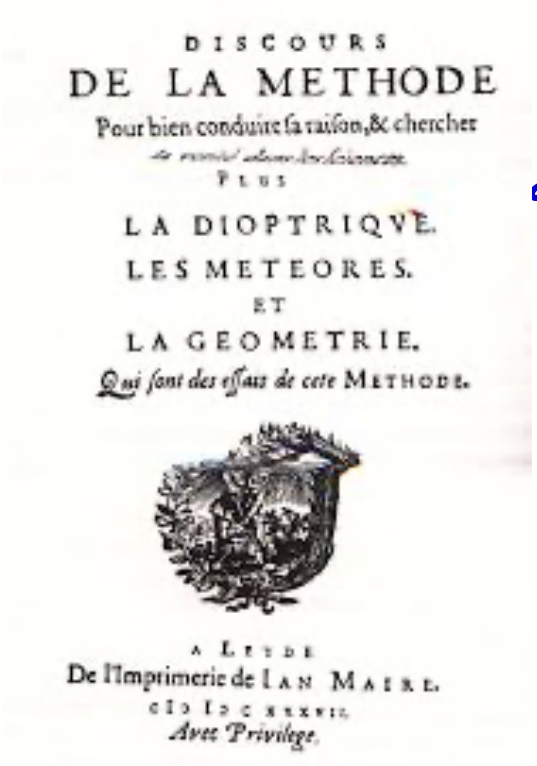

GEOMETRIA
INDIVISIBILIBVS
CONTINVORVM
Nova quadam ratione promota
AVCTORE
P. BONAVENTVRA CAVALERIO
MEDIOLANEN
MARTIVM VRSINVM
PENNA MARCHIONEN &c.
BONONIÆ. M.DC.XXX.


48 Trattato della Sfera. 49
FIGVRA II.
FIGVRA I.
O, & in essa il Circolo CD, benche rappresenti una linea retta, s'intenda, che sia vn Circolo Massimo, li Poli del quale siano li punti AB, per li quali punti passino quanti Circoli Massimi vogliamo, che per determinati faremo, che stiano distanti dieci gradi l'vno dall'altro nella detta Periferia CD: e li detti Circoli saranno li segnati BIA, BCA, BKA, e gl'altri che seguitano, che chiamaremo Circoli normali del Circolo CD, che lo segano ad angoli retti. Siano ancora attenenti Circoli Paralleli al Circolo CD, distanti l'vno dall'altro 10 gradi, che segaranno la Periferia ACBDA, e questi si chiamaranno Paralleli, e saranno li Circoli EF, GH, &c.
E benche nella Sfera Mondana si possa intendere, ch'ogni Circolo Massimo


4 GEOMETRIÆ
methodum præstensa, tantum per huius Libri fundamenta demonstrare, illam quoque methodum tanquam nouam, ac consideratione dignam, fuimus prosequuti. Nodum verò ipsum, cui negotium facesseret, non inuenimus in præcedentibus Libris relictum esse, quinimmo nos ipsum alicui Alexandro aut Frangendum, aut iuxta scrupolosissimi cuiusq; Geometræ vota dissoluendum, meritò reseruasse, non ineptè quispiam indicabit.
THEOREMA I. PROPOS. I.
FIguræ planæ quæcunq; in eisdem parallelis constitutæ, in quibus, ductis quibuscunq; eisdem parallelis æquidistantibus rectis lineis, conceptæ cuiuscumq; rectæ lineæ portiones sunt æquales, etiam inter se æquales erunt. Et figuræ solidæ quæcumq; in eisdem planis parallelis constitutæ, in quibus, ductis quibuscunq; planis eisdem planis parallelis æquidistantibus, conceptæ cuiuscunq; sic ducti plani in ipsis solidis figuræ planæ sunt æquales, pariter inter se æquales erunt. Dicantur autem figuræ æqualiter analogæ, tum planæ, tum ipsæ solidæ inter se comparatæ, ac etiam iuxta regulas lineas, seu plana parallela, in quibus esse supponuntur, cum hoc fuerit opus explic
Sint quæcunq; planæ figuræ, BZ & TA, in eisdem parallelis, AD, Y4, constitutæ, ductis autem ipsis, AD, Y4, quibuscunque parallelis, E6, LX, portiones ex. g. ipsius, E6,


268 GEOMETRIÆ
COROLLARIVM XII.
COROLLARIVM XIII.
COROLLARIVM XIV.
LIBER III. 269
CO-

Descartes



The discourse on reason explains his rationalistic approach to the study of nature

Searched for a general method of thinking in order to
- facilitate inventions
- find the truth in sciences

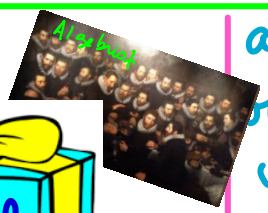
At that time, the sciences with a higher degree of systematic coherence were Astronomy and Mechanics, both "based" in Mathematics

Thus, Math was the most important tool to understand the universe.

Also, Math was itself an example that truth could be found in science.

- Platonists believed on the harmony of the universe.
- Cartesianists believed on a general method based on reason

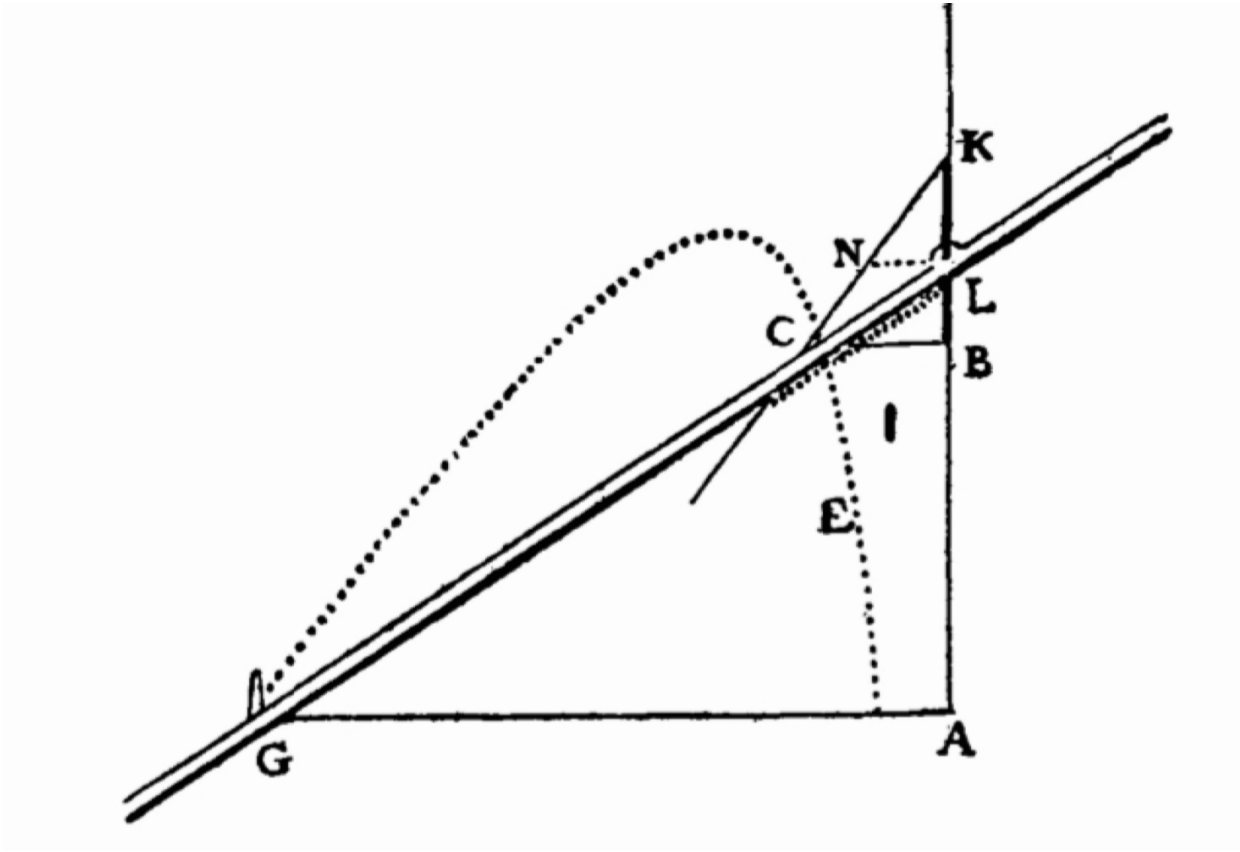
} found in Mathematics



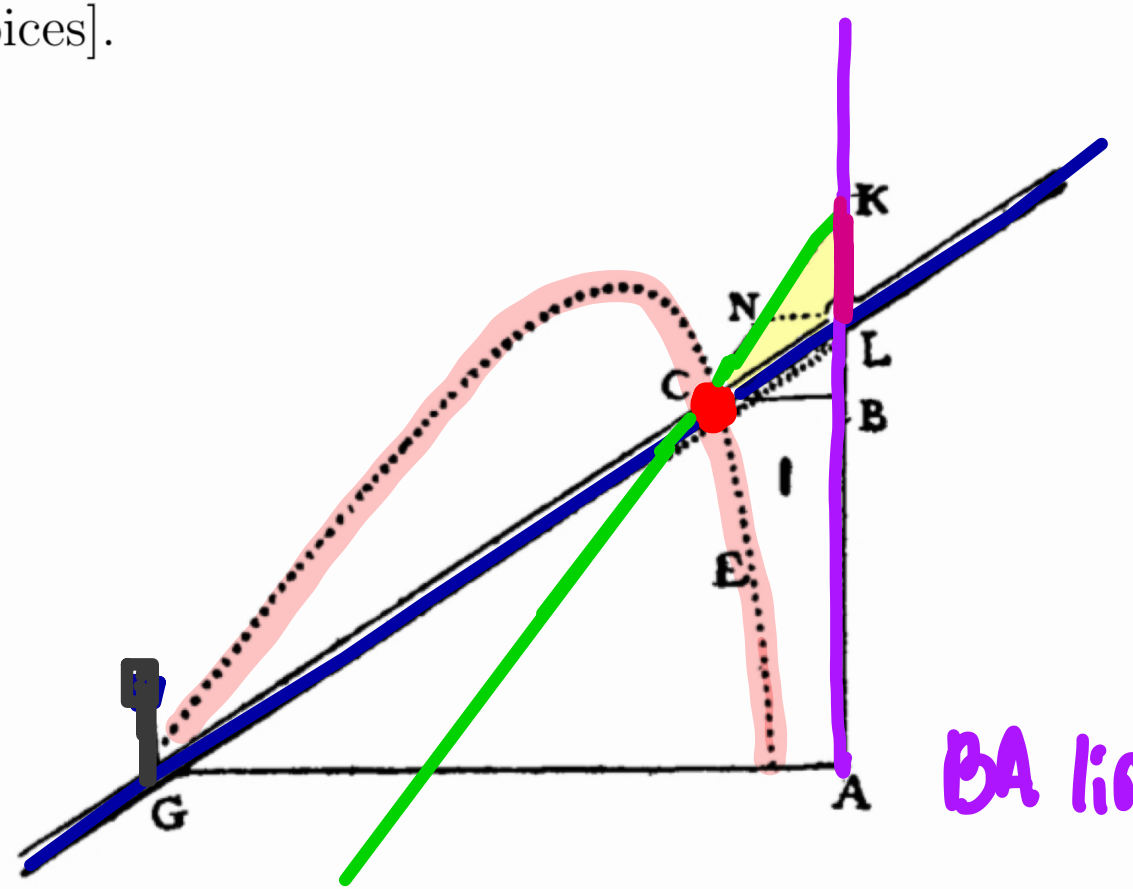
as an application of his general method of rationalistic unification, in this case, unifying algebra and geometry.
→ creating analitic geometry
(Although there are no axes or explit coordinates.

Descartes published "La Geometrie"

Here is Descartes' text, as published in *The Geometry of René Descartes*, David Eugene Smith and Marcia L. Latham, Dover Publications, 1954 (with some minor edits).



"Suppose the curve EC to be described by the intersection of the ruler GL and the rectilinear plane figure CNKL whose side KN is extended indefinitely in the direction of C, and which, being moved in the same plane in such a way that its side KL always coincides with some part of the line BA, imparts to the ruler GL a rotary motion about G (the ruler being hinged to the figure CNKL at L. If I wish to find out to what class this curve belongs, I choose a straight line, as AB, to which to refer all its points, and in AB I choose a point A at which to begin the investigation. [The class of the curve in independent of these choices].



Curves
↓
- circles
- lines

3D → conic sections

BA lin

"Then I take on the curve an arbitrary point, such as C, at which we will suppose the instrument applied to describe the curve. Then I draw through C the line CB parallel to GA. Since CB and BA are unknown and indeterminate quantities, I shall call one of them y and the other x. To the relation between these quantities I must consider also the known quantities which determine the description of the curve, as GA, which I shall call a; KL, which I shall call b, and NL parallel to GA, which I shall call c.

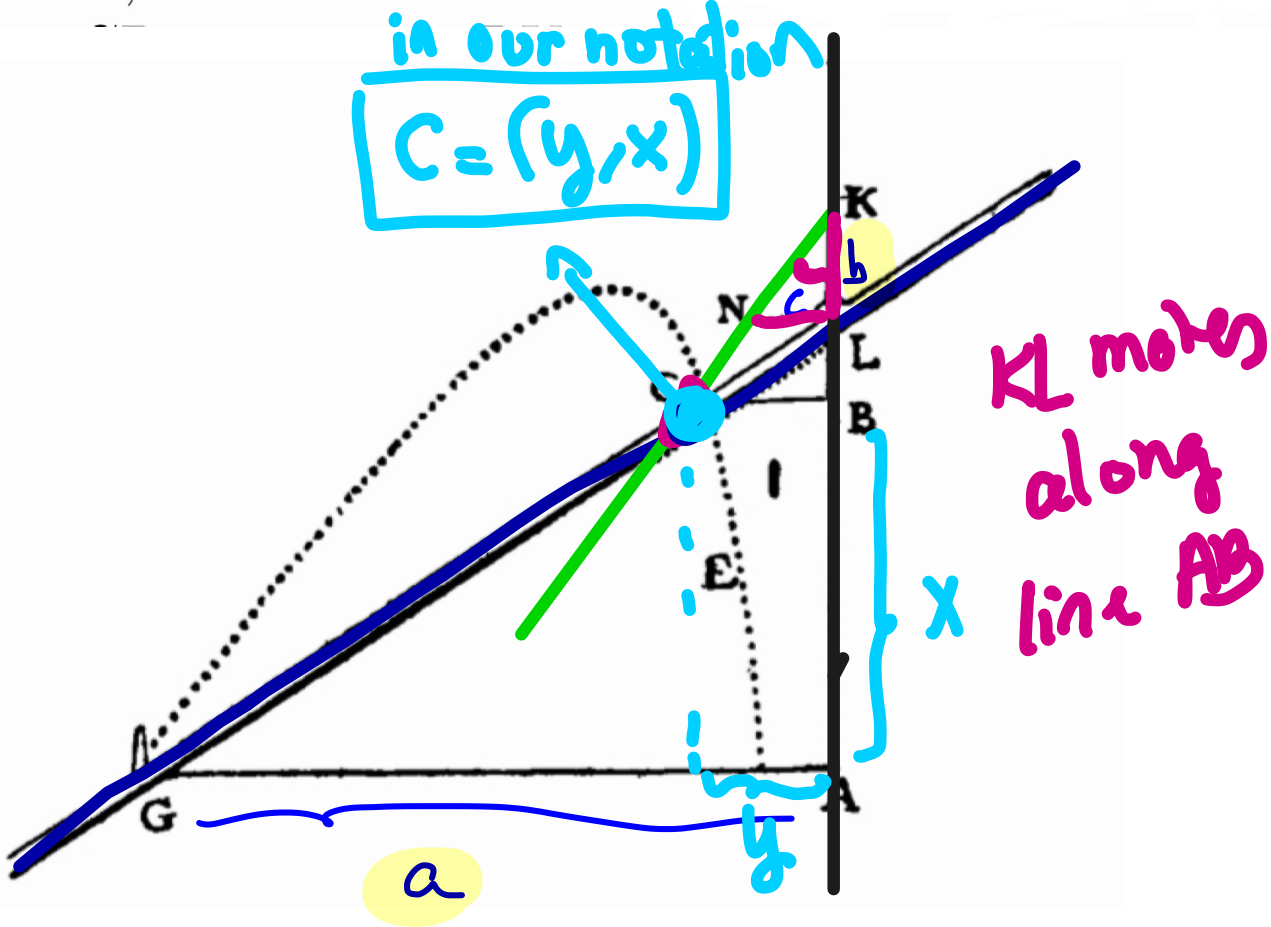
CB = y

BA = x

GA = a

KL = b

NL = c



Then I say that as NL is to LK, or as c is to b, so CB, or y, is to BK, which is therefore equal to $\frac{b}{c}y$. Then BL is equal to $\frac{b}{c}y - b$, and AL is equal to $x + \frac{b}{c}y - b$.

$\triangle NLK \underset{\text{similar}}{\sim} \triangle CBK$

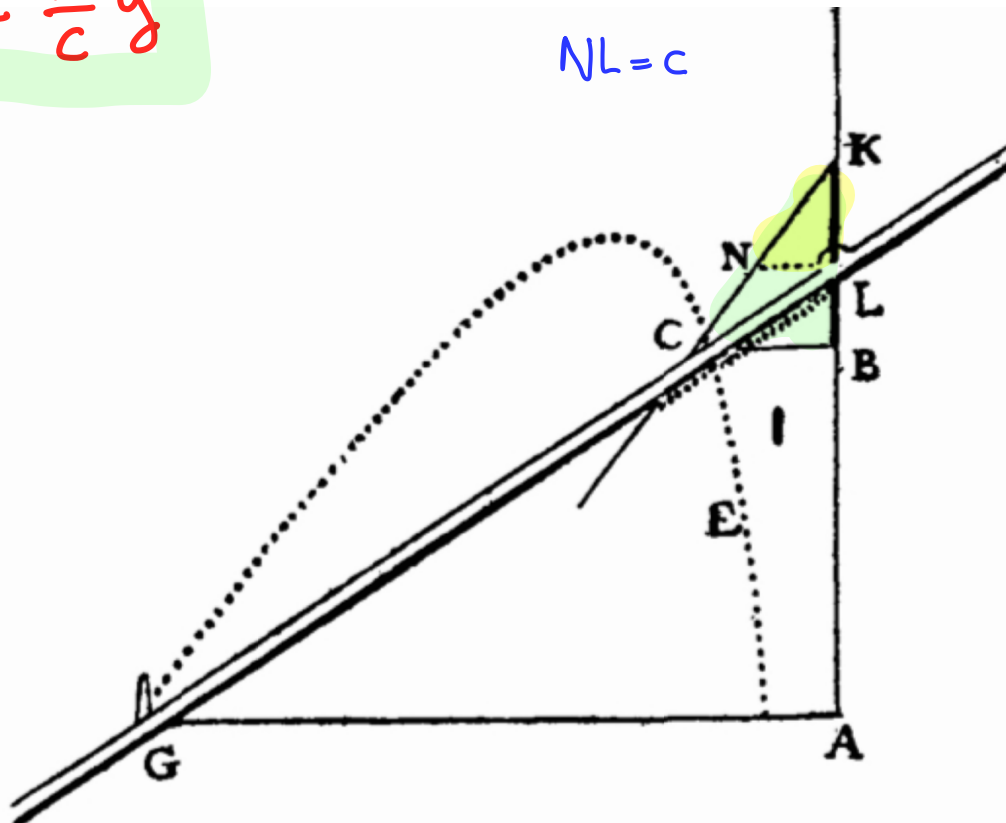
$\frac{NL}{LK} = \frac{CB}{BK} \Rightarrow \frac{c}{b} = \frac{y}{BK}$ $\Rightarrow$ $BK = \frac{b}{c}y$

CB = y
BA = x
GA = a
KL = b
NL = c

Note

$BL = BK - KL = \frac{b}{c}y - b$

$AL = AB + BL$
$= x + \frac{b}{c}y - b$



Moreover,

as CB is to LB, that is, as y is to $\frac{b}{c}y - b$, so AG or a is to LA or $x + \frac{b}{c}y - b$.

Multiplying the second by the third, we get $\frac{ab}{c}y - ab$ equal to $xy + \frac{b}{c}y^2 - by$, which is obtained by multiplying the first by the last. Therefore the required equation is

$$y^2 = cy - \frac{cx}{b}y + ay - ac.$$

From this equation we see that the curve EC belongs to the first class, it being, in fact, a hyperbola."

$\Delta CBL \sim \Delta GAL$ (similar)

$\frac{CB}{BL} = \frac{GA}{AL} \rightarrow \frac{y}{\frac{b}{c}y - b} = \frac{a}{x + \frac{b}{c}y - b}$

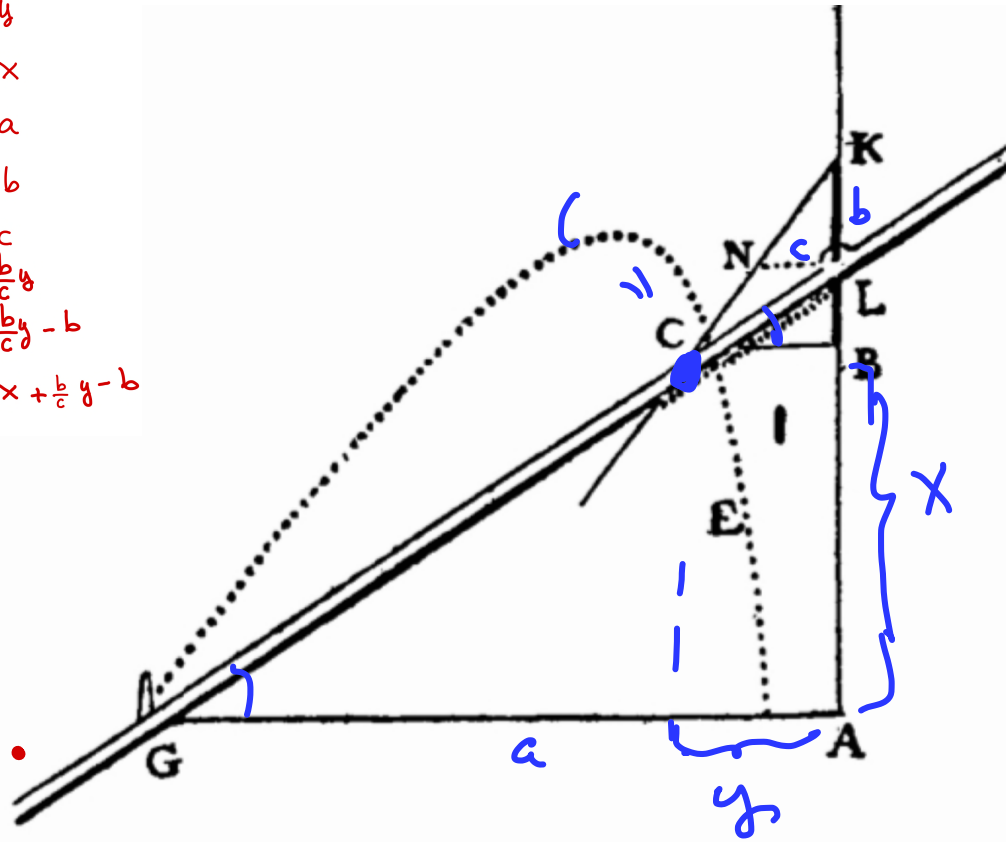
$xy + \frac{b}{c}y^2 - by = a\frac{b}{c}y - ba$

$yy \propto cy - \frac{cx}{b}y + ay - ac.$

$y^2 + (a-c)y + \frac{c}{b}xy + ac = 0$

Recall: Descartes swapped x and y.

CB = y
BA = x
GA = a
KL = b
NL = c
BK = $\frac{b}{c}y$
BL = $\frac{b}{c}y - b$
AL = $x + \frac{b}{c}y - b$



$$y^2 + (a-c)y + \frac{c}{b}xy + ac = 0$$

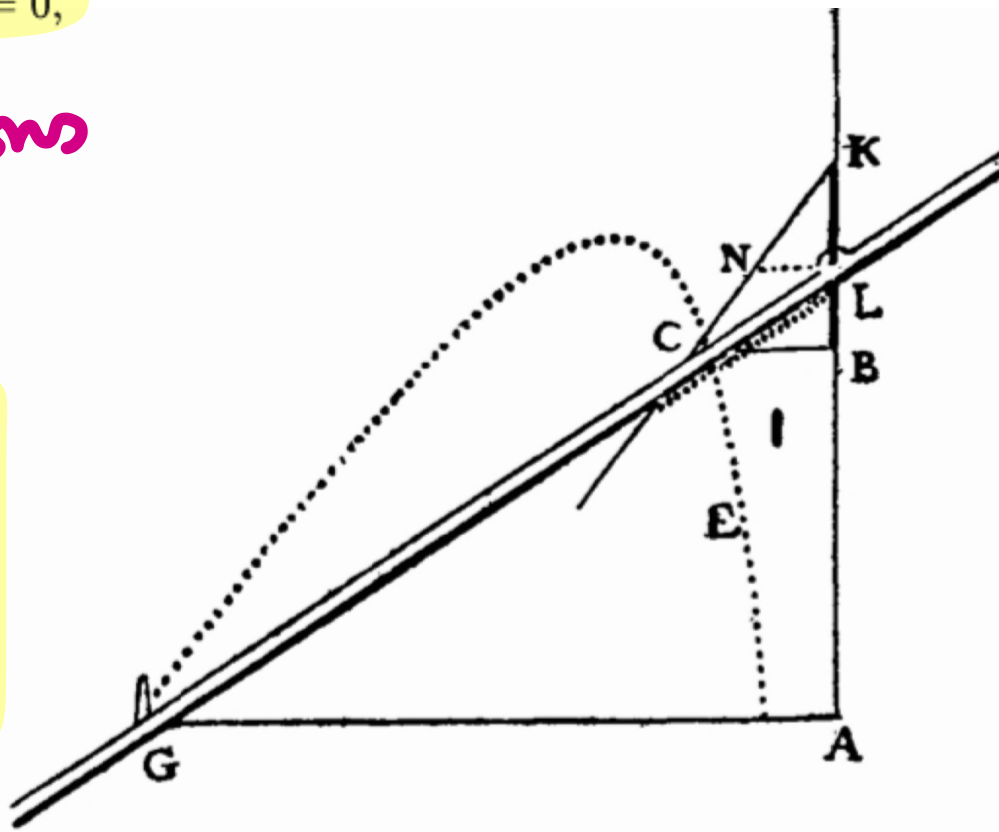
A hyperbola is the set of point in the plane whose difference of distance to two given points (the foci) is constant.

$$A_{xx}x^2 + 2A_{xy}xy + A_{yy}y^2 + 2B_x x + 2B_y y + C = 0,$$

conic sections

$$D := \begin{vmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{vmatrix} < 0.$$

$$D = \det\begin{pmatrix} 0 & c/2b \\ c/2b & 1 \end{pmatrix} = -\left(\frac{c}{2b}\right)^2$$



$$y^2 + (a-c)y + \frac{c}{b}xy + ac = 0$$

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

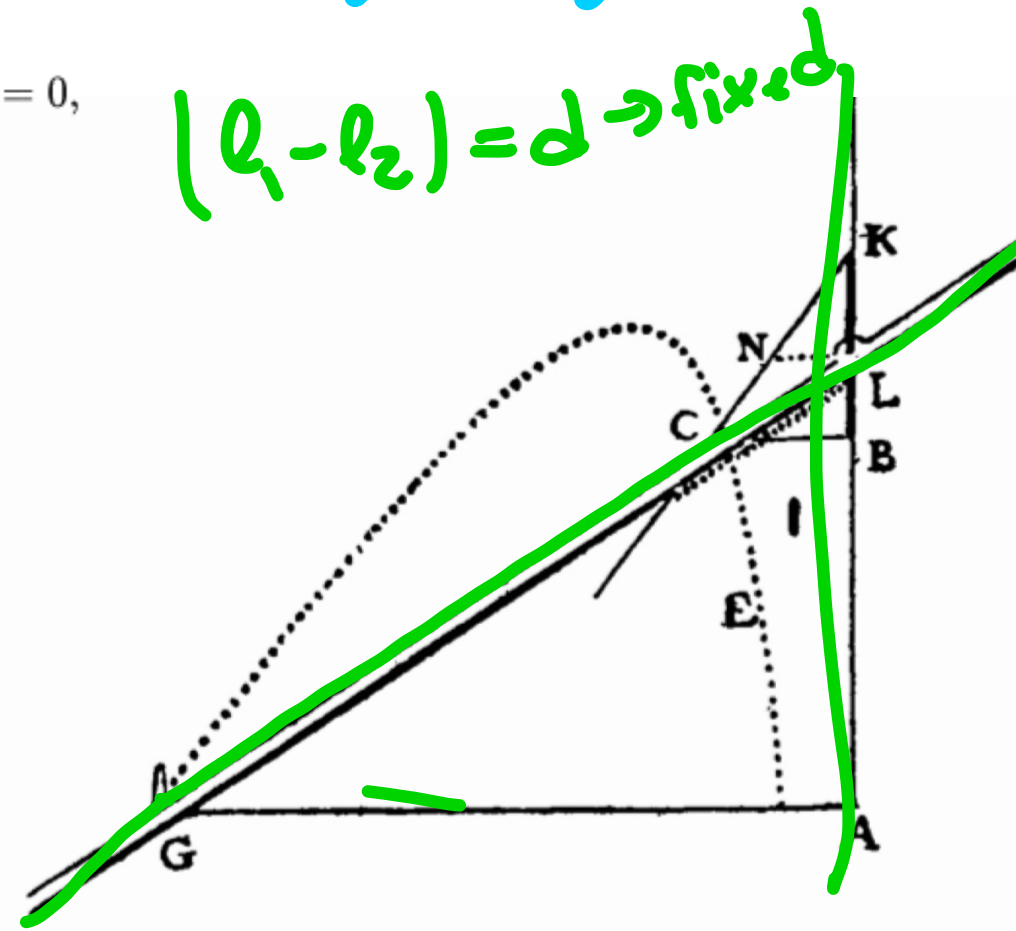
A **hyperbola** is the set of point in the plane whose difference of distance to two given points (the foci) is constant.

$$A_{xx}x^2 + 2A_{xy}xy + A_{yy}y^2 + 2B_x x + 2B_y y + C = 0,$$

$$D := \begin{vmatrix} A_{xx} & A_{xy} \\ A_{xy} & A_{yy} \end{vmatrix} < 0.$$

$$D = \det\begin{pmatrix} 0 & c/2b \\ c/2b & 1 \end{pmatrix} = -\left(\frac{c}{2b}\right)^2$$

$(l_1 - l_2) = d \rightarrow$ fixed



A large part of → consist on a theory of algebraic equations, including the "rule of signs" or "rule of Descartes"

GEOMETRIA,
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AMSTELODAMI,
Apud Ludovicum & Danielem Elzevirios,
cIↃ IↃc LIX.

The goal of the rule of signs is to determine the number of positive or negative roots. (Descartes called them "true" and "false" roots.)

Descartes rule of signs

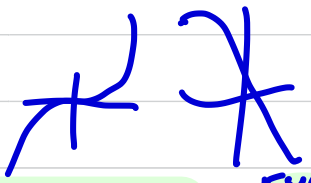
$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \quad a_i \text{ in } \mathbb{R},\ a_n \neq 0$

Set R_+ = number of positive roots of P

$R_+ \leq$ number of sign changes between consecutive coefficients $\neq 0$

R_- = number of negative roots of P

$\leq$ number of positive roots of $P(-x)$



Example $P(x) = 3x^3 + 5x^2 - 10x - 1$ $\rightsquigarrow$ 1 sign change } Exactly 1 positive root

odd deg 3 }

$P(-x) = -3x^3 + 5x^2 + 10x - 1$ $\rightsquigarrow$ 2 sign changes $\Rightarrow$ 0 or 2 pos roots

$\Rightarrow R_+(P(x)) \leq 2$ 0 or 2 neg roots

$R_-(P(x)) \leq 2$

Why there is an even number of neg roots

Example 2 $Q(x) = x^3 - 5$ 1 sign change $\rightarrow$ 1 pos root

$Q(-x) = -x^3 - 5$ no sign changes $\rightsquigarrow$ $Q(-x)$ no pos roots $\rightarrow Q(x)$ no neg roots