Homework 10 Solutions

Due: Thursday December 6th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: Suppose X and Y are random variables on some probability space $(\Omega, \mathcal{F}, P)$ with joint density

$$f_{(X,Y)} : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f_{(X,Y)} = \begin{cases} \frac{3}{4} + xy & \text{if } (x,y) \in (0,1)^2 \\ 0 & \text{otherwise.} \end{cases}$$

Compute

(a) $P(X < Y,\ Y < \frac{1}{2})$.

(b) $P(X \in (0,1/2) \,|\, Y \in (1/2,3/4))$.

(c) The density function $f_{(X|Y\in(0,1/2))}$ of the random variable $(X|Y \in (0,1/2))$.

Solution:

(a) $P(X < Y,\ Y < \frac{1}{2}) = \int_{\{(x,y)\in\mathbb{R}^2\ :\ x<y,\ y<\frac{1}{2}\}} f_{(X,Y)}\ dm^2 =$

$$\int_0^{\frac{1}{2}} \int_0^y \frac{3}{4} + xy\ dxdy = \int_0^{\frac{1}{2}} \frac{3}{4}y + \frac{y^3}{2}\ dy =$$

$$\frac{3}{32} + \frac{1}{128} = \frac{13}{128}.$$

(b) $P(X \in (0,1/2)|Y \in (1/2,3/4)) = \frac{P(X\in(1/2),Y\in(1/2,3/4)}{P(Y\in(1/2,3/4))}$. Now

$$P(Y \in (1/2,3/4)) = \int_{1/2}^{3/4} \int_0^1 \frac{3}{4} + xy\ dxdy = \int_{1/2}^{3/4} \frac{3}{4} + \frac{y}{2}\ dy$$

$$= \frac{3}{16} + \frac{9}{64} - \frac{1}{16} = \frac{17}{64}.$$

Also

$$P(X \in (0,1/2), Y \in (1/2,3/4)) = \int_{1/2}^{3/4} \int_0^{1/2} \frac{3}{4} + xy\ dxdy =$$

$$\int_{1/2}^{3/4} \frac{3}{8} + \frac{y}{8}\ dxdy = \frac{3}{32} + \frac{9}{256} - \frac{1}{64} = \frac{29}{256}.$$

So

$$P(X \in (0,1/2)|Y \in (1/2,3/4)) = \frac{29/256}{17/64} = \frac{29}{68}.$$

(c) The conditional density function is equal to

$$f_{(X|Y\in(0,1/2))}:\mathbb{R}\longrightarrow\overline{\mathbb{R}},\quad f_{(X|Y\in(0,1/2))}(x)=\frac{\int_0^{\frac{1}{2}} f_{(X,Y)}(x,y)\ dy}{P(Y\in(0,1/2))}.$$

Now

$$P(Y\in(0,1/2))=\int_0^{1/2}\int_0^1 \frac{3}{4}+xy\ dxdy=\int_0^{1/2}\frac{3}{4}+\frac{y}{2}\ dxdy$$
$$=\frac{3}{8}+\frac{1}{16}=\frac{7}{16}.$$

Also

$$\int_0^{\frac{1}{2}} f_{(X,Y)}(x,y)\ dy=\int_0^{\frac{1}{2}}\frac{3}{4}+xy\ dy=\frac{3+x}{8}$$

if $x\in(0,1)$. This integral is 0 if $x\notin(0,1)$. Hence

$$f_{(X|Y\in(0,1/2))}(x)=\begin{cases}\frac{16}{7}\left(\frac{3+x}{8}\right) & \text{if } x\in(0,1)\\ 0 & \text{otherwise}\end{cases}=\begin{cases}\frac{6+2x}{7} & \text{if } x\in(0,1)\\ 0 & \text{otherwise.}\end{cases}$$

For Problems 2,3,4 below we need the following two definitions.

Definition: Let $\mathcal{F}$ be a σ-field and let μ, ν be measures on $\mathcal{F}$. We say that ν is *absolutely continuous with respect to* μ if for each $N\in\mathcal{F}$ satisfying $\mu(N)=0$, we have $\nu(N)=0$. We write $\nu\ll\mu$ if ν is absolutely continuous with respect to μ.

Definition: Let $\mathcal{F}$ be a σ-field and let ν_1 and ν_2 on be measures on $\mathcal{F}$. We write $\nu_1\perp\nu_2$ if there are elements $E_1,E_2\in\mathcal{F}$ satisfying $E_1\cap E_2=\emptyset$ and

$$\nu_j(E)=\nu_j(E\cap E_j),\quad \forall\ E\in\mathcal{F},\ j=1,2.$$

Problem 2: Let $\mathcal{F}$ be a σ-field on a set Ω. Let μ, ν_1, ν_2 be measures on $\mathcal{F}$. Show that

(a) if $\nu_1,\nu_2\ll\mu$ then $\nu_1+\nu_2\ll\mu$,
(b) if $\nu_1,\nu_2\perp\mu$ then $(\nu_1+\nu_2)\perp\mu$
(c) and if $\nu_1\ll\mu$ and $\nu_2\perp\mu$ then $\nu_1\perp\nu_2$.

Solution:

(a) Let $N\in\mathcal{F}$ satisfy $\mu(N)=0$. Then $\nu_1(N)=\nu_2(N)=0$ by definition. Hence

$$(\nu_1+\nu_2)(N)=\nu_1(N)+\nu_2(N)=0+0=0.$$

Therefore $\nu_1+\nu_2\ll\mu$.

(b) Suppose $\nu_1, \nu_2 \perp \mu$. Then, by definition, there are elements $M_1, M_2, N_1, N_2 \in \mathcal{F}$ satisfying

$$M_1 \cap N_1 = M_2 \cap N_2 = \emptyset$$

and

$$\mu(E) = \mu(E \cap M_1) = \mu(E \cap M_2),\ \forall\ E \in \mathcal{F}$$
$$\nu_1(E) = \nu_1(E \cap N_1),\ \nu_2(E) = \nu_2(E \cap N_2),\ \forall\ E \in \mathcal{F}.$$

Hence

$$\mu(E) = \mu(E \cap M_1) = \mu(E \cap (M_1 \cap M_2))\ \forall\ E \in \mathcal{F}.$$

Also

$$(\nu_1 + \nu_2)(E) = \nu_1(E) + \nu_2(E) = \nu_1(E \cap N_1) + \nu_2(E \cap N_2)$$
$$\leq \nu_1(E \cap (N_1 \cup N_2)) + \nu_2(E \cap (N_1 \cup N_2))$$

for each $E \in \mathcal{F}$. Since $E \cap (N_1 \cup N_2) \subset E$, we get

$$(\nu_1 + \nu_2)(E) \geq \nu_1(E \cap (N_1 \cup N_2)) + \nu_2(E \cap (N_1 \cup N_2))$$

and hence

$$(\nu_1 + \nu_2)(E) = \nu_1(E \cap (N_1 \cup N_2)) + \nu_2(E \cap (N_1 \cup N_2))$$

for each $E \in \mathcal{F}$. Finally $(M_1 \cap M_2) \cap (N_1 \cup N_2) = \emptyset$. Hence $\nu_1 + \nu_2 \perp \mu$.

(c) Suppose $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$. Then, by definition, there exists $N, M \in \mathcal{F}$ satisfying $N \cap M = \emptyset$ and

$$\nu_2(E) = \nu_2(E \cap N),\ \mu(E) = \mu(E \cap M),\ \forall\ E \in \mathcal{F}.$$

Since $\mu(E) = \mu(E \cap M)$, we get that $\mu(E) = 0$ for each $E \in \mathcal{F}$ satisfying $E \subset M^c$. Hence $\nu_1(E) = 0$ for each $E \in \mathcal{F}$ satisfying $E \subset M^c$. Hence $\nu_1(E) = \nu_1(E \cap M^c) + \nu_1(E \cap M) = \nu_1(E \cap M)$. Since $N \cap M = \emptyset$, we get $\nu_1 \perp \nu_2$.

Problem 3: Let $\mathcal{M}$ be the set of Lebesgue measurable sets in $\mathbb{R}$ and let m be the standard Lebesgue measure. Define

$$\mu : \mathcal{M} \longrightarrow [0, \infty], \quad \mu(E) := m(E) + \sum_{n \in \mathbb{Z} \cap E} 2^{-n}.$$

and

$$\nu : \mathcal{M} \longrightarrow [0, \infty], \quad \nu(E) := m(E) + \sum_{n \in \mathbb{N} \cap E} n^2$$

(a) Show that $\nu \ll \mu$.
(b) Show that μ is not absolutely continuous with respect to ν.
(c) Find a measurable function

$$h : \mathbb{R} \longrightarrow [0, \infty]$$

so that

$$\nu(E) = \int_E h\ d\mu, \quad \forall\ E \in \mathcal{M}.$$

Solution:

(a) Suppose $\mu(E) = 0$. Then $m(E) + \sum_{n \in \mathbb{Z} \cap E} 2^{-n} = 0$. Hence $m(E) = 0$ and $\mathbb{Z} \cap E = \emptyset$. Therefore $\mathbb{N} \cap E = \emptyset$. Hence

$$\nu(E) = m(E) + \sum_{n \in \mathbb{N} \cap E} = 0 + \sum_{n \in \emptyset} n^2 = 0.$$

Hence $\nu \ll \mu$.

(b) Let $E = \{-1\} \in \mathcal{M}$. Then

$$\mu(E) = m(\{-1\}) + 2^{-(-1)} = 0 + 2 = 2.$$

However $\nu(E) = m(E) = 0$. Therefore μ is not absolutely continuous with respect to ν.

(c) Define

$$h : \mathbb{R} \longrightarrow \mathbb{R}, \quad h(x) = \begin{cases} \frac{n^2}{2^{-n}} & \text{if } n \in \mathbb{N} \\ 0 & \text{if } n \in \mathbb{Z} - \mathbb{N} \\ 1 & \text{otherwise.} \end{cases}$$

and

$$h_k : \mathbb{R} \longrightarrow \mathbb{R}, \quad h_k(x) = \begin{cases} \frac{n^2}{2^{-n}} & \text{if } n \in \mathbb{N},\ n \leq k \\ 0 & \text{if } n \in \mathbb{Z},\ n \notin [0,k], \\ 1 & \text{otherwise.} \end{cases}$$

Then h_k is a simple function for each $k \in \mathbb{N}$ and $h = \lim_{k \to \infty} h_k$.
Let $E \in \mathcal{M}$. Then since h_k is simple for each $k \in \mathbb{N}$, we have by the monotone convergence theorem

$$\int_E h \, d\mu = \lim_{k \to \infty} \int_E h_k \, d\mu = \lim_{k \to \infty} \left(m(E - \mathbb{Z}) + \sum_{n \in \mathbb{N} \cap E \cap [0,k]} h(n) 2^{-n} \right)$$

$$= m(E) + \sum_{n \in \mathbb{N} \cap E} \frac{n^2}{2^{-n}} 2^{-n} = m(E) + \sum_{n \in \mathbb{N} \cap E} n^2 = \nu(E).$$

Hence $h = \frac{d\nu}{d\mu}$ and so

$$\frac{d\nu}{d\mu} = \begin{cases} \frac{n^2}{2^{-n}} & \text{if } n \in \mathbb{N} \\ 0 & \text{if } n \in \mathbb{Z} - \mathbb{N} \\ 1 & \text{otherwise.} \end{cases}$$

Problem 4: Given an example of measures ν, μ on $\mathcal{M}$ satisfying the property $\nu \ll \mu$ but not satisfying

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0 \ \text{ so that } \forall\, E \in \mathcal{F},\ \mu(E) < \delta \implies \nu(E) < \epsilon.$$

Solution: Let $\mathcal{M} \subset 2^{\mathbb{R}}$ be the set of Lebesgue measurable sets and let

$$m : \mathcal{M} \longrightarrow [0, \infty]$$

be the usual Lebesgue measure. Define

$$h : \mathbb{R} \longrightarrow \mathbb{R}, \quad h(x) = x^2.$$

Define

$$m_h : \mathcal{M} \longrightarrow [0, \infty], \quad m_h(E) := \int_E h \ dm.$$

This is a measure. Also if $m(E) = 0$, we have $m_h(E) = \int_E h \ dm = 0$ since E is null and hence $m_h \ll m$. However, define

$$E_n := [n, n + \frac{1}{n}].$$

Then $m(E_n) = l(E_n) = \frac{1}{n}$. But

$$m_h(E_n) = \int_n^{n+\frac{1}{n}} x^2 \ dm = \left[\frac{x^3}{3}\right]_n^{n+\frac{1}{n}} = \frac{1}{3}\left((n + \frac{1}{n})^3 - n^3\right) \geq \frac{1}{3}.$$

Hence for each $\delta > 0$, we have $m(E_n) < \delta$ for each $n > \frac{1}{\delta}$ and $m_h(E_n) \geq \frac{1}{3}$.