MAT 542 Homework 1

(1) Let X be a CW complex and let $H_i \subset \pi_i(X)$ be a subgroup for each $i \in \mathbb{N}$. Construct a CW complex Y together with a continuous map $f : Y \longrightarrow X$ so that $\pi_i(Y) = H_i$ and so that
$$f_* : \pi_i(Y) \longrightarrow \pi_i(X)$$
is the inclusion map $H_i \hookrightarrow \pi_i(X)$ for all $i \in \mathbb{N}$.

(2) (Hatcher 4.1 Ex 11): Let X be a CW complex. Show that X is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is homotopic to a constant map.

(3) (Hatcher 4.1 Ex 15, modified). Show that $\pi_n(S^n) = \mathbb{Z}$. Here are some hints:
- Show that two maps
$$f_1, f_2 : (S^n, \star) \longrightarrow (S^n, \star)$$
are homotopic if $f_1^{-1}(U) = f_2^{-1}(U)$ and $f_1|_{f_1^{-1}(U)} = f_2|_{f_2^{-1}(U)}$ for some neighborhood U containing $\star$.
- Approximate continuous maps between spheres with smooth maps where $\star$ is a regular value.

(4) Give an example of a topological space which is weakly homotopic to a circle, but not homotopic to a circle. More generally, for any CW complex X, construct a topological space weakly homotopic to X but not homotopic to X.

Hint: reduce to the contractible case using mapping cones - Hatcher page 13

(5) Show that the following topological space X with topology τ is weakly homotopic to the circle:
$$X := \{0, 1, 2, 3\}, \quad \tau := \{\emptyset, \{0\}, \{3\}, \{0, 3\}, \{0, 1, 3\}, \{0, 2, 3\}, X\}.$$