Homework 8

Due: Thursday November 15th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Throughout this problem set, $(\mathbb{R}, \mathcal{M}, m)$ is the usual Lebesgue measure on $\mathbb{R}$ and $(\mathbb{R}^2, \sigma(\mathcal{M} \times \mathcal{M}), m \times m)$ is the product measure. For each $E \in \sigma(\mathcal{M} \times \mathcal{M})$, we have

$$(m \times m)(E) := \int_{\mathbb{R}} \phi \; dm = \int_{\mathbb{R}} \psi \; dm$$

where

$$\phi : \mathbb{R} \longrightarrow \overline{\mathbb{R}}, \; \phi(x) := m(E \cap (\{x\} \times \mathbb{R}))$$

and

$$\psi : \mathbb{R} \longrightarrow \overline{\mathbb{R}}, \; \psi(y) := m(E \cap (\mathbb{R} \times \{y\})).$$

Problem 1: For each $p, q \in [1, \infty)$ satisfying $p \neq q$, construct a sequence of Lebesgue measurable functions

$$f_n : \mathbb{R} \longrightarrow \mathbb{R}, \; n \in \mathbb{N}$$

so that $f_n \in \bigcap_{r \in [1,\infty)} L^r(\mathbb{R})$ and so that $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R})$ but not a Cauchy sequence in $L^q(\mathbb{R})$.

Problem 2: Let $(I_n)_{n \in \mathbb{N}}$ and $(I'_n)_{n \in \mathbb{N}}$ be a sequence of intervals in $\mathbb{R}$ and let $E \in \sigma(\mathcal{M} \times \mathcal{M})$. Suppose $E \subset \cup_{n \in \mathbb{N}} I_n \times I'_n$. Show that

$$(m \times m)(E) \leq \sum_{n=1}^{\infty} m(I_n) m(I'_n).$$

Problem 3: Show that any continuous function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is $\sigma(\mathcal{M} \times \mathcal{M})$-measurable.

Problem 4: Let $E \in \sigma(\mathcal{M} \times \mathcal{M})$. Show that for each $\epsilon > 0$ there is an open set $O \subset \mathbb{R}^2$ containing E satisfying $(m \times m)(O) \leq (m \times m)(E) + \epsilon$.

You may assume that open subsets of $\mathbb{R}^2$ *are in* $\sigma(\mathcal{M} \times \mathcal{M})$.