Homework 3

Due: Thursday September 20th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: Let $\mathcal{M}_{[0,1]}$ be the set of Lebesgue measurable subsets of $[0,1]$ and let

$$P : \mathcal{M}_{[0,1]} \longrightarrow \mathbb{R}, \quad P(A) := m^*(A)$$

be the outer measure. Consider the probability space

$$([0,1], \mathcal{M}_{[0,1]}, P)$$

(I.e the uniform probability measure on $[0,1]$). Construct $A_1, A_2, A_3, A_4 \in \mathcal{M}_{[0,1]}$ so that A_i, A_j, A_k are independent events for any distinct i, j, k but

$$P(\cap_{n=1}^{4} A_n) \neq \prod_{n=1}^{4} P(A_n).$$

Problem 2: Let $f_1, \cdots, f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be measurable functions. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous. Show that

$$h : \mathbb{R} \longrightarrow \mathbb{R}, \; h(x) := F(f_1(x), \cdots, f_n(x))$$

is measurable.

Problem 3: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function and $E \subset \mathbb{R}$ a null set with the property that for each $x \in \mathbb{R} - E$ and each $\epsilon > 0$ there exists $\delta > 0$ so that $|f(y) - f(x)| < \epsilon$ for all $y \in \mathbb{R} - E$ satisfying $|x - y| < \delta$.

Show that f is measurable.

Problem 4: Show that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is measurable iff $f^{-1}(B)$ is measurable for each Borel set B.