Homework 4

(1) (a) Let

$$D^n := \{x \in \mathbb{R}^n \ : \ |x| \leq 1\}, \quad S^n := \{x \in \mathbb{R}^{n+1} \ : \ |x| = 1\}$$

be the unit ball and sphere respectively. Let $f : S^n \longrightarrow S^n$ be a degree k map. Compute homology of the space

$$S^n \sqcup D^{n+1}/\sim, \quad x \sim y, \ \forall\, x \in S^n, \ y \in \partial D^{n+1}, \ f(y) = x$$

(I.e. the space obtained by gluing D^{n+1} to S^n via the map f).

(b) For any sequence of finitely generated abelian groups $(A_p)_{p \in \mathbb{N}}$, construct a topological space X satisfying $H_p(X) \cong A_p$ for each $p > 0$.

(2) Compute the homology groups of the 2-sphere S^2 with the north and south pole identified. More generally, compute the homology of $S^2/\sim$ where we identify k-disjoint points in S^2 with a single point.

(3) Construct a CW complex homeomorphic to $\mathbb{R}P^n$ and then compute $H_*(\mathbb{R}P^n)$.

(4) (a) Let M, N be two connected oriented smooth n-manifolds. The *connect sum* of M and N is the manifold $M\#N$ constructed as follows: Let D^n be the closed unit ball as above and let $(D^n)^o$ be its interior. Take two embeddings

$$i : D^n \hookrightarrow M, \quad j : D^n \hookrightarrow N$$

of the unit disk D^n which respect orientation. Then

$$M\#N = (M - i((D^n)^o)) \sqcup (M - j((D^n)^o))/\sim$$

where we identify $i(x)$ with $j(x)$ for all $x \in \partial D^n$.

Compute the $H_*(M\#N)$ in terms of $H_*(M)$ and $H_*(N)$ (You may assume the fact that any smooth oriented n-manifold X is homeomorphic to a CW complex and that $H_n(X) \cong \mathbb{Z}$ if X is closed and connected).

(b) Use the above computation combined with the classification of closed 2-manifolds to compute the homology of any oriented closed 2-manifold from that of the torus.

(5) Define the *Euler Characteristic* of a topological space X to be

$$\chi(X) := \sum_{p=0}^{\infty} (-1)^p \text{rank}(H_p(X)).$$

Let A, B be open subspaces of a topological space X and suppose

$$\oplus_{p \in \mathbb{N}} H_p(A \cap B), \ \oplus_{p \in \mathbb{N}} H_p(A), \ \oplus_{p \in \mathbb{N}} H_p(B), \ \oplus_{p \in \mathbb{N}} H_p(X)$$

all have finite rank. Compute $\chi(X)$ in terms of $\chi(A)$, $\chi(B)$ and $\chi(A \cap B)$.