Notes on Conformal Field Theory (incomplete)
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§1. Tate's linear algebra

1.1 Crossed modules and central extensions of Lie algebras. We will need Lie and associative algebra versions of crossed modules:

1.1.1 Definition. *(i) Let L be a Lie algebra. An L-crossed module is an L-module $L^\#$ together with a morphism $L^\# \xrightarrow{\partial} L$ of L-modules. For $\ell \in L$ we will denote the action of L on $L^\#$ as $[\ell, \cdot\,]$; so one has $\partial[\ell, \widetilde{\ell}] = [\ell, \partial\widetilde{\ell}]$, $\widetilde{\ell} \in L^\#$.*
(ii) Let R be an associative algebra. An R-crossed module is an R-bimodule $R^\#$ together with a morphism $R^\# \xrightarrow{\partial} R$ of R-bimodules. □

We have canonical pairings $\{,\} : Sym^2 L^\# \to L$, $\langle, \rangle : R^\# \otimes_R R^\# \to R^\#$ defined by formulas $\{m_1, m_2\} := [\partial m_1, m_2] + [\partial m_2, m_1]$, $\langle s_1, s_2\rangle := (\partial s_1)s_2 - s_1(\partial s_2)$. These are morphisms of L-modules and R-bimodules respectively; one has $\partial\{,\} = 0$, $\partial\langle,\rangle = 0$.

Crossed modules in both versions form categories in an obvious manner. For example, if $R_1 \xrightarrow{f} R_2$ is a morphism of associative algebras and $R_i^\#$ are R_i-crossed modules, then an f-morphism of crossed modules is an f-morphism $f^\# : R_1 \to R_2$ of bimodules such that $\partial f^\# = f\partial$. If R is an associative algebra, then R, considered as Lie algebra with commutator $ab - ba$, will be denoted R^{Lie}. If $R^\#$ is an R-crossed module, then it has also an R^{Lie}-crossed module structure $R^{\# Lie}$ with $[r, \widetilde{r}] = r\widetilde{r} - \widetilde{r}r$. One has $\{s_1, s_2\} = \langle s_1, s_2\rangle + \langle s_2, s_1\rangle$ for $s_i \in R^\# = R^\# = R^{\# Lie}$.

Below "DG algebra" means "differential graded algebra"; so "Lie DG algebra" is the same as differential graded Lie superalgebra.

1.1.2 Lemma. *(i) Let L (resp. R) be a Lie (resp. associative) DG algebra such that $L^i = 0$ ($R^i = 0$) for $i > 0$. Then $L^{-1} \xrightarrow{d} L^0$ (resp. $R^{-1} \xrightarrow{d} R^0$) is a Lie (resp. associative) algebra crossed module. For $m_1, m_2 \in L^{-1}$ (resp. $s_1, s_2 \in R^{-1}$) one has $\{m_1, m_2\} = d[m_1, m_2]$ (resp. $\langle s_1, s_2\rangle = d(s_1 s_2)$).*
(ii) Conversely, let $L^\# \xrightarrow{\partial} L$ (resp. $R^\# \xrightarrow{\partial} R$) be a crossed module, and $i : N \subset L^\#$ (resp. $i : T \subset R^\#$) be an L-submodule (resp. R-sub-bimodule) such that $\{L^\#, L^\#\} \subset N \subset \ker\partial$ (resp. $\langle R^\#, R^\#\rangle \subset T \subset \ker\partial$). Then $N \xrightarrow{i} L^\# \xrightarrow{\partial} L$ (resp. $T \xrightarrow{i} R^\# \xrightarrow{\partial} R$) is a dg Lie (resp. associative) dg algebra placed in degrees $-2, -1, 0$.
□

In other words, the lemma claims that DG algebras zero off degrees $-2, -1, 0$ and acyclic off degrees $-1, 0$ are in 1-1 correspondence with pairs $(L^\# \xrightarrow{\partial} L; N)$, where $L^\# \xrightarrow{\partial} L$ is a crossed module and $N \subset L^\#$ is a submodule as in (ii) above. For example, one may take N = image of $\{,\}$ (or image of $\langle,\rangle$ in the associative algebra version); we will say that the corresponding DG algebra is defined by our crossed module.
1.1.3 The simplest example of a Lie algebra crossed module is a central extension $\widetilde{L} \to L$ of a Lie algebra L (the bracket on $\widetilde{L}$ factors through an L-action); note that here $\{,\}$ vanishes. Conversely, let L be a DG Lie algebra. Then L^{-1}/dL^{-2},

equipped with the bracket $[\ell_1, \ell_2] := [d\ell_1, \ell_2]^{0,-1}$ is a Lie algebra, and $d : L^{-1}/dL^{-2} \to L^0$ is a morphism of Lie algebras such that $(H^{-1} \to L^{-1}/dL^{-2} \to d(L^{-1}))$ is a central extension of dL^{-1} by H^{-1}. Hence if $L^{\#} \xrightarrow{\partial} L$ is an L-crossed module such that ∂ is surjective, then $\ker\partial/\{L^{\#}, L^{\#}\} \to L^{\#}/\{L^{\#}, L^{\#}\} \to L$ is a central extension of L. If $tr : \ker\partial/\{L^{\#}, L^{\#}\} \to \mathbb{C}$ is any linear functional, then it defines, by push-out, a central $\mathbb{C}$-extension $L^{\#}_{tr}$ of L.

1.1.4 The following example of a crossed module will be used below. Let L be a Lie algebra, and let $L_+, L_- \subset L$ be ideals. Then we have an L-crossed module $L_+ \oplus L_- \xrightarrow{\partial} L$, $\partial(\ell_+, \ell_-) = \ell_+ + \ell_-$. We have isomorphism $i : L_+ \cap L_- \overset{\sim}{\to} \ker\partial$, $i(\ell) = (\ell, -\ell) \in L_+ \oplus L_-$. Or we may take an associative algebra R equipped with 2-sided ideals R_+, R_-, and get an R-crossed module $R_+ \oplus R_- \xrightarrow{\partial} R$. Note that $\{,\}$ vanishes on L_+ and L_- (and $\langle,\rangle$ vanishes on R_+ and R_-) and one has $\{\ell_+, \ell_-\} = i([\ell_-, \ell_+])$, $\langle r_+, r_-\rangle := -i(r_+r_-)$, $\langle r_-, r_+\rangle = i(r_-r_+)$.

If $L_+ + L_- = L$, then we get a central extension $L_+ \cap L_-/[L_+, L_-] \xrightarrow{i} \widetilde{L} \to L$ of L, where $\widetilde{L} = L_+ \oplus L_-/i([L_+, L_-])$. This central extension is equipped with obvious splittings $s_\pm : L_\pm \to \widetilde{L}$ such that $s_\pm(L_\pm)$ are ideals in $\widetilde{L}$; it is easy to see that $\widetilde{L}$ is universal among all central extensions of L equipped with such splittings. Note also that the embedding $s_+ : L_- \hookrightarrow \widetilde{L}$ yields an isomorphism $L_+/[L_+, L_-] \overset{\sim}{\to} \widetilde{L}/s_-(L_-)$ and we have the Cartesian square

$$\begin{array}{ccccc} \widetilde{L} & \longrightarrow & \widetilde{L}/s_-(L_-) & \overset{\sim}{\leftarrow} & L_+/[L_+, L_-] \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & L/L_- & \overset{\sim}{\leftarrow} & L_+/L_+ \cap L_- \end{array}$$

and the same for $\pm$ interchanged.

1.1.5 Now let $tr : L_+ \cap L_-/[L_+, L_-] \to \mathbb{C}$ be any linear functional. According to 1.1.3 it defines a central $\mathbb{C}$-extension $\widetilde{L}_{tr}$ of L. One has the splittings $s_+ : L_+ \to \widetilde{L}_{tr}$, $s_- : L_- \to \widetilde{L}_{tr}$ such that $s_\pm(L_\pm)$ are ideals and $(s_+ - s_-)\big|_{L_+\cap L_-} = tr$. Clearly L_{tr} is the unique extension equipped with this data.

1.1.6 The above constructions are functorial with respect to $(L, L_\pm)$. Hence if $L'_\pm \subset L$ are other ideals such that $L_\pm \subset L'_\pm$, then we get a canonical morphism $\widetilde{L} \to \widetilde{L}'$ between the corresponding central extensions of L. If $tr : L_+ \cap L_-/[L_+, L_-] \to \mathbb{C}$ extends to $tr : L'_+ \cap L'_-/[L'_+, L'_-] \to \mathbb{C}$, then $\widetilde{L}_{tr} = \widetilde{L}'_{tr}$. In particular, assume that $tr : L_+ \cap L_-/[L_+, L_-] \to \mathbb{C}$ extends to $tr : L_-/[L_-, L_-] \to \mathbb{C}$. Then we may take $L'_+ = L, L'_- = L_-$ to get the same extension $\widetilde{L}_{tr}$, hence we get the splitting $\widetilde{s}_+ : L \to \widetilde{L}_{tr}$ that extends our old $s_+ : L_+ \to \widetilde{L}_{tr}$. Explicitly, $\widetilde{s}_+(\ell_+ + \ell_-) = s_+(\ell_+) + s_-(\ell_-) + tr\ell_-$; clearly $\widetilde{s}_+ - s_- = tr : L_- \to \mathbb{C}$. In the same way, an extension of $tr : L_+ \cap L_- \to \mathbb{C}$ to L_+ determines the splitting $\widetilde{s}_- : L \to \widetilde{L}_{tr}$ that extends the old $s_- : L_- \to \widetilde{L}_{tr}$. If we have the trace functional on the whole L, i.e. $tr : L/[L, L] \to \mathbb{C}$, then $\widetilde{s}_+ - \widetilde{s}_- = tr : L \to \mathbb{C}$.

1.1.7 We will often use the following notation. If $\mathfrak{g}$ is a Lie algebra, V is a vector space, and $0 \to V \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ is a central V-extension of $\mathfrak{g}$, then for any $c \in \mathbb{C}$ we will denote by $\widetilde{\mathfrak{g}}_c$ a V-extension of $\mathfrak{g}$ which is the c-multiple of $\widetilde{\mathfrak{g}}$. So we have a canonical morphism $\widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}_c$ of central extensions of $\mathfrak{g}$ that restricted to V's is multiplication by c. For example, in situation 1.1.3 one has $(L^{\#}_{tr})_c = L^{\#}_{ctr}$.

1.2 Tate's vector spaces. For subspaces V_0, V_1 of a vector space V we will write $V_0 \prec V_1$ if $V_0/V_0 \cap V_1$ is of finite dimension, and $V_0 \sim V_1$ (V_i are commensurable) if $V_0 \prec V_1$ and $V_1 \prec V_0$. Clearly $\prec$ is partial order on a set of commensurability classes of subspaces.

1.2.1 A *Tate's topological vector space* (or, simply, *Tate's space*) V is a linearly topologized complete separated vector space V that admits a basis $\{V_\alpha\}$ of neighbourhoods of 0 with V_α mutually commensurable. Equivalently, V is the projective limit of a family of epimorphisms of usual vector spaces with finite dimensional kernels: $V = \lim\limits_{\overleftarrow{\alpha}} V/V_a$.

Let $L \subset V$ be a vector subspace. We will say that L is *bounded* if for any open $U \subset V$ one has $L \prec U$, and L is *discrete* if for some open U one has $U \cap L = 0$. Clearly simultaneously bounded and discrete subspaces are just finite dimensional ones.

A *lattice* $V_+ \subset V$ is a bounded open subspace; equivalently, this is a maximal (with respect to $\prec$) bounded closed subspace. The lattices form a maximal basis of neighbourhoods of 0 that consists of mutually commensurable subspaces.

A *colattice* $V_- \subset V$ is a maximal discrete subspace. Equivalently, this means that for (any) lattice V_+ one has $V_+ \cap V_- \sim 0$, $V_+ + V_- \sim V$ (or for some lattice V_+ one has $V_+ \oplus V_- \xrightarrow{\sim} V$).

Tate's vector spaces form an additive category $\mathcal{TV}$ with kernels and cokernels. The category $\mathcal{TV}$ is self-dual: Namely, for a Tate's space V its *dual* V^* is $Hom(V, \mathbb{C})$ with open subspaces in V^* equal to orthogonal complements to bounded subspaces in V. This V^* is a Tate's space, and $V^{**} = V$. Note that $V_+ \longmapsto V_+^\perp$ is 1-1 correspondence between lattices in V and V^*; and the same for colattices.

1.2.2 Let V be a Tate's vector space. One has a canonical $\mathbb{Z}$-torsor Dim_V together with a map $dim : \{$ Set of all lattices in $V\} \to Dim_V$ such that for a pair V_{+1}, V_{+2} of lattices one has $dimV_{+1} - dimV_{+2} := dim(V_{+1}/V_{+1} \cap V_{+2}) - dim(V_{+2}/V_{+1} \cap V_{+2}) \in \mathbb{Z}$. One has a natural map $codim : \{$ Set of all colattices in $V\} \to Dim_V$ defined by formula $codimV_- = dimV_+ + dim(V/V_+ + V_-) - dim(V_+ \cap V_-)$, where V_+ is any lattice. The $\mathbb{Z}$-torsor Dim_{V^*} coincides with the opposite torsor to Dim_V: one has $dimV_+^\perp = -dimV_+$. The group $Aut\ V$ acts on Dim_V; if V is neither bounded nor discrete, then the action is non-trivial.

1.2.3 Let V_1, V_2 be Tate's vector spaces. We will say that a linear operator $f \in Hom(V_1, V_2)$ is bounded if Imf is bounded, is discrete if $\ker f$ is open, and is finite if Imf is finite dimensional. Denote by Hom_+, Hom_- and Hom_{00} respectively the corresponding spaces of operators; put $Hom_0 := Hom_+ \cap Hom_-$. Clearly $Hom_+ + Hom_- = Hom$, $Hom_?$ (where $? = +, -, 0, 00$) is a 2-sided ideal in Hom (i.e., if for $V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3$ either f_1 or f_2 is in $Hom_?$, then $f_2 f_1$ is in $Hom_?$), and $Hom_- Hom_+ \subset Hom_{00}$.

Remark. Let $\mathcal{TV}_+$, $\mathcal{TV}_- \subset \mathcal{TV}$ be full subcategories of bounded, resp. discrete, spaces. Then $\mathcal{TV}_-$ coincides with the category of usual vector spaces, and $*$ identifies $\mathcal{TV}_+$ with the dual category $\mathcal{TV}_-^\circ$; in particular these are abelian categories. Consider the quotient categories $\mathcal{TV}/+, \mathcal{TV}/-, \mathcal{TV}/0$, whose objects are Tate's vector spaces, and Hom's are the corresponding quotients $Hom/\pm := Hom/Hom_\pm, Hom/0 := Hom/Hom_0$ (clearly $\mathcal{TV}/\pm$ are just the quotient categories $\mathcal{TV}/\mathcal{TV}_\pm$). These quotient categories are abelian. In fact, the projection $\mathcal{TV}/0 \to \mathcal{TV}/+ \oplus \mathcal{TV}/-$ is an equivalence of categories, and embeddings $\mathcal{TV}_\pm \hookrightarrow \mathcal{TV}$ composed with projections define equivalences $\mathcal{TV}_+/Vect \underset{\sim}{\rightarrow} \mathcal{TV}/-, \mathcal{TV}_-/Vect \underset{\sim}{\rightarrow} \mathcal{TV}/+$ (here $Vect = \mathcal{TV}_+ \cap \mathcal{TV}_-$ is the category of finite dimensional vector spaces).

1.2.4 For $V \in \mathcal{TV}$ consider the algebra $EndV$ equipped with 2-sided ideals $End_\pm \supset End_0 \supset End_{00}$. We will write $\mathfrak{gl} = \mathfrak{gl}V$ for $EndV^{Lie} = EndV$ considered as Lie algebra. Since $End_0^2 \subset End_{00}$, we have a canonical trace functional $tr : \mathfrak{gl}_0 \to \mathbb{C}$ which vanishes on $[\mathfrak{gl}_+, \mathfrak{gl}_-]$

According to 1.1.4, we get an End-crossed module $End_+ \oplus End_- \to End$. By 1.1.5, tr defines a central $\mathbb{C}$-extension $\widetilde{\mathfrak{gl}} \to \mathfrak{gl}$ of $\mathfrak{gl}$, together with canonical Lie algebra splittings $s_\pm : \mathfrak{gl}_\pm \to \widetilde{\mathfrak{gl}}$ such that $s_+ - s_- = tr$ on $\mathfrak{gl}_0$.

1.2.5 Let $T \subset V$ be a Tate's subspace (= a closed subspace with induced Tate structure), and V/T be the quotient. Denote by $P_T \overset{i}{\hookrightarrow} \mathfrak{gl}V$ the parabolic subalgebra of endomorphisms that preserve T; let $\pi = (\pi_T, \pi_{V/T}) : P_T \to \mathfrak{gl}T \times \mathfrak{gl}V/T$ be the obvious projection. Let $\widetilde{\mathfrak{gl}T \times \mathfrak{gl}V/T}$ be a central $\mathbb{C}$-extension of $\mathfrak{gl}T \times \mathfrak{gl}V/T$ which is the Baer sum of $\widetilde{\mathfrak{gl}}T$ and $\widetilde{\mathfrak{gl}}V/T$; one has $\widetilde{\mathfrak{gl}T \times \mathfrak{gl}V/T} = \widetilde{\mathfrak{gl}}T \times \widetilde{\mathfrak{gl}}V/T/\{(a_1, a_2) \in \mathbb{C} \times \mathbb{C} : a_1 + a_2 = 0\}$. Clearly $\widetilde{\mathfrak{gl}T \times \mathfrak{gl}V/T}$ coincides with the $\mathbb{C}$-extension constructed by the recipe of 1.1.4, 1.1.5 using the ideals $\mathfrak{gl}_+T \times \mathfrak{gl}_+V/T$, $\mathfrak{gl}_-T \times \mathfrak{gl}_-V/T$ and the trace functional $tr = tr_T + tr_{V/T}$.

Let $\widetilde{P}_T = i^*\widetilde{\mathfrak{gl}}V$ be the $\mathbb{C}$-extension of P_T induced by $\widetilde{\mathfrak{gl}}V$. Since $P_T = P_{T+} + P_{T-}$, where $P_{T\pm} = P_T \cap \mathfrak{gl}_\pm V$, this $\mathbb{C}$-extension coincides with the one constructed by means of ideals $P_{T\pm}$ and the trace functional $tr_V\big|_{P_T}$. Note that $\pi(P_{T\pm}) = \mathfrak{gl}_\pm T \times \mathfrak{gl}_\pm V/T$ and $tr_V\big|_{P_T} = tr \circ \pi$. By 1.1.6 this defines a canonical morphism $\widetilde{\pi} : \widetilde{P}_T \to \widetilde{\mathfrak{gl}T \times \mathfrak{gl}V/T}$ of $\mathbb{C}$-extensions that lifts π. In other words, $\widetilde{P}_T$ is canonically isomorphic to the Baer sum of $\mathbb{C}$-extensions induced by projections π_T, $\pi_{V/T}$ from $\widetilde{\mathfrak{gl}}T, \widetilde{\mathfrak{gl}}V/T$.

Let us consider an important particular case of this situation. Assume that $T = V_+$ is a lattice. Then we have a canonical splitting $s_+ : \mathfrak{gl}V_+ = \mathfrak{gl}_+V_+ \to \widetilde{\mathfrak{gl}}V_+$, $s_- : \mathfrak{gl}V/V_+ = \mathfrak{gl}_-V/V_+ \to \widetilde{\mathfrak{gl}}V/V_+$, hence a canonical splitting $s_{V_+} = s_+\pi_{V_+} + s_-\pi_{V/V_+} : P_{V_+} \to \widetilde{\mathfrak{gl}}_V$. Note that s_{V_+} actually depends on V_+: if V_+' is another lattice, then $s_{V_+} - s_{V_+'} : P_{V_+} \cap P_{V_+'} \to \mathbb{C}$ is given by formula $(s_{V_+} - s_{V_+'})(a) = tr_{V_+/V_+ \cap V_+'}(a) - tr_{V_+'/V_+ \cap V_+'}(a)$.

Similarly, if $T = V_-$ is a colattice, then we have the splittings $s_- : \mathfrak{gl}V_- = \mathfrak{gl}_-V_- \to \widetilde{\mathfrak{gl}}V_-$, $s_+ : \mathfrak{gl}V/V_- = \mathfrak{gl}_+V/V_- \to \widetilde{\mathfrak{gl}}V/V_-$, hence the splitting $s_{V_-} = s_-\pi_{V_-} + s_+\pi_{V/V_-} : P_{V_-} \to \widetilde{\mathfrak{gl}}_V$. On $P_{V_-} \cap P_{V_+}$ the difference $s_{V_+} - s_{V_-} : P_{V_-} \cap P_{V_+} \to \mathbb{C}$ is given by formula

$$(s_{V_+} - s_{V_-})(a) = tr_{V_- \cap V_+}(a) - tr_{V/V_- + V_+}(a).$$

The following subsection 1.3 could be omitted on first reading.

1.3 Elliptic complexes. Let $(V^\cdot, d)$ be a finite complex of Tate's vector spaces. We will call it elliptic, if for some (or any) subcomplex $(V_+^\cdot, d) \subset (V^\cdot, d)$ formed by lattices in $V^\cdot$ both $V_+^\cdot$ and $V^\cdot/V_+^\cdot$ have finite dimensional cohomology spaces.

Clearly, elliptic complexes have finite dimensional cohomology.

Remark. $V^\cdot$ is elliptic iff its image in the abelian category $\mathcal{TV}/0$ (see 3.2.2) is acyclic.

1.3.1 Let $(U^\cdot,d),(V^\cdot,d)$ be elliptic complexes. Then $Hom = Hom(U^\cdot,V^\cdot) := \prod Hom(U^i,V^i)$ carries a bunch of subspaces. First, one has the subspaces $Hom_\pm := \prod Hom_\pm(U^i,V^i)$, Hom_0, Hom_{00} that have nothing to do with differential. We may enlarge those spaces as follows. Put $Hom^d_\pm := \{f\in Hom : [f,d]\in Hom_\pm(U^\cdot,V^{\cdot+1})\}, Hom^d_0 := Hom^d_+\cap Hom^d_-$, $Hom_d := \{f\in Hom : [f,d]=0\}$ (= usual morphisms of complexes). Clearly $Hom_\pm\subset Hom^d_\pm, Hom_0\subset Hom^d_0$, and all $Hom^d_?$ are compatible with $\pm$ decomposition: one has $Hom^d_? = (Hom^d_?\cap Hom_+)+(Hom^d_?\cap Hom_-)$.

The following easy technical lemma is quite useful. Assume that we picked subcomplexes $U'^\cdot_+\subset U^\cdot_+\subset U$, $V'_+\subset V_+\subset V$ formed by lattices. Put $P := \{f\in Hom(U^\cdot,V^\cdot) : f(U'^\cdot_+)\subset V'^\cdot_+, f(U_+)\subset V_+\}$, $P_{+d} := \{f\in P : [f,d](U^\cdot)\subset V^{\cdot+1}_+\}$, $P_{-d} := \{f\in P : [f,d](U'^\cdot_+)=0\}, P_{od} = P_{+d}\cap P_{-d}$.

1.3.2 Lemma. *One has* $Hom^d_\pm = P_{\pm d}+Hom_{00}$, $Hom^d_0 = P_{0d}+Hom_{00}$.

Proof. Consider, e.g., the case of Hom^d_+. One has $Hom^d_+ = (P\cap Hom^d_+)+Hom_0$. An element $f\in P\cap Hom^d_+$ induces the linear map $\bar f : U^\cdot/U^\cdot_+\to V^\cdot/V^\cdot_+$ such that $\alpha=[\bar f,d]$ is of finite rank. One may find $\bar g$ of finite rank such that $[\bar g,d]=\alpha$. Lift $\bar g$ to an element $g\in P\cap Hom_0$; then $f-g\in P_{+d}$, and we are done. □

Now let us define the traces. Consider a single elliptic complex $(V^\cdot,d)$. We have a bunch of Lie subalgebras in $\mathfrak{gl} = \mathfrak{gl}V^\cdot = \Pi\mathfrak{gl}V^i$. Pick subcomplexes $V'^\cdot_+\subset V^\cdot_+\subset V^\cdot$ formed by lattices; we get the corresponding parabolic subalgebra $P\subset\mathfrak{gl}$ and its standard subalgebras. Define the trace functional tr: $P_{0d}\to\mathbb{C}$ by formula $trf := \Sigma(-1)^i(tr_{H^i(V/V_+)}+tr_{V^i_+/V'^i_+}+tr_{H^i(V'_+)})$. In particular, if V/V_+ and V'_+ are acyclic, then $tr = \Sigma(-1)^i tr_{V^i_+/V^{i\prime}_+}$. The algebra $\mathfrak{gl}_{00}$ also carries the trace $tr = \Sigma(-1)^i tr_{V^i}$. Clearly on $P_{0d}\cap\mathfrak{gl}_{00}$ these traces coincide, so, by 1.3.2, they define $tr:\mathfrak{gl}^d_0\to\mathbb{C}$.

1.3.3 Lemma. *The trace functional* $tr:\mathfrak{gl}^d_0\to\mathbb{C}$ *does not depend on the choice of* $V^\cdot_+, V'^\cdot_+$ *and vanishes on* $[\mathfrak{gl}^d_0,\mathfrak{gl}^d_0]$. □

Let $\widetilde{\mathfrak{gl}}$ be the central extension of $\mathfrak{gl}$ by $\mathbb{C}$ which is the alternating Baer sum of $\widetilde{\mathfrak{gl}}V^i$. Equivalently, to get $\widetilde{\mathfrak{gl}}$ take the ideals $\mathfrak{gl}_\pm\subset\mathfrak{gl}$ and the trace functional $tr=\Sigma(-1)^i tr_{V^i}$ on $\mathfrak{gl}_0$, and apply constructions 1.1.4, 1.1.5. We have canonical splittings $s_\pm:\mathfrak{gl}_\pm\to\widetilde{\mathfrak{gl}}$.

1.3.4 Lemma. *These splittings extend to canonical splittings* $s_\pm:\mathfrak{gl}^d_\pm\to\widetilde{\mathfrak{gl}}$*; one has* $s_+-s_-=tr:\mathfrak{gl}^d_0\to\mathbb{C}$.

Proof. Consider, say, the case of s_+. Let $\widetilde{\mathfrak{gl}}^d_+$ be $\widetilde{\mathfrak{gl}}$ restricted to $\mathfrak{gl}^d_+$. Note that $\mathfrak{gl}^d_+ = \mathfrak{gl}_+ + (\mathfrak{gl}_-\cap\mathfrak{gl}^d_+)$, so $\widetilde{\mathfrak{gl}}^d_+$ comes from constructions 1.1.4, 1.1.5 applied to $\mathfrak{gl}^d_+$, its ideals $\mathfrak{gl}_+$ and $\mathfrak{gl}_-\cap\mathfrak{gl}^d_+$ and the trace functional tr. We may even replace $\mathfrak{gl}_-\cap\mathfrak{gl}^d_+$ by the larger ideal $\mathfrak{gl}^d_0$ and, since tr extends to $\mathfrak{gl}^d_0$ by 1.3.3, according to 1.1.6 we get the desired section $s_+:\mathfrak{gl}^d_+\to\widetilde{\mathfrak{gl}}$. One treats s_- in a similar way; the formula $s_+-s_-=tr$ results from 1.1.6. □

1.4 Clifford modules. Let W be a Tate's space, and let (,) be a non-degenerate symmetric form on W (which is the same as symmetric isomorphism $W \xrightarrow{\sim} W^*$).

1.4.1 For a lattice $W_+\subset W$ let $W^\perp_+$ be the orthogonal complement with respect to (,). This is also a lattice, and the parity of $dimW^\perp_+ - dimW_+\in\mathbb{Z}$ does not depend on W_+ (and depends on $(W,(\ ,\ ))$ only). We will say that W is even or odd dimensional if $dimW_+^\perp - dimW_+$ is even or odd, respectively.

1.4.2 A Clifford module M is a module over Clifford algebra $Cliff(W,(\ ,\ ))$ such that W acts on M in a continuous way (in the discrete topology of M). This means that for any $m \in M$ there is a lattice W_+ such that $W_+m = 0$. Denote by $\mathcal{CM}_W$ the category of Clifford modules.

Let $W_+ \subset W$ be a lattice such that $(\ ,\ )|_{W_+} = 0$. Then the finite-dimensional vector space $W_+^\perp/W_+$ carries an induced non-degenerate form. If M is a Clifford module, then $M^{W_+} := \{m \in M : W_+m = 0\}$ is a $W_+^\perp$-invariant subspace of M, hence a $Cliff(W_+^\perp/W_+,(\ ,\ ))$-module.

1.4.3 Lemma. *The functor $\mathcal{CM}_W \to \mathcal{CM}_{W_+^\perp/W_+}$, $M \longmapsto M^{W_+}$, is an equivalence of categories. The inverse functor is given by formula $N \longmapsto Cliff(W) \bigotimes_{Cliff(W_+^\perp)} N$.*
□

In particular, we see that $\mathcal{CM}_W$ is a semisimple category. There is 1 irreducible object if W is even-dimensional, and 2 such if W is odd-dimensional.

Denote by $C\ell W$ the completion $\lim\limits_{\leftarrow} Cliff(W)/Cliff(W)\cdot W_+$, where W_+ runs the set of all lattices in W. It is easy to see that the multiplication extends to this completion by continuity, so $C\ell W$ is an associative algebra. Clearly, it acts on any Clifford module.

1.4.4 Let $L_+ \subset W$ be a maximal $(\ ,\ )$-isotropic lattice (so either $L_+^\perp = L_+$ or $dimL_+^\perp/L_+ = 1$ depending on parity of dimension of W). If L'_+ is another such lattice, put $\lambda(L_+ : L'_+) := \det(L_+/L_+ \cap L'_+)$. One has a canonical embedding $i : \lambda(L_+ : L'_+) \hookrightarrow C\ell W/C\ell W \cdot L'_+$, given by the formula $v_1 \wedge \cdots \wedge v_n \longmapsto \widetilde{v}_1 \cdots \widetilde{v}_n \bmod C\ell W \cdot L'_+$. Here $\{v_i\}$ is a basis of $L_+/L_+ \cap L'_+, \widetilde{v}_i$ are any liftings of v_i to elements of L_+. For a Clifford module M one has a canonical isomorphism $\lambda(L_+ : L'_+) \otimes M^{L'_+} \underset{\sim}{\rightarrow} M^{L_+}$, $v \otimes m \longmapsto i(v)m$.

Now let $L_- \subset L$ be a maximal isotropic colattice (so $codimL_- = dimL_+$ in case $dimW$ is even, or $codimL_- = dimL_+ + 1$ if $dimW$ is odd). Put $\lambda(L_+, L_-) = \det(L_+ \cap L_-)$. For a Clifford module M put $M_{L_-} := M/L_-M$. One has a canonical isomorphism $\lambda(L_+, L_-) \otimes M_{L_-} \underset{\sim}{\rightarrow} M^{L_+}$, defined by formula $v \otimes m \longmapsto v\widetilde{m}$, where $v \in \lambda(L_+, L_-) \subset Cliff(W)$, $m \in M_{L_-}$, and $\widetilde{m} \in M_{L_-}$, and $\widetilde{m} \in M$ is any element such that $\widetilde{m} \bmod L_-M = m$ and $v\widetilde{m} \in M^{L_+}$. If M is irreducible, then $dimM^{L_+} = dimM_{L_-} = 1$, and we may rewrite the above isomorphisms as

$$\lambda(L_+ : L'_+) = M^{L_+}/M^{L'_+}, \quad \lambda(L_+, L_-) = M^{L_+}/M_{L_-}.$$

1.4.5 The algebra $C\ell W$ carries a natural $\mathbb{Z}/2$-grading such that W lies in the degree 1 component. Denote by $\mathcal{CM}_W^{\mathbb{Z}/2}$ the corresponding category of $\mathbb{Z}/2$-graded Clifford modules. This is a semisimple category. If $dimW$ is odd, then it has a single irreducible object; if $dimW$ is even, then there are two irreducible objects that differ by a shift of $\mathbb{Z}/2$-grading.

If $dimW$ is even, then each $M \in \mathcal{CM}_W$ carries a natural $\mathbb{Z}/2$-grading defined up to a shift. Precisely, consider the set of all maximal isotropic lattices. This breaks into two components: lattices L_+, L'_+ lie in the same component iff $dimL_+/L_+ \cap L'_+$ is even. Denote the two element set of these components by $\mathbb{Z}/2_W$; we will consider it as $\mathbb{Z}/2$-torsor. Then any $M \in \mathcal{CM}_W$ carries a canonical $\mathbb{Z}/2_W$-grading determined by the property that $M^{L_+} \subset M^\alpha$ for $L_+ \in \alpha \in \mathbb{Z}/2_W$.

1.4.6 Let $C\ell^{Lie}W$ denote the Clifford algebra considered as Lie (super)algebra (with the above $\mathbb{Z}/2$-grading; the (super)commutator is defined by the usual formula $[a,b] = ab - (-1)^{\alpha\beta}ba$ for $a \in C\ell^{Lie}W^{\alpha}$, $b \in C\ell^{Lie}W^{\beta}$). Denote by $\mathfrak{a}W$ the normalizer of $W \subset C\ell^{Lie}W^1$ in $C\ell^{Lie}W$. This is a Lie subalgebra of $C\ell^{Lie}W$. As a vector space $\mathfrak{a}W$ is the completion in $C\ell W$ of the subspace of all degree ≤ 2 polynomials of elements of W. One has $\mathfrak{a}W^1 = W$. The Lie algebra $\widetilde{OW} := \mathfrak{a}W^0$ is called the spinor algebra of W. The subspace $\mathbb{C} \subset C\ell W$ coincides with center of $\mathfrak{a}W$. One has a canonical isomorphism $\mathfrak{a}W/\mathbb{C} = OW \rtimes W$. Here OW is the orthogonal Lie algebra of all (,)-skew symmetric elements in $\mathfrak{g}\ell W$; the projection $\pi : \widetilde{OW} \to \widetilde{OW}/\mathbb{C} = OW$ is given by the adjoint action on $W = \mathfrak{a}W^1$.

The Lie superalgebra $\mathfrak{a}W$ acts on any $M^{\cdot} \in C\mathcal{M}_W^{\mathbb{Z}/2}$ in an obvious manner. If $M^{\cdot}$ is irreducible, this action identifies $\mathfrak{a}W$ with the normalizer of W in the Lie superalgebra $End_{\mathbb{C}}M^{\cdot}$. Similarly, $\widetilde{OW}$ acts on any $M \in C\mathcal{M}_W$, and, in case M is irreducible, $\widetilde{OW}$ coincides with the normalizer of W in $End_{\mathbb{C}}M$.

1.4.7 Here is another construction of $\widetilde{OW}$. For $a \in \mathfrak{g}\ell W$ denote by ${}^t a \in \mathfrak{g}\ell W$ the adjoint operator with respect to (,); for $a \in \mathfrak{g}\ell_- W$ one has ${}^t a \in \mathfrak{g}\ell_+ W$. Consider now the ideal $\mathfrak{g}\ell_- W \subset \mathfrak{g}\ell W$ as an OW-module with respect to Ad-action. Then $\mathfrak{g}\ell_- W$ together with the surjective morphism $\mathfrak{g}\ell_- W \xrightarrow{\partial} OW$, $a \longmapsto a - {}^t a$, is an OW-crossed module. The pairing $\{,\} : \mathfrak{g}\ell_- W \times \mathfrak{g}\ell_- W \to \ker \partial$ (see 1.1.1) is given by formula $\{a_1, a_2\} = [a_1, {}^t a_2] + [a_2, {}^t a_1]$. Clearly $\ker \partial \subset \mathfrak{g}\ell_0 W$. The usual trace tr(1.2.4) vanishes on $\{\ker \partial, \ker \partial\}$; put $o\ tr = 1/2 tr$. By 1.1.3 we get a central $\mathbb{C}$-extension $\widetilde{OW}' = (\mathfrak{g}\ell_- W)_{o\ tr}$ of OW.

We define a canonical isomorphism $\alpha : \widetilde{OW}' \xrightarrow{\sim} \widetilde{OW}$ of central $\mathbb{C}$-extensions of OW as follows. One has a canonical identification $W \otimes W \simeq \mathfrak{g}\ell_{00} W$, $w_1 \otimes w_2$ corresponds to a linear operator $w \longmapsto (w_2, w)w_1$. This isomorphism extends by continuity to the isomorphism of completions $\varprojlim_{W_+} W \otimes (W/W_+) \simeq \mathfrak{g}\ell_- W$. Hence the map $\mathfrak{g}\ell_{00} W = W \otimes W \to Cliff(W, (,))$, $a_1 \otimes a_2 \longmapsto a_1 a_2$, extends by continuity to the map $\alpha^{\#} : \mathfrak{g}\ell_- W \to C\ell W$. Clearly $\alpha^{\#}$ maps $\mathfrak{g}\ell_- W$ to $\mathfrak{a}W^0 = \widetilde{OW}$. For $a_1, a_2 \in \mathfrak{g}\ell_- W, w \in W$ one has $[\alpha^{\#}(a), w] = \partial(a)(w)$, $[\alpha^{\#}(a_1), \alpha^{\#}(a_2)] = \alpha^{\#}([\partial a_1, a_2])$. For $b \in \ker \partial \cap \mathfrak{g}\ell_{00} W$ one has $b = 1/2(b + {}^t b) = \Sigma(w_i \otimes w_i' + w_i' \otimes w_i)$, hence $\alpha^{\#}(b) = \Sigma(w_i, w_i') = o\ tr\ b$; by continuity this holds for any $b \in \ker \partial$. This implies that $\alpha^{\#}$ yields a map $\alpha : \mathfrak{g}\ell_- W/\ker\ tr = \widetilde{OW}' \to \widetilde{OW}$, which is the desired isomorphism of $\mathbb{C}$-extensions of OW.

1.4.8 Let $L_+ \subset W$ be a maximal isotropic lattice; denote by $P_{L_+}O \subset OW$ the "parabolic" subalgebra of operators that preserve L_+. One has a canonical Lie algebra splitting $s_{L_+} : P_{L_+}O \to \widetilde{OW}$ defined by formula $s_{L_+}(a) = \alpha^{\#}(b)$, where $b \in \mathfrak{g}\ell_- W$ is any operator such that $\partial(b) = a, b(L_+) = 0, (a-b)(W) \subset L_+$. For any Clifford module M one has $s_{L_+}(a)(M^{L_+}) = 0$ (and $s_{L_+}(a)$ is a unique lifting of a to $\widetilde{OW}$ with this property).

Similarly, let $L_- \subset W$ be a maximal isotropic colattice. The corresponding parabolic subalgebra $P_{L_-}O \subset OW$ also has a canonical Lie algebra splitting $s_{L_-} : P_{L_-}O \to \widetilde{OW}$ defined by formula $s_{L_-}(a) = \alpha^{\#}(b)$, where $b \in \mathfrak{g}\ell_- W$ is an operator such that $\partial(b) = a$, $b|_{L_-} = a|_{L_-}, b(W) \subset L_-$. For a Clifford module one has $s_{L_-}(a)(M_{L_-}) = 0$ (i.e., $s_{L_-}(a)(M) \subset L_- M$).

According to 1.4.4 for $a \in P_{L_+}O \cap P_{L_-}O$ one has $(s_{L_-} - s_{L_+})(a) = tr_{L_- \cap L_+}(a) \in$

$\mathbb{C} \subset \widetilde{OW}$. If L'_+ is another maximal isotropic lattice, then for $a \in P_{L_+}O \cap P_{L'_+}O$ one has $(s_{L'_+} - s_{L_+})(a) = tr_{L_+/L_+\cap L'_+}(a)$.

1.4.9 Let V be any Tate's vector space. Then $W := V \oplus V^*$, equipped with the form $((v, v^*), (v', v^{*'})) := v^*(v') + v^{*'}(v)$, is an even-dimensional space. For any lattice $V_+ \subset V$ and a colattice $V_- \subset V$ a lattice $L(V_+) = V_+ \oplus V_+^\perp \subset W$ and a colattice $L(V_-) = V_- \oplus V_-^\perp \subset W$ are maximal isotropic ones; clearly one has a canonical isomorphisms

$$\lambda(L(V_+) : L(V'_+)) = \det(V_+/V_+ \cap V'_+)/\det(V'_+/V_+ \cap V'_+)$$
$$\lambda(L(V_+), L(V_-)) = \det(V_+ \cap V_-)/\det(V/V_+ + V_-).$$

The algebra $C\ell W$ gets a natural $\mathbb{Z}$-grading such that the subspaces V, V^* ($\subset W \subset C\ell W$) lie in degrees 1, -1, respectively. Any Clifford module M has a canonical Dim_V-grading such that $M^{L(V_+)}$ lies in degree $dimV_+$.

The embedding $i : \mathfrak{g}\ell V \hookrightarrow OW$, $\ell \longmapsto \ell \oplus (-{}^t\ell)$, lifts canonically to a morphism of $\mathbb{C}$-extensions $\widetilde{i} : \widetilde{\mathfrak{g}\ell}V \longrightarrow \widetilde{OW}$ constructed as follows. For $\ell_+ \in \mathfrak{g}\ell_+V$ choose a lattice $V_+ \supset Im\ell_+$. Then $i(\ell_+) \in P_{L(V_+)}O$. Put $\widetilde{i}_+(\ell_+) = s_{L(V_+)}i(\ell_+) \in \widetilde{OW}$; by 1.4.8 this element is independent of a choice of V_+. Similarly, for $\ell_- \in \mathfrak{g}\ell_-V$ choose a lattice $V'_+ \subset \mathrm{Ker}\ \ell_-$; then $i(\ell_-) \in P_{L(V'_+)}O$, and $\widetilde{i}_-(\ell_-) := s_{L(V'_+)}i(\ell_-) \in \widetilde{OW}$ depends on ℓ_- only. For $\ell_0 \in \mathfrak{g}\ell_0V$ one has $(\widetilde{i}_- - \widetilde{i}_+)(\ell_0) = tr_{L(V_+)/L(V_+)\cap L(V'_+)}(i\ell_0) = tr\ell_0$ by 1.4.8. According to 1.2.3 we get a canonical morphism $\widetilde{i} : \widetilde{\mathfrak{g}\ell}_{-1}V \longrightarrow \widetilde{OW}$ of $\mathbb{C}$-extensions such that $\widetilde{i}s_\pm = \widetilde{i}_\pm : \mathfrak{g}\ell_\pm V \longrightarrow \widetilde{OW}$ (here $\widetilde{\mathfrak{g}\ell}_{-1}V = (\widetilde{\mathfrak{g}\ell}V)_{-1}$, see 1.1.7).

The action of $\widetilde{\mathfrak{g}\ell}V$ on M preserves the Dim_V-grading. If M is irreducible, then it is natural to denote the $\widetilde{\mathfrak{g}\ell}_{-1}V$-module M^a, $a \in DimV$, as Λ^aV ("semi-infinite wedge power"). Note that $\wedge^aV$ (as well as M itself) is defined up to tensorization with 1-dimensional $\mathbb{C}$-vector space.

1.4.10 We will need a version "with formal parameter" of the above constructions. Namely, let $\mathcal{O} = \mathbb{C}[[q]]$ be our base ring. Consider a flat complete $\mathcal{O}$-module V (so $\varprojlim V/q^nV$). A Tate structure on V is given by Tate's $\mathbb{C}$-vector space structure on each V/q^nV such that each short exact sequence $0 \to V/q^mV \xrightarrow{q^n} V/q^{m+n}V \to V/q^nV \to 0$ is strongly compatible with the Tate structures (i.e., V/q^mV is a Tate's subspace of $V/q^{m+n}V$ and V/q^nV is the quotient space). A lattice $V_+ \subset V$ is an $\mathcal{O}$-submodule such that V/V_+ is $\mathcal{O}$-flat, $V_+ = \varprojlim V_+/q^nV_+$ and V_+/q^nV_+ is a lattice in V/q^nV for each n. One defines a colattice $V_- \subset V$ in a similar way. For a Tate $\mathcal{O}$-module V one defines its dual V^* in an obvious way; one has $V^*/q^nV^* = (V/q^nV)^*$, $V^{**} = V$.

Let W be Tate's $\mathcal{O}$-module and $(\,,) : W \times W \to \mathcal{O}$ be a non-degenerate symmetric form (i.e., a symmetric isomorphism $W \xrightarrow{\sim} W^*$). Let $Cliff(W)$ be the Clifford $\mathcal{O}$-algebra of $(\,,)$. A Clifford module M is a $Cliff(W)$-module such that M is flat as $\mathcal{O}$-module, $M = \varprojlim M/q^nM$, and W/q^nW acts on each M/q^nM in a continuous way (in discrete topology of M/q^nM). Such M carries the action of completed Clifford algebra

$$C\ell W = \varprojlim_n \varprojlim_{W_+^{(n)}} Cliff(W)/q^nCliff(W) + Cliff(W)W_+^{(n)}$$

(where $W_+^{(n)}$ is a lattice in W/q^nW). Clearly $M_0 ;= M/qM$ is Clifford module for $(W_0, (,)_0) := (W/qW, (,) mod q)$; if M' is another Clifford module, then $Hom(M, M')$ is a flat $\mathcal{O}$-module and $Hom(M, M')/qHom(M, M') = Hom(M_0, M'_0)$. In particular, if $(W_0, (,)_)$ is even-dimensional, then there exists a Clifford module M, unique up to isomorphism, such that M_0 is irreducible; one has $End M = \mathcal{O}$. All the facts 1.4.3-1.4.9 have an obvious $\mathbb{C}[[q]]$-version.

§2. Tate's residues and Virasoro-type extensions

2.1 Tate's construction of the local extension. Let F be a 1-dimensional local field, and $\mathcal{O}_F \subset F$ be the corresponding local ring. A choice of uniformization parameter $t \in \mathcal{O}_F$ identifies $\mathcal{O}_F$ with $\mathbb{C}[[t]]$, and F with $\mathbb{C}((t))$. Let E be an F-vector space of dimension $n < \infty$. Denote by $\mathcal{D}E$ the algebra of F-differential operators acting on E. A choice of a basis of E identifies $\mathcal{D}E$ with the algebra of matrix differential operators $a_N\partial_t^N + \cdots + a_1\partial_t + a_0$, $a_i \in Mat_n(F)$.

2.1.1 The space E, considered as $\mathbb{C}$-vector space, is actually a Tate's vector space in a canonical way. A basis of neighbourhoods of 0 is formed by $\mathcal{O}_F$-submodules of E that generate E as F-module. We will denote by $EndE, \mathfrak{g}\ell_\pm E$, etc., the corresponding algebras of endomorphisms of E, considered as Tate's $\mathbb{C}$-vector space.

Clearly $\mathcal{D}E \subset EndE$. We may restrict the central extension $\widetilde{\mathfrak{g}\ell}E$ of $\mathfrak{g}\ell E$ to $\mathcal{D}E^{Lie}$ to get a central extension $0 \to \mathbb{C} \to \widetilde{\mathcal{D}E} \to \mathcal{D}E^{Lie} \to 0$ of the Lie algebra $\mathcal{D}E^{Lie}$.

It is easy to compute a 2-cocycle of this extension explicitly. Namely, let us choose a parameter $t \in \mathcal{O}_F$ and an F-basis $\{v_i\}$ in E. Put $E_+ = \sum_i \mathcal{O}_F v_i$, $E_- = \sum_i t^{-1}\mathbb{C}[t^{-1}]v_i$: this is a lattice and a colattice in E and $E = E_+ \oplus E_-$. For $\ell \in \mathfrak{g}\ell E$ define the operator $\ell_+ \in \mathfrak{g}\ell_+ E$ by formula $\ell_+|_{E_+} = \ell|_{E_+}, \ell_+|_{E_-} = 0$. Clearly this map $\mathfrak{g}\ell E \to \mathfrak{g}\ell_+ E$, $\ell \longmapsto \ell_+$, lifts the canonical projection $\mathfrak{g}\ell E \to \mathfrak{g}\ell E/\mathfrak{g}\ell_- E = \mathfrak{g}\ell_+ E/\mathfrak{g}\ell_0 E$. Hence by 1.1.4 it defines a section $\sigma : \mathfrak{g}\ell E \to \widetilde{\mathfrak{g}\ell}E$; the corresponding 2-cocycle is given by formula $\ell_1, \ell_2 \longmapsto \alpha(\ell_1, \ell_2) = [\sigma(\ell_1), \sigma(\ell_2)] - \sigma([\ell_1, \ell_2]) = tr([\ell_{1+}, \ell_{2+}] - [\ell_1, \ell_2]_+)$. Take now $\ell_1 = At^a\frac{\partial_t^b}{b!}$, $\ell_2 = A't^{a'}\frac{\partial_t^{b'}}{b'!}$, where $A, A' \in Mat_n(\mathbb{C})$, $a, a' \in \mathbb{Z}, b, b' \in \mathbb{Z}_{\geq 0}$. Clearly $\alpha(\ell_1, \ell_2) = 0$ if $a - b \neq b' - a'$. Assume that $a - b = b' - a'$; since α is skew-symmetric we may assume that $n = a - b \geq 0$. Then one has

$$\alpha(\ell_1, \ell_2) = -Tr(AA')\sum_{i=0}^{n-1}\binom{i}{b'}\binom{i-n}{b}.$$

2.1.2 Let $\mathcal{A}E \subset \mathcal{D}E^{Lie}$ be a Lie subalgebra that consists of operators of order ≤ 1 with scalar symbol (i.e., the operators of type $a_0 + a_1\partial_t, a_0 \in End_F E, a_1 \in F$). Denote by $\mathcal{T}_F$ the Lie algebra of vector fields on F. One has a canonical short exact sequence of Lie algebras $0 \to End_F E^{Lie} \to \mathcal{A}E \xrightarrow{\sigma} \mathcal{T}_F \to 0$, $\sigma(a_0 + a_1\partial_t) = a_1\partial_t$. Let $\widetilde{\mathcal{A}E}$ be the $\mathbb{C}$-extension of $\mathcal{A}E$ induced from $\widetilde{\mathcal{D}E}$. The above formulas reduce to the following ones:

$$\alpha(At^a, Bt^b) = b\delta_a^{-b}trAB, \alpha(At^a, t^{b+1}\partial_t) = \frac{a-a^2}{2}\delta_a^{-b}trA,\ \alpha(t^{a+1}\partial_t, t^{b+1}\partial_t) = \frac{n}{6}(a^3-a)\delta_a^{-b}.$$

This is the Kac-Moody-Virasoro cocycle.

2.1.3 Consider the case $E = F$. One has an obvious embedding $\mathcal{T}_F \subset \mathcal{A}F$ which defines the $\mathbb{C}$-extension $\widetilde{\mathcal{T}}_F$ of $\mathcal{T}_F$ with cocycle $\alpha_{Vir}(t^{a+1}\partial_t, t^{b+1}\partial_t) = \frac{1}{6}(a^3 - a)\delta_a^{-b}$. This $\widetilde{\mathcal{T}}_F$ is called (a local) Virasoro algebra. For any $c \in \mathbb{C}$ consider the $\mathbb{C}$-extension $\widetilde{\mathcal{T}}_{Fc}$ (see 1.1.7). Since $\mathcal{T}_F$ is perfect, $\widetilde{\mathcal{T}}_{Fc}$ has no automorphisms. One knows that any central $\mathbb{C}$-extension of $\mathcal{T}_F$ is isomorphic (canonically) to a unique $\widetilde{\mathcal{T}}_{Fc}$ (one has $H^2(\mathcal{T}_F, \mathbb{C}) \simeq \mathbb{C}$).

2.1.4 Now consider for $j \in \mathbb{Z}$ a 1-dimensional F-vector space $\omega_F^{\otimes j}$ of j-differentials (the elements of $\omega_F^{\otimes j}$ are tensors $fdt^{\otimes j}, f \in F$). The Lie algebra $\mathcal{T}_F$ acts canonically on $\omega_F^{\otimes j}$ by Lie derivatives, i.e., we have a canonical embedding $\mathcal{T}_F \hookrightarrow \mathcal{A}\omega_F^{\otimes j}$. Denote by $\widetilde{\mathcal{T}}_F^{(j)}$ the corresponding $\mathbb{C}$-extensions of $\mathcal{T}_F$ induced from $\widetilde{\mathcal{A}\omega_F^{\otimes j}}$. The explicit formula for this action is $\varphi\partial_t(fdt^{\otimes j}) = (\varphi\partial_t(f) + jf\partial_t(\varphi))dt^{\otimes j}$, i.e., with respect to the basis $dt^{\otimes j}$ a field $t^{a+1}\partial_t$ acts as $t^{a+1}\partial_t + j(a+1)t^a$. The formulas 2.1.2 immediately show that a 2-cocycle for $\widetilde{\mathcal{T}}_F^{(j)}$ coincides with $(6j^2-6j+1)\alpha_{Vir}$. Hence $\widetilde{\mathcal{T}}_F^{(j)}$ coincides with $\widetilde{\mathcal{T}}_{F(6j^2-6j+1)}$.

2.2 A geometric construction of a global extension. Let us describe the above extensions in geometric language.

2.2.1 Let C be a smooth algebraic curve (not necessary compact). Denote by $\omega = \Omega^1_C$ the sheaf of 1-forms, and by $\mathcal{H} = H^1_{DR} = \Omega^1_C/d\mathcal{O}_C$ the de Rham cohomology sheaf (in the Zariski topology of C). For a vector bundle E on C let $\mathcal{D} = \mathcal{D}E$ denote the sheaf of differential operators on E, and $E^\circ := \omega E^*$. Then E is a left $\mathcal{D}$-module, E° is a right $\mathcal{D}$-module (so one has a canonical anti-isomorphism $t : \mathcal{D}E \to \mathcal{D}E^0$, see, e.g., [B]), and the pairing $E^0 \otimes E \xrightarrow{\langle\ \rangle} \omega$ quotients to the pairing $E^0 \bigotimes_{\mathcal{D}E} E \to \mathcal{H}$.

Let $\Delta : C \to C \times C$ be the diagonal; we will identify the sheaves on C with ones on $C \times C$ supported on the image of Δ. Consider the sheaf $E \boxtimes E^0 := p_1^*E \otimes p_2^*E^0$ on $C \times C$. Recall that one has a canonical isomorphism $\delta : E \boxtimes E^0(\infty\Delta)/E \boxtimes E^0 \underset{\sim}{\rightarrow} \mathcal{D}$. Explicitly, for a "kernel" $k(t_1,t_2) = e(t_1)e^0(t_2)f(t_1,t_2)$, $e \in E, e^0 \in E^0, f(t_1,t_2) \in \mathcal{O}_{C\times C}(\infty\Delta)$, the corresponding differential operator $\delta(k)$ acts on sections of E according to formula $(\delta(k)\ell)(t_1) = Res_{t_2=t_1}\langle k(t_1,t_2)\ell(t_2)\rangle = e(t_1)Res_{t_2=t_1}f(t_1,t_2)\langle e^0(t_2)\ell(t_2)\rangle$. Here $\ell \in E, \langle e^0(t_2)\ell(t_2)\rangle \in \omega, \langle k(t_1,t_2)\ell(t_2)\rangle \in E \boxtimes \omega(\infty\Delta)$; we take the residue along the t_2 variable. The right action of $\delta(k)$ on sections of E^0 is given by formula $(m\delta(k))(t_2) = Res_{t_1=t_2}f(x,t_2)\langle m(t_1)e(t_1)\rangle)e^0(t_2)$.

2.2.2 Put $\mathcal{P}E_n := \lim\limits_{\overleftarrow{i}} E \boxtimes E^0((n+1)\Delta)/E \boxtimes E^0(-i\Delta)$, $\mathcal{P}E = \cup\mathcal{P}E_n$, so we have an isomorphism $\delta : \mathcal{P}E/\mathcal{P}E_{-1} \underset{\sim}{\rightarrow} \mathcal{D}E$. Clearly $\mathcal{P}E$ is a $\mathcal{D}E$-bimodule (the left and right actions are the obvious actions along the first, resp. the second variable), and δ is a morphism of bimodules, i.e., $\mathcal{P}E$ is a DE-crossed module (see 1.1). Let $t : \mathcal{P}E \to \mathcal{P}E^0$ be minus the isomorphism "transposition of coordinates" (here minus comes since E, E^0 have "odd" nature). Then for $k \in \mathcal{P}E$ one has ${}^t\delta(k) = \delta({}^tk)$, and ^{t} is an "anti-isomorphism" between crossed modules.

The pairing $\langle\ \rangle : \mathcal{P}E \bigotimes_{\mathcal{D}E} \mathcal{P}E \to \mathcal{P}E_{-1}$ from 1.1, $\langle k_1,k_2\rangle = \delta(k_1)k_2 - k_1\delta(k_2)$, is given by formula

$$\langle k_1k_2\rangle(t_1,t_2) = (Res_{z=t_1} + Res_{z=t_2})\langle k_1(t_1,z)k_2(z,t_2)\rangle = \int_{\gamma_{t_1,t_2}} \langle k_1(t_1,z)k_2(z,t_2)\rangle.$$

Here $\langle k_1(t_1,z)k_2(z,t_2)\rangle$ is the 1-form of variable z (with values in $E_{t_1} \otimes E^0_{t_2}$), and γ_{t_1,t_2} is a loop round $z = t_1$ and $z = t_2$. The corresponding Lie algebra pairing $\{\ \} : S^2\mathcal{P}E \to \mathcal{P}E_{-1}$ is $\{k_1,k_2\} := \langle k_1,k_2\rangle + \langle k_2,k_1\rangle$. Let $tr : \mathcal{P}E_{-1} \to \omega$ be the composition $\mathcal{P}E_{-1} \to \mathcal{P}E_{-1}/\mathcal{P}E_{-2} = E \otimes E^0 \to \omega$. We have

$$tr\{k_1,k_2\} = (Res_1 - Res_2)\langle k_1(t_1,t_2)k_2(t_2,t_1)\rangle.$$

Here $k_2(t_2, t_1) = {}^t k_2 \in \mathcal{P}E^0$ is k_2 with coodinates transposed, $\langle k_1(t_1, t_2) k_2(t_2, t_1)\rangle$ is a 2-form with poles along the diagonal and $Res_1, Res_2 : \Omega^2_{C\times C}(\infty\Delta) \to \omega_C$ are residues around the diagonal along the first and second coordinates, respectively. Clearly, $Res_1 - Res_2$ vanishes on $\Omega^2_{C\times C}(\Delta)$ and has image in exact forms. In fact, there is a canonical map $\widetilde{Res} : \Omega^2_{C\times C}(\infty\Delta)/\Omega^2_{C\times C}(\Delta) \to \mathcal{O}_C$ such that $d\widetilde{Res} = Res_1 - Res_2$ (see [B Sch] (2.11)). An explicit formula for $\widetilde{Res}$ is

$$\widetilde{Res}(f(t_1, t_2)(t_1 - t_2)^{-i-1} dt_1 \wedge dt_2) = i!^{-1} \sum_{a+b=i-1} \partial^a_{t_1} \partial^b_{t_2} f(t_1, t_2)\big|_{t_1=t_2=t}.$$

Here $f(t_1, t_2) \in \mathcal{O}_{C\times C}$. Hence one has $tr\{k_1, k\} = d\widetilde{Res}\langle k_1, {}^t k_2\rangle$. Note that the symmetric pairing $\mathcal{P}E \otimes \mathcal{P}E \to \mathcal{O}_C,\ k_1, k_2 \longmapsto \{k_1, k_2\}^\sim := \widetilde{Res}\langle k_1, {}^t k_2\rangle$ vanishes on $\sum_{a+b=-1} \mathcal{P}E_a \otimes \mathcal{P}E_b$; in particular, it induces the pairing on $\mathcal{P}E_1/\mathcal{P}E_{-2}$.

According to 1.1.2, 1.1.3 we get a central extension $\widetilde{DE}$ of the Lie algebra DE^{Lie} by $\mathcal{H}$ defined by a following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{P}E_{-1} & \longrightarrow & \mathcal{P}E & \xrightarrow{\delta} & \mathcal{D}E & \longrightarrow & 0 \\
 & & \big\downarrow tr & & \big\downarrow & & \big\| & & \\
 & & \omega & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathcal{H} & \longrightarrow & \widetilde{\mathcal{D}E} & \longrightarrow & \mathcal{D}E^{Lie} & \longrightarrow & 0.
\end{array}$$

2.2.3 Denote by $\mathcal{A}E \subset \mathcal{D}E^{Lie}$ the Lie subalgebra of differential operators of order ≤ 1 with scalar symbol. In other words, $\mathcal{A}E$ is the Lie algebra of infinitesimal symmetries of (C, E): the elements of $\mathcal{A}E$ are pairs $(\tau, \widetilde{\tau})$, where $\tau \in \mathcal{P}_C$ is a vector field, and $\widetilde{\tau}$ is an action of τ on E (so $\widetilde{\tau}$ is an order 1 differential operator with symbol equal to τ).

The constructions of 2.2.2 give rise to a differential graded Lie algebra $\mathcal{A}E$ defined as follows. One has $\mathcal{A}^0 E = \mathcal{A}E$, $\mathcal{A}^{-1}E$ is pre-image of $\mathcal{A}E \subset \mathcal{D}E$ by the projection $\mathcal{P}E/\ker tr \xrightarrow{\delta} \mathcal{D}_E$ (so we have short exact sequence $0 \to \omega \to \mathcal{A}^{-1}E \xrightarrow{\delta} \mathcal{A}E \to 0$), and finally $\mathcal{A}^{-2}E = \mathcal{O}_C$; all the other components of $\mathcal{A}^{\cdot}E$ are zero ones. The differential $d : \mathcal{A}^{-2}E = \mathcal{O}_C \to \omega \subset \mathcal{A}^{-1}E$ is the de Rham differential, and $\mathcal{A}^{-1}E \to \mathcal{A}E$ is δ. The bracket components $[\ \ ]^{ij} : \mathcal{A}^i E \times \mathcal{A}^j E \to \mathcal{A}^{i+j}E$ are the following. $[\ \ ]^{00}$ is the usual bracket $[\ \ ]^{0-1}$ comes from $\mathcal{D}^{Lie}$-action on $\mathcal{P}E$, $[\ \ ]^{0,-2}$ is the action of $\mathcal{A}E$ on $\mathcal{O}_C$ via $\sigma : \mathcal{A}E \to \mathcal{T}_C$, and $[\ \ ]^{-1-1}$ is $\{\ ,\ \}^\sim$ defined above. So $\mathcal{A}^{\cdot}E$ contains de Rham complex $\Omega^{\cdot}_C[2]$ as an ideal, $\mathcal{A}^{\cdot}E/\Omega^{\cdot}_C[2]$ is acyclic and the central extension $\widetilde{\mathcal{A}}E = \mathcal{A}^{-1}E/d\mathcal{A}^{-2}E$ of $\mathcal{A}E$ by $\mathcal{H}$ (see 1.13) coincides with restriction of $\widetilde{\mathcal{D}}E$ to $\mathcal{A}E \subset \mathcal{D}E^{Lie}$.

2.2.4 Consider the case $E = \mathcal{O}_C$. An obvious embedding $\mathcal{P}_C \hookrightarrow \mathcal{A}\mathcal{O}_C$ defines the central $\mathcal{H}$-extension $\widetilde{\mathcal{P}}_C$ called a global Virasoro algebra. As in 2.1.3 for $c \in \mathbb{C}$ we will denote by $\widetilde{\mathcal{P}}_{Cc}$ the $\mathcal{H}$-extension of $\mathcal{P}_C$ which is c-multiple of $\widetilde{\mathcal{P}}_C$. Since $\mathcal{P}_C$ is perfect (see 2.5 below), the extensions $\widetilde{\mathcal{P}}_{Cc}$ have no automorphisms.

2.2.5 Consider for $j \in \mathbb{Z}$ the sheaf $\omega^{\otimes j}$. The natural action of $\mathcal{P}_C$ on $\omega^{\otimes j}$ by Lie derivatives defines a canonical embedding of Lie algebras $\mathcal{P}_C \hookrightarrow \mathcal{A}\omega^{\otimes j}$. Denote by $\widetilde{\mathcal{P}}_C^{(j)}$ the induced $\mathcal{H}$-extension $\widetilde{\mathcal{A}\omega}^{\otimes j}|_{\mathcal{P}_C}$. Given a local coordinate t, one may consider elements of $\widetilde{\mathcal{P}}_C^{(j)}$ as expressions

$$\varphi_{(f,g)}^{(j)} = \left[\frac{f(t_1)}{(t_2-t_1)^2} + j\frac{\partial_{t_1} f(t_1)}{t_2-t_1} + g(t_1)\right] dt_1^{\otimes j} dt_2^{\otimes 1-j},$$

where $f, g \in \mathcal{O}_C$, modulo the ones of type $\varphi_{(0,\partial_t h)}$. The map $\widetilde{\mathcal{P}}_C = \widetilde{\mathcal{P}}_C^{(0)} \to \widetilde{\mathcal{P}}_C^{(j)}$ defined by formula $\varphi_{(f,g)}^{(0)} \longmapsto \varphi_{(f,(6j^2-6j+1)g)}^{(j)}$ is a morphism of Lie algebras, and does not depend on a choice of a local coordinate t. Hence it defines a canonical isomorphism $\widetilde{\mathcal{P}}_{C(6j^2-6j+1)} \underset{\sim}{\rightarrow} \widetilde{\mathcal{P}}_C^{(j)}$ of $\mathcal{H}$-extensions of C (see [B Sch]). Unfortunately, we do not know any "coordinate-free" explanation of this isomorphism.

2.3 Compatibility with Tate's construction. Let $x \in C$ be a point. We may consider the constructions of 2.2 locally at x. Namely, let $\mathcal{O}_x^\wedge$ be the completed local ring of C at x, $\mathcal{O}_{(x,x)}^\wedge$ be the completed local ring of $C \times C$ at (x,x), $F_x \supset \mathcal{O}_x^\wedge$ the local field at x, so if t is a parameter at x then $\mathcal{O}_{(x,x)}^\wedge = \mathbb{C}[[t_1,t_2]]$. Denote by R the localization of $\mathcal{O}_{(x,x)}^\wedge$ with respect to $t_1^{-1}, t_2^{-1}, (t_1-t_2)^{-1}$. Put $\omega_{(x)} := F_x \otimes_{\mathcal{O}} \omega$, $E_{(x)} := F_x \otimes_{\mathcal{O}} E, \mathcal{D}_{(x)} = \mathcal{D}E_{(x)} := F_x \otimes_{\mathcal{O}} \mathcal{D}E_{(x)}, \mathcal{P}_{(x)} = \mathcal{P}E_{(x)} = E \otimes_{\mathcal{O}} R \otimes_{\mathcal{O}} E^0$: these are local versions of the objects in 2.2. We can manage all the constructions of 2.2 purely locally. In particular we get the central extension $\widetilde{\mathcal{D}}_{(x)}$ of $\widetilde{\mathcal{D}}_{(x)}^{Lie}$ by $\mathcal{H}_{(x)} = \omega_{(x)}/dF_x \overset{Res}{\underset{\sim}{\rightarrow}} \mathbb{C}$.

2.3.1 By 2.1, $E_{(x)}$ is a Tate's vector space, and we have the embedding $i_x : \mathcal{D}_{(x)} \hookrightarrow EndE_{(x)}$. For $k = k(t_1,t_2) \in \mathcal{P}_{(x)}$ let $k_-, k_+ \in EndE_{(x)}$ be the linear operator defined by formulas

$$[k_-(e)](t) = -Res_{t_2=0}\langle k(t,t_2)e(t_2)\rangle, [k_+(e)](t) = (Res_{t_2=t_1} + Res_{t_2=0})\langle k(t,t_2)e(t_2)\rangle.$$

Here $e(t) \in E_{(x)}, \langle k(t,t_2)e(t_2)\rangle \in E \otimes R \otimes \omega$, and the residues are taken along the second variable. According to 2.2.1 one has $i_x\delta(k) = k_- + k_+$. Denote by $i_{x\pm}^\# : \mathcal{P}_{(x)} \to EndE_{(x)}$ the maps $i_{x\pm}^\#(k) = k_\pm$.

2.3.2 Lemma. *(i) For $k \in \mathcal{P}_{(x)}$ one has $k_\pm \in End_\pm E_{(x)}$.*

(ii) The commutative diagram

$$\begin{array}{ccc} \mathcal{P}_{(x)} & \xrightarrow{i_x^\# = (i_{x+}^\#, i_{x-}^\#)} & End_+E_{(x)} \oplus End_-E_{(x)} \\ \downarrow \delta & & \downarrow \\ \mathcal{D} & \xrightarrow{i_x} & End\ E_{(x)} \end{array}$$

is an i_x-morphism of crossed modules (see 1.1).

(iii) For $k \in \ker\delta \subset \mathcal{P}_{(x)}$ one has $Res_x tr(k) = tri_x^\#(k) (= trk_+ = -trk_-)$.

(iv) Let us identify $E_{(x)}^0$ with $E_{(x)}^*$ via the pairing $(\ ,\ ) : E \times E^0 \to \mathbb{C}$, $(e,e^0) = Res\langle e, e^0\rangle$; this gives the anti-isomorphism $t : EndE_{(x)} \to EndE_{(x)}^0$. Then the diagram

$$\begin{array}{ccc} \mathcal{P}E_{(x)} & \xrightarrow{i_+^\#} & End_+E_{(x)} \\ t \downarrow \wr & & t \downarrow \wr \\ \mathcal{P}E_{(x)}^0 & \xrightarrow{i_-^\#} & End_-E_{(x)}^0 \end{array}$$

commutes.

Proof. Assume for simplicity of notation that $E = \mathcal{O}_C$, so $E_{(x)} = F_x$. The statement $k_- \in End_- F_x$ from (i) is clear, since k_- vanishes on the lattice $t^N \mathcal{O}_x^\wedge \subset F_x$ for N equal to the order of pole of $k(t_1, t_2)$ at divisor $t_2 = 0$. Now the fact that $k_+ \in End_+ F_x$ will follow from (iv). The statements (ii), (iii) are obvious. To prove (iv) let us compute the residues integrating the forms along cycles. Let $\gamma_\pm(t)$ be the following loops in the t_2-complex plane $t_1 = t$:

Then for any function $f \in F_x$ one has $[k_\pm(f)](t) = \frac{1}{2\pi i} \int_{\gamma_\pm(t)} k(t, t_2) f(t_2)$.

Denote by U a small neighbourhood of zero in $\mathbb{C} \times \mathbb{C}$ with coordinate cross and diagonal removed. One has the following 2-dimensional cycles $C_\pm$ in U. Fix a small real numbers $0 < \epsilon < r \ll 1$. Then $C_+ = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| = \epsilon,\ |z_2| = r\}$, $C_- = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| = r, |z_2| = \epsilon\}$; the orientation of C_+ is a standard orientation of $S^1 \times S^1$, and the one of C_- is minus the standard orientation.

The above formula for the action of $k_\pm$ implies that for a 1-form $g \in F_x^0 = \omega_{(x)}$ one has $(g, k_\pm(f)) = \int_{C_\pm} g(t_1) k(t_1, t_2) f(t_2)$. Since the transposition of coordinates identifies C_+ with C_-, this implies that $(g, k_+(f)) = (({}^t k)_-(g), f)$. $\square$

2.3.3 Now the morphism $i_x^\#$ 2.3.2(ii) of crossed modules together with compatibility 2.3.2(iii) defines the morphism of the corresponding $\mathbb{C}$-extensions $\widetilde{i}_x : \widetilde{\mathcal{D}}_{(x)} \to \widetilde{\mathfrak{g}\ell} E_{(x)}$, $\widetilde{i}_x(k) = s_+(k_+) + s_-(k_-)$, or, equivalently, the isomorphism of $\mathbb{C}$-extensions $\widetilde{D}_{(x)} \xrightarrow{\sim} \widetilde{\mathcal{D}E}_{(x)}$ (see 2.1.1).

2.3.4 Assume now that our curve C is compact. Let $X = \{x_i\} \subset C$ be a finite non empty set of points, and E be a vector bundle on $U = C \setminus X$. Put $E_{(X)} = \Pi E_{(x_i)}, \mathcal{D}_{(X)} = \Pi \mathcal{D}_{(x_i)}$. Denote by $\widetilde{\mathcal{D}}_{(X)}$, a $\mathbb{C}$-extension of $\mathcal{D}_{(X)}^{Lie}$ which is the Baer sum of $\mathbb{C}$-extensions $\mathcal{D}_{(x_i)}$, so $\widetilde{\mathcal{D}}_{(X)} = \Pi \widetilde{\mathcal{D}}_{(x_i)} / \{(a_i) \in \mathbb{C}^X : \sum a_i = 0\}$. Clearly $\widetilde{\mathcal{D}}_{(X)}$ coincides with the $\mathbb{C}$-extension $\widetilde{\mathcal{D}E}_{(X)}$ induced from $\widetilde{\mathfrak{g}\ell} E_{(X)}$ via the embedding $\mathcal{D}_{(X)} \hookrightarrow \prod End\ E_{(x_i)} \hookrightarrow End\ E_{(X)}$.

Put $\mathcal{D}_U := H^0(U, \mathcal{D}E_U)$ and consider the central extension $0 \to H^1_{DR}(U) \to \widetilde{\mathcal{D}}_U \to \mathcal{D}_U \to 0$ constructed in 2.2.2. One has the "localization around x_i" maps $\mathcal{D}_U \hookrightarrow \prod \mathcal{D}_{(x_i)}, \widetilde{\mathcal{D}}_U \to \prod \widetilde{\mathcal{D}}_{(x_i)}$. The composition $\widetilde{\mathcal{D}}_U \to \prod \widetilde{\mathcal{D}}_{(x_i)} \to \widetilde{\mathcal{D}}_{(X)}$ vanishes on $H^1_{DR}(U)$ (since $\sum_X Res_{x_i} = 0$). Hence it defines a canonical morphism $s_X : \mathcal{D}_U^{Lie} \to \widetilde{\mathcal{D}}_{(X)}$ that lifts the embedding $\mathcal{D}_U \hookrightarrow \mathcal{D}_{(X)}$.

This morphism could be constructed by pure linear algebra means. Namely, consider the colattice $E_U = H^0(U, E) \subset E_{(X)}$. Clearly $\mathcal{D}_U^{Lie} \subset P_{E_U} \subset \mathfrak{g}\ell E_{(X)}$, hence we have the splitting $s_{E_U | \mathcal{D}_U} : \mathcal{D}_U^{Lie} \to \widetilde{\mathcal{D}E}_{(X)} = \widetilde{\mathcal{D}}_{(X)}$ (see 1.2.5).

2.3.5 Lemma. *This splitting coincides with the above s_X.*

Proof. Let $\partial \in \mathcal{D}_U$ be a differential operator. Choose a section $k \in H^0(U \times U, E \boxtimes E^0(\infty\Delta))$ such that $\delta(k) = \partial$. Denote by $k_- = (k_-^{x_i}) \in Hom(E_{(X)}, E_U)$ the morphism given by formula $k_-(e_{x_i}) = \Sigma k_-^{x_i}(e_{x_i})$, $k_-^{x_i}(e_{x_i}) = -Res_{x_i}\langle k \cdot e_{x_i}\rangle \in E_U$. Here $e_{x_i} \in E_{x_i}, \langle k \cdot e_{x_i}\rangle \in H^0(U \times SpecF_{x_i}, E \boxtimes \omega(\infty\Delta))$ is a section obtained by convolution of k and e_{x_i} (where e_{x_i} is considered as a section of $\mathcal{O}_U \boxtimes E_{(x_i)}$ independent of first variable), and Res_{x_i} is residue along the second variable at x_i. Clearly k_- is a morphism of Tate spaces (here E_U is a discrete space).

Let $j = (j_{x_i}) : E_U \hookrightarrow E_{(X)}$ be the embedding. The residue formula implies that for $e \in E_U$ one has $k_-(j(e)) = \partial(e)$. Hence $j \circ k_- \in P_{E_U} \subset \mathfrak{gl}E_{(X)}$, one has $j \circ k_- \in \mathfrak{gl}_-E_{(X)}$, $\partial - j \circ k_- \in \mathfrak{gl}_+E_{(X)}$, and, according to 1.2.5, $s_{E_U}(\partial)$ coincides with $s_-(j \circ k_-) + s_+(\partial - j \circ k_-)$.

Now consider $j \circ k_-$ as a matrix $(j \circ k_-)_{x_j}^{x_i} \in Hom(E_{(x_i)}, E_{(x_j)})$. Let $j \circ k_-^{diag} = \Sigma(j \circ k_-)_{x_i}^{x_i} \in End\ E_{(X)}$ be the diagonal part of $j \circ k_-$. According to 2.3.2, one has $s_X(\partial) = s_-(j \circ k_-^{diag}) + s_+(\partial - j \circ k_-^{diag})$. Hence $s_X(\partial) - s_{E_U}(\partial) = tr(j \circ k_- - j \circ k_-^{diag})$. This is a trace of a matrix in $\mathfrak{gl}_0 E_{(X)}$ with zero diagonal component which is zero, q.e.d. □

2.3.6 We will often use the morphism s_X for appropriate subalgebras of $\mathcal{D}_U^{Lie}$, say, for $\mathcal{A}E_U$.

2.4 Spinors and theta-characteristics. Let W be a vector bundle on our curve C equipped with a symmetric non-degenerate pairing $(\ ,\ ) : W \times W \to \omega$.

2.4.1 One may consider $(\ ,\ )$ as an isomorphism $W \simeq W^0$, hence we have the involution ${}^t : \mathcal{D}W \to \mathcal{D}W$ such that ${}^t(\partial_1\partial_2) = {}^t\partial_2{}^t\partial_1$, and ^{t} acts on degree n symbols as multiplication by $(-1)^n$. Denote by $\mathcal{OD}W$ the anti-invariants of ^{t}; this is a Lie subalgebra of $\mathcal{D}W^{Lie}$.

The isomorphism $W \simeq W^0$ also defines an involution ${}^t : \mathcal{P}W \to \mathcal{P}W$ (see 2.2.2) such that ${}^t\delta = \delta^t$. Let $\mathcal{OP}W$ be the anti-invariants of ^{t} in $\mathcal{P}W$; put $o\delta = \delta\big|_{\mathcal{OP}W}$. The action of $\mathcal{D}W$ on $\mathcal{P}W$ defines the $\mathcal{OD}W$-action on $\mathcal{OP}W$, and $o\delta : \mathcal{OP}W \to \mathcal{OD}W$ is an $\mathcal{OD}W$-crossed module. The trace otr which is $-\frac{1}{2}$ of the composition $\ker o\delta \to W \otimes W^0 \xrightarrow{(\ ,\ )} \omega \longrightarrow \mathcal{H}$ defines by 1.1.3, a canonical central $\mathcal{H}$-extension $\widetilde{\mathcal{OD}W}$ of $\mathcal{OD}W$. In $\mathcal{OD}W$ we have a Lie subalgebra $\mathcal{OA}W = \mathcal{A}W \cap \mathcal{OD}W$ of infinitesimal symmetries of $(C, W, (\ ,\ ))$: this is an extension of $\mathcal{P}_C$ by an orthogonal Lie algebra $\mathcal{O}W \subset End\ W$. Denote by $\widetilde{\mathcal{OA}W}$ the central extension $\widetilde{\mathcal{OD}W}\big|_{\mathcal{OA}W}$. Note that if $rkW = 1$, i.e., if $W = \omega^{\otimes 1/2}$ is a theta-characteristic, then $\mathcal{O}\omega^{\otimes 1/2} = 0$, hence $\mathcal{OA}\omega^{\otimes 1/2} = \mathcal{T}_C$. The formula from 2.2.5 applied to $j = 1/2$ gives a canonical isomorphism $\widetilde{\mathcal{OA}\omega^{1/2}} = \widetilde{\mathcal{T}}_{C-1/2}$.

2.4.2 If E is any vector bundle, and $W = E \oplus E^0$ with obvious $(\ ,\ )$, then the Lie algebras embedding $j : \mathcal{D}E \to \mathcal{OD}W$, $\partial \longmapsto (\partial, -{}^t\partial)$, lifts to a morphism of crossed modules $j^\# : \mathcal{P}E \to \mathcal{OP}W$, $k \longmapsto (k, -{}^tk)$. For $k \in \ker\delta$ one has $otr(j^\# k) = -trk$. So we get a canonical morphism $\widetilde{j} : \widetilde{\mathcal{D}E}_{-1} \to \mathcal{OD}W$ of $\mathcal{H}$-extensions (see 1.1.7 for -1 index).

2.4.3 Let us consider a local version of the above construction. Now our curve is a punctured disc $SpecF_x$, so one has the identification $Res_x : \mathcal{H}(F_x) \underset{\sim}{\rightarrow} \mathbb{C}$. The

Tate $\mathbb{C}$-vector space $W_{(x)}$ carries a non-degenerate symmetric form $(\ , \ )_{\bullet}$ defined by formula $(w_1, w_2)_{\bullet} = Res_x(w_1, w_2)$. The action of $\mathcal{D}W_{(x)}$ on $W_{(x)}$ gives the embedding $oi_X : \mathcal{ODW}_{(x)} \hookrightarrow \mathcal{O}W_{(x)}$. It lifts to an oi_x-morphism $oi_x^{\#} : \mathcal{OP}W_{(x)} \longrightarrow \mathfrak{g}\ell_{-}W_{(x)}$ of crossed modules (for the latter crossed module see 1.4.7), $oi_x^{\#}(k) = k_{-}$, according to 2.3.2 (i),(ii),(iv). For $k \in \ker \delta$ one has $otr(k) = \frac{1}{2}trk_{-} = otr(k_{-})$ by 2.3.2 (iii), 1.4.7. Hence $oi_x^{\#}$ defines a canonical morphism of $\mathbb{C}$-extensions $\widetilde{oi_x} : \widetilde{\mathcal{ODW}_{(x)}} \hookrightarrow \widetilde{\mathcal{O}W_{(x)}}$.

2.4.4 Assume we are in a situation 2.3.4, i.e., we have a compact curve C, a finite set of points $X \subset C$, and our bundle $(W, (\ , \ ))$ on $U = C \setminus X$. We get a Tate vector space $W_{(X)} = \prod W_{(x_i)}$ with the form $(\ , \ )_{(X)} = \sum (\ , \ )_{(x_i)}$, a central $\mathbb{C}$-extension $\widetilde{\mathcal{ODW}_{(X)}} \subset \widetilde{\mathcal{O}W_{(X)}}$ of $\mathcal{ODW}_{(X)} = \prod \mathcal{ODW}_{(x_i)} \subset \mathcal{O}W_{(X)}$. Just as in 2.3.4 a localization at X morphism $\mathcal{ODW}_U := H^0(U, \mathcal{ODW}) \longrightarrow \mathcal{ODW}_{(X)}$ lifts canonically to a morphism $s_X : \mathcal{ODW}_U \longrightarrow \widetilde{\mathcal{ODW}_{(X)}}$; as in 2.3.5 this s_X coincides with the lifting $s_{W_U}\big|_{\mathcal{ODW}_U}$ from 1.4.8. Certainly s_X extends in an obvious manner to a morphism of Lie superalgebras $\mathcal{ODW}_U \ltimes W_U \to \mathfrak{a}W_{(X)}$ (here W_U has odd degree, for $\mathfrak{a}W_{(X)}$, see 1.4.6).

2.4.5 By Serre's duality W_U is a maximal isotropic colattice in $W_{(X)}$.

2.5 Simplicity of Lie algebra of vector fields. The following lemma will be of use:

2.5.1 Lemma. *Let C be a smooth curve. Then the Lie algebra $T = H^0(C, \mathcal{T}_C)$ of vector fields on C is simple.*

Proof. The case of compact C is clear, so we will assume that C is affine. Let $I \subset T$ be a non-zero ideal; we have to show that $I = T$. Let $\tau \in I$ be a non-zero vector field. Note that if $g \in \mathcal{O}(C)$ is a function such that $g\tau \in I$ and $f \in \mathcal{O}(C)$ is any function, then $\tau(f)g\tau = \frac{1}{2}([g\tau, f\tau] + [\tau, fg\tau]) \in I$. Let $A_\tau \subset \mathcal{O}(C)$ be the subalgebra of functions generated by all functions $\tau(f), f \in \mathcal{O}(C)$. The previous remark implies (by induction) that $A_\tau \tau \subset I$. One may describe A_τ explicitly, namely A_τ consists precisely of those $f \in \mathcal{O}(C)$ that take equal values at zeros of τ and $ord_x(f - f(x)) \geq ord_x(\tau)$ for any $x \in C$; this condition is non empty only for $x =$ zero of τ. (To see this, consider the morphism $\pi : C \to C' = SpecA_\tau$. Clearly A_τ is a curve. An easy local analysis at points at ∞ of C shows that π is finite. If $x, y \in C, x \neq y$, are not zeros of τ, then a finite jet at x, y of the functions $\tau(f), f \in \mathcal{O}(C)$, could be arbitrary ones, hence π is isomorphism on the complement of zeros of τ. An easy local analysis at zeros of τ finishes the proof). In particular, any function that vanishes at zeros of τ with large order of zero lies in A_τ. Hence I contains any vector field that vanishes at zeros of τ with sufficiently large order of zero (namely, twice that of τ). A trivial local analysis at zeros of τ (take brackets of elements of I with vector fields non-vanishing at zeros of τ) shows that $I = T$. □

2.5.2 Corollary. *If C is an affine curve, then T has no non-trivial finite dimensional representations.* □

§3. Localization of representations

3.1 Harish-Chandra modules. Recall some definitions.

3.1.1 Let K be a pro-algebraic group. A K-module M is a comodule over the coalgebra $\mathcal{O}(K)$. Equivalently, M is a vector space with an algebraic $K(\mathbb{C})$-action. Here "algebraic" means that M is a union of finite dimensional $K(\mathbb{C})$-invariant subspaces M_α such that $K(\mathbb{C})$ acts on M_α via an algebraic action of a factor group K/K_α of finite type. Any K-module is a Lie K-module in a natural way.

3.1.2 A Harish-Chandra pair $(\mathfrak{g}, K)$ consists of a Lie algebra $\mathfrak{g}$ and a pro-algebraic group K together with an "adjoint" action Ad of $K(\mathbb{C})$ on $\mathfrak{g}$ and a Lie algebra embedding $i : LieK \hookrightarrow \mathfrak{g}$ that satisfy the compatibilities:
(i) The embedding i commutes with adjoint actions of K.
(ii) The action Ad is "pro-algebraic": for any normal subgroup $K' \subset K$ such that K/K' has finite type the action of $K(\mathbb{C})$ on $\mathfrak{g}/i(LieK')$ is algebraic.
(iii) The $ad \circ i$-action of Lie K on $\mathfrak{g}$ coincides with the differential of the Ad-action.

3.1.3 Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair. A $(\mathfrak{g}, K)$-module, or a Harish-Chandra module, is a $\mathbb{C}$-vector space equipped with $\mathfrak{g}$- and K-module structures such that
(i) For $k \in K, h \in \mathfrak{g}, m \in M$ one has $Ad_k(h)m = khk^{-1}(m)$.
(ii) The two Lie K-actions on M (the one that comes from $\mathfrak{g}$-action via i, and the differential of K-action) coincide.

We denote by $(\mathfrak{g}, K)$-mod the category of $(\mathfrak{g}, K)$-modules.

3.1.4 Let T be any K-torsor. Denote $(\mathfrak{g}, K)_T = (\mathfrak{g}_T, K_T)$ the T-twist of $(\mathfrak{g}, K)$ with respect to adjoint action; this is a Harish-Chandra pair. If M is a $(\mathfrak{g}, K)$-module, then the T-twist M_T is a $(\mathfrak{g}_T, K_T)$-module, and $M \longmapsto M_T$ is equivalence of categories $(\mathfrak{g}, K)$-mod $\xrightarrow{\sim}$ $(\mathfrak{g}_T, K_T)$-mod.

3.1.5 The following version of the above definitions is quite convenient.

A pro-algebraic groupoid $\mathcal{V}$ is a groupoid such that for any object X the group $AutX$ carries a pro-algebraic structure and for any $f : X \to Y$ the map $Ad_f : AutY \xrightarrow{\sim} AutX$ preserves the pro-algebraic structures (the objects of $\mathcal{V}$ form a usual set with no algebraic structure). A $\mathcal{V}$-module is a functor $M : \mathcal{V} \to Vect_{\mathbb{C}}$ such that for any $X \in \mathcal{V}$ the $AutX$-action on M_X is algebraic.

A Harish-Chandra groupoid $(\mathfrak{g}, \mathcal{V})$ is a pro-algebraic groupoid $\mathcal{V}$ together with a functor $X \longmapsto (\mathfrak{g}_X, K_X)$ from $\mathcal{V}$ to the category of Harish-Chandra pairs equipped with a canonical identification of "group part" K_X of the functor with $AutX$; we assume that for $g \in AutX = K_X$ the "functorial" action of g on $\mathfrak{g}_X$ coincides with the Ad-action from 3.1.3.

One defines a representation of our Harish-Chandra groupoid (or simply a $(\mathfrak{g}, \mathcal{V})$-module) in the obvious manner. For any $X \in \mathcal{V}$ one has a canonical "fiber" functor $(\mathfrak{g}, \mathcal{V})$-mod $\to$ $(\mathfrak{g}_X, K_X)$-mod, $M \longmapsto M_X$. If $\mathcal{V}$ is connected, this functor is an equivalence of categories. Note that if T is a K_X-torsor, and $X_T \in \mathcal{V}$ is T-twist of X (i.e., X_T is an object of $\mathcal{V}$ equipped with isomorphism of K_X-torsors $T \xrightarrow{\sim} \mathrm{Hom}(X, X_T)$, then one has a canonical isomorphism $(\mathfrak{g}_{X_T}, K_{X_T}) = (\mathfrak{g}_X, K_X)_T$, $M_{X_T} = (M_X)_T$ (see 3.1.4).

3.1.6 We will need to consider the above objects depending on parameters.

Let S be a scheme, and K be a pro-algebraic group. A K-torsor on S is a projective limit of K/K'-torsors in the étale topology of S; here $K' \subset K$ is any normal subgroup such that K/K' has finite type.

Let $\mathcal{V}$ be a pro-algebraic groupoid. An S-object Y_S of $\mathcal{V}$ is a rule that assigns to each object $X \in \mathcal{V}$ on $AutX$-torsor $Y_S(X) = \underline{Hom}(X, Y_S)$ on S together with canonical identifications of $AutX$-torsors $Y_S(X) = Y_S(X')_{Hom(X,X')}$ (= the twist of $Y_S(X')$ by $AutX'$-torsor $Hom(X, X')$) for each $X, X' \in \mathcal{V}$; these identifications should satisfy an obvious compatibility condition for three objects $X, X', X'' \in \mathcal{V}$. In other words, Y_S is a functor from $\mathcal{V}$ to schemes over S such that the $AutX$-action defines on $Y_S(X)$ the structure of $AutX$-torsor, and for any connected component S' of S the objects X for which $Y_{S'}(X) = Y_S(X)_{S'}$ is non-empty are isomorphic. If M is a $\mathcal{V}$-module, then an S-object Y_S of $\mathcal{V}$ defines a locally free $\mathcal{O}_S$-module M_{Y_S} on S. If $Y_S(X)$ for $X \in \mathcal{V}$ is non-empty then M_{Y_S} coincides with $Y_S(X)$-twist of $M_X \otimes \mathcal{O}_S$.

Let now $(\mathfrak{g}, \mathcal{V})$ be a Harish-Chandra groupoid, and Y_S be an S-object of $\mathcal{V}$ (considered as pro-algebraic groupoid). We get a sheaf $\mathfrak{g}_{Y_S}$ of $\mathcal{O}_S$-Lie algebras; $\mathfrak{g}_{Y_S}$ is a projective limit of locally free $\mathcal{O}_S$-modules. For any $(\mathfrak{g}, \mathcal{V})$-module M the $\mathcal{O}_S$-module M_{Y_S} is a $\mathfrak{g}_{Y_S}$-module.

3.2 Lie algebroids. Let S be a scheme.

3.2.1 A Lie algebroid on S (which is an infinitesimal version of Lie groupoid) is a sheaf $\mathcal{A}$ of Lie algebras on S together with an $\mathcal{O}_S$-module structure on $\mathcal{A}$ and an $\mathcal{O}_S$-linear map $\sigma : \mathcal{A} \to \mathcal{T}_S$ such that σ is a morphism of Lie algebras, and the formula $[a, fb] = \sigma(a)(f)b + f[a, b]$ holds for $a, b \in \mathcal{A}, f \in \mathcal{O}_S$. Clearly $\mathcal{A}_{(0)} = \ker\sigma$ is a sheaf of $\mathcal{O}_S$-Lie algebras. In the case when S is smooth we will say that $\mathcal{A}$ is transitive if σ is surjective.

The Lie algebroids form a category $Lie(S)$ with final object $\mathcal{T}_S$. This category has products: for $\mathcal{A}, \mathcal{B} \in Lie(S)$ we have $\mathcal{A} \times \mathcal{B} = \mathcal{A} \times_T \mathcal{B}$ in the obvious notations. The categories $Lie(S)$ form a fibered category over the category of schemes. For a morphism $f : S' \to S$ of schemes and $\mathcal{A} \in Lie(S)$ the inverse image $f^*\mathcal{A} \in Lie(S')$ is defined by the formula $f^*\mathcal{A} = \mathcal{T}_{S'} \times f^*(\mathcal{A})$. Here $f^*(\mathcal{A}), f^*(\mathcal{T}_S)$ are inverse images in the categories of $\mathcal{O}$-modules, and the fibered product is $f^*(\mathcal{T}_S)$ taken with respect to projections $\mathcal{T}_{S'} \xrightarrow{df} f^*(\mathcal{T}_S) \xleftarrow{f^*(\sigma)} f^*(\mathcal{A})$.

3.2.2 Let $\mathcal{A}$ be a Lie algebroid. An $\mathcal{A}$-module is a sheaf $\mathcal{F}$ of $\mathcal{A}$-modules on S together with an $\mathcal{O}_S$-module structure such that for $a \in \mathcal{A}$, $f \in \mathcal{O}_S$, $m \in \mathcal{F}$ one has $a(fm) = \sigma(a)(f)m + f(am)$. We will also call such a structure an action of $\mathcal{A}$ on $\mathcal{O}_S$-module $\mathcal{F}$. If $\mathcal{A}, \mathcal{B}$ are Lie algebroids, $\mathcal{F}$ is an $\mathcal{A}$-module, G is a $\mathcal{B}$-module, then $\mathcal{F} \otimes_{\mathcal{O}_S} G$ is $\mathcal{A} \times \mathcal{B}$-module: for $(a, b) \in \mathcal{A} \times \mathcal{B}$, $m \in \mathcal{F}, n \in G$ one has $(a, b)(m \otimes n) = (am) \otimes n + m \otimes (bn)$.

3.2.3 Let $\mathcal{A}$ be a Lie algebroid, and $\mathfrak{g}$ an $\mathcal{O}_S$-Lie algebra equipped with an $\mathcal{A}$-action. An $\mathcal{A}$-morphism $\psi : \mathcal{A}_{(0)} \to \mathfrak{g}$ is a morphism of $\mathcal{O}_S$-Lie algebras that commutes with $\mathcal{A}$-action (here the $\mathcal{A}$-action on $\mathcal{A}_{(0)}$ is adjoint one). Note that if $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism of Lie algebroids, then $\mathcal{A}$ acts on $\mathcal{B}_{(0)}$ by $ad \circ \varphi$, and $\varphi_{(0)} : \mathcal{A}_{(0)} \to \mathcal{B}_{(0)}$ is an $\mathcal{A}$-morphism. Conversely, for an $\mathcal{A}$-morphism $\psi : \mathcal{A}_{(0)} \to \mathfrak{g}$ let $\mathcal{A}_\psi$ be the quotient of the semi-direct product $\mathcal{A} \ltimes \mathfrak{g}$ by the ideal $\mathcal{A}_{(0)} \hookrightarrow \mathcal{A} \ltimes \mathfrak{g}, a \longmapsto (a, -\psi(a))$. Then $\mathcal{A}_\psi$ is a Lie algebroid, $\mathcal{A}_{\psi_{(0)}} = \mathfrak{g}$, and we have a canonical morphism $\psi : \mathcal{A} \to \mathcal{A}_\psi$ with $\psi_{(0)} =$ old ψ. These constructions are mutually inverse: if $\mathfrak{g} = \mathcal{B}_{(0)}$, $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism of Lie algebroids, and $\psi = \varphi_{(0)}$, then we have a canonical morphism $i : \mathcal{A}_\psi \to \mathcal{B}$ which is an isomorphism if $\mathcal{A}$ is transitive.

3.2.4 Let $\mathcal{A}$ be a Lie algebroid. A central extension of $\mathcal{A}$ by $\mathcal{O}_S$ is a Lie algebroid $\widetilde{\mathcal{A}}$ together with a surjective morphism $\pi : \widetilde{\mathcal{A}} \to \mathcal{A}$ and a central element $1 \in \ker \pi$ such that the map $\mathcal{O}_S \underset{\sim}{\rightarrow} \ker \pi$, $f \longmapsto f \cdot 1$, is isomorphism. Note that the adjoint action of $\widetilde{\mathcal{A}}$ on $\widetilde{\mathcal{A}}_{(0)}$ quotients to an $\mathcal{A}$-action. We will call a central extension $\mathcal{L}$ of $\mathcal{T}_S$ by $\mathcal{O}_S$ an invertible Lie algebroid (so $\mathcal{L}_{(0)} = \mathcal{O}_S$).

3.2.5 Remarks. (i) Let B be any Lie algebroid, and let $tr : \mathcal{B}_{(0)} \to \mathcal{O}_S$ be a $\mathcal{B}$-morphism (we will call such tr a trace on $\mathcal{B}$). If $\mathcal{B}$ is transitive, then $\mathcal{B}_{tr}$ is an invertible algebroid.
(ii) Let $\widetilde{\mathcal{A}} \xrightarrow{\pi} \mathcal{A}$ be a central extension of $\mathcal{A}$ by $\mathcal{O}_S$, and $\gamma : \mathcal{A}_{(0)} \to \widetilde{\mathcal{A}}$ be an $\mathcal{O}$-linear section of π such that γ commutes with adjoint action of $\mathcal{A}$. Then $\gamma(\mathcal{A}_{(0)})$ is ideal in $\widetilde{\mathcal{A}}$, and $\widetilde{\mathcal{A}}/\gamma(\mathcal{A}_{(0)})$ is invertible algebroid.

3.2.6 The invertible Lie algebroids form a category $\mathcal{P}Lie(S)$ which is a Picard category, and, more generally, a "$\mathbb{C}$-vector space" in categories. This means that for $\alpha, \beta, \in \mathbb{C}$, $\mathcal{A}, \mathcal{B} \in \mathcal{P}Lie(S)$ we may form the linear combination $C = \alpha\mathcal{A} + \beta\mathcal{B} \in \mathcal{P}Lie(S)$: by definition $C = (\mathcal{A} \times \mathcal{B})_{tr_{\alpha,\beta}}$, where $tr_{\alpha,\beta}(f,g) = \alpha f + \beta g$. For $\mathcal{A} \in \mathcal{P}Lie(S)$ we have $Aut\mathcal{A} = \Omega_S^{1c\ell}$: for a closed 1 form ω the corresponding automorphism of $\mathcal{A}$ is $a \longmapsto a + \langle \omega\sigma(a) \rangle \cdot 1$. The trivial invertible algebroid is $\mathcal{T}_{SO} = \mathcal{T}_S \ltimes \mathcal{O}_S$ (where $O : \mathcal{T}_{S(0)} = 0 \to \mathcal{O}_S$ is the trivial trace map). The locally trivial invertible Lie algebroids form a full $\mathbb{C}$-linear subcategory canonically equivalent to the one of $\Omega^{1c\ell}$-torsors.

3.2.7 For $\mathcal{A} \in \mathcal{P}Lie(S)$ define $\mathcal{D}_{\mathcal{A}}$ to be the sheaf of associative $\mathbb{C}$-algebras on S together with a morphism of $\mathbb{C}$ Lie algebras $i : \mathcal{A} \to \mathcal{D}_{\mathcal{A}}$ such that $i\big|_{\mathcal{O}_S}$ is a morphism of associative algebras (in particular, $i(1)$ is 1 in $\mathcal{D}_{\mathcal{A}}$) and one has $i(f)i(a) = i(fa)$ for $f \in \mathcal{O}_S$, $a \in \mathcal{A}$, and universal with respect to these data. For example, if $\mathcal{A}$ is trivial, then $\mathcal{D}_{\mathcal{A}}$ is the usual algebra of differential operators on S. For arbitrary $\mathcal{A}$ this is a twisted differential operators ring, see, e.g. Appendix to [BK] for details. Clearly a $\mathcal{D}_{\mathcal{A}}$-module $\mathcal{F}$ is the same as an $\mathcal{A}$-module such that $1 \in \mathcal{A}$ acts on $\mathcal{F}$ as the identity operator. Since $\mathcal{D}_{\mathcal{A}}$ carries an obvious filtration with $gr\mathcal{D}_{\mathcal{A}} = Sym\mathcal{T}_S$, for a coherent $\mathcal{D}_{\mathcal{A}}$-module $\mathcal{F}$ we have its singular support $SS\mathcal{F}$ which is a closed conical subset in the cotangent bundle of S. A $\mathcal{D}_{\mathcal{A}}$-module $\mathcal{F}$ is called lisse if $SS\mathcal{F} = (0)$: this condition is equivalent to the fact that $\mathcal{F}$ is a vector bundle (as $\mathcal{O}_S$-module).

3.2.8 The standard example of a Lie algebroid is the current (or Atiyah) algebra $\mathcal{A}(E)$ of a vector bundle E. This is the Lie algebra of infinitesimal symmetries of E. The sections of $\mathcal{A}(E)$ are pairs $(\sigma(\tau), \tau)$, where $\sigma(\tau) \in \mathcal{T}_S$ and τ is an action of $\sigma(\tau)$ on E, or, equivalently, a first order differential operator on E with symbol $\sigma(\tau) \cdot id_E$. Clearly $\mathcal{A}(E)$ is transitive and $\mathcal{A}(E)_{(0)} = \mathfrak{g}\ell(E)$. If L is a line bundle, then $\mathcal{A}(L)$ is invertible algebroid; one has $\mathcal{A}(L_1 \otimes L_2) = \mathcal{A}(L_1) + \mathcal{A}(L_2)$, i.e., $\mathcal{A} : \mathcal{P}ic(S) \to \mathcal{P}Lie(S)$ is a morphism of Picard categories. The ring $\mathcal{D}_{\mathcal{A}(L)}$ coincides with the algebra $\mathcal{D}_L$ of differential operators on L. If E is any vector bundle, then tr: $\mathfrak{g}\ell(E) \to \mathcal{O}_S$ is a trace on $\mathcal{A}(E)$, and $\mathcal{A}(E)_{\mathrm{tr}} = \mathcal{A}(\det E)$: this canonical isomorphism comes from a natural action of $\mathcal{A}(E)$ on det E given explicitly by the Leibnitz rule $a(e_1 \wedge \ldots \wedge e_n) = ae_1 \wedge e_2 \wedge \ldots \wedge e_n + \cdots + e_1 \wedge \ldots \wedge ae_n$.

3.3 Localization of $(\mathfrak{g}, K)$-modules. Below we will explain the general pattern how to transform representations to $\mathcal{D}$-modules. We will start with some notations.

3.3.1 Let $(\widetilde{\mathfrak{g}}, \mathcal{V})$ be a Harish-Chandra groupoid. We will say that it is centered if for any $X \in \mathcal{V}$ there is a fixed central element $1 \in \widetilde{\mathfrak{g}}_X$, $1 \notin LieAutX$, that depends on X in a natural way. Put $\mathfrak{g}_X = \widetilde{\mathfrak{g}}_C/\mathbb{C}1$, so $\widetilde{\mathfrak{g}}_X$ is a central $\mathbb{C}$-extension of $\mathfrak{g}_X$.

Our $(\widetilde{\mathfrak{g}}, \mathcal{V})$ defines several Harish-Chandra groupoids with the same underlying proalgebraic groupoid $\mathcal{V}$. Namely, we have the groupoid $(\mathfrak{g}, \mathcal{V})$ that corresponds to $\mathfrak{g}_X$; for any $c \in \mathbb{C}$ we have the centered groupoid $(\widetilde{\mathfrak{g}}_c, \mathcal{V})$ with $\widetilde{\mathfrak{g}}_{cX}$ equal to c-multiple of the central extension $\widetilde{\mathfrak{g}}_X$ of $\mathfrak{g}_X$. Denote by $(\widetilde{\mathfrak{g}}, \mathcal{V})_c$-mod the category of $(\widetilde{\mathfrak{g}}_c, \mathcal{V})$-modules on which $1 \in \mathbb{C} \subset \widetilde{\mathfrak{g}}_c$ acts as identity.

3.3.2 Let S be a smooth scheme, K be a proalgebraic group and Y_S be a K-torsor over S. Denote by $\mathcal{A}Y_S$ the Lie algebroid of infinitesimal symmetries of (S, Y_S). Its sections are pairs (τ, τ_{Y_S}), where $\tau \in \tau_{Y_S}$ and $\mathcal{T}_{Y_S}$ is a lifting of τ to Y_S that commutes with K-action. Clearly $\mathcal{A}Y_{S(0)} = LieK_{Y_S}$ (= Y_S-twist of Lie $K\widehat{\otimes}\mathcal{O}_S$ with respect to the adjoint action of K); $\mathcal{A}Y_S$ is a transitive groupoid. If $(\mathfrak{g}, K)$ is a Harish-Chandra pair, then we have the $\mathcal{O}_S$-Lie algebra $\mathfrak{g}_{Y_S}$ (= Y_S-twist of $\mathfrak{g}\widehat{\otimes}\mathcal{O}_S$ with respect to the adjoint action). The Lie algebroid $\mathcal{A}Y_S$ acts on $\mathfrak{g}_{Y_S}$ in an obvious manner, and the canonical embedding $i : \mathcal{A}Y_{S(0)} = LieK_{Y_S} \hookrightarrow \mathfrak{g}_{Y_S}$ is an $\mathcal{A}Y_S$-morphism. According to 3.2.3 we get the transitive Lie algebroid $\mathcal{A}\mathfrak{g}_{Y_S} = \mathcal{A}Y_{Si}$ with $\mathcal{A}\mathfrak{g}_{Y_S(0)} = \mathfrak{g}_{Y_S}$. If M is a $(\mathfrak{g}, K)$-module, then M_{Y_S} (= Y_S-twist of $M \otimes \mathcal{O}_S$) is an $\mathcal{A}\mathfrak{g}_{Y_S}$-module.

Now let $(\mathfrak{g}, \mathcal{V})$ be a Harish-Chandra groupoid, and let Y_S be an S-object of $\mathcal{V}$. The above construction defines a transitive Lie algebroid $\mathcal{A}\mathfrak{g}_{Y_S}$ on S with $\mathcal{A}\mathfrak{g}_{Y_S(0)} = \mathfrak{g}_{Y_S}$. If M is a $(\mathfrak{g}, \mathcal{V})$-module, then M_{Y_S} is an $\mathcal{A}\mathfrak{g}_{Y_S}$-module in a natural way. Note that if $(\widetilde{\mathfrak{g}}, \mathcal{V})$ is a centered groupoid, then $\mathcal{A}\widetilde{\mathfrak{g}}_{Y_S}$ is a central $\mathcal{O}_S$-extension of $\mathcal{A}\mathfrak{g}_{Y_S}$.

3.3.3 Definition. *Let S be a smooth scheme and $(\widetilde{\mathfrak{g}}, \mathcal{V})$ be a centered Harish-Chandra groupoid. An S-localization data ψ for $(\widetilde{\mathfrak{g}}, \mathcal{V})$ is a collection $(Y_S, N, \varphi, \widetilde{\varphi}_{(0)})$ where*

(i) Y_S is an S-object of $\mathcal{V}$.

(ii) N is a transitive Lie algebroid on S.

(iii) $\varphi : N \to \mathcal{A}\mathfrak{g}_{Y_S}$ is a morphism of Lie algebroids.

(iv) $\widetilde{\varphi}_{(0)} : N_{(0)} \to \widetilde{\mathfrak{g}}_{Y_S}$ is a lifting of $\varphi_{(0)}$ such that for $n \in N, m \in N_{(0)}$ one has $\widetilde{\varphi}_{(0)}([n, m]) = [\varphi(n), \varphi_{(0)}(m)]$. □

3.3.4 A localization data ψ defines an invertible Lie algebroid $\mathcal{A}_\psi$ on S as follows. Consider a fiber product $\mathcal{A}\widetilde{\mathfrak{g}}_{Y_S}N = \mathcal{A}\widetilde{\mathfrak{g}}_{Y_S} \times_{\mathcal{A}\mathfrak{g}_{Y_S}} N$: this is a central $\mathcal{O}_S$-extension of N. This central extension splits over $N_{(0)}$ by means of the section $s : N_{(0)} \to \mathcal{A}\mathfrak{g}_{Y_S}N_{(0)}$, $s(m) = (\widetilde{\varphi}_{(0)}(m), m)$. Put $\mathcal{A}_\psi := \mathcal{A}\widetilde{\mathfrak{g}}_{Y_S}N/s(N_{(0)})$. Let $D_\psi = D_{\mathcal{A}_\psi}$ be the corresponding algebra of twisted differential operators.

3.3.5 Let $M \in (\widetilde{\mathfrak{g}}, \mathcal{V})_1$-mod be a Harish-Chandra module such that 1 acts as id_M. Then M_{Y_S} is an $\mathcal{A}\widetilde{\mathfrak{g}}_{Y_S}N$-module (via the projection $\mathcal{A}\widetilde{\mathfrak{g}}_{Y_S}N \to \mathcal{A}\widetilde{\mathfrak{g}}_{Y_S}$), and $\Delta_\psi M = M_{Y_S}/s(N_{(0)})M_{Y_S}$ is $\mathcal{A}_\psi$-module on which $1 \in \mathcal{A}_\psi$ acts as identity. Hence $\Delta_\psi M$ is a D_ψ-module. Clearly $\Delta_\psi : (\widetilde{\mathfrak{g}}, \mathcal{V})_1\text{-mod} \to D_\psi\text{-mod}$ is a right exact functor; we call it the S-localization functor that corresponds to ψ. Note that for a point $s \in S$ we have a Lie algebra map $N_{(0)s} \to \widetilde{\mathfrak{g}}_{Y_S}$ (where $N_{(0)s} = N_{(0)}/m_sN_{(0)}$), hence the fiber $\Delta_\psi(M)/m_s\Delta_\psi(M)$ coincides with coinvariants $M_{Y_s}/N_{(0)s}M_{Y_S}$.

3.3.6 The above constructions are functorial with respect to morphisms of localization data. Precisely, let $(\widetilde{\mathfrak{g}}', \mathcal{V}')$ be another centered Harish-Chandra groupoid, and $r : (\widetilde{\mathfrak{g}}, \mathcal{V}) \to (\widetilde{\mathfrak{g}}', \mathcal{V}')$ is a morphism of centered groupoids. One defines an

r-morphism of S-localization data $r^{\#} : \psi \to \psi'$ in an obvious manner. Such $r^{\#}$ defines the isomorphisms $r^{\#}_{\mathcal{A}} : \mathcal{A}_{\psi} \xrightarrow{\sim} \mathcal{A}_{\psi'}$, $r^{\#}_{D} : D_{\psi} \xrightarrow{\sim} D_{\psi'}$. For $M \in (\widetilde{\mathfrak{g}}, \mathcal{V})_1$-mod, $M \in (\widetilde{\mathfrak{g}}', \mathcal{V}')_1$-mod and an r-morphism $\ell : M \to M'$ we have $r^{\#}_{D}$-morphism $r^{\#}_{\Delta} : \Delta_{\psi}(M) \to \Delta_{\psi'}(M')$.

One has also functoriality with respect to base change. If $f : S' \to S$ is a morphism of smooth schemes, and ψ is an S-localization data for $(\widetilde{\mathfrak{g}}, \mathcal{V})$, then one gets an S'-localization data $f^*\psi$ for $(\widetilde{\mathfrak{g}}, \mathcal{V})$. One has $\mathcal{A}_{f^*\psi} = f^*\mathcal{A}_{\psi}$, and for $M \in (\widetilde{\mathfrak{g}}, \mathcal{V})_1$-mod one has a natural isomorphism $f^*\Delta_{\psi}(M) = \Delta_{f^*\psi}(M)$ of $D_{f^*\psi}$-modules.

3.3.7 An S-localization data ψ for $(\widetilde{\mathfrak{g}}, \mathcal{V})$ defines in an obvious way for each $c \in \mathbb{C}$ an S-localization data ψ_c for $(\widetilde{\mathfrak{g}}_c, \mathcal{V})$. One has $\mathcal{A}_{\psi_c} = c\mathcal{A}_{\psi}$ (see 3.2.6).

3.4 Localization along moduli of curves. This section collects some basic examples of the above localization constructions.

3.4.1 Let us describe a centered Harish-Chandra groupoid $(\widetilde{\mathcal{T}}, \mathcal{V})$ called the Virasoro groupoid. The underlying connected proalgebraic groupoid $\mathcal{V}$ is the groupoid of one-dimensional local fields with residue field equal $\mathbb{C}$ (the morphisms are isomorphisms of the local fields). Precisely, let $F \in \mathcal{V}$ be a local field, $\mathcal{O}_F \subset F$ the corresponding local ring, and $m_F \subset \mathcal{O}_F$ the maximal ideal. A choice of uniformizing parameter t identifies F with $\mathbb{C}((t))$ and $\mathcal{O}_F$ with $\mathbb{C}[[t]]$. The group $AutF = Aut\mathcal{O}_F$ is the projective limit of groups $Aut\mathcal{O}_F/m_F^n = AutF/Aut_nF$. These groups are obviously algebraic groups, our $AutF$ is a proalgebraic group, and $\mathcal{V}$ is a proalgebraic groupoid. Note that $AutF/Aut_1F = \mathbb{C}^*$, and $Aut_iF/Aut_{i+1}F$ is isomorphic to $\mathbb{C}$ for $i \geq 1$; in particular Aut_1F is the pro-unipotent radical of $AutF$. Explicitly, $Aut\mathbb{C}((t))$ coincides with the group of power series $a_1t + a_2t^2 + \cdots, a_1 \neq 0$, with multiplication law equal to composition of series.

Now for $F \in \mathcal{V}$ let $\mathcal{T}_F$ be the Lie algebra of vector fields on F and $\widetilde{\mathcal{T}}_F$ be the Virasoro $\mathbb{C}$-extension of $\mathcal{T}_F$ defined in 2.1.3. The Lie algebra $\mathcal{T}_F$ carries a canonical filtration $\mathcal{T}_{iF}$; for $F = \mathbb{C}((t))$ one has $\mathcal{T}_{iF} = t^{i+1}\mathbb{C}[[t]]\partial_t$. The subalgebra $\mathcal{T}_{-1F}$ preserves the lattice $\mathcal{O}_F \subset F$, hence we have a canonical splitting $s_{\mathcal{O}_F} : \mathcal{T}_{-1F} \to \widetilde{\mathcal{T}}_F$. Clearly $LieAutF = \mathcal{T}_{0F}$, and the embedding $s_{\mathcal{O}_F} : LieAutF \hookrightarrow \widetilde{\mathcal{T}}_F$ together with the natural $AutF$-action on $\widetilde{\mathcal{T}}_F$ define the Harish-Chandra pair $(\widetilde{\mathcal{T}}_F, AutF)$. This defines our centered Virasoro groupoid $(\widetilde{\mathcal{T}}, \mathcal{V})$.

3.4.2 Let S be a scheme. It is easy to see that an S-object Y_S of $\mathcal{V}$ is the same as a "family of formal discs" over S or, equivalently, a formal $\mathcal{O}_S$-algebra $\mathcal{O}_Y$ locally isomorphic to $\mathcal{O}_S[[t]]$. The corresponding Lie algebroid $\mathcal{A}Y_S$ consists of pairs (τ, τ_{Y_S}) where $\tau \in \mathcal{T}_S$ and $\tau_{Y_S} \in Der\mathcal{O}_{Y_S}$ is a τ-derivation of $\mathcal{O}_{Y_S}$.

3.4.3 Now let $\pi : C_S \to S$ be a family of smooth projective curves and $a : S \to C_S$ be a section of π. These define an S-localization data $\psi = \psi(C_S, a)$ for $(\widetilde{\mathcal{T}}, \mathcal{V})$ as follows. Our Y_S is the formal completion of C_S along $a(S)$, and N is the Lie algebroid of pairs (τ, τ_U) where $\tau \in \mathcal{T}_S$ and τ_U is a lifting of τ to $U = C_S \setminus a(S)$. Clearly $\mathcal{A}\mathcal{T}_{Y_S}$ is the Lie algebroid of pairs $(\tau, \tau_{Y_{S\setminus(a)}})$, where $\tau \in \mathcal{T}_S$ and $\tau_{Y_{S\setminus(a)}}$ is a lifting of τ to a meromorphic vector field on Y_S with possible pole at $a(S)$. Our $\varphi : N \to \mathcal{A}\mathcal{T}_{Y_S}$ is just the restriction of a vector field τ_U on $Y_S \setminus \{a\}$ = punctured neighbourhood of a. Now the lifting $\widetilde{\varphi}_{(0)} : N_{(0)} = \pi_*\mathcal{T}_{U/S} \to \widetilde{\mathcal{T}}_{Y_S}$ is the restriction to $\mathcal{T}_{U/S} \subset D_{U/S}$ of the morphism $s_a : \pi_*\mathcal{D}_{U/S} \to \widetilde{D}_{(a)}$ (here $D = D_{\mathcal{O}_{C/S}}$) defined in 2.3.4 (more precisely, in 2.3.4 we considered the case of a single curve, $S =$ point; the generalization to families is immediate). These $(Y_S, N, \varphi, \widetilde{\varphi}_{(0)})$ is our localization data $\psi(C_S, a)$. According to 3.3.4, 3.3.5, 3.3.7 for any $c \in \mathbb{C}$ we have the localization functor $\Delta_{\psi_c(C_S,a)} : (\widetilde{\mathcal{T}}, \mathcal{V})_c\text{-mod} \to \mathcal{D}_{\psi_c(C_S,a)}\text{-mod}$.

3.4.4 Here is an explicit description of $\mathcal{A}_{\psi(C_S,a)}$ and $\Delta_{\psi(C_S,a)}$. Choose (locally on S) a formal parameter t at a, so $\mathcal{O}_{Y_S} = \mathcal{O}_S[[t]]$. Consider the space B of triples $(\tau, \tau_U, \widetilde{\tau}_U^v)$, where $\tau \in \mathcal{T}_S$, τ_U is a lifting of τ to U, and $\widetilde{\tau}_U^v : S \to \widetilde{\mathcal{T}}_{\mathbb{C}((t))}$ is a lifting of a vertical component of τ_U, $\tau_U^v = \tau_U(t)\partial_t : S \to \mathcal{T}_{\mathbb{C}((t))}$. This B is a Lie algebroid on S in an obvious manner. We have a canonical morphism $\mathcal{T}_{U/S} \to B_{(0)}$, $\nu \longmapsto (o, \nu, s_a(\nu))$, see 2.3.4. One has $\mathcal{A}_{\psi(C_S,a)} = \mathcal{B}/\mathcal{T}_{U/S}$. Now let M be a $(\widetilde{\mathcal{T}}, \mathcal{V})_c$-module. One has $M_{Y_S} = M_{\mathbb{C}((t))} \otimes \mathcal{O}_S$. The algebroid $\mathcal{B}$ acts on M_{Y_S} by formula $(\tau, \tau_U, \widetilde{\tau}_U^v)(m \otimes f) = m \otimes \tau(f) + \widetilde{\tau}_U^v(m \otimes f)$. One has $\Delta_{\psi(C_S,a)}(M) = M_{Y_S}/\mathcal{T}_{U/S} M_{Y_S}$.

3.4.5 Variant. For any non empty finite set A we may consider the centered groupoid $(\widetilde{\mathcal{T}}^A, \mathcal{V}^A)$. Here $\mathcal{V}^A$ is the A-th power of $\mathcal{V}$ and $\widetilde{\mathcal{T}}^A_{\{F_a\}}$ is the Baer sum of $\mathbb{C}$-extension $\widetilde{\mathcal{T}}_{F_a}$, $a \in A$ (so $\widetilde{\mathcal{T}}^A_{\{F_a\}}$ is a $\mathbb{C}$-extension of $\prod_{a \in A} \mathcal{T}_{F_a}$). A family $\pi : C_S \to S$ of curves together with a disjoint set A of sections (where "disjoint" means that for $a_i \neq a_j \in A$ and any $s \in S$ one has $a_i(s) \neq a_j(s) \in C_S$) defines an S-localization data $\psi(C_S, A)$ for $(\widetilde{\mathcal{T}}^A, \mathcal{V}^A)$ in a way similar to 3.4.2. For example, the corresponding Lie algebroid N consists of pairs (τ, τ_U), where $\tau \in \mathcal{T}_S$ and τ_U is a lifting of τ to $U = C_S \setminus \coprod_{a \in A} a_i(S)$.

3.4.6 Remark. Let $B \subset A$ be a non-empty subset. The groupoids $(\widetilde{\mathcal{T}}^B, \mathcal{V}^B)$ and $(\widetilde{\mathcal{T}}^A, \mathcal{V}^A)$ are related by an obvious correspondence $(\widetilde{\mathcal{T}}^B, \mathcal{V}^B) \xleftarrow{\pi_B} (\widetilde{\mathcal{T}}^{A,B}, \mathcal{V}^A) \xrightarrow{i_A} (\widetilde{\mathcal{T}}^A, \mathcal{V}^A)$, where $\widetilde{\mathcal{T}}^{A,B}_{\{F_a\}} = \widetilde{\mathcal{T}}^B_{\{F_b\}_{b \in B}} \times \prod_{a \in A \setminus B} \mathcal{T}_{-1F_a} \hookrightarrow \widetilde{\mathcal{T}}^A_{\{F_a\}}$. Any family of curves $\pi : C_S \to S$ and a set A of disjoint sections defines an S-localization data $\psi(C_S, A, B)$ for $(\widetilde{\mathcal{T}}^{A,B}, \mathcal{V}^A)$ in an obvious manner together with corresponding morphisms $\psi(C_S, B) \xleftarrow{\pi_B^{\#}} \psi(C_C, A, B) \xrightarrow{i_A} \psi(C_S, A)$. These define the corresponding isomorphisms $D_{\psi_c(C_S,B)} \xrightarrow{\sim} D_{\psi_c(C_S,A,B)} \xrightarrow{\sim} D_{\psi_c(C_S,A)}$. For $M_B \in (\mathcal{T}^B, \mathcal{V}^B)_c$-mod, $M_A \in (\mathcal{T}^A, \mathcal{V}^A)$-mod a morphism $f : M_B \to M_A$ is, by definition, an i_A-morphism from M_B, considered as $(\widetilde{\mathcal{T}}^{A,B}, \mathcal{V}^A)$-module via π_B, to M_A. Since $\Delta_{\psi_c(C_S,B)} M_B = \Delta_{\psi_c(C_S,A,B)} M_B$, such an f defines a morphism $\Delta(f) : \Delta_{\psi_c(C_S,B)} M_B \to \Delta_{\psi_c(C_S,A)} M_A$. For example, if $M_A = Ind^{\widetilde{\mathcal{T}}^A}_{\widetilde{\mathcal{T}}^{A,B}}(M_B)$ and f is the canonical embedding, then $\Delta(f)$ is isomorphism.

Note that the above canonical identification $D_{\psi_c(C_S,A)} = D_{\psi_c(C_S,B)}$ for $B \subset A$ actually provides a canonical algebra $D_{\psi_c(C_S)}$ that depends on C_S only together with canonical isomorphisms $D_{\psi_c(C_S)} = D_{\psi_c(C_S,A)}$ for any set A of disjoint sections. To construct $D_{\psi_c(C_S)}$ we may assume, working locally in étale topology of S, that C_S has many sections. To construct $D_{\psi_c(C_S)}$ it suffices to define for any two sets A, A' of disjoint sections a canonical isomorphism $D_{\psi_c(C_S,A)} = D_{\psi_c(C_S,A')}$. Choose a non-empty set B of sections such that both $A \sqcup B, A' \sqcup B$ are sets of disjoint sections. Our isomorphism is $D_{\psi_c(C_S,A)} = D_{\psi_c(C_S,A \sqcup B)} = D_{\psi_c(C_S,B)} = D_{\psi_c(C_S,A' \sqcup B)} = D_{\psi_c(C_S,A')}$. One verifies easily that this does not depends on the choice of B. We will compute $D_{\psi_c(C_S)}$ explicitly in 3.5.6.

3.4.7 Variant. Often the Virasoro modules are integrable only with respect to the subgroup $\mathrm{Aut}_1 F$ (see 3.4.1). To localize them one needs to consider the groupoid $(\widetilde{\mathcal{T}}, \mathcal{V}_1)$. The objects of $\mathcal{V}_1$ are pairs (F, ν), where F is a local field and $\nu \in m_F/m_F^2$, $\nu \neq 0$, is a 1-jet of a parameter. One has $\mathrm{Aut}(F, \nu) = \mathrm{Aut}_1 F$. The Lie algebra $\widetilde{\mathcal{T}}_{(F,\nu)}$ is $\widetilde{\mathcal{T}}_F$. If $\pi : C_S \to S$ is a family of curves, $a : S \to C_S$ a section, and $\nu \in a^*\Omega^1_{C_S/S}$ a 1-jet of parameters at a, then we get an S-localization data $\psi(C_S, a, \nu)$ for $(\widetilde{\mathcal{T}}, \mathcal{V}_1)$. We may also consider many points, as in 3.4.5.

We have a "forgetting of ν" morphism $r : (\widetilde{\mathcal{T}}, \mathcal{V}_1) \to (\widetilde{\mathcal{T}}, \mathcal{V})$ and a corresponding r-morphism of localization data $\psi_c(C_S, a, \nu) \to \psi_c(C_s, a)$. This defines a canonical isomorphism $r_D : D_{\psi_c(C_S,a,\nu)} \underset{\sim}{\leftarrow} D_{\psi_c(C_S,a)}$ and for any $M \in (\mathcal{T}, \mathcal{V})_c$-mod the r_D-isomorphism $r_M : \Delta_{\psi_c(C_S,a,\nu)} M \underset{\sim}{\leftarrow} \Delta_{\psi_c(C_S,a)} M$.

3.4.7.1 Let C be a fixed curve, $a \in C$, and ν a 1-jet of parameter at a. Consider a constant $\mathbb{C}^*$-family $C_{\mathbb{C}^*} = C \times \mathbb{C}^*$ with constant point a, and put $\nu^\vee(u) = u\nu$ for $u \in \mathbb{C}^*$. We get the corresponding $\mathbb{C}^*$-localization data $\psi = \psi(C_{\mathbb{C}^*}, a, \nu^\vee)$. One has $D_\psi = D_{\psi(C_{\mathbb{C}^*},a,\nu^\vee)} = D_{\psi(C_{\mathbb{C}^*},a)} = D_{\mathbb{C}^*}$ – the usual ring of differential operators. In particular, we have $\lambda\partial_\lambda \in D_{\psi_c}$. Let us compute the action of $u\partial_u$ on $\Delta_{\psi_c}(M)$ for $M \in (\mathcal{T}, \mathcal{V}_1)_c$-mod. Choose a parameter t_a at a on C such that $dt(a) = \nu$. Then $t_{au} = ut$ is a $\mathbb{C}^*$-family of parameters which identifies our $\mathcal{O}_{Y_{\mathbb{C}^*}}$ with $\mathcal{O}_{\mathbb{C}^*}[[t]]$. We have $M_{Y_{\mathbb{C}^*}} = M_{\mathbb{C}((t))} \otimes \mathcal{O}_{\mathbb{C}^*}$, and $\Delta_{\psi_c}(M)$ is a quotient of $M_{Y_{\mathbb{C}^*}}$. For $m \in M_{\mathbb{C}((t))}$ denote by $\overline{m}$ its image in $\Delta_{\psi_c}(M)$. Put $L_0 = s_{\mathbb{C}[[t]]}(t\partial_t) \in \widetilde{\mathcal{T}}_{\mathbb{C}((t))}$. One has $u\partial_u(\overline{m}) = \overline{L_0 m}$. In particular, if M is a higher weight module (see 7.3.1), then $\Delta_\psi M$ is smooth along $\mathbb{C}^*$ with monodromy equal to the action of $T = \exp(2\pi i L_0)$ (see 7.3.2).

3.4.8 Now consider the case "vector symmetries". Our "Virasoro-Kac-Moody" centered Harish-Chandra groupoid $(\widetilde{\mathcal{A}}, \mathcal{VV})$ defined as follows. The objects of $\mathcal{VV}$ are pairs $(F, E_\mathcal{O})$ where F is a local field, and $E_\mathcal{O}$ is a free $\mathcal{O}_F$-module of finite rank; we put $E_F = F \otimes E_\mathcal{O}$. The morphisms are defined in an obvious manner. Clearly $\mathrm{Aut}(F, E_\mathcal{O})$ is extension of Aut F by $\mathrm{GL}(E_\mathcal{O}) = \mathrm{Aut}_{\mathcal{O}_F}(E_\mathcal{O})$; this is a proalgebraic group. We put $\widetilde{\mathcal{A}}(F, E_\mathcal{O}) = \widetilde{\mathcal{A}E_F}$, see 2.1.2. The canonical embedding $s_{E_\mathcal{O}}$: Lie Aut $(F, E_\mathcal{O}) \to \widetilde{\mathcal{A}E_F}$ defines the Harish-Chandra pair $(\widetilde{\mathcal{A}E_F}, \mathrm{Aut}(F, E_\mathcal{O}))$. This defines our centered groupoid $(\widetilde{\mathcal{A}}, \mathcal{VV})$.

Let S be a scheme. An S-object of $\mathcal{VV}$ is a pair (Y_S, E_{Y_S}), where Y_S is an S-object of $\mathcal{V}$ (see 3.4.2) and E_{Y_S} is a locally free $\mathcal{O}_{Y_S}$-module of finite rank.

Assume that S is smooth. Let $\pi : C_S \to S$ be a famly of smooth projective curves, $a : S \to C_S$ a section, and let E be a vector bundle on C_S. These define an S-localization data $\psi(C_S, E, a)$. Namely, the corresponding S-object of $\mathcal{VV}$ is the completion of C_S, E along a. The Lie algebroid N consists of triples $(\tau, \tau_U, \tau_{E_U})$, where $\tau \in \mathcal{T}_S$, τ_U is a lifting of τ to $U = C_S \setminus a(S)$, and τ_{E_U} is an action of τ_{E_U} on E_U. The morphisms $\varphi, \widetilde{\varphi}_{(0)}$, appear precisely as in 3.4.3 from 2.3.4.

As above, this localization data gives rise to a localization functor. The versions 3.4.5-3.4.7 are immediate.

3.4.9 Let us consider now the spinor or "fermionic" version. The corresponding centered Harish-Chandra groupoid $(\widetilde{\mathcal{OA}}, \mathcal{OV})$ is defined as follows. Its objects are triples $Q = (F, W_\mathcal{O}, (\ ,\ ))$, where F is a local field, W is a free $\mathcal{O}_F$-module of finite rank, and $(\ ,\ )$: $W_\mathcal{O} \times W_\mathcal{O} \to \omega_{\mathcal{O}_F}$ is a symmetric bilinear form with values in

1-forms of $\mathcal{O}_F$. We assume that (,) is maximally non-degenerate, i.e., the cokernel of the corresponding map $W_{\mathcal{O}} \to W^0_{\mathcal{O}} = \mathrm{Hom}_{\mathcal{O}_F}(W_{\mathcal{O}}, \omega_{\mathcal{O}_F})$ is either trivial (such Q is called even) or a 1-dimensional $\mathbb{C}$-vector space (such Q is called odd). The morphisms in $\mathcal{OV}$ are the obvious ones. For Q as above, put $W_F = F \otimes W_{\mathcal{O}}$; our (,) extends to non-degenerate form $(\ , \ ) : W_F \times W_F \to \omega_F$. Note that our condition means that $W_{\mathcal{O}}$ is a maximal isotropic lattice in W_F. We may consider W_F as a Tate's $\mathbb{C}$-vector space with form $(\ , \ )_\bullet = \mathrm{Res}(\ , \ )$ (see 2.4.3); then $W_{\mathcal{O}}$ is also a maximal isotropic $(\ , \ )_\bullet$-lattice so Q is even iff W_F is even-dimensional, see 1.4.1. We put $\widetilde{\mathcal{OA}}(Q) = \widetilde{\mathcal{OA}W}_F$ (see 2.4.1). The Lie algebra Lie Aut $Q \subset \mathcal{OA}W_F$ preserves $W_{\mathcal{O}}$, hence we have a canonical embedding $s_{W_{\mathcal{O}}}$: Lie Aut $Q \hookrightarrow \widetilde{\mathcal{OA}}(Q)$. This defines the Harish-Chandra pair $(\widetilde{\mathcal{OA}}(Q), \mathrm{Aut}\ Q)$, and we get the groupoid $(\widetilde{\mathcal{OA}}, \mathcal{OV})$.

Remark. Clearly Q is even (resp. odd) iff $(W_F, (\ , \ )_\bullet)$ is even (resp. odd) dimensional, see 1.4.1. The two objects of Q are isomorphic iff the W's have the same rank and parity.

Now let S be a smooth scheme. Let $\pi : C_S \to S$ be a family of smooth projective curves, $a : S \to C_S$ a section, W a vector bundle on C_S, and $(\ , \ ) : W \times W \to \omega_{C_S/S}$ a symmetric bilinear pairing. Assume that the cokernel of the corresponding map $W \to W^0 = \mathrm{Hom}(W, \omega_{C_S/S})$ is either trivial or supported on $a(S)$ and is an $\mathcal{O}_S$-module of rank 1. This collection $(C_S, a, W, (\ , \ ))$ defines an S-localization data ψ for $(\widetilde{\mathcal{OA}}, \mathcal{OV})$ in a way similar to 3.4.3, 3.4.8. Namely, the formal completion of W along a defines an S-object of $\mathcal{OV}$. The Lie algebroid N consists of triples $(\tau, \tau_U, \tau_{W_U})$, where $\tau \in \mathcal{T}_S$, $\tau_U \in \mathcal{T}_U$ is a lifting of τ to $U = C_S \setminus a(S)$, and τ_{W_U} is an action of τ_U on W_U that preserves (,). The corresponding map φ is obvious, and $\widetilde{\varphi}_{(0)}$ comes from 2.4.4.

One has immediate variants of this construction for the case of several points and points with 1-jet of a parameter (see 3.4.6, 3.4.7).

3.4.10 Note that we have a canonical morphism $r : (\widetilde{\mathcal{A}}_{-1}, \mathcal{VV}) \to (\widetilde{\mathcal{OA}}, \mathcal{OV})$ of centered Harish-Chandra groupoids. It assigns to $(F, E_{\mathcal{O}}) \in \mathcal{VV}$ the triple $(F, E_{\mathcal{O}} \oplus E^0_{\mathcal{O}}, (\ , \ ))$ where (,) is the obvious pairing. The morphism $\widetilde{\mathcal{A}}E_F \to \widetilde{\mathcal{OA}}(E_F \oplus E^0_F)$ was defined in 2.4.2. Now for a scheme S and a collection (C_S, a, E) from 3.4.8 we have $(C_S, a, E \oplus E^0, (\ , \ ))$ from 3.4.9. We have an obvious r-morphism of corresponding localization data $r^\# : \psi_c(C_S, a, E) \to \psi_{-c}(C_S, a, E \oplus E^0, (\ , \ ))$ (see 2.4), hence the isomorphism $r_D : D_{\psi_c(C_S,a,E)} \xrightarrow{\sim} D_{\psi_c(C_S,a,E\oplus E^0,(\ , \ ))}$.

3.5 Fermions and determinant bundles. In this section the rings of twisted differential operators D_ψ that appeared in 3.4 will be canonically identified with the rings $\mathcal{D}_L$ for some natural line bundles L (see 3.2.8). Equivalently, we will construct a D_ψ-module L which is a line bundle (as $\mathcal{O}$-module). This will be done by means of Clifford modules.

3.5.1 Let us start with the situation in 3.4.9. For $Q = (F, W_{\mathcal{O}}, (\ , \ )) \in \mathcal{OV}$ denote by M_Q the Clifford module (for Clifford algebra $C\ell(Q) = C\ell\ (W_F.(\ , \ )_\bullet)$, see 1.4) generated by a single fixed vector v with the only relation $W_{\mathcal{O}} v = 0$. If Q is even, then M_Q is irreducible; if Q is odd, then M_Q is the sum of two non-isomorphic irreducible modules. Note that M_Q carries a canonical Aut Q-action (the only one) that leaves v invariant. By 2.4.3 M_Q is an $\widetilde{\mathcal{OA}}W_F = \widetilde{\mathcal{OA}}_Q$-module. Clearly these actions are compatible, hence M_Q is an $(\widetilde{\mathcal{OA}}_Q, \mathrm{Aut}\, Q)$-module. This way we get the $(\widetilde{\mathcal{OA}}, \mathcal{OV})$-module M.

Let S be a smooth scheme, and $(C_S, a, W, (\ ,\ ))$ the geometric data from 3.4.9 that defines the corresponding S-localization data ψ for $(\mathcal{OV}, \widetilde{\mathcal{OA}})$. Let $Q_S = (F_S, W_{\mathcal{O}_{F_S}}, (\ ,\ ))$ be the corresponding S-object of $\mathcal{OV}$ (= the completion of our data along a), and M_{Q_S} be the corresponding $\mathcal{O}_S$-module with $\widetilde{\mathcal{OA}}_{Q_S}$- action. Certainly, M_{Q_S} is a Clifford module for the $\mathcal{O}_S$-Clifford algebra $C\ell(W_{F_S}, (\ ,\ )_\bullet)$ generated by the section v with the only relation $W_{\mathcal{O}_{F_S}} v = 0$. Note that $\pi_* W_U = \pi\big|_{U^*}(W\big|_U)$ is an S-family of maximal isotropic colattices in W_{F_S} (see 2.4.5). Put $L_\psi = M_{Q_S}/\pi_* W_U M_{Q_S}$. This is a line bundle on S if Q_S is even (which means that $(\ ,\ ): W \times W \to W_{C_S/S}$ is non-degenerate). If Q_S is odd, then L_ψ is a two-dimensional vector bundle which splits canonically as a sum of two line bundles on the 2-sheeted covering of S that corresponds to a choice of $\gamma \in W^\perp_{\mathcal{O}_{F_S}}/W_{\mathcal{O}_{F_S}}$ with $(\gamma, \gamma)_\bullet = 1$.

3.5.2 Lemma. *L_ψ is naturally a D_ψ-module: it is a D_ψ-module quotient of $\Delta_\psi M$.*

Proof. Consider the action of Lie algebroid $\widetilde{\mathcal{AOA}}_{Q_S} N$ (see 3.3.4) on M_{Q_S}. Since for $(a, n) \in \widetilde{\mathcal{AOA}}_{Q_S} N = \widetilde{\mathcal{AOA}}_{Q_S} \underset{\mathcal{AOA}_{Q_S}}{\times} \pi_* \mathcal{OA}W_U$ and $w \in \pi_* W_U$ one has $[(a, n), w] = n(w)$ (as operators on M_{Q_S}), we see that this action "quotients down" to L_ψ. It remains to show that L_ψ is actually an A_ψ-module. We need to prove that the $\mathcal{O}_S$-Lie subalgebra $s(N_{(0)}) \subset \widetilde{\mathcal{AOA}}_{Q_S} N$ acts trivially on L_ψ. Note that $s(N_{(0)}) = \pi_* \mathcal{OA}W_{U/S}$ acts on L_ψ $\mathcal{O}_S$-linearly, hence it suffices to consider the case when S is a point. Then $N_{(0)} = \mathcal{OA}W_U$ is an extension of $\mathcal{T}_U$ by the orthogonal Lie algebra $\mathcal{O}W_U$. Since both $\mathcal{O}W_U$ and $\mathcal{T}_U$ are perfect $\mathbb{C}$-Lie algebras, we see that $N_{(0)}$ is perfect, hence every 1-dimensional representation of $N_{(0)}$ is trivial. Since L_ψ is either 1-dimensional or a sum of two 1-dimensional $N_{(0)}$-invariant subspaces, we are done. □

Actually we have proven that L_ψ is a quotient of the D_ψ-module $\Delta_\psi(M)$. Certainly, 3.5.2 implies

3.5.3 Proposition. *One has a canonical isomorphism of twisted differential operators algebras $D_\psi = D_{L_\psi}$ if Q_S is even, and $D_{\psi_2} = D_{\det L_\psi}$ if Q_S is odd.* □

3.5.4 Remarks. (i) According to 1.4.4 the fibers L_{ψ_s}, $s \in S$, are canonically identified with det $H^0(C_s, W_s)$ if Q_S is even, i.e., if $(\ ,\ )$ is non degenerate (if Q_S is odd, one has det $L_{\psi_s} = \det^{\otimes 2} H^0(C_s, W_s)$). Hence the automorphism - id_W of our data acts on L_ψ as ± 1 depending on whether dim $H^0(C_s, W_s)$ is even or odd. This proves the theorem of Mumford that the parity of dim does not jump.
(ii) Of course we may consider the situation with several points $a_1, \ldots, a_n \in C$. By a reason similar to 3.4.6 one may see that the corresponding line bundle L_ψ actually does not depend on these points; certainly, we may delete only "even" points where $(\ ,\ )$ is non-degenerate. □

Now let us consider the situation 3.4.8 of vector symmetries. By 3.4.10 we have a canonical isomorphism $D_{\psi_c(C_S, a, E)} = D_{\psi_{-c}(C_S, a, E \oplus E^0, (\ ,\ ))}$. By 3.5.4(i) the fibers of the line bundle $L_\psi = L_\psi(C_S, a, E \oplus E^0, (\ ,\ ))$ coincide with det $H^0(C_s, E) \otimes$

$\det H^0(C_s, E_s^0) = \det H^0(C_s, E)/\det H^1(C_s, E) = \det R\Gamma(C_s, E)$. It is easy to see that $L_\psi = \det R\pi_* E =$ the determinant line bundle of E (about determinant line bundles, see e.g. [KM]). By 3.5.4 (ii) and a version of 3.4.6 for vector symmetries we may delete a point a above. Hence

3.5.5 Corollary. *One has a canonical isomorphism* $D_{\psi_c(C_S,E)} = D_{\det^{\otimes -c} R\pi_* E}$.$\square$

Consider finally the pure Virasoro situation. We have an obvious embedding of Harish-Chandra groupoids $r : (\mathcal{V}, \widetilde{\mathcal{T}}) \to (\mathcal{V}\mathcal{V}, \widetilde{\mathcal{A}})$, $F \longmapsto (F, \mathcal{O}_F)$, $\widetilde{\mathcal{T}} \hookrightarrow \widetilde{\mathcal{A}F}$ (see 2.1.3). If C_S is an S-family of curves, a is an S-point of C_S, we have an obvious r-morphism of localization data $\psi_{(C_S,a)} \longrightarrow \psi_{(C_S,a,\mathcal{O}_{C_S})}$ which identifies $D_{\psi_c(C_S,a)}$ with $D_{\psi_c(C_S,a,\mathcal{O}_{C_S})}$. Now 3.5.5 implies

3.5.6 Corollary. *One has a canonical isomorphism* $D_{\psi_c(C_S)} = D_{\det^{\otimes -c} R\pi_* \mathcal{O}_{C_S}}$.$\square$

3.6 Quadratic degeneration. In this section we will describe the determinant bundle of a family of curves that degenerates quadratically. Below $S =$ Spec $\mathbb{C}[[q]]$ is a formal disc, $0 \in S$ is the special point $q = 0$, $\eta =$ Spec $\mathbb{C}((q))$ is the generic point.

3.6.1 Lemma. *There is a canonical 1-1 correspondence between the following data (i) and (ii):*

(i) A proper S-family of curves, C_S such that C_η is smooth and C_0 has exactly one singular point a which is quadratic, together with formal coordinates t_1, t_2 at a such that $q = t_1 t_2$.

(ii) A proper smooth S-family of curves $C_S^\vee$ together with two disjoint points $a_1, a_2 \in C_S(S)$ and formal coordinates t_i at a_i.

Proof. Here is a construction of mutually inverse maps. Note that, according to Grothendieck, we may replace any proper S-curve B_S by the corresponding formal scheme $\widehat{B}_S =$ the completion of B_S along B_0.

(i) $\longmapsto$ (ii). Let C_S, t_1, t_2 be a (i)-data. The corresponding $C_S^\vee, a_i, t_i$ are the following ones. One has $C_0^\vee =$ normalization of C_0, so t_i define formal coordinates at points $a_1(0), a_2(0) \in C_S^\vee$. To define $C_S^\vee$ as a formal scheme, we have to construct the corresponding sheaf $\widehat{\mathcal{O}}_{C_S^\vee}$ of functions on $C_0^\vee$. We demand that on $U = C_S^\vee \setminus \{a_1, a_2\} = C_0 \setminus \{a\}$ our $\widehat{\mathcal{O}}_{C_S^\vee}$ coincides with $\widehat{\mathcal{O}}_{C_S}$. Note that any function $\varphi \in \widehat{\mathcal{O}}_{C_S}(V)$, where $V \subset U$, has Laurent series expansions $\varphi_i(t_i, q) \in \mathbb{C}((t_i))[[q]]$ at $a_i(0)$. We say that φ is regular at $a_i(0)$ if $\varphi_i(t_i, q) \in \mathbb{C}[[t_i, q]]$. This defines $\widehat{\mathcal{O}}_{C_S^\vee}$. The points a_i are defined by equations $t_i = 0$.

(ii) $\longmapsto$ (i). Let $C_S^\vee, a_i, t_i$ be (ii)-data. The zero fiber C_0 of our curve C_S is $C_0^\vee$ with points a_1, a_2 glued together. The sheaf $\widehat{\mathcal{O}}_{C_S}$ coincides with $\widehat{\mathcal{O}}_{C_S^\vee}$ on $U = C_0 \setminus \{0\} = C_0^\vee \setminus \{a_1, a_2\}$. For a Zariski open $V \subset U$ a function $\varphi \in \widehat{\mathcal{O}}_{C_S}(V)$ is regular at a if the t_i-Laurent series expansions $\varphi_i \in \mathbb{C}((t_i))[[q]]$ of φ at a_i lie in $\mathbb{C}[[t_1, t_2]] \subset \mathbb{C}((t_i))[[q]]$ and $\varphi_1 = \varphi_2 \in \mathbb{C}[[t_1, t_2]]$. Here the embedding $\mathbb{C}[[t_1, t_2]] \hookrightarrow \mathbb{C}((t_1))[[q]]$ is $t_1 \longmapsto t_1, t_2 \longmapsto q/t_1$, and the one $\mathbb{C}[[t_1, t_2]] \hookrightarrow \mathbb{C}((t_2))[[q]]$ is $t_1 \longmapsto q/t_2$, $t_2 \longmapsto t_2$. This defines $\widehat{\mathcal{O}}_{C_S}$. $\square$

Below we will say that a vector bundle E on a scheme X is *stratified* at $x \in X$ if we are given an isomorphism $E \simeq A \otimes_{\mathbb{C}} \mathcal{O}_X$ on a formal neighbourhood of x (here A is a vector space; $A = E_x$).

3.6.2 Lemma. *Let C_S and $C_S^\vee$ be the S-curves from 3.6.1. There is natural 1-1 correspondence between the data*

(i) A vector bundle E on C_S together with a stratification of E at a.

(ii) A vector bundle $E^\vee$ on $C_S^\vee$ together with a stratifications of $E^\vee$ at a_1, a_2 and an isomorphism of fibers $E^\vee_{a_1} \simeq E^\vee_{a_2}$. □

3.6.3 Proposition. *Let $(C_S, E), (C_S^\vee, E^\vee)$ be the related objects from 3.6.1, 3.6.2. Then there is a canonical stratification of the line bundle $\mathcal{L} = \det R\pi_* E / \det R\pi_*^\vee E^\vee$ on S.*

Remark. Here "stratification" = "stratification at 0" = (isomorphism $\mathcal{L} \simeq \mathcal{L}_0 \otimes \mathcal{O}_S$). Note that $\mathcal{L}_0 = \det R\Gamma(C_0, E_0)/\det R\Gamma(C_0^\vee, E_0^\vee)$ is naturally isomorphic to $\det^{-1} E_a$, so 3.6.3 is canonical isomorphism $\det R\pi_*^\vee(C^\vee, E^\vee) = \det E_a \det R\pi_*(C, E)$.

Proof. Construction. Let us compute our determinant bundles. Below we will use notations from the proof of 3.6.1. Put $A = E_a = E^\vee_{a_1} = E^\vee_{a_2}$. Our data identifies the formal completion $E_{\widehat{a}}$ of E at a with $A \otimes \mathbb{C}[[t_1, t_2]]$, and the formal completion of $E^\vee_{\widehat{a}_i}$ of $E^\vee$ at a_i with $A \otimes \mathbb{C}[[t_i, q]]$. The restrictions of E and $E^\vee$ to the formal scheme $\widehat{U} = (U, \widehat{\mathcal{O}}_U)$ coincide; put $P = H^0(U, E|_{\widehat{U}}) = \varprojlim H^0(U, E/q^n E)$. Also put $V = A \otimes \{\mathbb{C}((t_1))[[q]] \oplus \mathbb{C}((t_2))[[q]]\}$, $V_{+0} = A \otimes \{\mathbb{C}[[t_1, q]] \oplus \mathbb{C}[[t_2, q]]\}$, $V_{+1} = A \otimes \{\mathbb{C}[[t_1, t_2]]$. We may compute $R\pi_* E$, $R\pi_*^\vee E^\vee$ by means of "adelic" complexes for our formal schemes. Namely, $R\pi_*^\vee E^\vee = \operatorname{Cone}(P \oplus V_{+0} \to V)[-1]$, $R\pi_* E = \operatorname{Cone}(P \oplus V_{+1} \to V)[-1]$; here the map $P \to V$ is minus the Laurent series expansion map, the map $V_{+1} \to V$ is given by formula $a \otimes t_1^m t_2^n \longmapsto a \otimes \{q^n t_1^{m-n} + q^m t_2^{n-m}\}$ (see the proof of 3.6.1), and $V_{+0} \to V$ is the obvious embedding.

Note that V is a flat complete $\mathbb{C}[[q]]$-module with the obvious Tate structure (see 1.4.10), V_{+0}, V_{+1} are lattices in V and P is a colattice in V. So to compute our determinants we may use Clifford modules. Namely, take $W = V \oplus V^*$ with the standard form $(\ ,\ )$; let M be the corresponding Clifford module such that $M_0 = M/qM$ is an irreducible Clifford module for $(W_0, (\ ,\ )_0)$. Then $L(P) = P \oplus P^\perp$, $L(V_{i+}) = V_{i+} \oplus V_{i+}^\perp$ are maximal isotropic colattice and lattices respectively. A $\mathbb{C}[[q]]$-version of 1.4.9 shows that coinvariants $M_{L(P)}$ and invariants $M^{L(V_{i+})}$ are free $\mathbb{C}[[q]]$-modules of rank one, and there are canonical isomorphisms

$$\det R\pi_*^\vee E^\vee = M^{L(V_{0+})}/M_{L(P)}, \det R\pi_* E = M^{L(V_{1+})}/M_{L(P)}.$$

Hence $\det R\pi_* E / \det R\pi_*^\vee E^\vee = M^{L(V_{1+})}/M^{L(V_{0+})}$. In this description of the ratio of determinants all the "global" data that may vary (encoded in P) disappeared; we've got the standard "local" expression for it.

It remains to fix an isomorphism $\gamma : M^{L(V_{0+})} \to M^{L(V_{1+})} \otimes \det A$; the desired stratification of the ratio of determinants then will be $\gamma(v)/v$ for a non-zero generator v (clearly it does not depend on M). Let $a_1, \dots, a_\ell$ be a basis of A. Consider the vectors $e^k_{\alpha 1} = a_\alpha \otimes t_1^k, e^k_{\alpha 2} = a_\alpha \otimes t_2^k$, $k \in \mathbb{Z}$, $\alpha = 1, \dots, \ell$. This is a basis (in an obvious sense) of V; denote by $e^{k*}_{\alpha i} \in V^*$ the dual basis. The vectors $\{e^k_{\alpha i}\}, k \geq 0$, form a basis of V_{0+}, and the vectors $f^k_{\alpha 1} := e^k_{\alpha 1} + q^k e^{-k}_{\alpha 2}$, $f^k_{\alpha 2} := q^k e^{-k}_{\alpha 1} + e^k_{\alpha 2}, e^0_{\alpha 1} + e^0_{\alpha 2}$, $k \geq 1$, form a basis of V_{1+}. In a slightly informal way our γ could be defined as follows. A generator of $M^{L(V_{0+})}$ is an infinite wedge product $\bigwedge\limits_{\substack{k \geq 0 \\ \alpha, i}} e^k_{\alpha i}$, a generator of

$M^{L(V_{1+})}\otimes\det A$ is $\bigwedge\limits_{\substack{k\geq 1\\ \alpha,i}} f^k_{\alpha i}\wedge\bigwedge\limits_{\alpha}(e^0_{\alpha 1}+e^0_{\alpha 2})\otimes\bigwedge\limits_{\alpha} a_\alpha$, and γ just identifies these generators. To be precise, consider the elements $\gamma_n=\prod\limits_{\substack{1\leq k\leq n\\ \alpha}}(f^k_{\alpha 1}f^k_{\alpha 2}e^{k*}_{\alpha 2}e^{k*}_{\alpha 1})\in\mathrm{Cliff}(W)$. These γ_n do not depend on a choice of basis $\{a_\alpha\}$ in A, and it is easy to see that $\gamma_\infty=\lim\limits_n\gamma_n\in C\ell W$ is correctly defined. Let $V_{0++}\subset V_{0+}, V_{1++}\subset V_{1+}$ be sublattices with bases $\{e^k_{\alpha i}\}, k\geq 1$, and $\{f^k_{\alpha i}\}, k\geq 1$, respectively. It is easy to see that $\gamma_\infty(M^{L(V_{0++})})=M^{L(V_{1++})}$ (more precisely, $\gamma_n(M^{L(V_{0+})})\equiv M^{L(V_{1+})}\mathrm{mod} q^{n+1}M$). Since $M^{L(V_{0+})}=\bigwedge\limits_{\alpha,i}e^0_{\alpha i}\cdot M^{L(V_{0++})}, M^{L(V_{1+})}=\bigwedge\limits_{\alpha}(e^0_{\alpha 1}+e^0_{\alpha 2})\cdot M^{L(V_{1++})}$, we have $\bigwedge\limits_{\alpha}(e^{0*}_1-e^{0*}_2)\cdot\gamma_\infty M^{L(V_{0+})}=M^{L(V_{1+})}$. Put $\bigwedge\limits_{\alpha}(e^{0*}_1-e^{0*}_2)\cdot\gamma_\infty\otimes\bigwedge\limits_{\alpha}a_\alpha\in C\ell W\otimes\det A$. This γ does not depend on a choice of basis $\{a_\alpha\}$ of A, and the desired $M^{L(V_{0+})}\xrightarrow{\sim}M^{L(V_{1+})}\otimes\det A$ is just multiplication by γ. □

3.6.4 Let $C^\vee$ be a curve, $a_1,a_2\in C^\vee$, $a_1\neq a_2$, a pair of points, and t_i a formal parameter at a_i. Consider the constant S-family $C^\vee_S:=C^\vee\times S$; let $a_i\in C^\vee_S(S), t_i$ be the "constant" points and parameters. According to 3.6.1 these define an S-curve C_S with quadratic singularities along zero fiber and smooth generic fiber. Consider the trivial vector bundles $\mathcal{O}_{C_S},\mathcal{O}_{C^\vee_S}$; they correspond to each other by 3.6.2 correspondence. Note that $\det R\pi^\vee_*\mathcal{O}_{C^\vee_S}=\det R\Gamma(C^\vee,\mathcal{O}_{C^\vee})\otimes\mathcal{O}_S$ is obviously stratified, hence 3.6.3 defines the stratification of $\det R\pi_*\mathcal{O}_{C_S}$ which is a natural generator γ of the $\mathbb{C}[[q]]$-module $\det^{-1}R\Gamma(C^\vee,\mathcal{O}_{C^\vee})\otimes_{\mathbb{C}[[q]]}\det R\pi_*\mathcal{O}_{C_S}$. Let us compute γ in a couple of simple situations.

3.6.5 Assume that $C^\vee$ is a disjoint union of two copies of $\mathbb{P}^1$'s, $C^\vee=\mathbb{P}^1_1\coprod\mathbb{P}^1_2$, $a_1\in\mathbb{P}^1_1, a_2\in\mathbb{P}^1_2$ are "zero" points, t_i are standard parameters at a_i. Then the S-curve C_S is the compactification of the family of affine curves $\mathbb{A}^2\to S$, $q=t_2t_2$. This is a genus 0 curve, hence $R\pi_*\mathcal{O}_{C_S}=\mathcal{O}_S$, so we have a canonical trivialization α of $\det R\pi_*\mathcal{O}_{C_S}$ of "global" origin. In fact, it coincides with our γ. To see this, note that (in the notations of proof of 3.6.3) in our case P is colattice with basis $\{e^k_1,e^k_2\}, k\leq 0$, so one has $P\oplus V_{1++}=V=P\oplus V_{0++}$. The operator $(e^0_1+e^0_2)\cdot$ identifies $M^{L(V_{1++})}$ with $M^{L(V_{1+})}$, hence $\det R\pi_*\mathcal{O}_{C_S}=M^{L(V_{1++})}/M_{L(P)}$. The "global" trivialization α comes from the isomorphism $M^{L(V_{1++})}\xrightarrow{\sim}M_{L(P)}$, $m\longmapsto m \bmod L(P)M$. The trivialization γ comes from composition $M^{L(V_{1++})}\xrightarrow{\sim}M^{L(V_{0++})}\xrightarrow{\sim}M_{L(P)}$ where the first arrow is inverse to multiplication by γ_∞ and the second one is projection $m\longmapsto m \bmod L(P)M$. Since $f^k_i=e^k_i \bmod P$ for $k\geq 1$, the formula for γ_∞ shows that this composition coincides with projection $M^{L(V_{1++})}\to M_{L(P)}$, hence $\alpha=\gamma$.

3.6.6 Assume now that $C^\vee=\mathbb{P}^1, a_1=0, a_2=\infty$ and t_1,t_2 are standard parameters t and t^{-1} respectively. Then the curve C_S coincides with the standard Tate elliptic curve (see, e.g., [DR]), q is a standard parameter on moduli space of elliptic curves at ∞. The Tate curve carries a canonical relative 1-form ν (that corresponds to the standard invariant form on G_m via Tate's uniformization). One has $R^0\pi_*\mathcal{O}_{C_S}=\mathcal{O}_S$, $R^1\pi_*\mathcal{O}_{C_S}=(R^0\pi_*\omega_{C_S})^*$ by Serre duality (here ω_{C_S} is relative dualizing sheaf), hence $\det R\pi_*\mathcal{O}_{C_S}=R^0\pi_*\omega_{C_S}$ and ν is a canonical trivialization of $\det R\pi_*\mathcal{O}_{C_S}$. Let us calculate γ. The colattice P has basis $\{e^k_1+e^k_2\}, k\in\mathbb{Z}$. One has $\mathcal{O}_S=R^0\pi_*\mathcal{O}_{C_S}=\mathcal{O}_S(e^0_1+e^0_2)=P\cap V_{1+}$, $R^1\pi_*\mathcal{O}_{C_S}=V/P+V_{1+}=V/P+V_{1++}$. The relative differential ν in local coordinates t_i is $\frac{dt_1}{t_1} = -\frac{dt_2}{t_2}$, and the Serre duality morphism is the sum of local residues at a_i. Hence the functional $\nu \in (R^1\pi_*\mathcal{O}_{C_S})^* = (V/P + V_{1+})^* \subset V^*$ is $e_1^{0*} - e_2^{0*}$. As above, multiplication by $e_1^0 + e_2^0$ identifies $M^{L(V_{1++})}$ with $M^{L(V_1)}$, hence $\det R\pi_*\mathcal{O}_{C_S} = M^{L(V_{1++})}/M_{L(P)}$. The trivialization ν comes from the isomorphism $M^{L(V_{1++})} \to M_{L(P)}$, $m \longmapsto (e_1^0 m) \bmod L(P)M$. The trivialization γ comes from composition $M^{L(V_{1++})} \xrightarrow{\sim} M^{L(V_{0++})} \xrightarrow{\sim} M_{L(P)}$ where the first arrow is inverse to multiplication by γ_∞ isomorphism $M^{L(V_{0++})} \xrightarrow{\sim} M^{L(V_{1++})}$ and the second arrow is $m \longmapsto (e_1^0 m) \bmod L(P)M$. Since $f_1^k = (1-q^k)e_1^k \bmod P$, $f_2^k = (1-q^k)e_2^k \bmod P$ we see that $\gamma = [\prod_{k\geq 1}(1-q^k)^2]\nu$, or, in terms of Dedekind's η-function $\eta(q) = q^{1/24}\prod_{k\geq 1}(1-q^k)$, one has

$$\gamma = q^{-1/12}\eta(q)^2\nu.$$

One may reformulate this as follows. Recall that the line bundle $\lambda = \det R\pi_*\mathcal{O}_C = \pi_*\omega_C$ on moduli space of elliptic curves carries a canonical global integrable connection ∇ such that the discriminant Δ is a global horizontal section of $\lambda^{\otimes 12}$ (with respect to the corresponding connection on $\lambda^{\otimes 12}$). We see that our γ is a horizontal section of a connection $\nabla + \frac{1}{12}\frac{dq}{q}$.

§4. Fusion Categories

4.1 Recollections from symplectic linear algebra. Let V be a symplectic $\mathbb{R}$-vector space of dimension $2g$ with symplectic form $\langle\ ,\ \rangle$. To $(V,\langle\ ,\ \rangle)$ there corresponds a canonical transitive groupoid $\mathcal{T}_V$. In 1.1-1.3 below we give three different constructions of $\mathcal{T}_V$. Assume first that $V \neq 0$.

4.1.1 Let $H = H_V$ be the Siegel upper half plane of V. A point of H is a complex Lagrangian subspace $L \subset V_{\mathbb{C}} := V \otimes \mathbb{C}$ such that $i\langle x, \overline{x}\rangle > 0$ for $x \neq 0 \in L$. Equivalently, one may consider a point of H as a complex structure ℓ on V such that the form $\langle \cdot, i_\ell \cdot \rangle$ is symmetric and positive definite; here $i_\ell \in \operatorname{End} V$ is multiplication by $i \in \mathbb{C}$ with respect to ℓ (the 1-1 correspondence $\ell \longleftrightarrow L$ is $\ell \longmapsto L_\ell :=$ the i-eigenspace of i_ℓ, $L \longmapsto \ell_L :=$ the complex structure that comes from the isomorphism $V \underset{\sim}{\rightarrow} V_{\mathbb{C}}/L$). The space H is a complex variety, and the L's form a rank g holomorphic bundle $\mathcal{L}$ on H. Put $\lambda := \det \mathcal{L}$: this is a holomorphic line bundle on H. Denote by $\widetilde{H}$ the space of $\lambda^{\otimes 2} \setminus \{$ zero section $\}$; the projection $\widetilde{H} \to H$ is a $\mathbb{C}^*$-fibration. Let $\mathcal{H}$ be the space of C^∞-sections $H \to \widetilde{H}$. One has obvious maps

$$\mathcal{H} \longleftarrow \mathcal{H} \times H \longrightarrow \widetilde{H}, \quad \varphi \leftarrow (\varphi, h) \mapsto \varphi(h). \tag{4.1.1.1}$$

Since H is contractible, these are homotopy equivalences. Note that for any $a \in \widetilde{H}$ the map $i_a : S^1 \hookrightarrow \widetilde{H}$, $i_a(e^{i\theta}) := e^{i\theta} a$, is a homotopy equivalence which defines a canonical identification

$$\pi_1(\widetilde{H}, a) = \mathbb{Z}. \tag{4.1.1.2}$$

For a topological space X let $\mathcal{T}(X)$ be the fundamental groupoid of X: its objects are points of X, and its morphisms are homotopy classes of paths. Put $\mathcal{T}'_V := \mathcal{T}(\widetilde{H})$.

4.1.2 Denote by $\Lambda = \Lambda_V$ the grassmannian of real non-oriented Lagrangian subspaces of V; the planes form a canonical Lagrangian sub-bundle $\mathcal{L}_{\mathbb{R}}$ of $V_\Lambda := V \times \Lambda$. Put $\lambda_{\mathbb{R}} := \det \mathcal{L}_{\mathbb{R}}$: this is a real line sub-bundle of $\Lambda^g V_\Lambda$. Let Λ' be the space $\lambda_{\mathbb{R}} \setminus \{\text{zero section}\}/\pm 1$: the map $x \longmapsto x^2$ identifies Λ' with the "positive ray" of $\lambda_{\mathbb{R}}^{\otimes 2}$. The obvious projection $\Lambda' \longrightarrow \Lambda$ is an $\mathbb{R}_+^*$-torsor, hence a homotopy equivalence. One has a canonical map

$$v : \Lambda' \longrightarrow \mathcal{H} \tag{4.1.2.1}$$

defined by the formula $v(x^2)(h) = \lambda^2$, where $\lambda \in \det L_h \subset \wedge^g V_{\mathbb{C}}$ is the unique vector such that $\operatorname{vol}(x \wedge \lambda) = 1$ (here vol$= \frac{\langle\ ,\ \rangle}{g!} \in \Lambda^{2g} V^*$ is the canonical volume). The map v induces an isomorphism of fundamental groupoids. Put $\mathcal{T}''_V := \mathcal{T}(\Lambda)$. According to (1.1.1), 1.2.1) we have a canonical equivalence of groupoids

$$\alpha : \mathcal{T}''_V \xrightarrow{\sim} \mathcal{T}'_V. \tag{4.1.2.2}$$

4.1.3 Here is the third construction of $\mathcal{T}_V$. For 3 Lagrangian planes one defines, according to Kashiwara, their Maslov index $\tau(L_1, L_2, L_3)$ as the signature of the quadratic form B on $L_1 \oplus L_2 \oplus L_3$ given by the formula $B(x_1, x_2, x_3) = \langle x_1, x_2 \rangle +$

$\langle x_2, x_3\rangle + \langle x_3, x_1\rangle$ (see [LV] ( )). Let $\mathcal{T}_V'''$ be the following groupoid. Its set of objects is Λ. For $L_1, L_2 \in \Lambda$ we put $\mathrm{Hom}_{\mathcal{T}_V'''}(L_1, L_2) = \mathbb{Z}$, and the composition of morphisms $L_1 \xrightarrow{n} L_2 \xrightarrow{m} L_3$ is given by the formula $m \circ n := m + n + \tau(L_1, L_2, L_3)$. Since τ satisfies a cocycle formula [LV] ( ), the composition is associative.

Let us define a canonical isomorphism

$$\beta : \mathcal{T}_V''' \xrightarrow{\sim} \mathcal{T}_V'' \tag{4.1.3.1}$$

which is the identity on objects. To construct β we need to choose for each pair $L_1, L_2 \in \Lambda$ a canonical path $\gamma_{L_1,L_2} \in \mathrm{Hom}_{\mathcal{T}_V''}(L_2, L_1)$ such that

$$\gamma_{L_3L_2} \circ \gamma_{L_2L_1} = \gamma_{L_3L_1} + \tau(L_1, L_2, L_3). \tag{4.1.3.2}$$

Then one defines β by the formula $\beta(n) = n + \gamma_{L_1,L_2}$ for $n \in \mathrm{Hom}_{\mathcal{T}_V{}'''}(L_2, L_1) = \mathbb{Z}$ (recall that $\mathrm{Hom}_{\mathcal{T}_V''}(L_2, L_1)$ is a $\mathbb{Z}$-torsor by 1.1.2).

To define $\gamma_{L_1L_2}$ consider the subset $U_{L_1L_2} \subset \Lambda$ that consists of L's such that $L_1 + L_2 \supset L \supset L_1 \cap L_2 = L \cap L_1 = L \cap L_2$. A plane $L \in U_{L_1,L_2}$ defines a quadratic form φ_L on $L_1/L_1 \cap L_2$ by the formula $\varphi_L(a) = \langle b, a\rangle$ where $b \in L_2$ is a vector such that $b + a \in L$. In this way one gets a 1-1 correspondence between $U_{L_1L_2}$ and the set of all non-degenerate forms on $L_1/L_1 \cap L_2$. Let $U^+_{L_1L_2} \subset U_{L_1L_2}$ be the subspace that corresponds to positive-definite forms, so $U^+_{L_1L_2}$ is contractible. Now γ_{L_1,L_2} is the unique homotopy path from L_2 to L_1 which, apart from its ends, lies in $U^+_{L_1L_2}$. One verifies (4.1.3.2) immediately.

4.1.4 Below we will denote by $\mathcal{T}_V$ either of the groupoids $\mathcal{T}_V', \mathcal{T}_V'', \mathcal{T}_V'''$ identified via (4.1.2.2), (4.1.3.1). In case $V = 0$, the groupoid $\mathcal{T}_V$, by definition, has a single object 0 with End $0 = \mathbb{Z}$. For any V and $y \in \mathcal{T}_V$ we will denote by γ_0 the generator $1 \in \mathbb{Z} = \mathrm{Aut}\ y$.

4.1.5 The groupoid $\mathcal{T}_V$ has the following functorial properties. Let V be a symplectic space, $N \subset V$ a vector subspace such that $\langle\ \ \rangle|_N = 0$, and let $N^\perp$ be the $\langle\ \ \rangle$-orthogonal complement to N. Then $N^\perp/N$ has an obvious symplectic structure. Since the pre-image of a Lagrangian plane in $N^\perp/N$ is a Lagrangian plane in V, we have an embedding $\Lambda_{N^\perp/N} \hookrightarrow \Lambda_V$, which defines a canonical equivalence of groupoids $\mathcal{T}''_{N^\perp/N} \xrightarrow{\sim} \mathcal{T}_V$.

4.1.6 Now let V_1, V_2 be symplectic spaces. One has an obvious map $\Lambda_{V_1} \times \Lambda_{V_2} \longrightarrow \Lambda_{V_1 \oplus V_2}$, $(L_1, L_2) \longmapsto L_1 \oplus L_2$, and a similar map $\widetilde{H}_{V_1} \times \widetilde{H}_{V_2} \longrightarrow \widetilde{H}_{V_1 \oplus V_2}$, which comes from multiplication $\det^{\otimes 2} L_1 \times \det^{\otimes 2} L_2 \longrightarrow \det^{\otimes 2} L_1 \otimes \det^{\otimes 2} L_2 = \det^{\otimes 2}(L_1 \oplus L_2)$. These define morphisms between corresponding fundamental groupoids, compatible with the canonical equivalences (4.1.2.2). Hence we have a canonical morphism $\mathcal{T}_{V_1} \times \mathcal{T}_{V_2} \longrightarrow \mathcal{T}_{V_1 \oplus V_2}$.

4.2 The Teichmüller groupoid. This groupoid appears in two equivalent versions: a "combinatorial" or "topological" version, and a "holomorphic" version.

4.2.1 An object of the "topological" Teichmüller groupoid *Teich'* is an oriented surface S (possibly non-connected and with boundary) together with a set of points $P_S = \{x_\alpha\}$ of the boundary ∂S such that each connected component of ∂S contains exactly one x_α (we will denote this component ∂S_{x_α}). The morphisms are isotopy classes of diffeomorphisms.

Let us define an "enhanced" groupoid $\widetilde{Teich}'$. For a surface S denote by $H(S)$ the image of the canonical map $H^1_c(S,\mathbb{R}) \longrightarrow H^1(S,\mathbb{R})$ (which is the same as cohomology of a smooth compactification of S). An orientation of S defines a symplectic structure on $H(S)$ (intersection pairing). Now an object of $\widetilde{Teich}'$ is a triple (S, P_S, y), where $(S, P_S) \in Teich'$ and $y \in \mathcal{T}_{H(S)}$. A morphism $(S, P_S, y) \longrightarrow (S', P_{S'}, y')$ is a pair (φ, γ), where $\varphi : (S, P_S) \longrightarrow (S', P_{S'})$ is a morphism in $Teich'$, and $\gamma : \varphi_*(y) \longrightarrow y'$ is a morphism in $\mathcal{T}_{H(S')}$; the composition of morphisms is obvious.

The projection $\widetilde{Teich}' \to Teich'$, $(S, P_S, y) \longmapsto (S, P_S)$, is surjective. For any $(S, P_S, y) \in \widetilde{Teich}'$ the group $\mathrm{Aut}_{\widetilde{Teich}'}(S, P_S, y)$ is a central extension of $\mathrm{Aut}_{Teich'}(S, P_S)$ by $\mathbb{Z}(= \mathrm{Aut}_{\mathcal{T}_{H(S)}}(y))$. So we may say that $\widetilde{Teich}'$ is a central extension of $Teich'$ by $\mathbb{Z}$. We will denote the generator of this $\mathbb{Z}$ by γ_0.

Consider the functor $Teich' \longrightarrow Sets$, $(S, P_S) \longmapsto P_S =$ set of boundary components of S. Clearly $Teich'$ is a fibered category over the groupoid of finite sets. For a finite set A denote by $Teich'_A$ the fiber over A (the objects of this groupoid are pairs $((S, P_S), \nu)$, where $(S, P_S) \in Teich'$, and $\nu : P_S \xrightarrow{\sim} A$ is a bijection). For a bijection $f : A \xrightarrow{\sim} B$, $X \in Teich'_A$, $Y \in Teich'_B$ we will denote by $\mathrm{Hom}_f(X, Y)$ the set of f-morphisms (i.e., the ones that induce f on the sets of boundary components). We put $\mathrm{Aut}^0(S, P_S) = \mathrm{Aut}_{id_{P_S}}(S, P_S)$. We will use the same notations for $\widetilde{Teich}'$.

For $(S, P_S) \in Teich'$ and $x_\alpha \in P_S$ we denote by $d_{x_\alpha} \in \mathrm{Aut}^0(S, P_S)$ the Dehn twist around ∂S_{x_α}. Since d_{x_α} acts as the identity on $H(S)$ it lifts to the element $(d_{x_\alpha}, id_y) \in \mathrm{Aut}^0_{\widetilde{Teich}'}(S, P_S, y)$, which we will also denote by d_{x_α}. These d_{x_α} lie in the center. In particular, we have a canonical morphism $\mathbb{Z}^{P_S} \longrightarrow \mathrm{Aut}^0(S, P_S)$, $(n_{x_\alpha}) \longmapsto \prod d_{x_\alpha}^{n_{x_\alpha}}$; $\mathbb{Z} \times \mathbb{Z}^{P_S} \longrightarrow \mathrm{Aut}^0(S, P_S, y)$, $(n_y, n_{x_\alpha}) \longmapsto \gamma_0^{n_y} \times \prod d_{x_\alpha}^{n_{x_\alpha}}$.

4.2.2 Here is a "holomorphic" definition of the Teichmüller groupoid. An object of $Teich''$ is a complex curve C (smooth, projective, possibly reducible) together with a finite set of points $P_C = \{y_\alpha\} \subset C$ equipped with non-zero co-tangent vectors $\nu_\alpha \in \Omega^1_{C,y_\alpha}$. The morphisms are 1-parameter C^∞-class families of such objects connecting two given ones, these families being considered up to homotopy. In other words, $Teich''$ is the Poincaré groupoid of the modular stack $\mathcal{M}$ of the above structures. In the same way, $\widetilde{Teich}''$ is the Poincaré groupoid of the modular stack $\widetilde{\mathcal{M}}$ of the data $(C, y_\alpha, \nu_\alpha, y)$, where $(C, y_\alpha, \nu_\alpha) \in \mathcal{M}$, and $y \in \det^{\otimes 2}(H^0(C, \Omega^1_C)) \setminus \{0\}$. Clearly, the second modular stack is a $\mathbb{C}^*$-fibration over the first one, hence $\widetilde{Teich}''$ is a $\mathbb{Z}(= \pi_1(\mathbb{C}^*))$-extension of $Teich''$.

4.2.3 The groupoids $Teich'$ and $Teich''$, are canonically equivalent, as are $\widetilde{Teich}'$ and $\widetilde{Teich}''$. To define this equivalence, take $(S, P_S) \in Teich'$. Consider the data $(\mu; \{r_\alpha\})$, where μ is a complex structure on S, and $r_\alpha : S^1 = \{z \in \mathbb{C} : |z| = 1\} \xrightarrow{\sim} \partial S_{x_\alpha}$ is a parametrization such that $r_\alpha(1) = x_\alpha$ and r_α extends μ-holomorphically to the ring $\{z \in \mathbb{C} : 1 \le |z| \le 1 + \epsilon\}$. We may glue a collection of unit discs $D_\alpha = \{z \in \mathbb{C} : |z| \le 1\}$ (with their standard complex structure) to S using r_α. Denote the corresponding complex curve $C = C(S, P_S; (\mu, r_\alpha))$. It is equipped with the set of points $y_\alpha = 0 \in D_\alpha$, and the cotangent vectors $\nu_\alpha = dz_0 \in \Omega^1_{C,O}$. Hence $C(S, P_S; (\mu, r_\alpha)) \in Teich''$. It is easy to see that for given

(S, P_S) the data $(\mu; \{r_\alpha\})$ form a contractible space. So $(S, P_S) \in Teich'$ defines a canonical "homotopy point" in $Teich''$. In this way we get a morphism of groupoids $Teich' \longrightarrow Teich''$ which is an equivalence of categories.

To lift this equivalence to $\widetilde{Teich}' \longrightarrow \widetilde{Teich}''$, note that $H(S) = H^1(C, \mathbb{R})$. The complex structure on C defines the Hodge subspace $H^0(C, \Omega^1_C) \subset H(S)_{\mathbb{C}}$, which is a point h_C on the corresponding Siegel half plane (see 4.1.1). Now let us interpret $\mathcal{T}_{H(S)}$ as a fundamental groupoid of the space denoted by $\mathcal{H}$ in (4.1.1.1). For $y \in \mathcal{T}_{H(S)}$ put $y_C := y(h_C) \in \det^{\otimes 2}(H^0(C, \Omega^1_C)) \setminus \{0\}$. Our equivalence $\widetilde{Teich}' \longrightarrow \widetilde{Teich}''$ is given by the formula $(S, P_S, y) \longmapsto (C, y_\alpha, \nu_\alpha, y_C)$.

4.2.4 The above equivalence transforms γ_y to the loop $\theta \longmapsto (C, y_\alpha, \nu_\alpha, e^{i\theta} y)$, and transforms the Dehn twist d_{x_β} to the loop $\theta \longmapsto (C, y_\alpha, e^{i\theta} \delta^\alpha_\beta \nu_\alpha, y)$.

4.3 Operations in Teich. We will need the following ones:
(i) One has a functor "*disjoint union*" $\coprod : Teich \times Teich \rightarrow Teich$. According to 1.1.6 it lifts in a canonical way to a functor $\coprod : \widetilde{Teich} \times \widetilde{Teich} \rightarrow \widetilde{Teich}$. Clearly $Teich, \widetilde{Teich}$ are strictly commutative monoidal categories, and the projection $Teich \rightarrow Sets$, $(S, P_S) \longmapsto P_S$, commutes with $\coprod$.
(ii) *Deleting of a point.* For a finite set A and $\alpha \in A$ one has a canonical functor $del_\alpha : Teich_A \rightarrow Teich_{A \setminus \{\alpha\}}, \widetilde{Teich}_A \longrightarrow \widetilde{Teich}_{A \setminus \{\alpha\}}$. In "holomorphic" language (4.2.2) this functor just deletes y_α, ν_α. In "topological" language (4.2.1) one should delete the component ∂S_{χ_α} by glueing a "cup" to ∂S_{χ_α}.

(iii) *Sewing.* Let A be a finite set, and $\alpha, \beta \in A$, $\alpha \neq \beta$, two elements. One has a canonical Sewing Functor $\mathcal{S}_{\alpha,\beta} : Teich_A \rightarrow Teich_{A \setminus \{\alpha,\beta\}}$, $\widetilde{Teich}_A \rightarrow \widetilde{Teich}_{A \setminus \{\alpha,\beta\}}$. Let us define $\mathcal{S}_{\alpha,\beta}$ in combinatorial language first. For a surface $(S, A) \in Teich'$ choose a diffeomorphism $\varphi : \partial S_{x_\alpha} \overset{\sim}{\rightarrow} \partial S_{x_\beta}$, $\varphi(x_\alpha) = x_\beta$, reversing orientations. Our $\mathcal{S}_{\alpha,\beta}(S, A) \in Teich'_{A \setminus \{\alpha,\beta\}}$ is S with two boundary components identified by means of φ. Since the φ's form a contractible space, this surface does not depend on the choice of φ. Note that either $H(S) = H(\mathcal{S}_{\alpha,\beta}(S, A))$ (if α and β lie in different connected components of S), or $H(S)$ coincides with a subquotient of $H(\mathcal{S}_{\alpha,\beta}(S, A))$ in a manner described in 4.1.5. In any case one has a canonical equivalence $\mathcal{T}_{H(S)} \overset{\sim}{\rightarrow} \mathcal{T}_{H(\mathcal{S}_{\alpha,\beta}(S,A))}$. This defines $\mathcal{S}_{\alpha,\beta} : \widetilde{Teich}'_A \rightarrow \widetilde{Teich}'_{A \setminus \{\alpha,\beta\}}$.

4.3.1 To define $\mathcal{S}_{\alpha,\beta}$ in holomorphic language, take $(C, y_\gamma, \nu_\gamma) \in Teich''_A$. Consider a curve $C_{\alpha,\beta}$ with a single quadratic singularity obtained from C by "clutching" y_α and y_β together. One knows that curves with a single quadratic singularity form a smooth part of the divisor of singular curves in the modular stack $\overline{\mathcal{M}}_{A \setminus \{\alpha,\beta\}}$ of curves with at most quadratic singularities. The fiber of the normal bundle N to this divisor at $C_{\alpha,\beta}$ is canonically identified with $T_{C,y_\alpha} \otimes T_{C,y_\beta}$. Hence $\nu_\alpha^{-1} \cdot \nu_\beta^{-1}$ is a non-zero vector of this normal bundle. It defines a "point at infinity" of the modular stack $\mathcal{M}_{A \setminus \{\alpha,\beta\}}$ of smooth curves (for a detailed account of "points at infinity" see [D]), which is a correctly defined (up to unique canonical isomorphism) object $\mathcal{S}_{\alpha,\beta}(C, y_\gamma, \nu_\gamma) \in Teich''_{A \setminus \{\alpha,\beta\}}$. To lift $\mathcal{S}_{\alpha,\beta}$ to a functor between $\widetilde{Teich}''$'s, notice that the line bundle λ over $\mathcal{M}$ with fibers $\lambda_C := \det H^0(C, \Omega^1_C)$ extends canonically to a line bundle λ over $\overline{\mathcal{M}}$: if C' has quadratic singularities, one has $\lambda_{C'} := \det H^0(C, \omega_{C'})$, where $\omega_{C'}$ is the dualizing sheaf. Define the $\mathbb{C}^*$-bundle $\widetilde{\widetilde{\mathcal{M}}} \to \overline{\mathcal{M}}$ to be $\lambda^{\otimes 2} \setminus \{\text{zero section}\}$. Recall that for any $C' \in \overline{\mathcal{M}}$ one has a canonical isomorphism $\lambda_{C'}^{\otimes 2} = \lambda_{\widetilde{C}'}^{\otimes 2}$, where $\widetilde{C}'$ is the normalization of C' (recall that $\omega_{C'}/\omega_{\widetilde{C}'}$ is a skyscraper sheaf, supported at singular points, trivialized canonically up to sign using residues). Hence the fibers of $\widetilde{\mathcal{M}}$ over $(C, y_\gamma, \nu_\gamma)$ and $\mathcal{S}_{\alpha,\beta}(C, y_\gamma, \nu_\alpha)$ are nearby fibers of the same $\mathbb{C}^*$-fibration, and therefore one has a canonical identification of their fundamental groupoids. This defines the desired lifting $\mathcal{S}_{\alpha,\beta} : \widetilde{Teich}''_A \to \widetilde{Teich}''_{A\setminus\{\alpha,\beta\}}$. It is easy to verify that the equivalence 4.2.3 identifies the above "topological" and "holomorphic" constructions of $\mathcal{S}_{\alpha,\beta}$.

4.3.2 It is convenient to consider both sewing and deleting of points simultaneously. To do this, consider a category, $Sets^{\#}$, whose objects are finite sets, and whose morphisms $f : A \to B$ are pairs (i_f, ϕ_f), where $i_f : B \hookrightarrow A$ is an embedding, and $\phi_f = \{\phi_{f\delta}\}$ is a collection of two-element mutually non-intersecting subsets $\phi_{f\delta}$ of $A \setminus i_f(B)$. The composition is obvious: if $g : B \to C$ is another morphism, then $g \circ f = (i_f \circ i_g, \phi_f \cup \phi_g)$. For f as above we put $A_f^1 := \coprod_\delta \phi_{f\delta}$, $A_f^0 = A \setminus (i_f(B) \cup A_f^1)$, so $A = i_f(B) \coprod A_f^0 \coprod A_f^1$.

Now for any morphism $f : A \to B$ we have a canonical functor $f_* : Teich_A \to Teich_B$, $\widetilde{Teich}_A \to \widetilde{Teich}_B$ that deletes points in A_f^0 and sews pairwise points in all $\phi_{f\delta}$'s. One has $(g \circ f)_* = g_* \circ f_*$, and each f_* is a composition of elementary deletings of a single point, and glueing of a single pair. Clearly these f_*'s define a cofibered categories $Teich^{\#}, \widetilde{Teich}^{\#}$ over $Sets^{\#}$ with old fibers $Teich_A, \widetilde{Teich}_A$, respectively.

Note that all these categories are strictly commutative monoidal categories with respect to "disjoint union" operation $\coprod$; all the functors commute with $\coprod$.

4.4 Representations of Teich; central charge. Let A be a finite set. Denote by $\mathcal{R}_A$ the category of finite dimensional $\mathbb{C}$-representations of $Teich_A$ (i.e., the objects of $\mathcal{R}_A$ are functors $L : Teich_A \to Vect$), and by $\widetilde{\mathcal{R}}_A$ the same for $\widetilde{Teich}_A$. More generally, if Q is a component (i.e., a strictly full subcategory) of $Teich_A$, we denote by $\mathcal{R}_{A,Q}$ the category of representations of Q, identified with the full subcategory of $\mathcal{R}_A$ that consists of representations supported on Q (i.e., vanish off Q). For a representation $V \in \widetilde{\mathcal{R}}_A$ and $X \in \widetilde{Teich}_A$ we denote by V_X the value of V at X.

4.4.1 Definition. A representation $V \in \widetilde{\mathcal{R}}_A$ has multiplicative central charge $a \in \mathbb{C}^*$ if for any $X \in \widetilde{Teich}$ the canonical element $\gamma_0 \in AutX$ acts on V_X as multiplication by a. □

For any $a \in \mathbb{C}^*$ denote by $\mathcal{R}_{aA} \subset \widetilde{\mathcal{R}}_A$ the full subcategory of representations of central charge a. In particular, $\mathcal{R}_{1A} = \mathcal{R}_A$.

For any morphism $f : A \to B$ in $Sets^{\#}$ the functor $f_* : \widetilde{Teich}_A \to \widetilde{Teich}_B$ defines the corresponding functor $f^* : \widetilde{\mathcal{R}}_B \to \widetilde{\mathcal{R}}_A$; one has $f^*(\mathcal{R}_{aB}) \subset \mathcal{R}_{aA}$. The functors f^* define a category $\widetilde{\mathcal{R}}^*$ fibered over $Sets^{\#}$ with fibers $\widetilde{\mathcal{R}}_A$, together with fibered subcategories $\mathcal{R}_a^{\#} \subset \widetilde{\mathcal{R}}^{\#}$ with fibers $\mathcal{R}_{aA}$.

4.4.2 Here is an explicit description of representations. From a combinatorial point of view a representations $V \in \widetilde{\mathcal{R}}_A$ assigns to each surface $(S, A) \in Teich_A$ a local system V_S on the Lagrangian Grassmannian $\Lambda_{H(S)}$ (see 4.1.2), and to each $\varphi \in \mathrm{Hom}((S,A),(S',A))$ a lifting of the corresponding diffeomorphism $\Lambda_{H(S)} \xrightarrow{\sim} \Lambda_{H(S')}$ to $V_S \xrightarrow{\sim} V_{S'}$. This V lies in $\mathcal{R}_{aA}$ if the monodromy matrix of the loop $\gamma_0 = 1 \in \mathbb{Z} = \pi_1(\Lambda_{H(S)})$ coincides with multiplication by a.

4.4.3 From a holomorphic point of view our V is a local system on the modular stack $\widetilde{\mathcal{M}}_A$; V lies in $\mathcal{R}_{aA}$ if the monodromy around the fiber of the projection $\pi : \widetilde{\mathcal{M}}_A \to \mathcal{M}_A$ equals multiplication by a.

Recall that $\mathbb{C}$-local systems on smooth algebraic manifolds can be identified with algebraic vector bundles with integrable connections (= lisse D-modules) having regular singularities at infinity (see [D], [Bo]). So our V is a lisse D- module on $\widetilde{\mathcal{M}}_A$ with regular singularities at ∞. Assume that $V \in \mathcal{R}_{aA}$. Choose $c \in \mathbb{Z}$ ("additive central charge") such that $exp(2\pi ic) = a$. Let $D_{\lambda^c} = \mathcal{D}_{c\mathcal{A}(\lambda)}$ be the ring of differential operators on the "line bundle" $\lambda^{\otimes c}$. This is a twisted differential operator ring on $\mathcal{M}_A$ (see 3.2.6-3.2.8). Recall that D_{λ^c}-modules can be identified canonically with D-modules on $\widetilde{\mathcal{M}}_A$, monodromic along the fibers of π with monodromy a (see, e.g., [V]). In particular, V is a lisse D_{λ^c}-module on $\mathcal{M}_A$ having regular singularities at ∞.

4.5 Axioms of a fusion category. We will start with preliminary data.

4.5.1 Let $\mathcal{A}$ be an abelian $\mathbb{C}$-category ("category of modules"). We assume that $\mathcal{A}$ is semisimple, for $X \in \mathcal{A}$ the $\mathbb{C}$-vector space $EndX$ is finite dimensional, and there are finitely many isomorphism classes of irreducibles. Denote by $IrrA$ the set of isomorphism classes of irreducible objects in $\mathcal{A}$.

We should also have the following data:

– a contravariant functor ("duality") $* : \mathcal{A}^\circ \to \mathcal{A}$ together with a natural isomorphism $** \xrightarrow{\sim} id_{\mathcal{A}}$

– a distinguished irreducible object ("vacuum module") $\not{1}$ together with an isomorphism $\nu : \not{1} \xrightarrow{\sim} *\not{1}$ such that $*(\nu) \circ \nu = id_{\not{1}}$.

– an automorphism d of the identity functor $id_{\mathcal{A}}$, called the Dehn automorphism, such that $d* = *d$ and $d_{\not{1}} = 1$. Clearly to give d is the same as giving a collection of numbers $d_j = d_{I_j} \in \mathbb{C}^*$ for $j \in Irr\mathcal{A}$ (here I_j is an irreducible object of class j; recall that $AutI_j = \mathbb{C}^*$).

4.5.2 For any finite set B we have a category $\mathcal{A}^{\otimes B}$: this is an abelian $\mathbb{C}$-category equipped with a polylinear functor $\otimes : \mathcal{A}^B = \prod_{b\in B} A_b \longrightarrow \mathcal{A}^{\otimes B}$, $(X_b)_{b\in B} \longrightarrow \bigotimes_{b\in B} X_b$, which is universal in an obvious sense (see [D] § for an extensive discussion in a less trivial situation). The category $\mathcal{A}^{\otimes B}$ is semisimple. Its irreducible objects are tensor products of irreducibles in $\mathcal{A}$, so $Irr\mathcal{A}^{\otimes B} = (Irr\mathcal{A})^B$. Any isomorphism $\varphi : B \to B'$ induces a canonical equivalence $\mathcal{A}^{\otimes B} \to \mathcal{A}^{\otimes B'}$, $\otimes X_b \longmapsto \otimes X_{\varphi^{-1}(b')}$.

4.5.3 We put $A^{\otimes\emptyset} = Vect$. One may identify $\mathcal{A}^{\otimes\{1,2\}} = \mathcal{A}^{\otimes 2}$ with the category of $\mathbb{C}$-linear functors $F = \mathcal{A}^0 \to \mathcal{A}$. Namely, to an object $X \otimes Y \in \mathcal{A}^{\otimes 2}$ there corresponds the functor $F_{X\otimes Y}$ defined by formula $F_{X\otimes Y}(Z) = Hom(Z,X) \otimes Y$. We define a canonical object ("regular representation") $R \in \mathcal{A}^{\otimes 2}$ as an object that corresponds to the functor $* : \mathcal{A}^0 \to \mathcal{A}$. Here is an explicit construction of R. For each $j \in Irr\mathcal{A}$ pick an irreducible object I_j of class j. Then one has a canonical isomorphism $R = \bigoplus_{j\in Irr\mathcal{A}} I_j \otimes *I_j$. Note that R is symmetric: for the transposition

$\sigma = \{1,2\}$ acting on $\mathcal{A}^{\otimes 2}$ one has a canonical isomorphism $\sigma(R) = R$. So for any two element set B we have a canonical object $R_B \in \mathcal{A}^{\otimes B}$.

4.5.4 For finite sets A, B and a morphism $f : A \to B$ in $Sets^{\#}$ (see 4.3.2) we define a $\mathbb{C}$-linear functor $f^* : \mathcal{A}^{\otimes B} \to \mathcal{A}^{\otimes A}$ by the formula

$$f^*(\bigotimes_{b\in B} X_b) = \left[\bigotimes_{a\in i_f(B)} X_{i_f^{-1}(a)}\right] \otimes \left[\bigotimes_{a\in A_f^0} ident_a\right] \otimes \left[\bigotimes_{\phi_{f\delta}\in\phi_f} R_{\phi_{f\delta}}\right].$$

Clearly $(g\circ f)^* = f^* \circ g^*$, so the f^*'s define a fibered category $\mathcal{A}^{\#}$ over $Sets^{\#}$ with fibers $\mathcal{A}^{\#}_A = \mathcal{A}^{\otimes A}$. The tensor product functor $\otimes : \mathcal{A}^{\otimes B_1} \times \mathcal{A}^{\otimes B_2} \longrightarrow \mathcal{A}^{\otimes (B_1 \coprod B_2)}$ defines on $\mathcal{A}^{\#}$ the structure of commutative monoidal category such that the projection $\mathcal{A}^{\#} \to Sets^{\#}$ is a monoidal functor.

4.5.4 Definition. *A* **fusion structure** *on $\mathcal{A}$ is a collection of functors $\langle\ \ \rangle : \mathcal{A}^{\otimes A} \times \widetilde{Teich}_A \longrightarrow Vect$, $(X,S) \longmapsto \langle X\rangle_S$ (here A is any finite set), together with natural isomomorphisms (i), (ii):*

(i) $\langle X\otimes Y\rangle_{S\sqcup T} = \langle X\rangle_S \otimes \langle Y\rangle_T$ for $X \in \mathcal{A}^{\otimes A}, Y \in \mathcal{A}^{\otimes B}, S \in \widetilde{Teich}_A, T \in \widetilde{Teich}_B$.

(ii) $\langle f^*X\rangle_T = \langle X\rangle_{f_*T}$ for any morphism $f : A \to B$ in $Sets^{\#}$, $X \in \mathcal{A}^{\otimes B}, T \in \widetilde{Teich}_A$.

These isomorphisms should be compatible in an obvious sense. We also demand that:

a. For fixed $S \in \widetilde{Teich}_A$ the functor $\langle\ \ \rangle_S : \mathcal{A}^{\otimes A} \longrightarrow Vect$ is additive.

b. $\langle\ \ \rangle$ transforms Dehn automorphism to Dehn twist, i.e., for a finite set A, an element $\alpha \in A$ and a collection of objects $X_\gamma \in \mathcal{A}$, $\gamma \in A$, the automorphisms of $\langle \otimes X_\gamma\rangle_S$ induced by $\bigotimes_{\gamma\neq\alpha} id_{X_\gamma} \otimes d_{X_\gamma} \in Aut \otimes X_\gamma$ and by $d_\alpha \in Aut S$ coincide.

c. $\langle\ \ \rangle$ is non degenerate in the sense that for any non-zero $X \in \mathcal{A}$ there exists $Y \in \mathcal{A}$ such that $\langle X\otimes Y\rangle_{S_0} \neq 0$ where S_0 is a 2-sphere with two punctures.

We will say that $(\mathcal{A}, \langle\ \ \rangle)$ is a fusion category of multiplicative central charge $a \in \mathbb{C}^*$ if for any $X \in \mathcal{A}^{\otimes A}$ the representation $\langle X\rangle$ of $\widetilde{Teich}$ lies in $\mathcal{R}_{aA}$. □

4.5.5 Clearly (ii) just means that $X \longmapsto \langle X\rangle$ is a cartesian functor $\mathcal{A}^{\#} \to \widetilde{\mathcal{R}}^{\#}$ between categories fibered over $Sets^{\#}$. Since any morphism in $Sets^{\#}$ is a successive deleting of points and sewing of couples of points, we may rewrite (ii) as two compatibilities. Namely

(ii)′ $\langle X\rangle_{del_\alpha S} = \langle X\otimes ident_\alpha\rangle_S$ for any finite set A, $\alpha \in A$, $X \in \mathcal{A}^{\otimes A\setminus\{\alpha\}}$, $S \in \widetilde{Teich}_A$.

(ii)″ $\langle X\rangle_{\mathcal{S}_{\alpha,\beta}S} = \langle X\otimes R_{\alpha\beta}\rangle_S$ for any finite set A, a pair of elements $\alpha,\beta \in A, \alpha \neq \beta$, $X \in \mathcal{A}^{\otimes A\setminus\{\alpha,\beta\}}$, $S \in \widetilde{Teich}_A$.

4.5.6 Here is a reformulation of 4.5.5(ii)″ in "holomorphic" language 4.4.3. For $X \in \mathcal{A}^{\otimes A\setminus\{\alpha,\beta\}}$ our $\langle X\rangle$ is a lisse D_{λ^c}-module with regular singularities at infinity. As was explained in 4.3.1 we have a canonical surjective smooth map $\pi : \mathcal{M}_A \to N \setminus \{\text{zero section}\}$, where N is the normal bundle to the (smooth part of) the divisor at infinity of $\mathcal{M}_{A\setminus\{\alpha,\beta\}}$. We have the canonical specialization function Sp that assigns to a lisse D_{λ^c}-module with regular singularities at infinity on $\mathcal{M}_{A\setminus\{\alpha,\beta\}}$, the one on $N \setminus \{\text{zero section}\}$. Hence we have the D_{λ^c}-module $\pi^* Sp\langle X\rangle$ on $\mathcal{M}_A$, and 4.5.5 (ii)′ is an isomorphism $\pi^* Sp\langle X\rangle = \langle X\otimes R_{\alpha\beta}\rangle$.

4.6 Fusion functors. Let $(\mathcal{A}, \langle \quad \rangle)$ be a fusion category. Let A, B be finite sets. Any object $S \in \widetilde{Teich}_{A \sqcup B}$ defines a functor $\mathcal{F}_S = \mathcal{F}_S^{A,B} : \mathcal{A}^{\otimes A} \to \mathcal{A}^{\otimes B}$ by the formula $Hom(\mathcal{F}_S(X), Y) = \langle X \otimes *Y \rangle^*$, $X \in \mathcal{A}^{\otimes A}, Y \in \mathcal{A}^{\otimes B}$. We will call $\mathcal{F}_S$ the fusion functor along S. The automorphisms of S act as automorphisms of $\mathcal{F}_S$. Note that if $B = \emptyset$ then $A^{\otimes B} = Vect$ and $\mathcal{F}_S = \langle \quad \rangle_S$. If $A = \emptyset$, then $\mathcal{F}$ is a functor $\widetilde{Teich}_B \to \mathcal{A}^{\otimes B}$, i.e., an $\mathcal{A}^{\otimes B}$-valued representation of $\widetilde{Teich}_B$.

Let C be a third finite set, $T \in \widetilde{Teich}_{B \sqcup C}$. We define $T \circ S \in \widetilde{Teich}_{A \sqcup C}$ as the surface obtained from $T \sqcup S$ by sewing the B-boundary components.

4.6.1 Lemma. *There is a canonical isomorphism of functors $\mathcal{F}_{T \circ S} = \mathcal{F}_S \circ \mathcal{F}_T : \mathcal{A}^{\otimes A} \to \mathcal{A}^{\otimes C}$.*

Proof. For $X \in \mathcal{A}^{\otimes A}, Z \in \mathcal{A}^{\otimes C}$ one has

$$\begin{aligned} Hom(\mathcal{F}_{T \circ S}(X), Z) = \langle X \otimes *Z \rangle^*_{T \circ S} &\underset{4.5.4(ii)}{=} \langle X \otimes R^{\otimes B} \otimes *Z \rangle^*_{T \sqcup S} \\ &\underset{4.5.4(i)}{=} \bigoplus_{I_{\vec{j}} \in Irr \mathcal{A}^{\otimes B}} \langle X \otimes *I_{\vec{j}} \rangle^*_S \otimes \langle I_{\vec{j}} \otimes *Z \rangle^*_T \\ &= \bigoplus Hom(\mathcal{F}_S(X), I_{\vec{j}}) \otimes Hom(\mathcal{F}_T(I_{\vec{j}}), Z) = Hom(\mathcal{F}_T \circ \mathcal{F}_S(X), Z). \end{aligned}$$

The last equality comes since

$$\mathcal{F}_S(X) = \oplus Hom(\mathcal{F}_S(X), I_{\vec{j}})^* \otimes I_{\vec{j}}.$$

□

Now assume that $A = \{0\}$, $B = \{\infty\}$ are one point sets. Let $Teich'^0_{\{0,\infty\}} \subset Teich'_{\{0,\infty\}}$ be the full subcategory of "cylinders". So $Teich'^0_{\{0,\infty\}}$ is a connected groupoid; for $(S, 0, \infty) \in Teich^0_{\{0,\infty\}}$ the group (of its automorphisms) is a free abelian group with generator $d_0 = d_\infty^{-1}$. Denote by $S_0 = (S_0, 0, \infty)$ the object of $Teich^0_{\{0,\infty\}}$ such that for any $(S, 0, \infty) \in Teich^0_{\{0,\infty\}}$ one has $Hom(S_0, S) = \{$ set of homotopy classes of paths in S connecting 0 and $\infty\}$. This is a canonical object of $Teich'^0_{\{0,\infty\}}$. Its "holomorphic" counterpart is $(\mathbb{P}^1, 0, \infty, dt(0), dt^{-1}(\infty)) \in Teich''^0_{\{0,\infty\}}$, where t is a standard parameter on $\mathbb{P}^1$. One identifies this point of $Teich''$ with S_0 canonically by drawing the path $\mathbb{R}_{\geq 0}$ from 0 to ∞. Note that since $H(S) = 0$ for $S \in Teich'^0_{\{0,\infty\}}$ we have an obvious embedding $Teich'^0_{\{0,\infty\}} \hookrightarrow \widetilde{Teich}'_{\{0,\infty\}}$; the "holomorphic" counterpart of this section comes since the line bundle λ is canonically trivialized over the "moduli space" of genus zero curves. So we will consider S_0 as a canonical object of $\widetilde{Teich}_{\{0,\infty\}}$. Note that if A is any finite set and $T \in \widetilde{Teich}_{A \sqcup \{0\}}$, then one has an obvious canonical isomorphism $S_0 \circ T = T$. According to 4.6.1 this gives a canonical isomorphism of functors $\mathcal{F}_{S_0} \circ \mathcal{F}_T = \mathcal{F}_T$. In fact, one has

4.6.2 Lemma. *There is a canonical identification of the functor $\mathcal{F}_{S_0} : \mathcal{A} \to \mathcal{A}$ with the identity functor $id_{\mathcal{A}}$ that generates the above isomorphisms $\mathcal{F}_{S_0} \circ \mathcal{F}_T = \mathcal{F}_T$ for all $T \in \widetilde{Teich}_{A \sqcup \{0\}}$.*

Proof. Assume that we know that $\mathcal{F}_{S_0}$ is an equivalence of categories. Then the desired isomorphism $\mathcal{F}_{S_0} = id_{\mathcal{A}}$ would be $\mathcal{F}_{S_0}^{-1}(\mathcal{F}_{S_0} \circ \mathcal{F}_{S_0} = \mathcal{F}_{S_0})$. Since $\mathcal{A}$ is semisimple, to see that $\mathcal{F}_{S_0}$ is an equivalence it suffices to prove that $\mathcal{F}_{S_0}$ induces the identity map of the Grothendieck group $K(\mathcal{A})$. The irreducible I_i form a basis in $K(\mathcal{A})$. Put $\mathcal{F}_{S_0}(I_i) = f_i^j I_j$; we have to show that $f_i^j = \delta_i^j$. We know that $f_i^j \in \mathbb{Z}_{\geq 0}$. Since $f_i^j = \langle I_j \otimes *I_i \rangle_{S_0}^*$ we see, by 4.5.4c, that any row or column of the matrix f_i^j is non-zero. Since $\mathcal{F}_{S_0}^2 = \mathcal{F}_{S_0}$, these properties imply that $\mathcal{F}_{S_0} = id_{K(\mathcal{A})}$ (just note that $\mathcal{F}_{S_0}^2(I_i) = \mathcal{F}_{S_0}(I_i)$ implies $\mathcal{F}_{S_0}$ induces a transposition of the set of those I_j's that $f_i^j \neq 0$; hence $\mathcal{F}_{S_0}$ is a surjective endomorphism of $K(\mathcal{A})$, and hence it is the identity). □

4.6.3 Assume now that S is a connected surface of genus 0 and B is a one point set. Then the corresponding functors $\mathcal{F}_S : \mathcal{A}^{\otimes A} \longrightarrow \mathcal{A}$, together with $*$ and d from 4.5.1, define on $\mathcal{A}$ the structure of a balanced rigid tensor category (see, e.g. [K]). Here are some details. Denote by S_n the surface obtained from a unit disc by cutting out n holes with centers on the real line; the marked points lie on the real line to the right:

$S_3 :$ $\quad O^{x_1} \quad O^{x_2} \quad O^{x_3} \quad x_\infty$

Put $\mathcal{F}_{S_n}(X_1 \otimes \cdots \otimes X_n) = X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_n$. The axiom 1.5.4 (ii)$''$ implies immediately that the operation $\widehat{\otimes} : \mathcal{A}^n \to \mathcal{A}$ is strictly associative: one has $X_1 \widehat{\otimes} X_2 \widehat{\otimes} X_3 = (X_1 \widehat{\otimes} X_2) \widehat{\otimes} X_3 = X_1 \widehat{\otimes}(X_2 \widehat{\otimes} X_3)$. Consider the following diffeomorphism σ of S_2 that fixes ∂S_{2x_∞} and interchanges ∂S_{2x_1} and ∂S_{2x_2} (we move the holes in a way that the marked point remain on the very right of the hole):

This diffeomorphism induces a natural isomorphism $\sigma_{X_1 X_2} : X_1 \widehat{\otimes} X_2 \xrightarrow{\sim} X_2 \widehat{\otimes} X_1$. It is easy to see that σ satisfies the braid relations, and also one has a relation $\sigma^2 = d_{x_\infty} d_{x_1}^{-1} d_{x_2}^{-1}$ in $Aut S_2$. These imply the hexagon axiom for $\widehat{\otimes}$, and the axiom $\sigma_{X_1, X_2}^2 = d_{X_1 \widehat{\otimes} X_2} \circ (d_{X_1} \widehat{\otimes} d_{X_2})^{-1}$ of balanced tensor categories.

4.7 The fusion algebra. The above tensor structure on $\mathcal{A}$ defines a commutative ring structure on the Grothendieck group $K(\mathcal{A})$. One calls $K(\mathcal{A})$ the fusion algebra of $\mathcal{A}$. Note that $K(\mathcal{A})$ has a distinguished basis $\{I_j\}$ of irreducibles. By 4.5.5 (ii)$'$ the base element 1 that corresponds to vacuum module is the unit in $K(\mathcal{A})$.

Now 4.6.2 implies that $(K(\mathcal{A}), \{I_j\})$ is a *based ring* in the sense of [L] 1.1. According to [L] 1.2, $K(\mathcal{A}) \otimes \mathbb{Q}$ is a semisimple algebra. Hence $K(\mathcal{A}) \otimes \mathbb{C}$ has another canonical basis – the one that consists of mutually orthogonal idempotents.

Let T be a torus (= oriented genus one surface). Choose a basis γ_1, γ_2 in $H_1(T, \mathbb{Z})$ compatible with the orientation, so that γ_1, γ_2 are cycles on T that intersect at one point a. Consider the vector space $\langle \nVdash \rangle_T$. Note that if we cut T along γ_1, then γ_2 will become a path that connects two copies of a on the components of the boundary, hence it identifies this surface with the surface S_0 of 4.6.2. According to 4.5.5 (ii)$''$, 4.6.2, the corresponding decomposition 4.5.5(ii)$''$ gives the basis in $\langle \nVdash \rangle_T$ numbered by irreducibles in $\mathcal{A}$, i.e., we have the isomorphism $i_{\gamma_1,\gamma_2} : K(\mathcal{A}) \otimes \mathbb{C} \rightarrow \langle \nVdash \rangle_T$ that transforms I_j's to this basis. Interchanging γ_1 and γ_2 we get the isomorphism $i_{\gamma_2,-\gamma_1} : K(\mathcal{A}) \otimes \mathbb{C} \underset{\sim}{\rightarrow} \langle \nVdash \rangle_T$. The composition $i^{-1}_{\gamma_2,-\gamma_1} \circ i_{\gamma_1,\gamma_2} \in AutK(\mathcal{A}) \otimes \mathbb{C}$ is called the *Fourier transform*. According to the Verlinde conjecture, proved by Moore-Zeiberg, the Fourier transform maps a canonical basis $\{I_j\}$ of irreducibles to a basis proportional to the one given by the idempotents.

§6. Algebraic field theories

6.1 Axioms. Let $c \in \mathbb{C}$ be any complex number. An **algebraic rational field theory** (in dimension 1) of central charge c consists of data (i) - (iv) subject to axioms a-g below:
6.1.1

(i) A fusion category $\mathcal{A}$ of multiplicative central charge $\exp(2\pi i c)$ (see 4.5.4)

(ii) An additive "realization" functor $r : \mathcal{A} \to (\widetilde{\mathcal{T}}, \mathcal{V}_1)_c$-mod (see 3.4.7).

We assume that for any $X \in \mathcal{A}$

a. $r(X)$ *is a higher weight module*, i.e., the "coordinate module" $r(X)_{\mathbb{C}((t)),dt(o)}$ is a (direct) sum of generalized eigenspaces $r(X)_{\mathbb{C}((t)),\lambda} = \{m \in r(X)_{\mathbb{C}((t))} : (L_0 - \lambda)^N m = 0 \text{ for } N \gg 0\}$ for the operator L_0 (see 3.4.7, 7.3.1). *Each* $r(X)_{\mathbb{C}((t))\lambda}$, $\lambda \in \mathbb{C}$, *is a finite dimensional vector space.*

b. $r(d_X) = T_{r(X)}$, *where* d_X *is the Dehn automorphism* (see 4.5.1) *and* T *is the monodromy automorphism* (see 7.3.2).

Note that these axioms imply that $r(\mathbb{1})$ is actually a $(\widetilde{\mathcal{T}}, \mathcal{V})_c$-module (since $T_{r(\mathbb{1})} = id_{r(\mathbb{1})}$.

(iii) A fixed "vacuum" vector $1 \in \mathrm{Hom}_{\mathcal{V}}(\mathbb{C}, r(\mathbb{1}))$.

We assume that

c. *1 is a non-zero vector invariant with respect to the action of* $s_{\mathcal{O}_F}(\mathcal{T}_{-1F}) \subset \widetilde{\mathcal{T}}_F$ (see 3.4.1).

6.1.2. Now let S be a smooth scheme, $\pi : C_S \to S$ a family of smooth projective curves, $A \subset C_S(S)$ a finite disjoint set of sections, and $\{\nu_a\}_{a \in A}$ 1-jets of parameters at points in A. This collection defines S-localization data ψ_c for $(\widetilde{\mathcal{T}}_c^A, \mathcal{V}_1^A)$ (see 3.4.7, 3.4.5). The corresponding algebra of twisted differential operators D_{ψ_c} coincides with D_{λ^c} (see 3.5.6). Hence, by 3.3.5, we have the S-localization functor $\Delta_{\psi_c} \circ r^{\otimes A}$: $\mathcal{A}^{\otimes A} \longrightarrow D_{\lambda^c}$-mod. On the other hand, by 4.5.4, 4.4.3, the fusion structure on $\mathcal{A}$ defines the functor $\langle \quad \rangle_{C_S} : \mathcal{A}^{\otimes A} \longrightarrow D_{\lambda^c}$-mod such that for any $\otimes X_a \in \mathcal{A}^{\otimes A}$ the corresponding D_{λ^c}-module $\langle \otimes X_a \rangle_{C_S}$ is lisse with regular singularities at infinity. Our next piece of data is

(iv) A morphism of functors $\gamma : \Delta_{\psi_c} \circ r^{\otimes A} \longrightarrow \langle \quad \rangle_{C_S}$.

For $X \in \mathcal{A}^{\otimes A}$ denote by $r(X)_{A,C_S} = r(X)_{A,\nu_A,C_S}$ the $\mathcal{O}_S$-module that corresponds to the S-object "formal completion of C_S at A with 1-jet of parameters ν_A" of $\mathcal{V}_1^A$ (see 3.4.3, 3.4.6, 3.4.7). If $X = \otimes X_a$, then $r(X)_{A,C_S} = \otimes_{\mathcal{O}_S} r(X_a)_{a,C_S}$. Recall that $\Delta_{\psi_c} \circ r^{\otimes A}(X)$, considered as an $\mathcal{O}_S$-module, is a quotient of $r(X)_{A,C_S}$. For any section φ of $r(X)_{A,C_S}$ put $\langle \varphi \rangle_{C_S} = \gamma(\varphi) \in \langle X \rangle_{C_S}$. This is the "correlator of the field φ along C_S".

The following axioms should hold:

d. γ *commutes with base change*, i.e., γ is a morphism of D_{λ^c}-modules on the modular stack $\mathcal{M}_A$.

e. *For* $a \in A$, *objects* $X \in \mathcal{A}^{\otimes A \setminus \{a\}}$ *and a section* $\varphi \in r(S, r(X)_{A,C_S})$ *one has* $\langle \varphi \rangle_{C_S} = \langle \varphi \otimes 1_a \rangle_{C_S}$. Here $\langle \varphi \rangle_{C_S}$ is a section of $\langle X \rangle_{C_S}$ (we forget about the point a), and $\langle \varphi \otimes 1_a \rangle_{C_S}$ is a section of $\langle X \otimes \mathbb{1}_a \rangle_{C_S}$; the two D_{λ_c}-modules are identified via 4.5.5 (ii)$'$.

6.1.3 Now consider the two pointed curve $C_0 = (\mathbb{P}^1, 0, \infty, dt(0), dt^{-1}(\infty))$. We have coordinates t at 0 and t^{-1} at ∞. For any object $X \in \mathcal{A}$ consider the pairing

$$\langle\,\rangle_{C_0} : r(*X)_{\mathbb{C}((t))} \otimes r(X)_{\mathbb{C}((t^{-1}))} = r(*X)_{C_{0\hat{0}}} \otimes r(X)_{C_{0\hat{\infty}}} \longrightarrow \langle *X \otimes X \rangle_{C_0} \underset{4.6.2}{=} \mathrm{End}\, X$$

Here we write simply $\mathbb{C}((t))$ for $(\mathbb{C}((t)), dt(0)) \in \mathcal{V}_1$. This pairing is a morphism of End X-bimodules, hence it defines a linear map

$$i : r(*X)_{\mathbb{C}((t))} \longrightarrow \mathrm{Hom}_{\mathrm{End}\ X}(r(X)_{\mathbb{C}((t^{-1}))}, \mathrm{End}\ X) =: r(X)^*_{\mathbb{C}((t^{-1}))}.$$

Note that $r(X)^*_{\mathbb{C}((t^{-1}))}$ is a $\widetilde{\mathcal{T}}_{\mathbb{C}((t^{-1}))}$- module in an obvious manner. Denote by $*r(X)_{\mathbb{C}((t^{-1}))} \subset r(X)^*_{\mathbb{C}((t^{-1}))}$ the sum of generalized eigenspaces of the operator $L_0 \in \widetilde{\mathcal{T}}_{\mathbb{C}((t))}$. The pairing $\langle\ \rangle_{C_0}$ is $\mathcal{T}(\mathbb{P}^1 \setminus \{0, \infty\})$-invariant (by definition of Δ_ψ, see 3.4.4), hence i commutes with the the L_0-action. By axiom a above we see that $i(r(*X)_{\mathbb{C}((t))} \subset *r(X)_{\mathbb{C}((t^{-1}))}$. Our next axiom is

f. *The map* $i : r(*X)_{\mathbb{C}((t))} \longrightarrow *r(X)_{\mathbb{C}((t^{-1}))}$ *is an isomorphism of vector spaces.*

It suffices to verify f for irreducible X's only.

6.1.4 Our final axiom g ("factorization at infinity") describes the asymptotic expansion of correlators near the boundary of the moduli space. So consider the following situation.

Let $\pi : C_S \to S = \mathrm{Spec}\ \mathbb{C}[[q]]$ be a proper flat family of curves such that the generic fiber C_η is smooth and the special fiber C_0 has exactly one singular point which is quadratic. Let $B = \{b_i\}$ be a finite non-empty set of sections of π such that the points $b_i(0) \in C_0$ are pairwise different, and let $\nu_i \in b_i^* \omega_{C_S/S}$ be a 1-jet of coordinates at the b_i's. Then $\mathcal{C} = (C_\eta, b_i, \nu_i)$ is a $\mathbb{C}((q))$-point of $\mathcal{M}_B$.

Let t_1, t_2 be formal coordinates at a such that $t_1 t_2 = q$. According to 3.6.1 we get a smooth S-curve $C_S^\vee$ with points $a_1, a_2 \in C_S^\vee(S)$ and formal coordinates t_i at a_i. Put $A = B \bigsqcup \{a_1, a_2\}$. Then $\mathcal{C}^\vee = (C_\eta^\vee, b_i, a_1, a_2; \nu_i; q^{-1} dt_1(a_1), dt_2(a_2))$ is a $\mathbb{C}((q))$-point of $\mathcal{M}_A$.

The S-curves C_S and $C_S^\vee$ define the corresponding determinant line bundles on S. According to 3.6.3 their ratio is canonically stratified, hence the corresponding rings of differential operators are canonically identified; we denote this algebra D_{λ^c}.

For any object $X \in \mathcal{A}^{\otimes B}$ we get the lisse D_{λ^c}-modules $\langle X \rangle_{\mathcal{C}}$ and $\langle X \otimes R \rangle_{\mathcal{C}^\vee}$ on η with regular singularities at $q = 0$. According to 4.5.6 we have a canonical isomorphism between their specializations to $q = 0$ (these are D-modules on the punctured tangent line at $q = 0$). Since Sp_0 is an equivalence of categories, we have a canonical isomorphism of D_{λ^c}-modules $\langle X \rangle_{\mathcal{C}} = \langle X \otimes R \rangle_{\mathcal{C}^\vee}$.

To formulate axiom g we need to consider a special vector in $r(R)$. Recall that $R = \bigoplus_{I_j \in \mathrm{Irr}\ A} I_j \otimes *I_j$. Choose a basis $\{e_j^K\}$ in each $r(I_j)_{\mathbb{C}((t))}$ compatible with grading by generalized eigenspaces of L_0. Here, as above, we write simply $\mathbb{C}((t))$ for $(\mathbb{C}((t)), dt(0)) \in \mathcal{V}_1$.

Below we will use the following notation: if $F \in \mathcal{V}$ is any local field, t_F a parameter in F, $X \in \mathcal{A}$ and $e \in r(X)_{\mathbb{C}((t))}$, then $e_{(F,t_F)} \in r(X)_{F, dt_F(0)}$ is a vector that corresponds to e via the isomorphism $(\mathbb{C}((t)), dt(0)) \overset{\sim}{\rightarrow} (F, dt_F(0))$, $t \longmapsto t_F$.

According to axiom f. above, we get the dual basis $\{*e_j^K\}$ of $r(*I_j)_{\mathbb{C}((t))}$, namely $*e_j^K = i^{-1} e_j^{K*}$, where $e_j^{K*} \in *r(I_j)_{(\mathbb{C}((t^{-1})), t^{-1})}$ is the dual basis to $e^K_{j(\mathbb{C}((t^{-1})), t^{-1})}$.

Now let $\varphi = \varphi(q)$ be any section of $r(X)_{B, \nu_B, C} = r(X)_{B, \nu_B, \mathcal{C}^\vee}$ over S. Consider the correlator $a_j^K = \langle \varphi \otimes e^K_{j(\mathbb{C}((t_1)), q^{-1} t_1)} \otimes *e^K_{j(\mathbb{C}((t_2)), t_2)} \rangle_{\mathcal{C}^\vee}$: this is a section of $\langle X \otimes I_j \otimes *I_j \rangle_{\mathcal{C}^\vee}$. Note that $\langle X \otimes I_j \otimes *I_j \rangle_{\mathcal{C}^\vee}$ is a finite dimensional $\mathbb{C}((q))$-vector space. One has

6.1.5 Lemma. *The series* $\sum_K a_j^K$ *converges; its limit* $\langle \varphi \otimes c_j \rangle_{\mathcal{C}^\vee} \in \langle X \otimes I_j \otimes *I_j \rangle_{\mathcal{C}^\vee}$ *does not depend on a particular choice of basis* $\{e_j^K\}$. □

Assuming the lemma, our final axiom is

g. *One has* $\langle \varphi \rangle_{\mathcal{C}} = \langle \varphi \otimes \sum_j C_j \rangle_{\mathcal{C}^\vee} = \sum_j \langle \varphi \otimes C_j \rangle_{\mathcal{C}^\vee}$ *via the above canonical isomorphism*

$$\langle x \rangle_{\mathcal{C}} = \langle X \otimes R \rangle_{\mathcal{C}^\vee} = \oplus \langle X \otimes I_j \otimes *I_j \rangle_{\mathcal{C}^\vee}.$$

Proof of 6.1.5. The independence of a choice of basis is straightforward. To prove that our series converges it is convenient to add a parameter u, and consider a base scheme $\widetilde{S} = \mathrm{Spec}(\mathbb{C}[u, u^{-1}]) \times S$ together with an $\widetilde{S}$-point of $\mathcal{M}_A$ defined by the family $\mathcal{C}_u^\vee = (C_{\widetilde{S}}^\vee, b_i, a_1, a_2; \nu_i, udt_1, dt_2)$. We get the lisse D_{λ^c}-module $\langle X \otimes I_j \otimes *I_j \rangle_{\mathcal{C}_u^\vee}$ on $\widetilde{S}$, and a collection of sections $a_j^K(u,q) = \langle \varphi(q) \otimes e^K_{j(\mathbb{C}((t_1)), ut_1)} \otimes *e^K_{j(\mathbb{C}((t_2)), t_2)} \rangle_{\mathcal{C}_u^\vee} \in \Gamma(\widetilde{S}, \langle X \otimes I_j \otimes *I_j \rangle_{\mathcal{C}^\vee})$. The old picture is just the restriction of this one to the diagonal $u = q^{-1}$. Our D-module has regular singularities along the divisor $u = \infty$, so we may extend it to a vector bundle V to $\widetilde{S}^- = \mathrm{Spec}(\mathbb{C}[u^{-1}]) \times S$ invariant with respect to operator $u\partial_u$. Our lemma would follow if we show that for any $N \gg 0$ one has $a_j^K(u,q) \in u^{-N} V$ for all but finitely many K's. The action of the operator $u\partial_u$ on $a_j^K(u,q)$ was computed in 3.4.7.1. Namely, we have $u\partial_u(a_j^K(u,q)) = \langle \varphi(q) \otimes L_0(e_j^K)_{(\mathbb{C}((t_1)), ut_1)} \otimes *e_j^K \rangle_{\mathcal{C}_u^\vee}$, hence $a_j^K(u,q)$ is a generalized eigenvector of $u\partial_u$ with eigenvalue equal to an eigenvalue of L_0 at e_j^K. Axiom a. above implies that for any $\overline{\mu} \in \mathbb{C}/\mathbb{Z}$ and $c \in \mathbb{R}$ the space $\bigoplus_{\substack{\mu = \overline{\mu} \bmod \mathbb{Z} \\ \mathrm{Re}\ \mu > c}} r(I_j)_{\mathbb{C}(t)\mu} \subset r(I_j)_{\mathbb{C}((t))}$ is finite dimensional. On the other hand, since $\langle X \otimes I_j \otimes *I_j \rangle_{\mathcal{C}_u^\vee}$ is a lisse module, there are only finitely many $\overline{u} \in \mathbb{C}/\mathbb{Z}$ such that one has a section which is a generalized eigenvector of $u\partial_u$ with eigenvalue mod $\mathbb{Z}$ equal to $\overline{u}$. This implies that for any $c \in \mathbb{R}$ all but finitely many a_j^K's are generalized eigenvectors of $u\partial_u$ with Re (eigenvalue) $< c$. This implies that all but finitely many of them lie in $u^{-N} V$. □

6.1.6 Remark. We may consider the situation when a smooth curve degenerates to a curve with several quadratic singular points. One trivially reformulates axiom g for this situation; it is easy to see that this generalized version follows from axiom g. above (the case of one singular point).

6.1.7 Here is an example of how axiom g works. Let C be a fixed curve, $A \subset C$ a finite set, $\{\nu_a\}$, $a \in A$, 1-jets of coordinates at a's, $X \in \mathcal{A}^{\otimes A}$, and $\varphi \in r(X)_{a,C}$. Let $x \in C \setminus A$ be a point, t_x a parameter at x and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ distinct complex numbers. Let $x_i(q)$ be $\mathbb{C}[[q]]$ points of C defined by the formula $x_i(o) = x, t_x(x_i(q)) = \lambda_i q$. Put $t_i = t_{x/q} - \lambda_i$: these are parameters at x_i's for $q \neq 0$. Let $Y_1, \ldots, Y_n$ be objects in $\mathcal{A}$, $\psi_i \in r(Y_i)_{\mathbb{C}((t))}$. We would like to compute $\langle \varphi \in \psi_{1(\mathbb{C}((t_1)), t_1)} \otimes \cdots \otimes \psi_{n(\mathbb{C}((t_n)), t_n)} \rangle_C \in \langle X \otimes Y_1 \otimes \cdots \otimes Y_n \rangle_{(C, A, \{x_i\}, \nu_A, dt_i(x_i))}$. To do it one should blow up the point $(x, 0) \in C_S = C \times S$; denote this curve C'_S. Clearly $A, \{x_i\}$ are S-points of C'_S, and we have parameters $t_x, q/t_x$ at the (only) singular point of C'_0. The corresponding S-curve $C_S'^\vee$ is constant: one has $C_S'^\vee = C_S \coprod \mathbb{P}^1_S$; the formal parameters at $a_1 = x \in C_S$, $a_2 = \infty \in \mathbb{P}^1_S$ are t_x, t^{-1}, respectively. We see that C'_S comes from $(C \coprod \mathbb{P}^1; x, \infty; t_s, t^{-1})$ via the construction 3.6.4. The points $A, \{x_i\}$ on $C_S'^\vee$ are also constant, as well as coordinates t_i: one has $x_i = \lambda_i \in \mathbb{P}^1, t_i = t - \lambda_i$. Hence

$$\langle X \otimes Y_1 \otimes \cdots \otimes Y_n \rangle_{(C;A,\{x_i\};\nu_A,dt_i(x_i))} = \bigoplus_j \langle Y_1 \otimes \cdots \otimes Y_n \otimes I_j \rangle_{(\mathbb{P}^1;\lambda_i,\infty;dt(\lambda_i),q^{-1}dt^{-1}(\infty))}$$
$$\otimes \langle *I_j \otimes X \rangle_{(C;x,A;dt_x(x),\nu_A)}$$

and

$$\langle \varphi \otimes \psi_{1(\mathbb{C}((t_1)),t_1)} \otimes \cdots \otimes \psi_{n(\mathbb{C}((t_n)),t_n)} \rangle_C = \langle \psi_{1(\mathbb{C}((t-\lambda_1)),t-\lambda_1)} \otimes \cdots \otimes \psi_{n(\mathbb{C}((t-\lambda_n)),t-\lambda_n)}$$
$$\otimes e_j^K(\mathbb{C}((t^{-1})), q^{-1}t^{-1}) \rangle_{\mathbb{P}^1} \otimes \langle *e^K_{j\,(\mathbb{C}((t_x)),t_x)} \otimes \varphi \rangle_C.$$

6.2 Global vertex operators. Assume we have an algebraic field theory as in 6.1. Let C be a smooth compact curve, $A \subset C$ a finite set of points and ν_a, $a \in A$, a 1-jet of parameters at a's.

6.2.1 For an object $X \in \mathcal{A}^{\otimes A}$ we have a finite dimensional vector space $\langle X \rangle_C$ and a linear map $\langle\ \ \rangle_C : r(X)_{A_C} \longrightarrow \langle X \rangle_C$. Also for any n-tuple of points $x_1, \cdots, x_n \in C \setminus A$, $x_i \neq x_j$ for $i \neq j$, we have a linear map $\langle\ \ \rangle_C : r(X)_{A,C} \otimes r(\Bbbk)_{x_1,C} \otimes \ldots \otimes r(\Bbbk)_{x_n,C} =$
$r(X \otimes \Bbbk \otimes \cdots \otimes \Bbbk)_{A\cup\{x_1,\cdots,x_n\},C} \longrightarrow \langle X \otimes \Bbbk \otimes \cdots \otimes \Bbbk \rangle_C = \langle X \rangle_C$, where the last equality is 4.5.5 (ii)$'$. Note that we need not fix here 1-jets of parameters at x_i's since $r(\Bbbk)$ is a $(\widetilde{\mathcal{T}}, \mathcal{V})_c$-module (see axiom b). We may rewrite this as a linear map

$$V^A_{x_1,\cdots,x_n} : \otimes r(\Bbbk)_{x_i,C} \longrightarrow r(X)^*_{A,C} \otimes \langle X \rangle_C.$$

This construction may be rearranged in several ways:

6.2.2 Let the points $x_1, \cdots, x_n$ vary. On C^n we have a locally free $\mathcal{O}_{C^n}$-module $r(\Bbbk)^{\otimes n}_{C^n}$ with fibers $r(\Bbbk)^{\otimes n}_{C^n(x_1,\cdots,x_n)} = \otimes r(\Bbbk)_{x_i,C}$. On $U = (C \setminus A)^n \setminus \{\text{diagonals}\}$ we have a morphism $V^A : r\Bbbk)^{\otimes n}_U \longrightarrow \mathrm{Hom}_{\mathbb{C}}(r(X)_{A,C}, \langle X \rangle_C \otimes \mathcal{O}_U)$ of $\mathcal{O}_U$-modules such that the value of V^A at $(x_1, \cdots, x_n)$ coincides with $V^A_{x_1,\cdots,x_n}$. For any open set $W \subset U$ we get a map

$$V^A_H : \Gamma(W, r(\Bbbk)^{\otimes n}_W \otimes \Omega^n_W) \longrightarrow r(X)^*_{A,C} \otimes \langle X \rangle_C \otimes H^n_{DR}(W)$$

which is a composition of $V \otimes id_{\Omega^n_W}$ and the canonical projection $\Gamma(W, \Omega^n_W) \to H^n_{DR}(W)$.

6.2.3 Assume that $A = A_1 \sqcup A_2$ and $X = X_1 \otimes X_2,\ X_i \in \mathcal{A}^{\otimes A_i}$. Then $r(X)_{A,C} = r(X_1)_{A_1,C} \otimes r(X_2)_{A_2,C})$, $\quad r(X)^*_{A,C} = \mathrm{Hom}(r(X_1)_{A_1,C}, r(X_2)^*_{A_2,C})$. Let us fix a formal parameter t_a at α such that $dt_a(a) = \nu_a$. These identify $r(X_i)_{A_i,C}$ with "coordinate modules" $r(X_i)_{\mathbb{C}((t_{A_i}))}$ and $r(X_2)^*_{A_i,C}$ with a completion $r(*X_2)^{\wedge}_{\mathbb{C}((t_{A_2}))}$ of $r(*X_2)_{\mathbb{C}((t_{A_2}))}$. So we may rewrite the above $V_{x_1,\cdots,x_n}$ as

$$V^{A_1,A_2}_{x_1,\cdots,x_n} :\ \otimes r(\Bbbk)_{x_i,C} \otimes \langle X_1 \otimes X_2 \rangle^*_C \longrightarrow \mathrm{Hom}(r(X_1)_{\mathbb{C}((t_{A_1}))}, r(*X_2)^{\wedge}_{\mathbb{C}((t_{A_2}))}).$$

The linear operators in the image of this map are called vertex operators.

6.2.4 Now assume that $X_1 = Y,\ X_2 = *\mathcal{F}^{A_1,A_2}_C(Y)$, where $\mathcal{F}^{A_1,A_2}_C : \mathcal{A}^{\otimes A_1} \to \mathcal{A}^{\otimes A_2}$ is the fusion functor from 4.6. Then $\langle X_1 \otimes X_2 \rangle^*_C = \mathrm{Hom}(\mathcal{F}^{A_1,A_2}_C(X_1), *X_2)$ has a canonical element id_{*X_2}; hence we get

$$V^{A_1,A_2}_{x_1,\cdots,x_n} : \otimes r(\Bbbk)_{x_i,C} \ \longrightarrow\ \mathrm{Hom}(r(Y)_{\mathbb{C}((t_{A_1}))}, r(\mathcal{F}^{A_1,A_2}_C(Y))^{\wedge}_{\mathbb{C}((t_{A_2}))}).$$

Here are the first properties of vertex operators in this setting, that follow directly from the axioms.

6.2.5 For $j \in \{1, \dots, n\}$ and $\varphi \in \bigotimes_{i \neq j} r(\Bbbk)_{x_i, C}$ one has $V^{A_1,A_2}_{x_1,\dots,\widehat{x}_j,\dots,x_n}(\varphi) = V^{A_1,A_2}_{x_1,\dots,x_n}(\varphi \otimes 1_{x_j})$.

6.2.6 Put $\mathcal{T}(C \setminus A, x_1, \dots, x_n) = \{\tau \in \mathcal{T}(C \setminus A) : \tau(x_i) = 0\} \subset \mathcal{T}(C \setminus A)$. Then the linear map $V^{A_1,A_2}_{x_1,\dots,x_n}$ commutes with the $\mathcal{T}(C \setminus A, x_1, \dots, x_n)$-action. Here $\mathcal{T}(C \setminus A, x_1, \dots, x_n)$ acts on the left hand side via $\mathcal{T}(C \setminus A, x_1, \dots, x_n) \to \mathcal{T}_{(x_i)o} \subset \widetilde{\mathcal{T}}_{(x_i)}$ (= Virasoro algebra at x_i) and on the right hand side via the map $\mathcal{T}(C \setminus A) \to \widetilde{\mathcal{T}}_{(A)}$ from 2.3.4. In particular, any vertex operator F transforms via a finite dimensional representation of $\mathcal{T}(C \setminus A, x_1, \dots, x_n)$ and F is fixed by a Lie subalgebra of $\mathcal{T}(C \setminus A)$ that consists of fields vanishing to sufficiently high order at the x_i's.

6.2.7 Let C' be another curve, $A' = A_2 \sqcup A_3 \subset C'$ a finite set of points, $t_{a'}$ formal parameters at $a' \in A'$, and $\{x'_1, \cdots, x'_m\} \subset C' \setminus A'$. Let $(C \circ C')_q$ be the $\mathbb{C}[[q]]$-curve with zero fiber obtained from $C \sqcup C'$ by clutching together the points of A_2 in C, C', and where the q-deformation comes from using parameters $t_{a_2}, t_{a'_2}$ according to 3.6.4. Then $A_1 \sqcup A_3 \sqcup \{x_1, \dots, x_n\} \sqcup \{x'_1, \dots, x'_m\}$ is a finite set of $\mathbb{C}[[q]]$-points of $(C \circ C')_q$, and hence we have our vertex operators map $V^{A_1,A_3}_{x_1,\dots,x_n,x'_1,\dots,x'_m} : \otimes r(\Bbbk)_{x_i,C} \otimes r(\Bbbk)_{x'_j,C'} \longrightarrow \mathrm{Hom}(r(Y)_{\mathbb{C}((t_{A_1}))}, r(\mathcal{F}^{A_1,A_3}_{(C\circ C')_q}(Y)^{\wedge}_{\mathbb{C}((t_{A_3}))})$. On the other hand, it is easy to see that "topologically" $(C \circ C')_q$ coincides with "topological" composition $C_q \circ C'$ from 4.6.1, where

$$C_q = (C, dt_{a_1}(a_1), q^{-1} dt_{a_2}(a_2)) \in \mathcal{M}_A, \ a_1 \in A_1, a_2 \in A_2.$$

Hence, by 4.6.1, one has $\mathcal{F}^{A_1,A_3}_{(C\circ C')_q} = \mathcal{F}^{A_2,A_3}_{C'} \circ \mathcal{F}^{A_1,A_2}_{C_q}$.

Our next property, that follows directly from axiom g, is: for any $\varphi \in \otimes r(\Bbbk)_{x_i,C}$, $\varphi' \in \otimes r(\Bbbk)_{x'_j,C'}$ one has

$$V^{A_1,A_3}_{x_1,\cdots,x_n,x'_1,\cdots,x'_m}(\varphi \otimes \varphi') = V^{A_2,A_3}_{x'_1,\cdots,x'_m}(\varphi') \circ V^{A_1,A_2}_{x_1,\cdots,x_n},$$

where composition of "infinite matrixes" is understood in a way similar to 6.1.5.

6.3 Local vertex operators. Assume we have a field theory as in 6.1.

6.3.1 Let C be a smooth curve. Denote by $\widetilde{C}$ the cotangent bundle of C with zero section removed; so a point of $\widetilde{C}$ is a pair (x, ν_x), $x \in C$, ν_x is a 1-jet of coordinates at x. Any object $X \in \mathcal{A}$ defines a locally free $\mathcal{O}_{\widetilde{C}}$-module $r(X)_{\widetilde{C}}$ with fibers $r(X)_{(x,\nu_x)} = r(X)_{x,\nu_x,C}$. A choice of a family of local parameters defines a trivialization of $r(X)_{\widetilde{C}}$. More precisely, let t be a function on a formal neighbourhood of the diagonal $\Delta : \widetilde{C} \hookrightarrow \widetilde{C} \times C$, $\Delta(x, \nu_x) = (x, \nu_x, x)$, such that $t|_\Delta = 0$, $d_{x_2} t(x, \nu_x, x) = \nu_x$ (so $t_{(x,\nu_x)} = t(x, \nu_x, \cdot)$ is a formal parameter at x); such a t defines a trivialization $s^t : r(X)_{\widetilde{C}}) \mathrel{\tilde{\rightarrow}} r(X)_{\mathbb{C}((t))} \otimes \mathcal{O}_{\widetilde{C}}$.

This $r(X)_{\widetilde{C}}$ is a $D_{\widetilde{C}}$-module in a canonical way; the D-module structure comes from the $\mathcal{T}_{\mathbb{C}((t))^{-1}}$-action on $r(X)_{\mathbb{C}((t))}$. Explicitly, a vector field $\tau \in \mathcal{T}_{\widetilde{C}} \subset D_{\widetilde{C}}$ acts on $r(X)_{\widetilde{C}}$ as follows. Choose (locally) a family t of local parameters as above. Let ∇_0 be the flat connection that corresponds to the trivialization S^t. Let $\widetilde{\tau}^t \in \widetilde{\mathcal{T}}_{\mathbb{C}((t))} \otimes \mathcal{O}_{\widetilde{C}}$ be the section defined by formula $\widetilde{\tau}^t = \mathcal{S}_{\mathbb{C}[[t]]}(\mathcal{T}_{(x_1,\nu_{x_1})}(t)\partial_t)$: here $\mathcal{T}_{(x_1,\nu_{x_1})}$ is a vector field on $\widetilde{C} \times C$ equal to τ in the $\widetilde{C}$-directions and to 0 in the C

directions (hence $\mathcal{T}_{(x_1,\nu_{x_1})}(t)$ is a function on the formal neighbourhood of Δ), and $\mathcal{S}_{\mathbb{C}[[t]]} : \mathcal{T}_{\mathbb{C}[[t]]} \to \widetilde{\mathcal{T}}_{\mathbb{C}((t))}$ was defined in 3.4.1. Now for a section φ of $r(X)_{\widetilde{C}}$ one has $\tau(\varphi) = \nabla_0(\tau)(\varphi) - \widetilde{\tau}^t(\varphi)$, where $\widetilde{\tau}^t(\varphi)$ is the $\widetilde{\mathcal{T}}_{\mathbb{C}((t))}$-action on $r(X)_{\mathbb{C}((t))}$.

6.3.2 Remarks. (i) One may explain the $D_{\widetilde{C}}$-module structure on $r(X)_{\widetilde{C}}$ as follows. We have two natural actions of the Lie algebra $\mathcal{T}_C$ on $r(X)_{\widetilde{C}}$. The first one – "Lie derivative" – comes since $r(X)_{\widetilde{C}}$ is a natural sheaf, hence symmetries of C (and infinitesimal ones also) act on it. The second is an $\mathcal{O}$-linear action that comes because the fibers of $r(X)_{\widetilde{C}}$ are Virasoro modules (using the splitting $\mathcal{S}_{\mathcal{O}_x^\wedge}$). Now the D-module action of vector fields is the difference of these two actions.
(ii) For any étale map $f : C' \to C$ one has a canonical isomorphism $f_r^*(X)_{\widetilde{C}} = r(X)_{\widetilde{C}'}$ of $D_{\widetilde{C}'}$-modules.
(iii) If $d_X = id_X$ (see 4.5), e.g., if $X = \Bbbk$, then $r(X)$ is actually a $(\widetilde{\mathcal{T}}, \mathcal{V})$-module, hence $r(X)_{\widetilde{C}}$ comes from a canonical D-module $r(X)_C$ on C.

6.3.3 For $X_1, \cdots, X_n \in \mathcal{A}$ consider the D-module $\boxtimes_i r(X_i)_{\widetilde{C}} = r(X_1)_{\widetilde{C}} \boxtimes \ldots \boxtimes r(X_n)_{\widetilde{C}}$ on $\widetilde{C}^n$. If C is compact, we also have a lisse D-module $\langle X_1 \otimes \cdots \otimes X_n \rangle_{\widetilde{C}}$ on $\widetilde{C} \setminus \{\text{diagonals}\}$ with regular singularities along the diagonals; the fiber of $\langle X_1 \otimes \cdots \otimes X_n \rangle_{\widetilde{C}}$ over $(x_1, \nu_1, \cdots, x_n, \nu_n)$ is $\langle X_1 \otimes \cdots \otimes X_n \rangle_{(C,\{x_i\},\{\nu_i\})}$. By 6.1.2 we have a canonical morphism of $D_{\widetilde{C}^n}$-modules $\langle\quad\rangle_{\widetilde{C}} : \boxtimes r(X_i)_{\widetilde{C}} \to j_*\langle \otimes X_i \rangle_{\widetilde{C}}$, where $j : \widetilde{C}^n \setminus \{\text{diagonals}\} \hookrightarrow \widetilde{C}$.
6.3.4 For a moment let us drop the compactness assumption on C; we will work locally. For $X \in \mathcal{A}$ let $r(X)^\wedge{}_{\mathcal{C},C^n}$ be the completion of $r(X)_{\widetilde{C}} \boxtimes \mathcal{O}_{C^n}$ around the diagonal $\Delta : \widetilde{C} \to \widetilde{C} \times C^n$, $\Delta(x, \nu_x) = (x, \nu_x; x, \cdots, x)$. A choice of a family of local parameters $t = (t_{x,\nu_x})$ identifies sections of $r(X)^\wedge_{\widetilde{C},C^n}$ with formal power series $\Sigma m_{i_1,\cdots,i_n} t_1^{i_1} \cdots t_n^{i_n}$, where $m_{i_1,\cdots,i_n}$ are sections of $r(X)_{\widetilde{C}}$ and $t_i(x_0, \nu_{x_0}, x_1, \cdots, x_n) = t_{(x_0,\nu_{x_0})}(x_i)$. Then $r(X)^\wedge_{\widetilde{C},C^n}$ is a (non quasicoherent) $D_{\widetilde{C}\times C^n}$-module in an obvious manner. Let $\mathcal{O}^\#_{\widetilde{C}\times C^n} \supset \mathcal{O}_{\widetilde{C}\times C^n}$ denote the sheaf of functions having (meromorphic) singularities at diagonals $x_i = x_j$, $i, j \geq 0$. Put $r(X)^\#_{\widetilde{C},C^n} := \mathcal{O}^\#_{\widetilde{C}\times C^n} \otimes_{\mathcal{O}_{\widetilde{C}\times C^n}} r(X)^\wedge_{\widetilde{C},C^n}$: this is also a $D_{\widetilde{C}\times C^n}$-module. A section of $r(X)^\#_{\widetilde{C}\times C^n}$ is a formal series

$$\prod (t_i - t_j)^{-a_{ij}} (\Sigma m_{i_1\cdots i_n} t_1^{i_1} \cdots t_n^{i_n}),\ a_{ij} \geq 0.$$

Now let us define the "local" vertex operators:

6.3.5 Lemma. *There is a canonical morphism of $D_{\widetilde{C}\times C^n}$-modules*

$$\mu : r(\Bbbk)_C \boxtimes \cdots \boxtimes r(\Bbbk)_C \boxtimes r(X)_{\widetilde{C}} \longrightarrow r(X)^\#_{\widetilde{C},C^n}$$

such that (assuming C is compact) for any $(x, \nu_x; y_1, \nu_{y_1}; \cdots; y_m, \nu_{y_m}) \in \widetilde{C} \times \widetilde{C}^m$, $x \neq y_i$, $y_i \neq y_j$ for $i \neq j$, objects $Y_i \in \mathcal{A}$, an element $\psi_x \in r(X)_{x,\nu_x}$, $\psi_{y_i} \in r(Y_i)_{y_i,\nu_{y_i}}$ and a section $\varphi_1, \cdots, \varphi_n$ of $r(\Bbbk)_C$ in a neighbourhood of x one has

$$\langle \varphi_1 \otimes \cdots \otimes \varphi_n \otimes \psi_x \otimes \cdots \otimes \psi_{y_m} \rangle_{\widetilde{C}} = \langle \mu(\varphi_1 \otimes \cdots \otimes \varphi_n \otimes \psi_x) \otimes \psi_{y_1} \otimes \cdots \otimes \psi_{y_n} \rangle_{\widetilde{C}}$$

(as meromorphic functions on a formal neighbourhood of $(x, \ldots, x) \in C^n$ with values in $\langle X \otimes Y_1 \otimes \cdots \otimes Y_m \rangle_{(C,\{x,y_i\},\{\nu_x,\nu_{y_i}\})}$ identified with $\langle \Bbbk \otimes \cdots \otimes \Bbbk \otimes X \otimes Y_1 \otimes \cdots \otimes Y_m \rangle$ via 4.5.5 (ii)′).

Proof - construction. We will write an explicit formula for μ. To do this consider first $\mathbb{P}^1$ with the standard parameter t. So t defines a family of local parameters $t_x = t - x$ on $\mathbb{P}^1 \setminus \{\infty\}$, and hence we have a trivialization $s^t : r(\nVdash_{\mathbb{P}^1 \setminus \{\infty\}} = r(\nVdash)_{\mathbb{C}((t))} \otimes \mathcal{O}_{\mathbb{P}^1 \setminus \{\infty\}}$. For $\varphi \in r(\nVdash)_{\mathbb{C}((t))}$ we denote by φ^t the corresponding "constant" section of $r(\nVdash)_{\mathbb{P}^1 \setminus \{\infty\}}$.

Now for $\varphi_1, \cdots, \varphi_n \in r(\nVdash)_{\mathbb{C}((t))}$ and $x_1, \ldots, x_n \in \mathbb{P}^1 \setminus \{\infty\}$, $x_i \neq x_j$ for $i \neq j$, consider the vertex operator $V^{0,\infty}_{x_1,\cdots,x_n}(\varphi_1^t \otimes \cdots \otimes \varphi_n^t) : r(X)_{\mathbb{C}((t))} \longrightarrow r(X)^{\wedge}_{\mathbb{C}((t))}k$ from 6.2.4 (here we identified the module $r(X)_{\mathbb{C}((t^{-1}))}$ at ∞ with $r(X)_{\mathbb{C}((t))}$ via $t^{-1} \longmapsto t$). In fact, this operator lies in End $r(X)$.

[Proof. For any $a \in \mathbb{C}^*$ one has $t_{ax} = a(t - x)$; hence the automorphism $x \longmapsto ax$ of $\mathbb{P}^1$ acts on $r(\nVdash)_{\mathbb{P}^1}$ (according to 6.3.2) by the formula $\varphi^t \longmapsto (a^{L_0}\varphi)^t$. This implies immediately that if $L_0\varphi_i = n_i\varphi_i$, then $V^{0,\infty}_{x_1,\cdots,x_n}(\otimes\varphi_i^t)(L_0 e) = (L_0 + n_1 + \cdots + n_n)V^{0,\infty}_{x_1,\cdots,x_n}(e)$. Hence $V^{0,\infty}_{x_1,\cdots,x_n}(\otimes\varphi_i^t)$ maps L_0-generalized eigenspaces in $r(X)_{\mathbb{C}((t))}$ to ones in $r(X)^{\wedge}_{\mathbb{C}((t))}$; since the sum of these equals $r(X)_{\mathbb{C}((t))}$, we see that $V^{0,\infty}_{x_1,\cdots,x_n}(\otimes\varphi_i^t)$ maps $r(X)_{\mathbb{C}((t))}$ to $r(X)_{\mathbb{C}((t))}$.]

Clearly, $V^{0,\infty}_{x_1,\cdots,x_n}(\varphi_1^t \otimes \cdots \otimes \varphi_n^t)$ is a meromorphic function on $(\mathbb{P}^1 \setminus \{0,\infty\})^n \setminus$ {diagonals} with values in End $r(X)_{\mathbb{C}((t))}$. Put $\mu(\varphi_1^t \otimes \cdots \otimes \varphi_n^t \otimes \psi_0) = V^{0,\infty}_{x_1,\cdots,x_n}(\varphi_1^t \otimes \cdots \otimes \varphi_n^t)(\psi_0)$ for $\psi_0 \in r(X)_{\mathbb{C}((t))}$: we will consider $\mu(\quad)$ as a formal power series in variables $t_1, \cdots, t_n, t_i = t(x_i)$, with poles along diagonals $t_i = t_j$, with values in $r(X)_{\mathbb{C}((t))}$.

Now consider our curve $\widetilde{C}$. Choose a family of parameters t. It defines a trivialization $r(\nVdash)_C \boxtimes \cdots \boxtimes r(\nVdash)_C \boxtimes r(X)_{\widetilde{C}} \xrightarrow{\sim} r(\nVdash)^{\otimes n}_{\mathbb{C}((t))} \otimes r(X)_{\mathbb{C}((t))} \otimes \mathcal{O}_{\widetilde{C} \times C^n}$ in a formal neighbourhood of the diagonal. We put $\mu(\varphi_1^t \otimes \cdots \otimes \varphi_n^t \otimes \psi_{x,t}) = \mu(\varphi_1^t \otimes \cdots \otimes \varphi_n^t \otimes \psi_{\mathbb{C}((t))t})_{x,t}$ in obvious notations (so we write down the above μ on our curve in the coordinates t_x for each $x \in C$). It is easy to see that μ, so defined, is independent of choice of the family of parameters and is a morphism of D-modules.

To prove the correlators formula in 6.3.5 one proceeds as in 6.1.7: we should consider the curve C'_c as in 6.1.7 over $\mathbb{C}[[q]]$ and apply axiom g. □

We will often write $\mu(\varphi_1 \otimes \cdots \otimes \varphi_n \otimes \psi) = \varphi_1(x_1) \cdots \varphi_n(x_n)\psi(x) \in \prod_{i,j}(x_i - x_j)^{-N}\mathbb{C}[[x_1 - x, \cdots, x_n - x]] \otimes r(X)_x$. The composition property 6.2.7 for global vertex operators implies this associativity property of μ:

6.3.6 One has

$\varphi_1(x_1) \cdots \varphi_n(x_n)\psi(x) =$

$\varphi_1(x_1)(\varphi_2(x_2)(\cdots(\varphi_n(x_n)\psi(x))\cdots) \in \mathbb{C}((x_1 - x((\cdots((x_n - x))\ldots))) \otimes r(X)_x$.

Also if one of the φ_i's is equal to 1, we may delete it.

6.4 Chiral algebra. Consider the three step complex $\mathcal{L}_{C^\bullet} = (\mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{L}_0)$ of sheaves for the Zariski or étale topology of C. Here $\mathcal{L}_2 = r(\nVdash)_C$, $\mathcal{L}_1 = \omega \otimes_{\mathcal{O}_C} r(\nVdash)_C$, the differential $d : \mathcal{L}_2 \to \mathcal{L}_1$ is the de Rham differential, and $\mathcal{L}_0 = \mathcal{L}_1/d\mathcal{L}_2 = \mathcal{H}^1_{DR}(r(\nVdash)_C)$ is the sheaf of de Rham cohomology with coefficients in the D_C-module $r(\nVdash)_C$, and $d : \mathcal{L}_1 \to \mathcal{L}_0$ is the projection.

6.4.1 For sections γ_1, γ_2 of $\mathcal{L}_1$ we define a section $\gamma_1 * \gamma_2$ of $\mathcal{L}_1$ by the formula $\gamma_1 * \gamma_2 = Res_1\mu(\gamma_1 \otimes \gamma_2)$, and a section $\{\gamma_1, \gamma_2\} \in \mathcal{L}_2$ by the formula $\{\gamma_1, \gamma_2\} =$

$\widetilde{Res}\mu(\gamma_1 \otimes \gamma_2)$. Here $\gamma_1 \otimes \gamma_2$ is a section of $\mathcal{L}_1 \boxtimes \mathcal{L}_1 = \Omega^2_{C\times C} \otimes_{\mathcal{O}_{C\times C}} (r(\Bbbk)_C \boxtimes r(\Bbbk)_C)$, $\mu(\gamma_1 \otimes \gamma_2)$ is a section of $\omega_C \boxtimes \mathcal{L}_1 = \Omega^2_{C\times C} \otimes p_2^* r(\Bbbk)_C$ with poles along the diagonal, Res_1 is residue around the diagonal along the first variable, and $\widetilde{Res}$ was defined in 2.2.4. Now the lemma 6.3.5 implies immediately that $d(\{\gamma_1, \gamma_2\}) = \gamma_1 * \gamma_2 + \gamma_2 * \gamma_1$ and for $\varphi \in \mathcal{L}_2$ one has $(d\varphi) * \gamma = 0$. Define the bracket $[\ ,\ ] : \mathcal{L}_\bullet \otimes \mathcal{L}_\bullet \to \mathcal{L}_\bullet$ by the formula $[d\gamma_1, d\gamma_2]_{0,0} = d(\gamma_1 * \gamma_2)$, $[d\gamma_1, \gamma_2]_{0,1} = -[\gamma_2, d\gamma_1]_{1,0} = \gamma_1 * \gamma_2$, $[\gamma_1, \gamma_2]_{1,1} = \{\gamma_1, \gamma_2\}$ for $\gamma_i \in \mathcal{L}_1$. The associativity property 6.3.6 implies

6.4.2 Lemma. *This bracket provides $\mathcal{L}$ with the structure of DG Lie algebra.* □

This DG Lie algebra (or rather its zero component $\mathcal{L}_0$) is called the chiral Lie algebra of our field theory.

6.4.3 Consider a canonical embedding $i : \mathcal{O}_C \to r(\Bbbk)_C$ of D_C-modules, $i(f) = f \cdot 1$.

Denote by $C_\bullet$ the three step complex $C_2 = \mathcal{O}_C \xrightarrow{d} C_1 = \omega_C \to C_0 = \mathcal{H}$; here $\mathcal{H} = \mathcal{H}^1_{DR}$ and the differential $C_1 \to C_0$ is the canonical projection. We get a canonical morphism $i : C_\bullet \to \mathcal{L}_\bullet$ of complexes, $i(f) = f \cdot 1$. One may see that i is actually an embedding (for i_0 this will follow from 6.4.6), and obviously $i(C_\bullet)$ lies in the center of the chiral algebra.

6.4.4 For any $x \in \mathcal{A}$ consider the $D_{\widetilde{C}}$-module $r(X)_{\widetilde{C}}$. The formula $\gamma(m) = Res_1\mu(\gamma \otimes m)$ for $\gamma \in \mathcal{L}_0$, $m \in r(X)_{\widetilde{C}}$ defines a canonical action of $\mathcal{L}_0$ on $r(X)_{\widetilde{C}}$ that commutes with the $D_{\widetilde{C}}$-action.

6.4.5 For any local field F we may consider the "local" version $\mathcal{L}_{F^\bullet}$ of the above $\mathcal{L}_{C^\bullet}$. This is a differential graded Lie algebra constructed in a way similar to 6.4.1. If $F = F_x$ is a local field at a point $x \in C$, then $\mathcal{L}_{F_x^2} = F_x \otimes_{\mathcal{O}_C} \mathcal{L}_{C^2}$, $\mathcal{L}_{F_x^1} = F_x \otimes_{\mathcal{O}_C} \mathcal{L}_{C^1}$, $\mathcal{L}_{F_x^0} = H^1_{DR}(F_x, r(\Bbbk)_C) = \mathcal{L}_{F_x^1}/d\mathcal{L}_{F_x^2}$. For any $X \in \mathcal{A}$ we have a canonical map $\mathcal{L}_{F^0} \otimes r(X)_F \to r(X)_F$, $\gamma \otimes m \longmapsto \gamma(m) = Res_0\mu(\gamma \otimes m)$. Here $\mu(\gamma \otimes m) \in H^1_{DR}(F) \otimes r(X)_F$ and one has (cf. 6.4.4):

6.4.6 Lemma. *This map defines a representation of the Lie algebra $\mathcal{L}_{F^0}$ on $r(X)_F$. The central subalgebra $\mathbb{C} \overset{i}{\hookrightarrow} \mathcal{L}_{F^0}$, $i(a) = a\frac{dt}{t}$, (see 6.4.3) acts on $r(X)_F$ by the formula $i(a)(m) = am$.* □

In particular, $i(\mathbb{C}) \neq 0$; this implies, by degeneration arguments, that $i : C_0 \to \mathcal{L}_0$ is an embedding in the "global" situation.

Now assume that C is compact, $x_1, \cdots, x_n \in C$, $x_i \neq x_j, \nu_i$ are 1-jets of parameters at x_i's, and $X_1, \cdots, X_n \in \mathcal{A}$. Put $U = C \setminus \{x_1, \cdots, x_n\}$. Consider the pairing $\langle\quad\rangle_C : r(X_1)_{x,\nu_1,C} \otimes \cdots \otimes r(X_n)_{x_n,\nu_n,C} \longrightarrow \langle X_1 \otimes \cdots \otimes X_n \rangle_{C,x_i,\nu_i}$. We have an obvious "localization" morphism $\mathcal{L}_0(U) \to \mathcal{L}_0(F_{x_i})$, hence a natural action of $\mathcal{L}_0(U)$ on $\otimes r(X_i)_{x_i,\nu_i,C}$.

6.4.7 Lemma. *The morphism $\langle\quad\rangle_C$ is $\mathcal{L}_0(U)$-invariant.*

Proof. Stokes formula: we rewrite for $\ell \in \mathcal{L}_0(U) = \Omega^1 \otimes r(\Bbbk)_U$ the sum $\Sigma\langle \varphi_1 \cdots \ell(\varphi_i) \cdots \varphi_n \rangle$ as $\Sigma\ \mathrm{Res}_{x=x_i} \langle \ell(x)\varphi(x_1) \cdots \varphi(x_n) \rangle$. □

6.5 Stress-energy tensor. TO BE REWRITTEN! POSSIBLE MISTAKES!

For any local field F consider the linear map $\mathcal{T}_{F-2}/\mathcal{T}_{F-1} \to r(\Bbbk)_F/\mathbb{C} \cdot 1$, $\tau \longmapsto \tau(1)$ (see 3.4.1; recall that 1 is fixed by $\mathcal{T}_{F-1}$ by axiom c). The one-dimensional space $\mathcal{T}_{F-2}/\mathcal{T}_{F-1}$ canonically coincides with the fiber at 0 of $\mathcal{T}^{\otimes 2}$. Tensoring this map with the dual line, we get for any curve C a canonical section T of $\omega_C^{\otimes 2} \otimes$

$\mathcal{O}_C(r(\Bbbk)_C/\mathcal{O}_C)$. This section is called the stress-energy tensor. Multiplication by T defines a canonical map $\mathcal{T}_C \to \omega_C \otimes \mathcal{O}_C(r(\Bbbk)_C/\mathcal{O}_C) = \mathcal{L}_1/C_1 \xrightarrow{d} \mathcal{L}_0/C_0$ (see 6.4.3).

6.5.1 Lemma. *(i) The composition $\mathcal{T} \to \mathcal{L}_0/C_0$ is a morphism of Lie algebras.*
(ii) The corresponding "local" projective action (see 6.4.5, 6.4.6) of $\mathcal{T}_F \subset \mathcal{L}_{0F}/\mathbb{C}$ on $r(X)_F$ coincides with the canonical Virasoro action.

Remark. One should have a canonical isomorphism between the induced extension of $\mathcal{T}$ by $C_0 = \mathcal{H}$ and the Virasoro extension from §2, but we do not know how to establish it at a moment.

Proof. Let us sketch a proof of (ii); one proves (i) in a similar way. We may assume that $F = \mathbb{C}((t))$. Let us compute the action of the operator $L_K := t^{K+1}\partial_t \cdot T \subset \mathcal{L}_{\mathbb{C}((t))^o}/\mathbb{C}$ on $r(X)_{\mathbb{C}((t))}$. Take $e \in r(X)_{\mathbb{C}((t))}, e^* \in r(*X)_{\mathbb{C}((t^{-1}))}$. Consider the function $\nu(z) = \langle \frac{1}{t-z}\partial_{t-z}(1_z) \cdot e \cdot e^* \rangle_{\mathbb{P}^1}$; here $z \in \mathbb{P}^1 \setminus \{0, \infty\}$, $\langle \quad \rangle_{\mathbb{P}^1}$ is the correlator for fields $\frac{1}{t-z}\partial_{t-z}(1_z) \in r(\Bbbk)_{\mathbb{C}((t-z)),t-z}, e, e^*$ at points $z, 0, \infty$. By definition, the matrix coefficient $\langle L_K(e), e^* \rangle$ is equal to $\mathrm{Res}_{z=0} z^{K+1}\nu(z)dz$. We have the invariance property $\langle \frac{1}{t-z}\partial_{t-z}(1_z) \cdot e \cdot e^* \rangle + \langle (1_z) \cdot \frac{1}{t-z}\partial_t e \cdot e^* \rangle + \langle (1_z) \cdot e \cdot \frac{1}{t-z}\partial_t e^* \rangle = 0$. Deleting 1_z by $ax \cdot e$, we get $\langle L_K(e), e^* \rangle = -Res_{z=0}(\langle \frac{1}{t-z}\partial_t e \cdot e^* \rangle + \langle e \cdot \frac{1}{t-z}\partial_t e^* \rangle) \cdot Z^{K+1}dz$. To compute $\frac{1}{t-z}\partial_t e$ one should expand $\frac{1}{t-z}$ around $t = 0$, and to compute $\frac{1}{t-z}\partial_t e^*$ one should expand $\frac{1}{t-z}$ at $t = \infty$.

Hence

$$\langle L_K e, e^* \rangle = -Res_{z=0} z^{K+1}(-\langle \sum_{n \geq 0} z^{-n-1} t^n \partial_t e, e^* \rangle + \langle e, \sum_{n \geq 0} z^n t^{-n-1} \partial_t e^* \rangle)dz = \langle t^{K+1}\partial_t e, e^* \rangle,$$

since $\langle t^a \partial_t e, e^* \rangle + \langle e, t^a \partial_t e^* \rangle = 0$. We see that $L_K = t^{K+1}\partial_t$, q.e.d. □

6.6 Theta functions. Consider the vector spaces $\langle \Bbbk \rangle_C$, where C is a smooth connected compact curve (with empty set of distinguished points). They are fibers of a lisse λ^c-twisted D-module $\langle \Bbbk \rangle$ on the moduli space of smooth curves. For a point $x \in C$ we have $\langle \Bbbk \rangle_C = \langle \Bbbk_x \rangle_{C,x}$, hence one has a canonical map $\gamma_x : r(\Bbbk)_{x,C} \to \langle \Bbbk \rangle_C$. The image $\gamma_C = \gamma_x(\frac{1}{x})$ is independent of the choice of x (since $\partial_x(\gamma_x(1_x)) = 0$). As C varies, the γ_C form a holomorphic section of $\langle \Bbbk \rangle$.

Here is an explicit formula for γ on the moduli space of elliptic curves. Consider the usual uniformization of the moduli space by the upper half plane H with parameter z; then $q = exp(2\pi iz)$ is the standard parameter at infinity. The family of elliptic curves degenerates when $q \to 0$ in the standard way described in 3.6.6. Hence on H we get a canonical trivialization $\langle \Bbbk \rangle_H = \oplus \mathbb{C}_{I_j}$, horizontal with respect to the trivialization of λ^c described in 3.6.6. In this trivialization we have $\gamma(q) = \sum \gamma^\vee_{I_j}(q)$, where $\gamma^\vee_{I_j}(q) = tr_{I_{j\mathbb{C}((t))}} q^{-L_0}$ by axiom g. The "global" trivialization of λ^c given by $\eta(q)^c$ differs from the above trivialization by $q^{c/24}$ (see 3.6.6). In this global η-trivialization the components of γ are $\gamma_{I_j}(q) = q^{c/24} tr_{I_{j\mathbb{C}((t))}} q^{-L_0}$. We see that these are holomorphic functions on H and for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ the function $\gamma_{I_j}(\frac{az+b}{cz+d})$ is a linear combination with constant coefficients of other γ_{I_i}'s.

§7. Lisse representations

7.1 Singular support, lisse modules. Let $\mathfrak{g}$ be a Lie algebra, and $U = U(\mathfrak{g})$ its universal enveloping algebra. Then U is a filtered algebra ($U_0 = \mathbb{C}, U_1 = \mathbb{C} + \mathfrak{g}, U_i = U_1^i$ for $i > 0$), $grU = \oplus_i U_i/U_{i-1} = S^\bullet(\mathfrak{g})$. For $\varphi \in U_i$ its symbol $\sigma_i(\varphi)$ is φ mod $U_{i-1} \in S^i\mathfrak{g}$; if $\varphi \in U_i \setminus U_{i-1}$ we will write $\sigma(\varphi) = \sigma_i(\varphi)$.

7.1.1 Let M be a finitely generated $\mathfrak{g}$-module. Recall that a good filtration $M_\bullet$ on M is a $U_\bullet$-filtration such that $M = \cup M_i$, $\cap M_i = 0$ and $grM_\bullet$ is a finitely generated $S^\bullet(\mathfrak{g})$-module. For example, if $M_0 \subset M$ is a finite dimensional vector subspace that generates M, then $M_i = U_i M_0$ is a good filtration. Any two good filtrations $M_\bullet, M'_\bullet$ on M are comparable, i.e., for some a one has $M_{\bullet - a} \subset M'_\bullet \subset M_{\bullet + a}$.

Define the singular support SSM of M to be the support of the $S^\bullet(\mathfrak{g})$-module $grM_\bullet$, where $M_\bullet$ is a good filtration on M. This is a Zariski closed canonical subset of $SpecS^\bullet(\mathfrak{g}) = \mathfrak{g}^*$; it does not depend on the choice of a good filtration $M_\bullet$. If η is a generic point of SSM, then the length of the $S^\bullet(\mathfrak{g})$-module $(grM_\bullet)_\eta$ only depends on M; denote it $\ell_\eta(M)$. We will say that M is finite at η if $\ell_\eta(M) < \infty$: this means that $(grM_\bullet)_\eta$ is killed by an ideal of finite codimension in $S^\bullet(\mathfrak{g})_\eta$.

7.1.2 Remarks. (i) If M is generated by a single vector, $M \simeq U/I$, then $SS(M)$ is the zero set of symbols of elements of I.

(ii) A more precise way to speak about this subject needs the microlocalization language, see e.g. [La], Appendix.

The algebra $grU = S^\bullet(\mathfrak{g})$ carries a Poisson bracket defined by the formula $\{f_i, g_j\} = \widetilde{f}_i\widetilde{g}_j - \widetilde{g}_j\widetilde{f}_i$ mod U_{i+j-2}; here $f_i \in S^i(\mathfrak{g}), \widetilde{f}_i \in U_i, f_i = \widetilde{f}_i$ mod U_{i-1}, and the same for g_j, $\{f_i, g_j\} \in S^{i+j-1}(\mathfrak{g})$. One has the following integrability theorem, due to O. Gabber [Ga]:

7.1.3 Theorem. *Let M be a finitely generated U-module finite at any generic point of SSM. Then SSM is involutive, i.e., if $f, g \in S^\bullet(\mathfrak{g})$ vanish on SSM, then so does $\{f, g\}$.* □

7.1.4 Definition. *A finitely generated module M is lisse if $SSM = \{0\}$. More generally, we will say that M is lisse along a vector subspace $\ell \subset \mathfrak{g}$ if $SSM \cap \ell^\perp = \{0\}$.* □

Note that any quotient of a lisse module is lisse. Any extension of a lisse module by a lisse module is lisse. Any finite dimensional M is lisse; the converse is true if dim $\mathfrak{g} < \infty$.

Explicitly, a module M is lisse if amd only if for a finite dimensional subspace $V \subset M$ that generates M and any $g \in \mathfrak{g}$ there exists $N \gg 0$ such that $g^N V \subset U_{N-1}V$.

7.2 Finiteness property. Let $k \subset \mathfrak{g}$ be a Lie subalgebra. We will say that a $\mathfrak{g}$-module M is a $(\mathfrak{g}, k)$-module if k acts on M in a locally finite way (i.e., for any $x \in M$ one has $dimU(k)x < \infty$). If such an M is finitely generated, then it carries a good k-invariant filtration (e.g., take a finite dimensional k-invariant subspace $M_0 \subset M$ that generates M and put $M_i = U_i M_0$). Hence $SSM \subset k^\perp = (\mathfrak{g}/k)^* \subset \mathfrak{g}^*$.

7.2.1 Lemma. *Let M be a finitely generated $(\mathfrak{g}, k)$-module and $n \subset \mathfrak{g}$ be a vector subspace such that $dim \mathfrak{g}/n{+}k < \infty$ and M is lisse along n. Then $dim M/nM < \infty$.*

Proof. Let $M_\bullet$ be a K-invariant good filtration on M, so $grM_\bullet$ is a finitely generated $S^\bullet(\mathfrak{g}/k)$-module. Consider the induced filtration on M/nM. It suffices to see that $dimgr(M/nM) < \infty$. But $gr(M/nM)$ is a quotient of $grM/ngrM$ (since $gr_i M/nM = M_i/M_{i-1} + (M_i \cap NM)$, $(grM/ngrM)_i = M_i/M_{i-1} + nM_{i-1}$). The latter is a finitely generated module with zero support over the finitely generated algebra $S^\bullet(\mathfrak{g}/k+n)$, hence it is finitely generated. □

We will use 7.3.1 as follows. Assume we are in a situation 3.3, so we have a Harish-Chandra pair $(\mathfrak{g}, K)$, an S-localization data $\psi = (S^\#, N, \varphi, \varphi_0)$ for $(\mathfrak{g}, K)$ and the corresponding S-localization functor $\Delta_\psi : (\widetilde{\mathfrak{g}}, K)$-mod $\to \mathcal{D}_\psi$-mod. Certainly, any $(\widetilde{\mathfrak{g}}, K)$-module M is a $(\widetilde{\mathfrak{g}}, k)$-module and SSM is an Ad K-invariant closed subset of $k^\perp$. Now 7.2.1 (together with 3.3.4) implies:

7.2.2 Corollary. *Assume that the following finiteness condition holds:*
(*) The sheaf $\mathfrak{g}_S^\#/k_S^\# + \varphi(N_{(0)})$ is $\mathcal{O}_S$-coherent.

Then for a lisse $(\widetilde{\mathfrak{g}}, K)$-module M the $\mathcal{D}_\psi$-module $\Delta_\psi(M)$ is lisse (see 3.2.7). More generally, if a $(\widetilde{\mathfrak{g}}, K)$-module M is lisse along any subspace $\varphi_0(N_{(0)s}) \subset \widetilde{\mathfrak{g}}$, $s \in S^\#$, then $\Delta_\psi(M)$ is a lisse $\mathcal{D}_\psi$-module. □

The following corollaries of 7.1.3 will be useful.

7.2.3 Lemma. *Let M be a $(\mathfrak{g}, k)$-module such that SSM has finite codimension in $k^\perp$. Then SSM is involutive.* □

7.2.4 Corollary. *Assume that a Harish-Chandra pair $(\mathfrak{g}, K)$ has the property that any Zariski closed Ad K-invariant subset of $k^\perp$ is either $\{0\}$ or has finite codimension. Then for any $(\mathfrak{g}, K)$-module M the $SS(M)$ is involutive.* □

7.3 Lisse modules over Virasoro algebra. Consider the Virasoro algebra $\widetilde{\mathcal{T}}_c$: this is the central $\mathbb{C}$-extension of Lie algebra $\mathcal{T} = \mathbb{C}((t))$ that corresponds to the 2 cocycle $\langle f\partial_t, g\partial_t\rangle_c = cRes(f'''g\frac{dt}{t})$. It carries the filtration $\widetilde{\mathcal{T}}_{cn}$: for $n \geq 1$, $\widetilde{\mathcal{T}}_{cn} = t^{n+1}\mathbb{C}[[t]]\partial_t$, for $n \leq 0$, $\widetilde{\mathcal{T}}_{cn} = \mathbb{C} + t^{n+1}\mathbb{C}[[t]]\partial_t$. Put $L_i := t^{i+1}\partial_t \in \widetilde{\mathcal{T}}_c$. One also has the following Lie subalgebras of $\widetilde{\mathcal{T}}_c$:

$$n_+ = \widetilde{\mathcal{T}}_{c1} \subset b_+ = \mathbb{C}[[t]]t\partial_t \subset P_+ = \mathbb{C}[[t]]\partial_t, \quad n_- = \mathbb{C}[t^{-1}]\partial_t \subset b_- = \mathbb{C}[t^{-1}]t\partial_t,$$

so $b_+ = LieK$, $n_+ =$ Lie K_1 (see 3.4.1). One has $b_+ \oplus n_- \oplus \mathbb{C} = \widetilde{\mathcal{T}}_c$, $b_+ \cap b_- - f = \mathbb{C}L_0$.
7.3.1 A higher weight $\mathcal{T}$-module of central charge c is a $(\widetilde{\mathcal{T}}_c, b_+)$-module M such that $1 \in \mathbb{C} \subset \widetilde{\mathcal{T}}_c$ acts as id_M and any $m \in M$ is killed by some $\widetilde{\mathcal{T}}_{cn}$ for $n \gg 0$. Denote by $\mathcal{T}_{c+}$-mod the category of such modules. Note that any $M \in \mathcal{T}_{c+}$-mod is a $(\widetilde{\mathcal{T}}_c, K_1)$-module. We will say that M is L_0-diagonalizable if M coincides with the direct sum of L_0-eigenspaces.

Let M be a higher weight module. Denote by $*M$ the space of those linear functionals φ on M that are finite with respect to the action of ${}^tL_0$. The operators $L_i := {}^tL_{-i}$ define the $\widetilde{\mathcal{T}}_c$-action on $*M$. Clearly $*M$ is a higher weight module called the (contravariant) dual to M. One has an obvious morphism $M \to **M$ which is an isomorphism if amd only if the generalized eigenspaces of L_0 on M are finite dimensional. In particular this holds when M is a finitely generated module.

7.3.2 Remark. For $M \in \mathcal{T}_{c+}$-mod consider the monodromy operator $T = exp(2\pi i L_0)$. Clearly T commutes with the Virasoro action, i.e., $T \in Aut M$. Hence one has a canonical direct sum decomposition $M = \bigoplus_{\overline{a}\in\mathbb{C}/\mathbb{Z}} M_{\overline{a}}$, where $M_{\overline{a}}$ is the generalized $exp(2\pi i a)$-eigenspace of M. Denote by $\mathcal{T}_{c+\overline{a}}$-mod the subcategory of those M's that $M = M_{\overline{a}}$. Clearly $\mathcal{T}_{c+}\text{-mod} = \prod_{a\in\mathbb{C}/\mathbb{Z}} \mathcal{T}_{c+\overline{a}}\text{-mod}$.

7.3.3 Lemma. *For any finitely generated $M \in \mathcal{T}_{c+}$-mod there are exactly three possibilities for SSM: it is either equal to $\{0\}$, or to $\widetilde{\mathcal{T}}_{c0}^{\perp} = (\mathbb{C} + b_+)^{\perp}$, or to $\widetilde{\mathcal{T}}_{c-1}^{\perp} = (\mathbb{C} + P_+)^{\perp}$.*

Proof. Clearly $SSM \subset \widetilde{\mathcal{T}}_{c0}^{\perp}$. It is Ad K-invariant (the Ad K_1-invariance is obvious; for any $t \in \mathbb{C}$ the operator $exp(tL_0)$ acts on M, hence SSM is also Ad $exp(tL_0)$-invariant). It is easy to see that any Ad K-invariant Zariski closed subset of $\widetilde{\mathcal{T}}_{c0}^{\perp}$ is either $\{0\}$ or coincides with one of the vector spaces $\widetilde{\mathcal{T}}_{c-n}^{\perp}$, $n \geq 0$. According to 7.2.4 this $\widetilde{\mathcal{T}}_{c-n}$ is the Lie subalgebra of $\widetilde{\mathcal{T}}_c$; this implies 7.3.3. □

For a higher weight module M consider the subspace $M^{\mathfrak{n}_+}$ of singular vectors. Clearly $M^{\mathfrak{n}_+} \neq 0$ and it is L_0-invariant, so we have a decomposition $M^{\mathfrak{n}_+} = \bigoplus_{h\in\mathbb{C}} M^{\mathfrak{n}_+}_{(h)}$ by generalized eigenspaces of L_0. We will say that a singular vector v has generalized weight h if $v \in M^{\mathfrak{n}_+}_{(h)}$ (i.e., if $(L_0 - h)^n v = 0$ for $n \gg 0$), and that v has weight h if $L_0 v = hv$. As usual, the Verma module $M_{ch} = M_h \in \widetilde{\mathcal{T}}_{c+}$-mod is a module generated by a single "vacuum" singular vector v_h of weight h with no other relations. This M_h is the free $U(\mathfrak{n}_-)$-module generated by v_h, hence any submodule of M_h generated by a singular vector is a Verma module. Denote by $L_{ch} = L_h$ the (only) irreducible quotient of M_h. Any irreducible higher weight module is isomorphic to some L_h, and the L_h's with different h's non-isomorphic. One has $*L_h = L_h$.

The following basic facts are due to Feigin-Fuchs [FF].

7.3.4 Proposition. *Let $M = M_h$ be a Verma module, $N \subset M$ is a non-zero submodule. Then*

(i) N is generated by ≤ 2 singular vectors, i.e., N is either a Verma submodule or a sum of two Verma submodules.
(ii) N is an intersection of ≤ 2 Verma submodules.
(iii) M/N has finite length.
(iv) The spaces $M^{\mathfrak{n}_+}_{(h')}$ have dimension ≤ 1, therefore, by (i), the irreducible constituents of M have multiplicity 1. □

7.3.5 Lemma. *Let $P \in \widetilde{\mathcal{T}}_{c+}$-mod be a finitely generated module. Then*

(i) P admits a filtration of finite length ℓ with successive quotients isomorphic to a quotient of a Verma module.
(ii) The maximal semisimple quotient of P has length $\leq \ell$.
(iii) Any submodule of P is finitely generated.

Proof. Note that P is a quotient of some module Q induced from a finite dimensional b_+-module. Such Q has a filtration with successive quotients isomorphic to Verma modules. This implies (i) and reduces (ii), (iii) to the case of Verma module which follows from 7.3.4 (i). □

7.3.6 Lemma. *Let $M = M_h$ be a Verma module, $N \subset M$ be a non-zero submodule, $L = M/N$. One has*

(i) $SSM = \widetilde{\mathcal{T}}_{c0}^{\perp} = \mathfrak{n}_-^*$

(ii) SSL *is either* $\{0\}$ *or equals to* $\widetilde{\mathcal{T}}_{c-1}^{\perp}$

(iii) If $SSL = 0$*, then* L *is irreducible and* N *is generated by two singular vectors.*

(iv) If N *is a proper Verma submodule, then the coinvariants* $L_{[\mathfrak{n}_-,\mathfrak{n}_-]}$ *are infinite dimensional.*

Proof. (i) is obvious. To prove (ii) take a non-zero $\varphi \in U(\mathfrak{n}_-)$ such that $\varphi v_h \in N$. The symbol $\sigma(\varphi)$ vanishes on SSL, hence $SSL \neq \mathfrak{n}_-^*$, and we are done by 7.3.3.

(iii) By 7.3.4 (iii) any reducible L has a quotient such that the corresponding N is a Verma submodule. Since a quotient of a lisse module is lisse, (iii) is reduced to a statement that for any proper Verma submodule $N = M_{h'} \subset M_h$ one has $SSM_h/M_{h'} \neq 0$. By 7.2.1 this follows from (iv).

(iv) The commutant $[\mathfrak{n}_-,\mathfrak{n}_-]$ is Lie subalgebra of $\mathfrak{n}_-$ with basis $L_{-3}, L_{-4}, L_{-5}, \ldots$. The quotient $\mathfrak{n}_-/[\mathfrak{n}_-,\mathfrak{n}_-]$ is abelian Lie algebra with basis L_{-1}, L_{-2}. To prove (iv) note that $M_{h[\mathfrak{n}_-,\mathfrak{n}_-]}$ is a free module over $U(\mathfrak{n}_-/[\mathfrak{n}_-,\mathfrak{n}_-]) = \mathbb{C}[L_{-1}, L_{-2}]$ with generator $\overline{v}_h$, and $(M_h/M_{h'})_{[\mathfrak{n}_-,\mathfrak{n}_-]}$ is a quotient of $M_{h[\mathfrak{n}_-,\mathfrak{n}_-]}$ modulo the $\mathbb{C}[L_{-1}, L_{-2}]$ submodule generated by the image $\overline{v}_{h'}$ of $v_{h'}$ (since $M_{h'} = U(\mathfrak{n}_-)v_{h'}$). Since $\overline{v}_{h'} = P\overline{v}_h$, where P is a polynomial of weight $h' - h \neq 0$, we see that our coinvariants $(M_h/M_{h'})_{[\mathfrak{n}_-,\mathfrak{n}_-]} = \mathbb{C}[L_{-1}, L_{-2}]/P\mathbb{C}[L_{-1}, L_{-2}]$ are infinite dimensional. □

7.3.7 We will say that an irreducible module $L_h \in \widetilde{\mathcal{T}}_{c+}$-mod is minimal, or a Belavin-Polyakov-Zamolodchikov module, if the conditions (i), (ii) below hold:

(i) For some integers p, q such that $1 < p < q$, $(p, q) = 1$, one has

$$c = c_{p,q} = 1 - 6(p-q)^2/pq$$

(clearly p, q are uniquely defined by c)

(ii) For some integers n, m, $0 < n < p, 0 < m < q$ one has

$$h = h_{n,m} = \frac{1}{4pq}[(nq - mp)^2 - (p-q)^2].$$

Clearly $h_{n,m} = h_{p-n,q-m}$. For given $c = c_{p,q}$ there is exactly $\frac{1}{2}(p-1)(q-1)$ different minimal irreducible modules. Note that $L_{c_{p,q},0}$ is always minimal (since $0 = h_{1,1}$).

7.3.8 Proposition. *([FF]) An irreducible module* L_h *is minimal iff both the following conditions hold:*

(i) L_h *is dominant which means that* L_h *is not isomorphic to a subquotient of any* $M_{h'}, h' \neq h$.

(ii) The kernel N_h *of the projection* $M_h \to L_h$ *is generated by exactly 2 singular vectors (see 7.3.4 (i)).* □

7.3.9 Remarks. (i) For $h = h_{nm}, c = c_{pq}$ the singular vectors from 7.3.8 (ii) have weights $h - nm, h - (p-n)(q-m)$. They are different by 7.3.4 (iv) (or by a direct calculation).

(ii) It is easy to see, using contravariant duality, that L_h is dominant iff M_h is a projective object in the category of L_0-diagonalizable higher weight modules.

Equivalently, this means that $M_h^\wedge = \lim\limits_{\leftarrow} M_h^{(n)}$ is a projective covering of L_h in the category $\widetilde{\mathcal{T}}_{c+}$-mod. Here $M_h^{(n)}$ is the higher weight module generated by the singular vector v that satisfies the only relation $(L_0 - h)^n v = 0$.

7.3.10 Proposition. *For an irreducible module $L = L_h = M_h/N_h$ the following conditions are equivalent:*

(i) L is lisse

(ii) L is minimal

(iii) The coinvariants $L_{[\mathfrak{n}_-,\mathfrak{n}_-]}$ are finite-dimensional

(iv) The invariants $L^{[\mathfrak{n}_-,\mathfrak{n}_-]}$ are finite dimensional

(v) For some non-zero $\varphi \in U([\mathfrak{n}_-,\mathfrak{n}_-])$ one has $\varphi v_h \in N_h$

Proof. One has (i) $\Longrightarrow$ (iii) by 7.2.1, (iii) $\Longleftrightarrow$ (iv) by contravariant duality, (ii) $\Longleftrightarrow$ (iii) by [FF], (v) $\Longrightarrow$ (i) by 7.3.5 (ii) (since $\sigma(\varphi)$ vanishes on SSL, one has $SSL \neq \widetilde{\mathcal{T}}_{c-1}^{\perp}$. It remains to show that (ii) $\Longrightarrow$ (v). So let L_h be minimal. Put $T = U(\mathfrak{n}_-,\mathfrak{n}_-])v_h \subset M_h$. We wish to see that the projection $T \to L_h$ is not injective. This follows since the asymptotic dimension of T is larger than the one of L_h. Precisely, according to the character formula for L (see [K] prop. 4) the function $\log tr_L(exp(2\pi t L_0))$ is asymptotically equivalent as $t \to 0$ to $\pi\alpha/12t$ for some constant $\alpha < 1$. On the other hand, one has $\log tr_T(exp(-2\pi t L_0)) = \log tr_{M_h}(exp(2\pi t L_0)) + \log(1 - exp(-2\pi t)) + \log(1 - exp(-4\pi t))$ (since as L_0-module M_h is isomorphic to $v_h \otimes S(L_{-1}, L_{-2}, \cdots)$, where the generators L_{-i} of the symmetric algebra have weights i, and T is isomorphic to $v_h \otimes S(L_{-3}, L_{-n}, \cdots)$). This function is asymptotically equivalent to $\pi/12t$. Since the spectrum of L_0 is real, this implies that $T \to L_h$ is not injective. □

7.3.11 Remark. For $c = c_{p,q}, h = h_{11} = 0$ one may prove that (ii) $\Longrightarrow$ (i) in a very elementary way. Namely, by 7.3.8 (ii) one knows that L_0 is minimal iff N_0 does not coincide with the submodule N' of M_0 generated by $L_{-1}v_0$. Choose minimal i such that for certain $\varphi \in U(\mathfrak{n}_-)_i$ one has $\varphi v_0 \in N_0 \setminus N'$. Then the symbol of φ is prime to L_{-1}, hence, by 7.3.5 (ii), L_0 is lisse. This remark, due essentially to Drinfeld, was a starting point for the results of this paragraph. □

7.3.12 Proposition. *The following conditions on a higher weight module M are equivalent*

(i) M is a finitely generated lisse module

(ii) M is isomorphic to a finite direct sum of minimal irreducible modules.

(iii) One has dim $M^{[\mathfrak{n}_-,\mathfrak{n}_-]} < \infty$

Proof. By 7.3.10 we know that (i) $\Longleftarrow$ (ii) $\Longrightarrow$ (iii). We will use the following facts:

(*) Let L_h be a minimal irreducible module. Then any quotient of length 2 of $M_h^{(n)}$ (see 7.3.9 (ii)) is actually a quotient of $M_h = M_h^{(1)}$ (i.e., is L_0-diagonalizable).

(**) If L_{h_1}, L_{h_2} are minimal and $h_1 \neq h_2$, then M_{h_1} and M_{h_2} have no common irreducible component.

Here (*) follows from the fact that $N_h \subset M_h$ coincides with the 1st term of Jantzen filtration, see [FF]; for (**) see [FF]. Note that (*) implies, by 7.3.8, 7.3.9 (ii), that

(***) $Ext^1(L_{h_1}, L_{h_2}) = 0$ for any minimal L_{h_1}, L_{h_2}.

Now we may prove that (i) $\Longrightarrow$ (ii). By 7.3.10 it suffices to show that a lisse module M is semisimple. Consider the maximal semisimple quotient $P = M/N$

(see 7.3.5 (ii)). We have to show that $N = 0$. By 7.3.5 (iii) there is an irreducible quotient $Q = N/T$ of N, so we have a non-trivial extension $0 \to Q \to M/T \to P \to 0$ with lisse M/T. According to 7.3.9 (ii) and (**) we see that there exists at most one minimal L_h such that $Ext^1(L_h, Q) \neq 0$. By (*) and 7.3.9 (i) for such L_h one has dim $Ext^1(L_h, Q) = 1$. This implies that M/T is isomorphic to a direct sum of minimal irreducible modules and a length 2 module which is a non-trivial extension of a minimal module L_h by Q. By 7.3.9 (ii) and (*) this extension is a quotient of a Verma module. By 7.3.5 (ii) it is non-lisse, hence M/T is non-lisse. Contradiction.

Let us prove that (iii) $\Longrightarrow$ (ii). Let M be a module such that dim $M^{[\mathfrak{n}_+,\mathfrak{n}_+]} = r < \infty$. Let $M' \subset M$ be a maximal semisimple submodule of M. By 7.3.10 M' is a direct sum of minimal irreducible modules. Clearly the length of M' is $\leq r$, so it suffices to show that$M' = M$. Note that any non-zero submodule $N \subset M$ intersects M' non-trivially (if $N \cap M' = 0$ then, shrinking N if necessary, we may assume that N is a quotient of a Verma module. If N has finite length, then it contains an irreducbile submodule, which lies in M'. If N has infinite length, then, by 7.3.4, dim $N^{\mathfrak{n}_+} = \infty$; since $N^{\mathfrak{n}_+} \subset M^{[\mathfrak{n}_+,\mathfrak{n}_+]}$ this is not true). Assume that $M/M' \neq 0$. Replacing M by an appropriate submodule that contains M we may assume that M/M' is a quotient of a Verma module, in particular M/M' is L_0-diagonalizable. Consider the dual extension $0 \to *(M/M') \to *M \to *M' \to 0$. One has $*M' = \oplus L_{h_i}$, hence, by 7.3.8, 7.3.9 (ii) the projection $\oplus M_{h_i} \to \oplus L_{h_i} = *M'$ lifts to the map $\oplus M^{(2)}_{h_i} \to *M$. This map is surjective (otherwise the dual to its cokernel would intersect M' trivially), hence $*M$ has finite length. Replacing $*(M/M')$ by its irreducible quotient we may assume that M/M' is irreducible.

As above (see the proof (i) $\Longrightarrow$ (ii))$*M$ is a direct sum of irreducible minimal modules plus a length two non-trivial extension of a minimal module L_h. By 7.39 (ii), 7.3.4 (ii) and (*) above this length two extension is a quotient of M_h by a Verma submodule. By 7.3.6 (iv) the coinvariant $(*M)_{[\mathfrak{n}_-,\mathfrak{n}_-]}$ are of infinite dimension. Since $(*M)_{[\mathfrak{n}_-,\mathfrak{n}_-]} = (M^{[\mathfrak{n}_-,\mathfrak{n}_-]})^*$, we are done. $\square$

7.3.13 Now for $n \geq 1$ consider the product of Virasoro algebras $\widetilde{\mathcal{T}}^n_c$: this is a central $\mathbb{C}$-extension of $\mathcal{T}^n$ with cocycle $\langle (f_i\partial_t), (g_i\partial_t)\rangle_c = \sum_i \langle f_i\partial_t, g_i\partial_t\rangle_c$ (see 3.4.1). The above theory extends to $\widetilde{\mathcal{T}}^n_c$ in an easy manner. Namely, we have a standard subalgebra $\mathfrak{n}_+ = \prod \mathfrak{n}_{+i} \subset \mathfrak{b}_+ = \prod \mathfrak{b}_{+i} \subset \mathfrak{p}_+ = \prod \mathfrak{p}_{+i}$, $\mathfrak{n}_{-i} \subset \mathfrak{b}_- = \prod \mathfrak{b}_{-i}, \mathfrak{f} = \mathfrak{b}_+ \cap \mathfrak{b}_- = \mathbb{C}^n$ etc. of $\widetilde{\mathcal{T}}^n_c$. One defines the corresponding category $\mathcal{T}^n_{c+}$-mod of higher weight modules in an obvious manner. We have an obvious functor $\otimes : \prod \mathcal{T}_{c+}$-mod $\to \mathcal{T}^n_{c+}$-mod, $(M_1, \ldots, M_n) \longmapsto M_1 \otimes \ldots \otimes M_n$. Clearly $SSM_1 \otimes \ldots \otimes M_n = SSM_1 \times SSM_2 \times \cdots \times SSM_n$.

For $\hbar = (h_i) \in \mathbb{C}^n$ we have the corresponding Verma module $M_\hbar = \otimes M_{h_i}$ and its unique irreducible quotient $L_\hbar = \otimes L_{h_i}$; any irreducible higher weight module is isomorphic to a unique $L_\hbar$. It follows from 7.3.4 (iv) that any submodule $N \subset M_\hbar$ is tensor product $\otimes N_i$ of submodules $N_i \subset M_{h_i}$, so the structure of N is clear from 7.3.4. The lemma 7.3.5 (with its proof) remains valid for $\mathcal{T}^n_{c+}$-mod. The version of 7.3.6 for $\widetilde{\mathcal{T}}^n_c$ case (with obvious modifications) follows immediately from the case $n = 1$. A module $L_\hbar = \otimes L_{h_i}$ is called minimal if all L_{h_i} are minimal (see 7.3.7). The analog of 7.3.8 (with "2 singular vectors" replaced by "$2n$ singular vectors") remains obviously valid, as well as 7.3.9. The proposition 7.3.10 remains valid and follows directly from the case $n = 1$. The proposition 7.3.12 remains valid together with its proof.

§8. Minimal models

These were defined by Belavin, Polyakov and Zamolodchikov [BPZ]. Let us start with a general representation-theoretic construction.

8.1 Fusion functors for Virasoro algebra. Let C be a compact smooth curve, $A, B \subset C$ be two finite sets of points such that $A \cap B = \emptyset, A \neq \emptyset$. For a central charge $c \in \mathbb{C}$ we have Virasoro algebra $\widetilde{\mathcal{T}}_c^A$ which is central $\mathbb{C}$-extension of $\mathcal{T}^A = \prod_{a \in A} \mathcal{T}_a$ (where $\mathcal{T}_a$ = vector fields on punctured formal disc at a) and similar algebras $\widetilde{\mathcal{T}}_c^B$, $\widetilde{\mathcal{T}}_c^{A \sqcup B}$. One has a canonical surjective map $\widetilde{\mathcal{T}}_c^A \times \widetilde{\mathcal{T}}_c^B \to \widetilde{\mathcal{T}}_c^{A \sqcup B}$ (which is factorization by $\{(a, -a)\} \subset \mathbb{C} \times \mathbb{C}$); the morphisms $\widetilde{\mathcal{T}}_c^A \to \widetilde{\mathcal{T}}_c^{A \sqcup B} \longleftarrow \widetilde{\mathcal{T}}_c^B$ are injective. One also has the canonical embedding $i_{A \sqcup B} : \mathcal{T}(U) \longrightarrow \widetilde{\mathcal{T}}_c^{A \sqcup B}$, where $U = C \setminus (A \sqcup B)$, and the ones $i_A : \mathcal{T}(C \setminus A) \to \widetilde{\mathcal{T}}_c^A, i_B : \mathcal{T}(C \setminus B) \to \widetilde{\mathcal{T}}_c^B$. There is also a canonical morphism $j_B : \mathcal{T}(C \setminus A) \to \widetilde{\mathcal{T}}_c^B$ which is composition of the obvious embedding $\mathcal{T}(C \setminus A) \to \mathcal{T}_{-1}^B$ and the section $s_{\mathcal{O}_B} : \mathcal{T}_{-1}^B \to \widetilde{\mathcal{T}}_c^B$. The restriction $i_{A \sqcup B}\big|_{\mathcal{T}(C \setminus A)} : \mathcal{T}(C \setminus A) \longrightarrow \widetilde{\mathcal{T}}_c^{A \sqcup B}$ coincides with $i_A + j_B$.

8.1.1 Assume we have a positive divisor $d = \sum n_b b \geq 0$ supported on B. Let $\mathcal{T}(C \setminus A, d) \subset \mathcal{T}(C \setminus A, d)$ be the Lie subalgebra of vector fields vanishing of order $\geq n_b + 1$ at any $b \in B$. Clearly one has $\mathcal{T}(C \setminus A, d_1) \subset \mathcal{T}(C \setminus A, d_2)$ for $d_1 \geq d_2$, and $\mathcal{T}(C \setminus A, 0)/\mathcal{T}(C \setminus A, d) = \mathcal{T}_0^B/\mathcal{T}_d^B$, where $\mathcal{T}_d^B = \prod \mathcal{T}_{n_b, b}$. Let $\epsilon_d : \widetilde{\mathcal{T}}_c^B \longrightarrow \widetilde{\mathcal{T}}_c^A / i_A(\widetilde{\mathcal{T}}(C \setminus A, d))$ be the composition

$$\widetilde{\mathcal{T}}_c^B \longrightarrow \widetilde{\mathcal{T}}_c^B / s_{\mathcal{O}_B}(\mathcal{T}_{B,d}) \longrightarrow \widetilde{\mathcal{T}}_c^{A \sqcup B} / i_{A \sqcup B}(\mathcal{T}(U)) + s_{\mathcal{O}_B}(\mathcal{T}_{B,d}) \xleftarrow{\sim} \widetilde{\mathcal{T}}_c^A / i_A(\mathcal{T}(C \setminus A, d)).$$

The maps t_d are compatible, so we have $\epsilon = \lim\limits_{\overleftarrow{d}} \epsilon_d : \widetilde{\mathcal{T}}_c^B \longrightarrow \lim\limits_{\overleftarrow{d}} \mathcal{T}^A / i_A(\mathcal{T}(C \setminus A, d))$.

8.1.2 Now we are able to define the (contravariant) fusion functor $\mathcal{F}_C : \widetilde{\mathcal{T}}_c^A - \mathrm{mod} \to \widetilde{\mathcal{T}}_c^B - \mathrm{mod}$.

Let M be any $\widetilde{\mathcal{T}}_c^A$-module (so $1 \in \mathbb{C} \subset \widetilde{\mathcal{T}}_c^A$ acts as id_M). Put $\mathcal{F}_C(M) := \bigcup_d M^* i_A(\mathcal{T}(C \setminus A, d)) \subset M^*$; therefore an element of $\mathcal{F}_C(M)$ is a linear functional on M invariant with respect to some $i_A(\mathcal{T}(C \setminus A, d))$. For $\tau \in \widetilde{\mathcal{T}}_c^B$, $\ell \in \mathcal{F}_C(M)$ put $\tau(\ell) = {}^t\epsilon(\tau)(\ell)$. It is easy to see that this formula is correct, $\tau(\ell)$ lies in $\mathcal{F}_C(M) \subset M^*$ and $(\tau, \ell) \longmapsto \tau(\ell)$ is $\widetilde{\mathcal{T}}_c^B$-action on $\mathcal{F}_C(M)$. This way $\mathcal{F}_C(M)$ becomes $\widetilde{\mathcal{T}}_c^B$-module. One has an easy

8.1.3 Lemma.

(i) One has $\mathcal{F}_C(M) = \bigcup_\alpha \mathcal{F}_C(M)^{\mathcal{T}_{B,d}}$, *and* $\mathcal{F}_C(M)^{\mathcal{T}_{B,d}} = (M_{\mathcal{T}(C \setminus A, d)})^*$.

(ii) Let N be any $\widetilde{\mathcal{T}}_c^B$-module s.t. $N = \bigcup_\alpha N^{\mathcal{T}_{B,d}}$. *Then* $Hom(N, \mathcal{F}_C M) = [(M \otimes N)_{\mathcal{T}(U)}]^*$ *(here we consider $M \otimes N$ as $\widetilde{\mathcal{T}}_c^{A \sqcup B}$-module via the surjection $\widetilde{\mathcal{T}}_c^A \times \widetilde{\mathcal{T}}_c^B \longrightarrow \widetilde{\mathcal{T}}_c^{A \sqcup B}$).* □

¿From now on let us fix a central charge $c = c_{p,q}$ from the list 7.3.7(i). We will assume that our virasoro modules have central charge c. Let M be a finitely generated higher weight $\widetilde{\mathcal{T}}_c^A$-module.

8.1.4 Corollary. *(i) $\mathcal{F}_C(M)$ is finitely generated lisse higher weight $\widetilde{\mathcal{T}}_c^B$-module.*
(ii) For any finitely generated higher weight $\widetilde{\mathcal{T}}_c^B$-module N one has $(M\otimes N)_{\mathcal{T}(U)} = (M\otimes \overline{N})_{\mathcal{T}(U)}$, where $\overline{N}$ is the maximal lisse quotient of N.

Proof. (i) Use 8.1.3 (i), 7.2.1, 7.3.12 (inversion 7.3.13).
(ii) First note that the maximal lisse quotient $\overline{N}$ exists and has finite length by 7.3.5, 7.3.8, 7.3.12. By 8.1.3 (ii), 8.1.4 (i) one has $(M\otimes N)^*_{\mathcal{T}(U)} = \mathrm{Hom}(N, \mathcal{F}_C(M)) = \mathrm{Hom}(\overline{N}, \mathcal{F}_C(M)) = (M\otimes \overline{N})^*_{\mathcal{T}(U)}$, q.e.d. □

For $h=(h_b)\in\mathbb{C}^B$ let $L_h^B = \bigotimes_{b\in B} L_{c,h_b}$ be the irreducible $\widetilde{\mathcal{T}}_c^B$-module of higher weight h.

8.1.5 Corollary. *One has a canonical isomorphism $M_{\mathcal{T}(C\setminus A)} = (M\otimes L_0^B)_{\mathcal{T}(U)}$.*

Proof. Clearly $M_{\mathcal{T}(C\setminus A)} = (Ind_{\mathcal{T}(C\setminus A)}^{\mathcal{T}(U)} M)_{\mathcal{T}(U)}$. But $Ind_{\mathcal{T}(C\setminus A)}^{\mathcal{T}(U)} M$ coincides, as $\mathcal{T}(U)$-module, with $\widetilde{\mathcal{T}}_c^{A\sqcup B}$-module $M\otimes P_o^B$, where $P_{oB} = \bigotimes_{b\in B} P_{c,o,b}, P_{c,o}$ is a quotient of Verma module $M_{c,0}$ modulo relation $L_{-1}v_0 = 0$. Clearly L_o^B is maximal lisse quotient of P_o^B (see 7.3.8), and 8.1.5 follows from 8.1.4 (ii). □

8.1.6 Corollary. *Let d_1 be the divisor $\sum_{b\in B} b$. Consider the action of Lie algebra $\mathcal{T}(C\setminus A,0)/\mathcal{T}(C\setminus A,d_1) = \mathcal{T}_0^B/\mathcal{T}_{d_1}^B = \mathbb{C}^B$ on coinvariants $M_{\mathcal{T}(U,d_1)}$. This action is semisimple. For $h=(h_b)\in\mathbb{C}^B$ the (h_b)-component $M^{(h_b)}$ is equal to the coinvariants $(M\otimes L_h^B)_{\mathcal{T}(U)}$. This space vanishes unless all h_b lie in the list 7.3.7 (ii).*

Proof. Similar to 8.1.5; the semi-simplicity of $\mathbb{C}^B$-action follows from 7.3.12 (ii). □

8.1.7 Corollary. *Assume that B consists of two points b_1, b_2. Let $\mathcal{T}(C\setminus A,B)' \subset \mathcal{T}(C\setminus A,0)$ be the Lie subalgebra of vector fields that project to $\{(a,-a)\}\subset\mathbb{C}^2$ via the projection to $\mathcal{T}(C\setminus A,0)/\mathcal{T}(C\setminus A,d_1)=\mathbb{C}^2$. Then $M_{\mathcal{T}(C\setminus A,B)'} = \oplus(M\otimes L_{c,hb_1}\otimes L_{c,hb_2})_{\mathcal{T}(U)}$, where L_{ch} runs the list 7.3.7 (ii) of irreducible lisse modules.*

Proof. Similar to 8.1.6. □

8.2 Localization of lisse modules. Let $\pi: C_S\to S$ be a family of smooth projective curves, $A\subset C_S(S)$ be a finite non-empty disjoint set of sections, ν_a are 1-jets of parameters at $a\in A$. By 3.4.3-3.4.7 these define the S-localization data for $(\widetilde{\mathcal{T}}_c^A, v_1)$. Consider the corresponding S-localization functor $\Delta_{\psi_c}: (\widetilde{\mathcal{T}}_c^A, v_1)_c$-mod $\to D_{\lambda^c}$-modules on S. Assume as above that M is a lisse $(\widetilde{\mathcal{T}}_c^A, v_1)_c$-module.

8.2.1 Lemma. *The D_{λ^c}-module $\Delta_{\psi_0}(M)$ is lisse with regular singularities at infinity.*

Proof. Lissing follows from 7.2.2; the statement on regular singularities follows from 8.2.5 below. □

8.2.2 Assume now that $S = Spec\mathbb{C}[[q]], \pi: C_S\to S$ be a projective family of curves such that the generic fiber C_η is smooth and the closed fiber C_0 has the only singular point b which is quadratic, $A\subset C_S(S)$ be a finite non-empty disjoint set of sections, and $\{\nu_a\}$ be a 1-jet of coordinates at $a\in A$.

This collection defines an S-localization data "with logarithmic singularities at $q = 0$" for $(\widetilde{\mathcal{T}}_c^A, v_1)$. (The definition of "$S$-loc. data ψ with log. sing. at $q = 0$" coincides with 3.3.3 but we replace the condition that N is a transitive Lie algebroid by the one that a canonical map $\sigma : N \to \mathcal{T}_S$ has image equal to $\mathcal{T}_S^0 = q\mathcal{T}_S = \mathbb{C}[[q]]q\partial_q$. As in 3.3 such data defines an $\mathcal{O}_S$-extension $\mathcal{A}^0_{\psi_c}$ of $\mathcal{T}_S^0$ and the corresponding associative algebra $D^0_{\psi_c}$ which is isomorphic to the subalgebra of $D_{\mathbb{C}[[q]]}$ generated by $\mathbb{C}[[q]] = \mathcal{O}_S$ and $q\partial_q$. We have the corresponding S-localization functor $\Delta_{C_S} : (\widetilde{\mathcal{T}}_c^A, v_1)\text{-mod} \to D^0_{\psi_c}\text{-mod}$. The definition of this ψ repeats word-for-word 3.4.3-3.4.7: we get the loc. data with logarithmic singularities just because $\mathcal{T}_S^0$ consists precisely of those vector fields that could be lifted to $C_S \setminus A(S)$. Note that the "vertical" part $N_{(0)} = \ker\sigma \subset N$ is a free $\mathcal{O}_S$-module and $N_{(0)}/qN_{(0)}$ coincides with the Lie algebra $\mathcal{T}(C_0^\vee \setminus A, B)'$, where $C_0^\vee$ is the normalization of C_0 and $B = \{b_1, b_2\}$ is the preimage of b (see 8.1.7). According to 3.5 the algebra $D^0_{\psi_C}$ coincides with the algebra $D^0_{\lambda^c_{C_S}}$ of differential operators on the determinant bundle $\lambda^c_{C_S}$ generated by "$q\partial_q$" and $\mathcal{O}_S$.

Now let t_1, t_2 be formal coordinates at b such that $q = t_1 t_2$. Let $C_S^\vee$ be the corresponding smooth S-curve (our b's are the a's in 3.6.1). We have canonical points $b_1, b_2 \in C_S^\vee(S)$ with parameters t_1, t_2. Take 1-jets of parameters $q^{-1}dt_1, dt_2$ (see 6.1.4) at b's. These, together with A, ν_A, define $\mathbb{C}((q))$-localization data for $(\widetilde{\mathcal{T}}_c^{A\sqcup B}, v_1)$. The corresponding algebra coincides with $D_{\lambda^c_{C_\eta^\vee}}$, so we have the localization functor $\Delta_{C_\eta^\vee} : (\widetilde{\mathcal{T}}_c^{A\sqcup B}, v_1)\text{-mod} \to D_{\lambda^c_{C_\eta^\vee}}\text{-mod}$.

8.2.3 Let $\mathcal{H}$ be a lisse $D_{\lambda^c_{C_\eta^\vee}}$-module, i.e. a finite dimensional $\mathbb{C}((t))$-vector space with D-action. An h-lattice $\mathcal{H}_h \subset \mathcal{H}$, where $h \in \mathbb{C}$, is a $\mathbb{C}[[t]]$-lattice in $\mathcal{H}$ invariant with respect to the action of $D^0_{\lambda^c_{C_S^\vee}}$ and such that the operator $q\partial_q \in D^0_{\lambda^c_{C_S^\vee}}/q$ acts on $\mathcal{H}_h/q\mathcal{H}_h$ as multiplication by h. Certainly, such $\mathcal{H}_h$ exists iff $\mathcal{H}$ has regular singularities at 0 with monodromy equal to h mod $\mathbb{Z}$; if $\mathcal{H}_h$ exists, it is unique, so we'll call it "the" h-lattice.

From now on let M be a lisse $\widetilde{\mathcal{T}}_c^A$-module.

8.2.4 Lemma. *For any $h \in \mathbb{C}$, $\Delta_{C_\eta^\vee}(M \otimes L_{hb_1} \otimes L_{hb_2})$ is a lisse module that admits the h-lattice $\Delta_{C_\eta^\vee}(M \otimes L_h \otimes L_h)_h$.*

Proof. "lisse" follows from 8.1.4 (ii), 7.2.1. The existence of h-lattice follows easily from 3.4.7.1. □

According to 3.6.3 we have a canonical isomorphism $D_{\lambda^c_{C_S}} = D_{\lambda^c_{C_S^\vee}}$. Denote this algebra D_{λ^c}. So, by 8.2.4, we have for any $h \in \mathbb{C}$ a $D^0_{\lambda^c}$-module $D_{\lambda^c_{C_\eta^\vee}}(M \otimes L_n \otimes L_h)_h$, which is zero if L_h is not lisse (i.e. if $h \neq h_{nm}$ from 7.3.7 (ii)) by 8.1.4 (ii).

On the other hand, we have the $D^0_{\lambda^c}$-module $\Delta_{C_S}(M)$.

8.2.5 Proposition. *There is a canonical isomorphism of $D^0_{\lambda^c}$-modules*

$$\Delta_{C_S}(M) = \bigoplus_h \Delta_{C_\eta^\vee}(M \otimes L_h \otimes L_h)_h.$$

Proof. First, note that $\Delta_{C_S}(M)$ is a coherent $\mathcal{O}_S$-module by a version of 7.2.2 "with logarithmic singularities". Namely, $\Delta_{C_S}(M)$ is a coherent $D^0_{\lambda^c}$-module, and its singular support $\subset Spec(grD^0_{\lambda^c})$ is 0 section since M is lisse; hence $\Delta_{C_S}(M)$ is $\mathcal{O}_S$-coherent.

Let e_i be a basis of $L_{h\mathbb{C}((t))}$ that consists of L_0-eigenvectors, so $L_0 e_i = (h - n_i)e_i$ for $n_i \in \mathbb{Z} \geq 0$; let e_i^* be the dual basis in $L_{h\mathbb{C}((t))} = *L_{h\mathbb{C}((t))}$. It is easy to see that $\Delta_{C_\eta^\vee}(M \otimes L_h \otimes L_h)_h \subset \Delta_{C_\eta^\vee}(M \otimes L_h \otimes L_h)$ is $\mathcal{O}_S$-submodule generated by images of elements $q_m^{-n_i} \otimes e_i \otimes e_j^*$, where $m \in M_{A,C_S}, e_i \in L_{h(\mathbb{C}((t_1)),q^{-1}t)}, e_j^* \in L_{h(\mathbb{C}((t_2)),t_2)}$ (see 6.1.4 for notations).

To prove 8.2.5 it suffices to construct a morphism of $D^0_{\lambda^c}$-modules $\Delta_{C_S}(M) \to \oplus \Delta_{C_\eta^\vee}(\quad)_h$ which induces isomorphism mod q (since both are coherent $\mathcal{O}_S$-modules, and the one on the right hand has no q-torsion, this morphism will be isomorphism).

The h-component of this morphism just maps the image of $m \in M_{A,C_S} = M_{A,C_S^\vee}$ in $\Delta_{C_S}(M)$ to the image of $\sum_i m \otimes e_i \otimes e_i^*$ in $\Delta_{C_\eta^\vee}(M \otimes L_h \otimes L_h)$. It is easy to see that this formula defines a correctly defined morphism of $D^0_{\lambda^c}$-modules (cf. 6.1.5). It induces isomorphism modulo q by 8.1.7 (since $\Delta_{C_S}(M)/q = M_{N_{(0)}/qN_{(0)}} = M_{\mathcal{T}(C_0^\vee \setminus A,B)'}$, see 8.2.2). □

8.3 Definition of minimal theories. Now we may define the minimal theory. Pick central charge $c = c_{p,q}$ from the list 7.3.7(i).

The fusion category $\mathcal{A} = \mathcal{A}_{p,q}$ is category of finitely generated lisse higher weight modules for Virasoro algebra $\widetilde{\mathcal{T}}_c$ of central charge c. By 7.3.12 it satisfies the conditions listed in the beginning of 4.5.1. The data from 4.5.1 ar the following ones:

The duality functor $* : \mathcal{A}^0 \to \mathcal{A}$ is contravariant duality (see 7.3.1).

The vacuum module $\not\Vdash$ is $L_{c,o}$; the isomorphism $*\not\Vdash = \not\Vdash$ is canonical one (that identifies the vacuum vectors).

The Dehn automorphism d is equal to the monodromy automorphism $T = exp 2\pi i L_o$ from 7.3.2.

We will define a canonical fusion structure on $\mathcal{A}$ simultaneously with the structures 6.1 of algebraic field theory. Namely, our realization functor $r : A \to (\widetilde{\mathcal{T}}_c, v_1)$-mod is "identity" embedding. The vacuum vector $1 \in r(\not\Vdash) = L_0$ is v_0.

Let $\pi : C_S \to S, A \subset C_S(S), \nu_A$, be as in 6.1.2. Assume that $A \neq \emptyset$. For any $X \in \mathcal{A}^{\otimes A}$ the D_{λ^c}-module $\Delta_{\psi_c}(X)$ is lisse holonomic with regular singularities at ∞. We put $\langle X \rangle_{C_S} = \Delta_{\psi_c}(X)$ and γ from 6.1.2 (iv) is identity map.

Assume now that $A = \emptyset$. We should define a canonical lisse D_{λ^c}-module $\langle \not\Vdash \rangle_{C_S}$. Let us make the base change and consider $\pi_C : C_C = C_S \times_S C_S \to C_S$: this is a family of curves with a canonical (diagonal) section a. Consider the D_{λ^c}-module $\langle \not\Vdash \rangle_{C_C}$; this is a lisse D_{λ^c}-module on C_S generated by the holomorphic section $\langle 1 \rangle_{C_C}$. Note that $\langle 1 \rangle_{C_C}$ is horizontal along the fibers of $\pi : C_S \to S$. Hence there exists a (unique) D_{λ^c}-module $\langle \not\Vdash \rangle_{C_S}$ on S together with a holomorphic section $\langle 1 \rangle_{C_S}$ such that $\langle \not\Vdash_a \rangle_{C_C} = \pi^* \langle \not\Vdash \rangle_{C_S}$, $\langle 1 \rangle_{C_C} = \pi^* \langle 1 \rangle_{C_S}$.

Note that the axioms 4.5.4 (ii) and 6.1.2e hold by 8.1.5. The axiom 6.1.3f holds automatically. It remains to define the isomorphism 4.5.5 (ii) that will satisfy the axiom g from 6.1. This was done in 8.2.5 above (note that since $*L_h = L_h$, we have $R = \oplus L_h \otimes L_h$).

By the way, the covariant fusion functor $\mathcal{F}_C^{A,B}$ from 4.6 is $*\mathcal{F}_C$ for contravariant $\mathcal{F}_C$ from 8.1 (by 8.1.3 (iii)).