Homework 7

Due: Thursday November 8th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: **Definition:** A subset $K \subset V$ of a vector space V is *convex* if for each $p_1, p_2 \in K$, we have that $tp_1 + (1-t)p_2 \in K$ for each $t \in [0,1]$ (I.e. the line joining p_1 and p_2 is contained in K).

Let $(V, \langle , \rangle)$ be a Hilbert space and let $K \subset V$ be a closed convex subset. Let $x \in V$. Show that there exists a unique point $p \in K$ satisfying

$$\|x - p\| \leq \inf\{\|p' - x\| \ : \ p' \in K\}.$$

Hint: we proved this statement when K was a subspace.

Problem 2: For which $p \in [1, \infty]$ is the sequence

$$f_n : \mathbb{R} \longrightarrow \mathbb{R}, \quad f_n(x) = x^{-\frac{1}{3}} \mathbf{1}_{[n,n^4]}, \quad n \in \mathbb{N}$$

a Cauchy sequence in $L^p(\mathbb{R}, \mathcal{M}, m)$?

Problem 3: For each distinct $q, p \in [1, \infty]$ show that $L^p(E)$ is not contained in $L^q(E)$ where $E = (0, \infty)$ (cases $p = \infty$ or $q = \infty$ may require separate treatment).

Problem 4: **Definition:** A subset E of a metric space (X, d) is *dense* if for each $\epsilon > 0$ and $x \in X$, there exists $e \in E$ satisfying $d(x, e) < \epsilon$.

Show that $L^\infty(\mathbb{R})$ does not have a countable dense subset.