- During late 16th and 17th century, a group of people used mathematics to understand the universe
→ called natural phylosophers

Galileo
Astronomy
Physics of moving bodies

Descartes
algebra + geometry
- understand comets, light, etc

Mersenne.
Got scholars from different areas to cooperate

T. Harriot
Applied Math to optics, navegation,..

Kepler
Use of "Greek" conic sections to understand solar system.

Questions (not new!)
- Are time and space infinitely divisible?
- If an object moves in such a way that its velocity changes, how does one understand what is velocity?

Problem (studying motion) modern formulation

Given the formula of the distance a body covers as a function of time $d(t)$, describe velocity and acceleration at a given instant

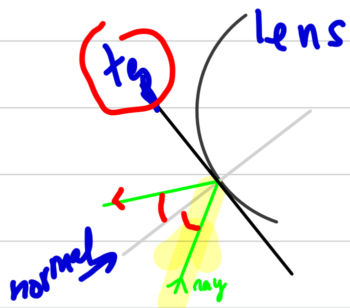
Given $a(t)$ (acceleration) find velocity and distance

Calculus Motivation: 4 types of problems.

1. Given the distance covered by a body as a function of time
   a) find velocity and acceleration at a given instant.
   b) Given the acceleration as a function of time, find velocity and distance travelled.

   Study of motion

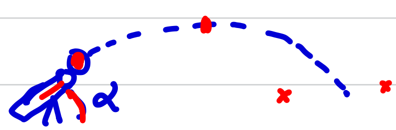
2. Find the tangent to a curve. (important for applications, mainly optics)



for instance, to study the passage of light through a lens, one must know the angle between ray and tangent.

Even the meaning of tangent was open!

3. Finding max or min of a function.
   Application · Find the best angle to shoot a cannon ball from a cannon
   · Motion of planets



4. Finding lengths of curves.

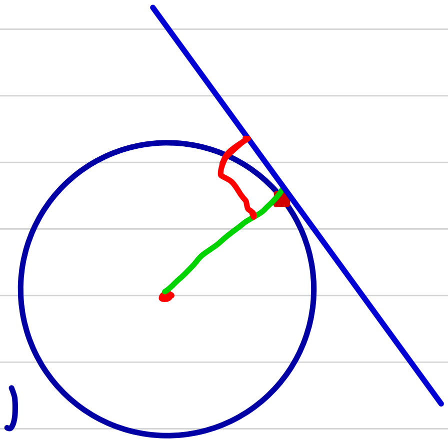
Many instances of these problems were solved in particular cases.
There was a need of a unifying theory.

Problem: Find the tangent line to a curve at a point.

For any curve

What is a tangent?

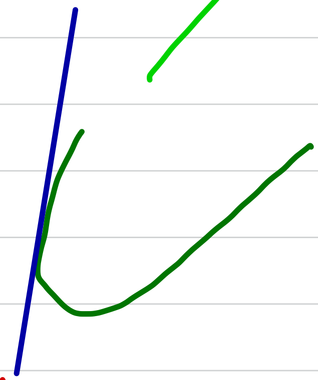
Euclid: line that touch the circle. (unique!)



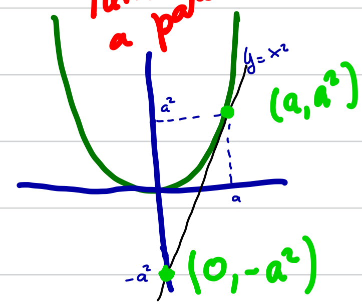
In Latin "to touch" → tangere (recall 'tangible')

Apollonius

parabola



Tangent to a parabola



The line is tangent if it is impossible to fit another line into the angle it makes with the parabola.

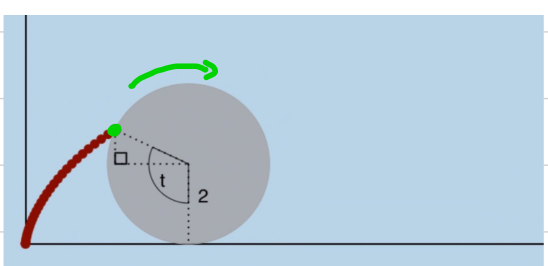
Analogous methods for other conic sections

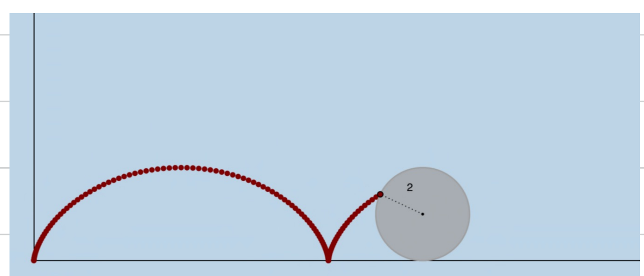
Mersenne → cycloid (not a polynomial curve)
- What is the area under it?
- What is its length?
- Find tangent line

Galileo	Fermat
Mersenne	Pascal
Roberval	Wallis
Wren	

Cycloid is not a conic section (not even defined by a polynomial)

cycloid





$$x = r(t - \sin t)$$
$$y = r(1 - \cos t)$$

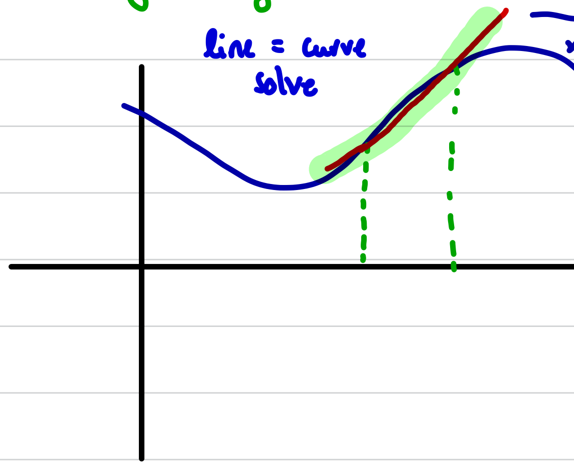
Problem: Find the tangent line to a curve at a point.

- Descartes and Fermat discoveries produced many more curves.

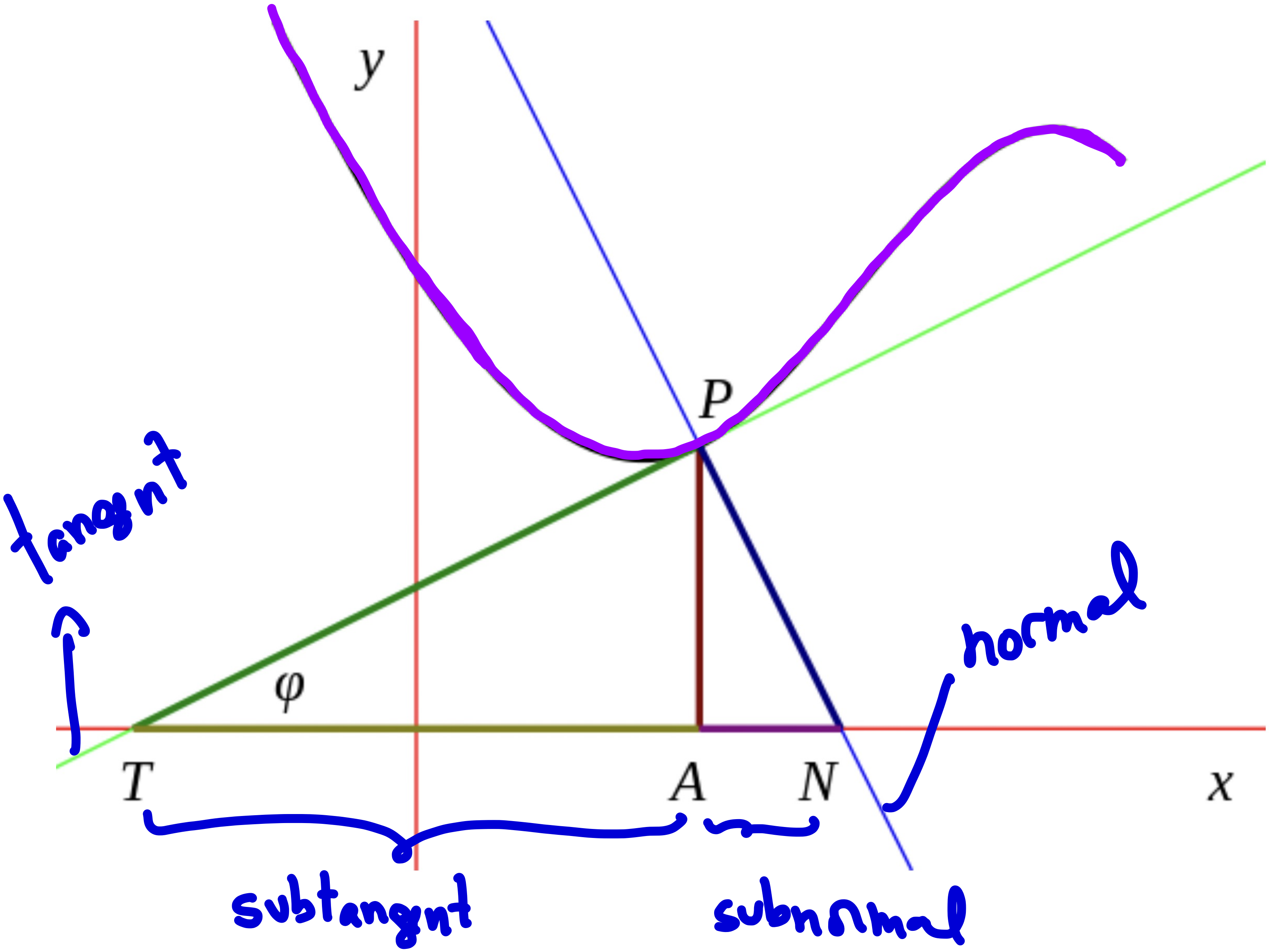
- Curves defined algebraically

Fermat: Any equation in two variables defines a curve.
Realized that the problem of finding tangents was related to the problem of finding max and min. values.

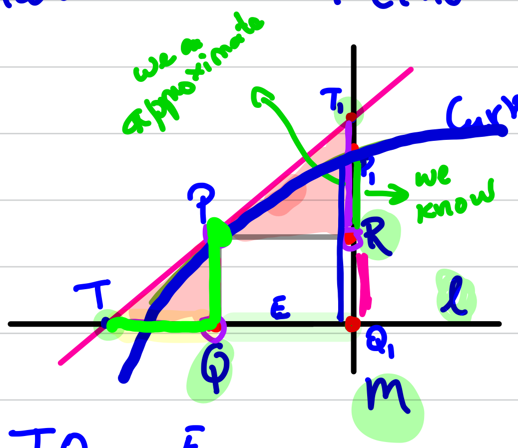
→ Key insight "double root"



- curve line intersects in 2 nearby points
- find the intersection of curve and line
- tangent occurs when the two intersection point become one


y
P
tangent
φ
T
A
N
x
normal
subtangent
subnormal

Fermat ~ 1660 Method of finding Maxima and Minima.



- In pink, the tangent to the curve at P.
- Draw lines ℓ, m.
- The tangent intersects ℓ in a point T.
- Consider point Q.
- We want to find dist(T, Q)

$$\frac{TQ}{PQ} = \frac{\varepsilon}{T_1R}$$

But T_1R is almost P_1R, then $\displaystyle \frac{TQ}{PQ} = \frac{\varepsilon}{P_1Q_1 - QP}$

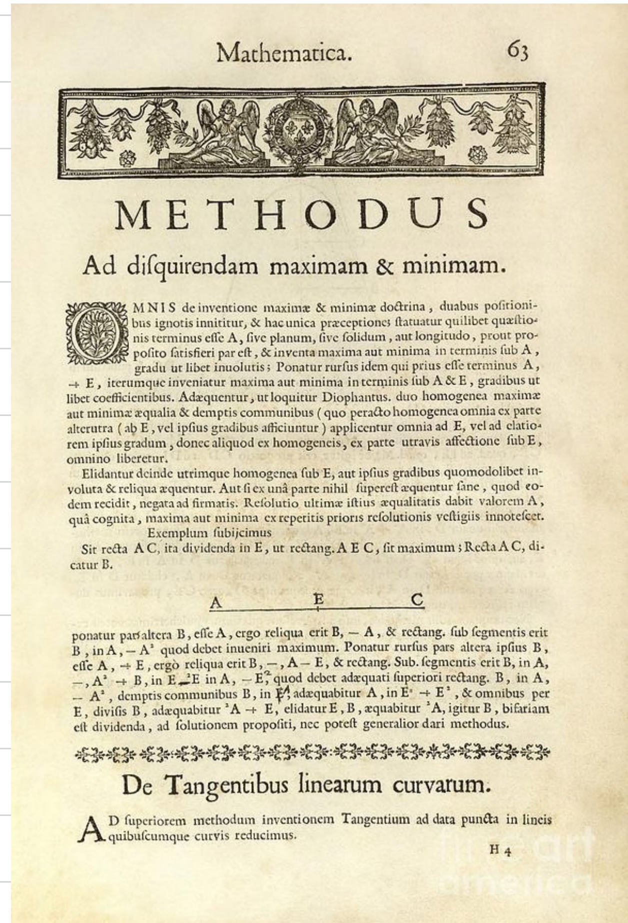
In modern notation, if line TQ is x-axis
line Q_1R is y-axis

$$Q = x$$
$$PQ = f(x)$$

$$TQ = \frac{\varepsilon \cdot PQ}{P_1Q - QP} = \frac{\varepsilon \cdot f(x)}{f(x+\varepsilon) - f(x)}$$

What is the slope of the tangent line?

TQ is called the subtangent of the curve at P


Mathematica. 63

METHODUS

Ad disquirendam maximam & minimam.

OMNIS de inventione maximæ & minimæ doctrina, duabus positionibus ignotis innititur, & hac unica præceptione; statuatur quilibet quæstionis terminus esse A, sive planum, sive solidum, aut longitudo, prout proposito satisfieri par est, & inventa maxima aut minima in terminis sub A, gradu ut libet inuolutis; Ponatur rursus idem qui prius esse terminus A, + E, iterumque inveniatur maxima aut minima in terminis sub A & E, gradibus ut libet coefficientibus. Adæquentur, ut loquitur Diophantus. duo homogenea maximæ aut minimæ æqualia & demptis communibus (quo peracto homogenea omnia ex parte alterutra (ab E, vel ipsius gradibus afficiuntur) applicentur omnia ad E, vel ad elatiorem ipsius gradum, donec aliquod ex homogeneis, ex parte utravis affectione sub E, omnino liberetur.

Elidantur deinde utrimque homogenea sub E, aut ipsius gradibus quomodolibet involuta & reliqua æquentur. Aut si ex unâ parte nihil superest æquentur sane, quod eodem recidit, negata ad firmatis. Resolutio ultimæ istius æqualitatis dabit valorem A, quâ cognita, maxima aut minima ex repetitis prioris resolutionis vestigiis innotescet.

Exemplum subijcimus

Sit recta A C, ita dividenda in E, ut rectang. A E C, sit maximum; Recta A C, dicatur B.

A E C

ponatur pars altera B, esse A, ergo reliqua erit B, — A, & rectang. sub segmentis erit B, in A, — A^2 quod debet inueniri maximum. Ponatur rursus pars altera ipsius B, esse A, + E, ergò reliqua erit B, —, A — E, & rectang. Sub. segmentis erit B, in A, —, A^2 + B, in E — 2E in A, — E^2, quod debet adæquati superiori rectang. B, in A, — A^2, demptis communibus B, in E, adæquabitur A, in E^2 + E^2, & omnibus per E, divisis B, adæquabitur 2A + E, elidatur E, B, æquabitur 2A, igitur B, bifariam est dividenda, ad solutionem propositi, nec potest generalior dari methodus.

De Tangentibus linearum curvarum.

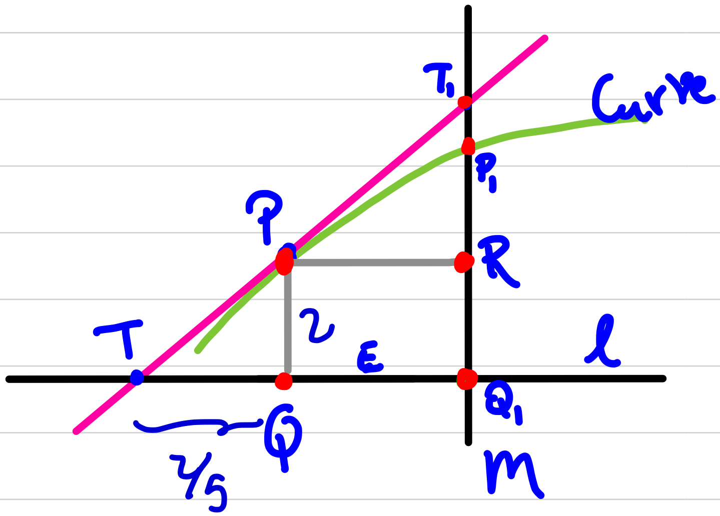
AD superiorem methodum inventionem Tangentium ad data puncta in lineis quibuscumque curvis reducimus.

H 4

Example Using Fermat's method, find the tangent to the parabola defined by

$$f(x) = 3x^2 - x \quad \text{at} \quad P = (1, 2)$$

$x=1$



$$TQ = \frac{\varepsilon \cdot f(x)}{f(x+\varepsilon) - f(x)}$$

$$TQ = \frac{\varepsilon \cdot 2}{\underbrace{3(1+\varepsilon)^2 - 1 - \varepsilon}_{f(x+\varepsilon)} - 2} = \frac{2\varepsilon}{6\varepsilon + 3\varepsilon^2 - \varepsilon} = \frac{2\varepsilon}{5\varepsilon + 3\varepsilon^2}$$

$$= \frac{2}{5 + 3\varepsilon} \xrightarrow{\varepsilon \to 0} \boxed{TQ = \frac{2}{5}}$$

$$\text{Slope} = \frac{2}{2/5} = 5$$

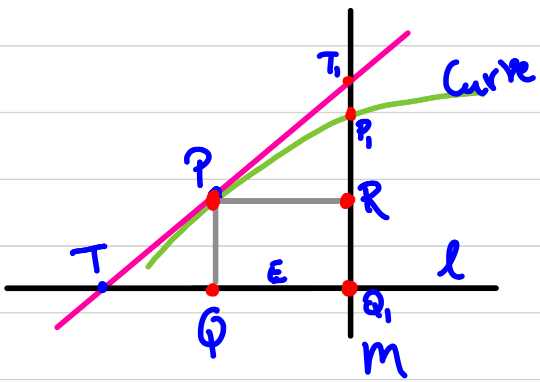
$$f'(1) = 5 \; !$$

Fermat use this method to find max and min

$$f(x) = 3x^2 - x \qquad TQ = \frac{E \cdot f(x)}{f(x+E) - f(x)}$$

$$P = (1, 2)$$

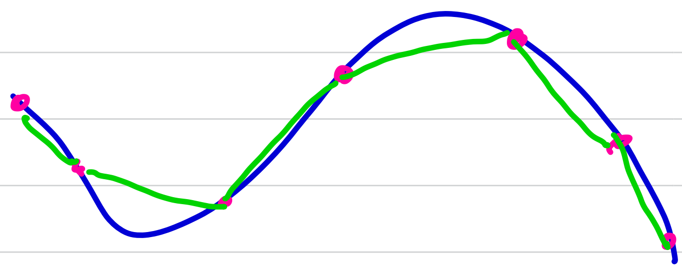
$$TQ = \frac{E \cdot 2}{3(1+E)^2 - (1+E) - 2} = \frac{2E}{6E + 3E^2 - E} = \frac{2}{5 + 3E}$$



Since $TQ = 2/5$

$$\Rightarrow \text{Slope } \frac{PQ}{TQ} = \frac{2}{2/5} = 5$$

How do you find approximate the length of a curve? (without calculus)



$y = f(x)$, $x \in [a,b]$

Fermat

Computation of tangents, max, min

Cavalieri

Computation of areas and volumes.

Newton

- flowing quantities and their rates of flow } fluxions

In Principia gave a mathematical analysis of the laws of motion and the workings the solar system.

Leibniz

- Infinitely small quantities } infinitesimals

x = quantity

dx = differential

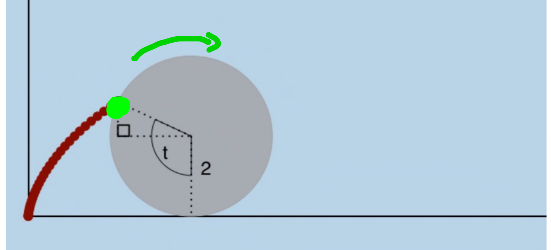
amount by which x changed in an infinitesimal amount of time.

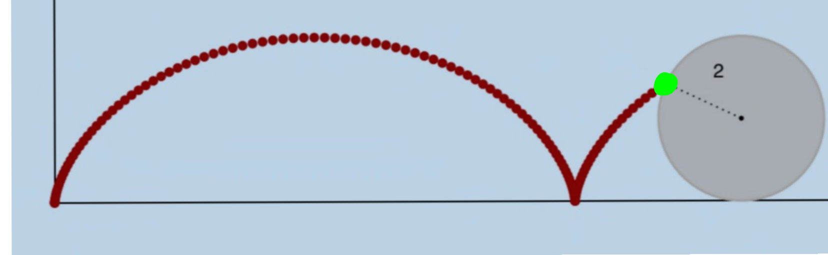
<u>Important</u> It is possible to develop a method to compute (calculate) these kind of things

calculus

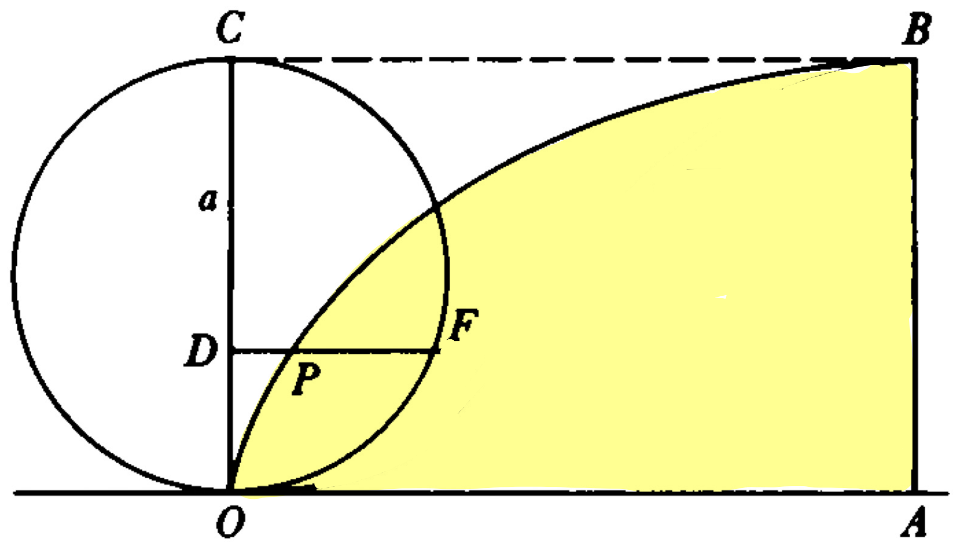
Calculus gives a method to find answers to the four problems

Cycloid



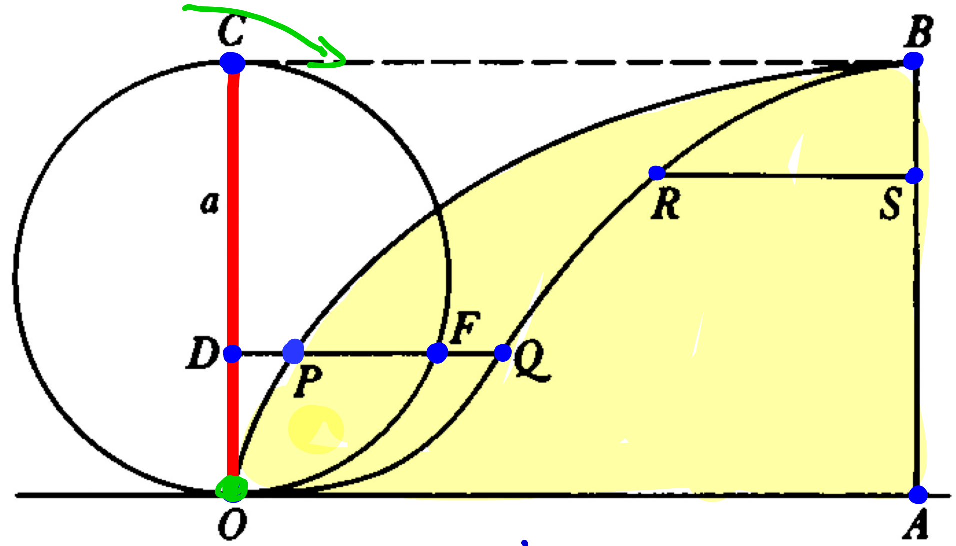


Goal: Find area



- Consider P a point in the cycloid
- Denote by D and F, resp. the intersection points of the horizontal line through P and the y-axis and the original circle.

1634 Roberval method to find the area under one arch of the cycloid



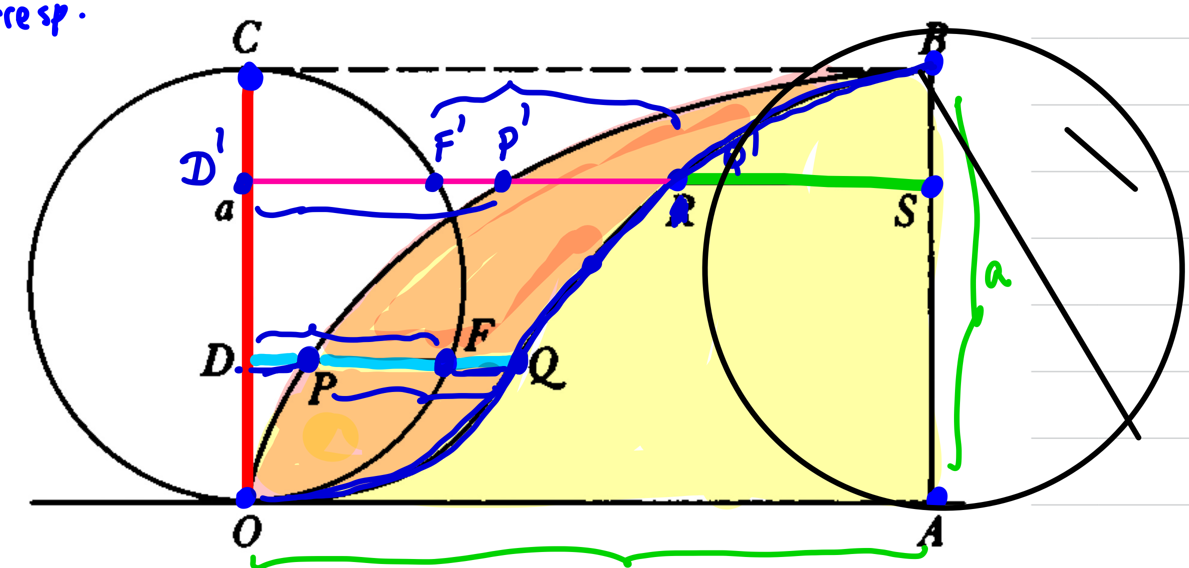
OC diam.

P pt in cycloid

For each P, Q pt s.t $PQ = DF$

Find $Q \,/\, PQ = DF$

<u>Claim</u> Curve OQB divides rect OABC in 2 equal parts (each segm. DQ in curve OQBC has a corresp. segment RS in OABQ)



Area of rect OABC is twice area of circle

Then area under OQB is equal to area of circle

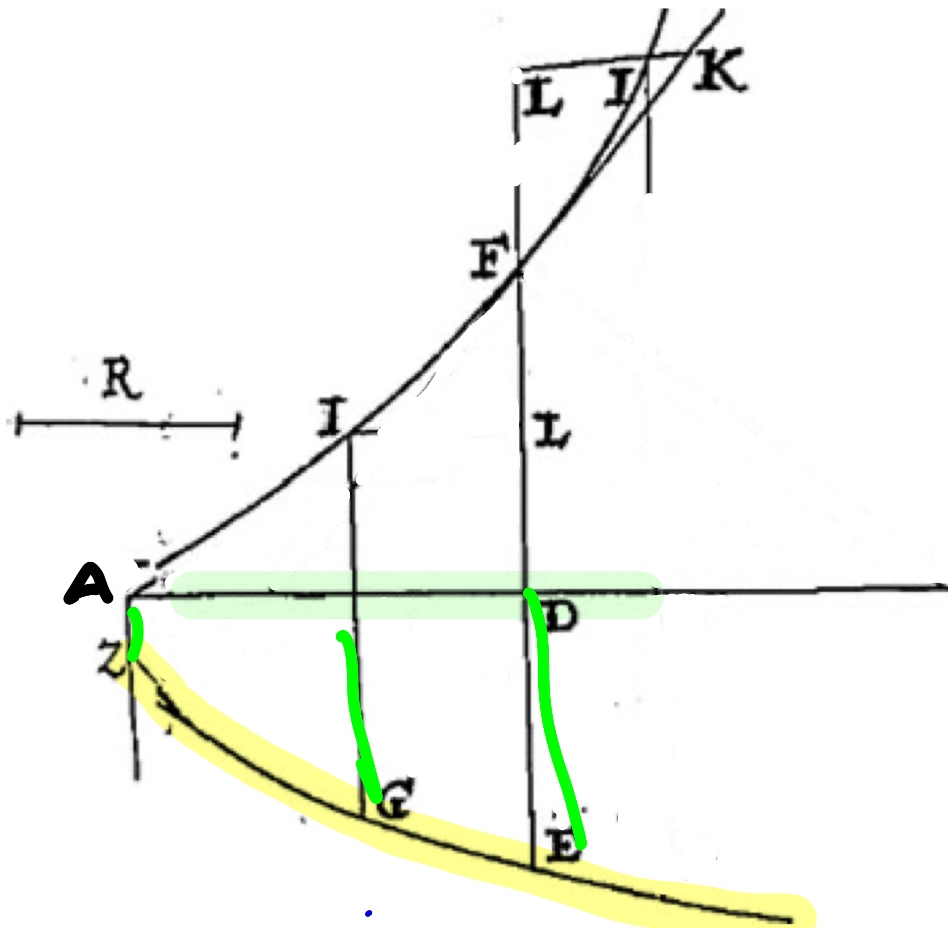
Area between OPB and OQB = area of semicircle

length ? = $2\pi r$

$\Rightarrow$ Area under cycloid is $\frac{3}{2}$ area of circle.

Let ZGE be any curve of which the axis is AD, and let ordinates applied to this axis AZ, PG, DE continually increase from the initial coordinate AZ;

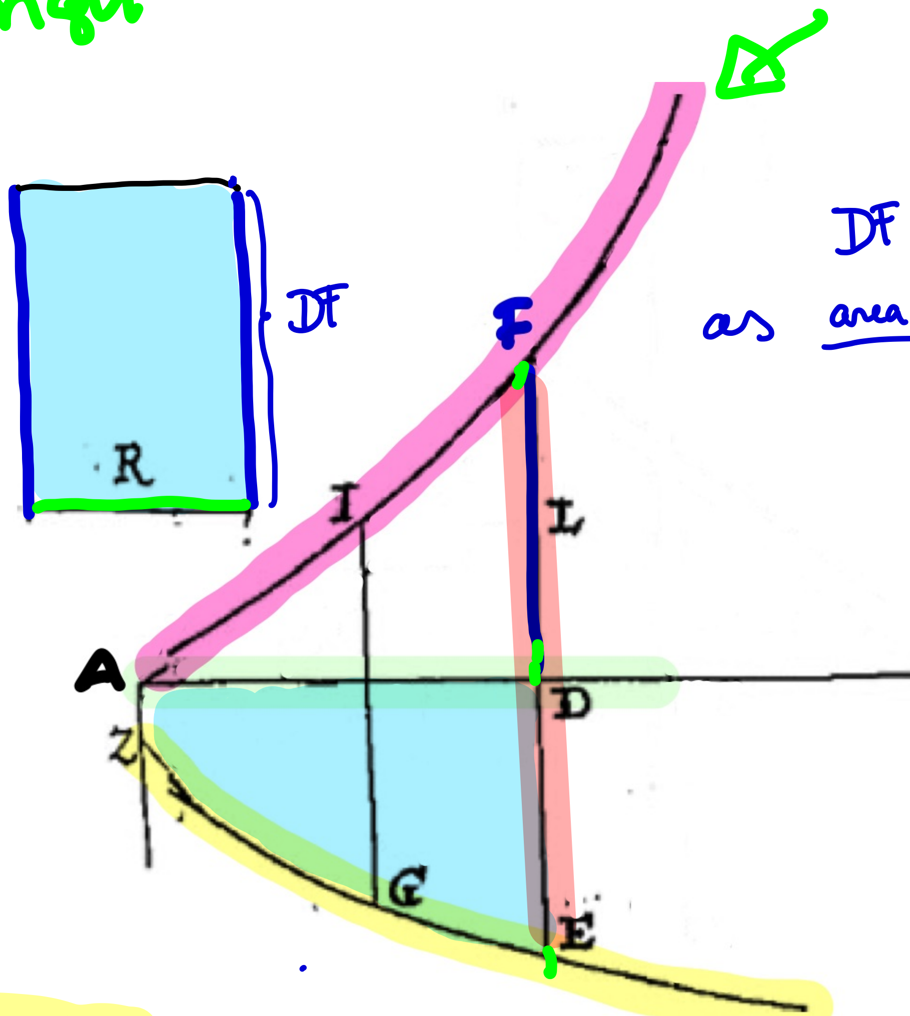
In modern notation ZGE is the graph of a strictly decreasing function f(x)



Barrow - FTC

let AIF be a line [another curve] such that, if any line EDF is drawn perpendicular to AD cutting the curves in the points E, F, and AD in D the rectangle contained by DF and a given length *R* is equal to the intercepted space ADEZ.[40]

Area = length



DF is defined as $\frac{\text{area}(\text{ADEZ})}{R}$

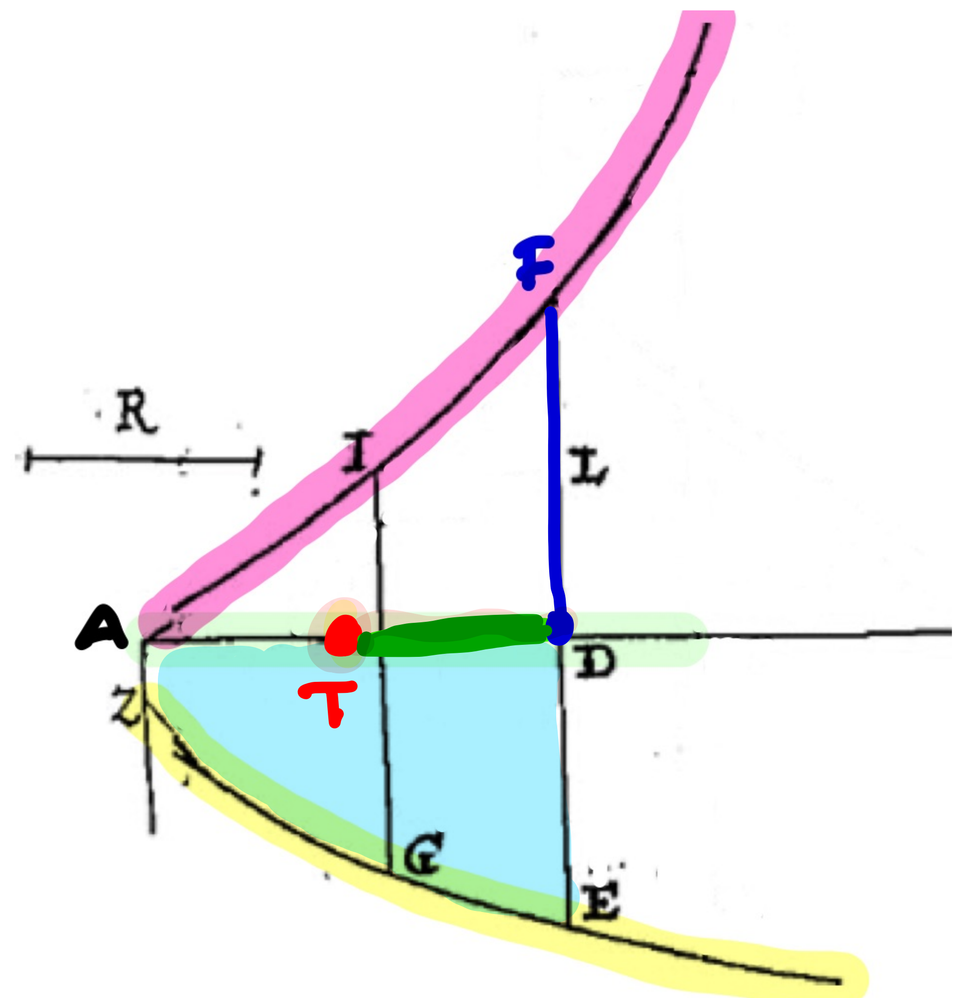
To fix ideas, assume R=1.
In this case
DF = Area(ADEZ)

Barrow-FTC

also let DE/DF = *R*/DT, and join DT.

Find a point T in AD

such that $DT = \frac{R \cdot DF}{DE} = \frac{\text{area}(ADEZ)}{DE}$



Barrow - FTC

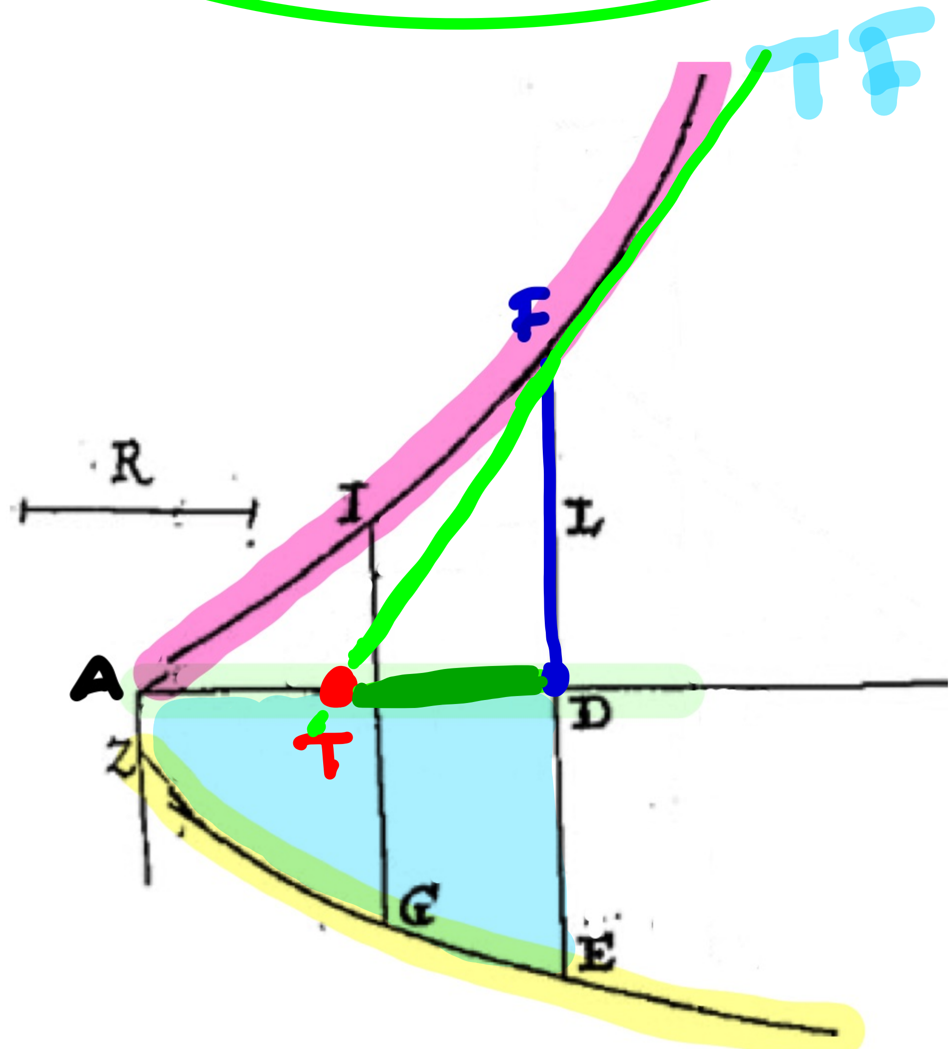
also let DE/DF = *R*/DT, and join DT. Then TF will touch the curve AIF,

Find a point T in AD such that $DT = R \cdot \frac{DF}{DE} = \frac{\text{area}(ADEZ)}{DE}$

(TF is the tangent to the curve AIF at F

- Area
- Tangent

DT is the subtangent.



Barrow - FTC