Hoop conjecture in spherically symmetric spacetimes

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We give general sufficient conditions for the existence of trapped surfaces due to concentration of matter in spherically symmetric initial data sets satisfying the dominant energy condition. These results are novel in that they apply and are meaningful for arbitrary spacelike slices, that is, they do not require any auxiliary assumptions such as maximality, time symmetry, or special extrinsic foliations, and most importantly they can easily be generalized to the nonspherical case once an existence theory for a modified version of the Jang equation is developed. Moreover, our methods also yield positivity and monotonicity properties of the Misner-Sharp energy.

I. INTRODUCTION

The hoop conjecture [1] concerns the folklore belief that if enough matter and/or gravitational energy are present in a small enough region (small in all three spatial dimensions), then the system must collapse to a black hole. This belief is often realized by establishing a statement of the following form. Let Ω be a compact spacelike hypersurface satisfying an appropriate energy condition in a spacetime $\mathcal{M}$. There exists a universal constant $C > 0$ such that if $\mathrm{Mass}(\Omega) > C \cdot \mathrm{Size}(\Omega)$, then Ω must contain a closed trapped surface. Of course finding the correct notions of $\mathrm{Mass}(\Omega)$ and $\mathrm{Size}(\Omega)$ is one of the primary difficulties with this conjecture. The conclusion of the above statement guarantees that the spacetime $\mathcal{M}$ contains a singularity (or more precisely is null geodesically incomplete) by the Hawking-Penrose singularity theorems [2], and assuming cosmic censorship must, therefore, contain a black hole. It should also be pointed out that modulo certain technical restrictions, trapped surfaces lead to gravitational confinement according to Israel's result [3]. Therefore, in asymptotically flat spacetimes the existence of a trapped surface almost certainly implies the existence of a black hole.

There have been many results realizing a version of the hoop conjecture in this spirit. Most notable are those of O'Murchadha, Malec, and others [4–12], which address concentration of matter in spherical symmetry and give necessary and sufficient conditions in some instances, but impose auxiliary conditions on the spacelike slice such as the condition of maximality, time-symmetry, or that it arises from an extrinsic foliation. On the other hand, there are the very important results of Schoen and Yau [13,14] which also address the issue of concentration of matter, but without extra assumptions. While their results are very impressive in that they do not require spherical symmetry, they suffer from the opposite problem in that they are not meaningful for slices with small extrinsic curvature, in particular, for maximal or time-symmetric slices. There

have been relatively fewer results on the concentration of pure gravitational radiation, see [9,15].

In this paper we also address the topic of concentration of matter. Our goal is to establish sufficient conditions for the existence of trapped surfaces in spherically symmetric initial data, which apply and are meaningful both in the maximal and general cases. Our methods are quite general in that they can easily be generalized to the nonspherically symmetric case once an appropriate existence theory (analogous to that developed by Schoen and Yau in [16]) for a modified version of the Jang equation has been established. Moreover our techniques yield positivity and monotonicity properties for the Misner-Sharp energy, as a natural and interesting byproduct.

An initial data set for the gravitational field consists of a 3-manifold M on which is defined a positive definite metric g and symmetric 2-tensor k representing the extrinsic curvature. The metric and extrinsic curvature must of course satisfy the constraint equations:

$$16\pi\mu = R + (\mathrm{Tr}_g k)^2 - |k|^2,$$
$$8\pi J_i = \nabla^j(k_{ij} - (\mathrm{Tr}_g k)g_{ij}),$$

where R denotes scalar curvature and μ, J are, respectively, the energy and momentum densities for the matter fields. If the initial data are spherically symmetric then we may take $M \simeq \mathbb{R}^3$ and write

$$g = g_{11}(r)dr^2 + \rho^2(r)d\chi^2,$$
$$k^{ij} = n^i n^j k_a(r) + (g^{ij} - n^i n^j)k_b(r),$$

where

$$n = n^1 \partial_r + n^2 \partial_{\psi^2} + n^3 \partial_{\psi^3} = \sqrt{g^{11}}\partial_r$$

is the unit normal to spheres S_r centered at the origin (the ball enclosed by S_r will be denoted B_r), and

$$d\chi^2 = (d\psi^2)^2 + \sin^2\psi^2(d\psi^3)^2$$

is the round metric on $\mathbb{S}^2$. We also assume that the metric is regular at the origin, so that $\rho(0) = 0$, $\rho_{,r}(0) = 1$, and

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$g_{11}(0) = 1$. The sphere S_r is future (past) trapped if the family of outgoing future (past) directed null geodesics, orthogonal to S_r, is converging at each point. This is equivalent to the following condition satisfied by the null expansions:

$$\theta_+ := H_{S_r} + \text{Tr}_{S_r} k < 0 \quad \text{(future trapped)},$$

$$\theta_- := H_{S_r} - \text{Tr}_{S_r} k < 0 \quad \text{(past trapped)},$$

where $H_{S_r} = n^i_{;i}$ denotes the mean curvature and $\text{Tr}_{S_r} k = (g_{ij} - n_i n_j) k^{ij}$. The outer boundary of a region in M which contains future (past) trapped surfaces is called a future (past) apparent horizon, and satisfies $\theta_+ = 0$ ($\theta_- = 0$). Our main result is as follows.

Theorem 1.—Let (M, g, k) be a spherically symmetric initial data set satisfying the dominant energy condition $\mu \geq |J|$. If

$$\min_{B_r}(\mu \mp J(n)) + \frac{3}{32\pi} \theta_+ \theta_-(r) > \frac{3}{2} \frac{\text{Rad}(B_r)}{\text{Vol}(B_r)}, \quad (1.1)$$

where the radius and volume are given by

$$\text{Rad}(B_r) = \int_0^r \sqrt{g_{11}}, \qquad \text{Vol}(B_r) = 4\pi \int_0^r \sqrt{g_{11}} \rho^2,$$

then B_r contains a future (past) trapped surface.

The first term on the left-hand side of (1.1) shows the intuitively obvious fact, that formation of trapped surfaces depends not only on matter concentration but also on the direction that the matter is flowing. Namely, inward flowing material hastens (delays) the formation of future (past) trapped surfaces, whereas outgoing material delays (hastens) formation. More interesting is the second term on the left-hand side, which indicates that the bending of light rays at the boundary of B_r, can by itself cause surfaces to be trapped on the interior. This phenomenon was first observed by Yau [14]. However as we have pointed out, the result of [14] as well as the earlier version of Schoen and Yau [13] are not meaningful when $\text{Tr}_g k$ is small. To be more precise let us recall their result, which states that if

$$\min_B(\mu - |J|) > \frac{3\pi^2}{2} \frac{1}{\text{Rad}_{SY}(B)},$$

then B contains a trapped surface. Here spherical symmetry is not assumed and $\text{Rad}_{SY}(B)$ is the square of a "homotopy radius." Thus their result requires matter density to be large on a "large region." However our basic intuition suggests that this is not the ideal situation which results in collapse, that is, as the hoop conjecture asserts we would rather like to show that if matter density is large on a "small region" then trapped surfaces exist. So it is not surprising that their result is only meaningful for a fairly restricted class of initial data (as pointed out by Bizon, Malec, and O'Murchadha [4]). They show (Theorem 1 of

their paper) that one cannot have a large set with large positive scalar curvature. Since the matter density is related to the scalar curvature via the Hamiltonian constraint, the only way we can have a large matter density and small scalar curvature (which is required from their Theorem 1) is that the trace of the extrinsic curvature $\text{Tr}_g k$ is large; in fact the trace must be not only large but significantly larger than $|k|$. This means that it may be difficult to find data which satisfy their condition, and, in particular, their result can say nothing about the time-symmetric or maximal cases. On the other hand, our result compares nicely with that of Malec and O'Murchadha [17] who showed that under the assumption of spherical symmetry and maximality ($\text{Tr}_g k = 0$),

$$4\pi \int_0^r (\mu \mp J(n))\rho^2 > \text{Rad}(B_r)$$

implies that B_r contains a future (past) trapped surface. Unfortunately, it is difficult to see how their arguments might generalize to the nonspherically symmetric case.

II. THE GENERALIZED JANG EQUATION

Our methods are based in large part on the generalized Jang equation [18], which we now explain. Many of the difficult issues and questions involving initial data are easier to express and solve if it happens that the scalar curvature of the given metric g is nonnegative, $R \geq 0$. Unfortunately there is no guarantee that this will be the case for an arbitrary set of initial data, except under the added assumptions of maximality and the dominant energy condition. It is for this reason that Jang [19] introduced the quasilinear elliptic equation for a scalar f depending on g and k, which bears his name:

$$H_\Sigma - \text{Tr}_\Sigma K = 0, \quad (2.1)$$

where Σ denotes the graph $t = f(x)$ inside the product manifold $(M \times \mathbb{R}, g + dt^2)$, H_Σ is the mean curvature, and K is a trivially extended version of k (extended to all of $M \times \mathbb{R}$). That is, he showed that if f solves (2.1) then the scalar curvature of the related metric $\bar{g} = g + df^2$ (this is the induced metric on the graph Σ) has nice positivity properties. In fact, Schoen and Yau [16] successfully employed the Jang equation in their solution of the positive energy conjecture, to reduce the general case to the case of time symmetry. Moreover they developed a full existence theory for this equation, and showed that regular solutions always exist if the initial data do not contain apparent horizons. The converse statement, that if a regular solution does not exist then the data must contain an apparent horizon, naturally led to their result [13] concerning the hoop conjecture.

These successful applications of the Jang equation led many to suggest that it could also be used to study the

Penrose inequality. However as pointed out by Malec and O'Murchadha [10], serious and immediate difficulties arise when attempting such an application. These difficulties motivated the author, together with H. Bray [18], to propose a modified version of the Jang equation, specifically designed for the Penrose inequality. This generalized Jang equation has the same geometric structure as that of (2.1), however the mean curvature of the graph Σ is now calculated inside the warped product manifold $(M \times \mathbb{R}, g + \phi^2 dt^2)$ where ϕ is a nonnegative scalar, and the extended tensor K is now a nontrivial extension of k (see [20]). An important feature of the generalized Jang equation, like the original, is that it yields nice positivity properties for the scalar curvature of the induced metric on Σ. More precisely, if $\bar{R}$ denotes the scalar curvature of $\bar{g} = g + \phi^2 df^2$ then we find [18] that

$$\bar{R} = 16\pi(\mu - J(w)) + |h - K|_{\Sigma}|^2_{\bar{g}} + 2|q|^2_{\bar{g}}$$
$$- 2\phi^{-1}\mathrm{div}_{\bar{g}}(\phi q), \tag{2.2}$$

where h is the second fundamental form of Σ, and the 1-forms w and q are given by

$$w_i = \frac{f_{,i}}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}, \qquad q_i = w^j(h - K|_{\Sigma})_{ij}.$$

According to the dominant energy condition this expression shows that $\bar{R}$ is almost nonnegative, with only a divergence term standing in the way. In fact, the extra degree of freedom given by the scalar ϕ will be used to remove the problematic divergence term in the next section. Moreover we have shown in our investigation of the Penrose inequality [20], in analogy with the theory developed by Schoen and Yau [16] for the classical Jang equation, that regular solutions of the modified Jang equation exist in spherical symmetry away from apparent horizons if we choose

$$\phi = \rho_{,s} \tag{2.3}$$

where

$$\partial_s = \frac{\sqrt{1 - v^2}}{\sqrt{g_{11}}} \partial_r, \qquad v = \frac{\sqrt{\phi^2 g^{11}} f_{,r}}{\sqrt{1 + \phi^2 g^{11} f_{,r}}}.$$

Note that

$$s = \int_0^r \frac{\sqrt{1 - v^2}}{\sqrt{g_{11}}} = \int_0^r \sqrt{g_{11} + \phi^2 f_{,r}^2}$$

is the radial arclength parameter for the $\bar{g}$ metric. In particular we have the following result.

Theorem 2—Let (M, g, k) be a spherically symmetric initial data set satisfying the dominant energy condition $\mu \geq |J|$. If a ball B_r centered at the origin does not contain an apparent horizon, then there exists a regular solution f in B_r of the modified Jang equation with the scalar ϕ given by (2.3).

III. EXISTENCE OF APPARENT HORIZONS

Here we shall give the proof of Theorem 1, which will proceed by contradiction. Assume that the ball B_r does not contain an apparent horizon. Then Theorem 2 guarantees the existence of a regular solution to the generalized Jang equation with ϕ given by (2.3). In particular we must have $v(0) = 0$, and $-1 < v < 1$. Therefore ϕ and $\rho_{,r}$ are strictly positive on B_r since

$$4\sqrt{g^{11}} \frac{\rho_{,r}}{\rho} = 2H_{S_r} = \theta_+ + \theta_- > 0,$$

as $\theta_+ > 0$ and $\theta_- > 0$ when B_r contains no horizons. Let

$$m(r) = \sqrt{\frac{A_{\bar{g}}(S_r)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{S_r} H^2_{S_r, \bar{g}}\right)$$
$$= \frac{1}{2}\rho(r)(1 - \rho_{,s}^2(r))$$

denote the Geroch energy [21] of a sphere S_r inside the Jang surface Σ. A well-known calculation shows that

$$m_{,s} = \frac{1}{4}\rho_{,s}\rho^2\bar{R},$$

so that the formula (2.2) for $\bar{R}$ yields

$$m(r) = m(r) - m(0) = \int_0^r m_{,s}ds = \int_0^r \frac{1}{4}\rho_{,s}\rho^2\bar{R}ds$$
$$= \frac{1}{16\pi}\int_{B_r}\rho_{,s}\bar{R}d\omega_{\bar{g}}$$
$$\geq \int_{B_r}\rho_{,s}(\mu - J(w) - (8\pi\phi)^{-1}\mathrm{div}_{\bar{g}}(\phi q))d\omega_{\bar{g}}, \tag{3.1}$$

where $d\omega_{\bar{g}}$ is the volume element on the Jang surface Σ. We may then apply the divergence theorem [as a result of the special choice of ϕ given by (2.3)] and the calculation (see [20])

$$\phi\bar{g}(q, n_{\bar{g}})d\sigma_{\bar{g}} = -2\frac{\rho_{,r}v}{\sqrt{g_{11}}}\left(\sqrt{g^{11}}\frac{\rho_{,r}}{\rho}v - k_b\right)d\sigma_g,$$

where $n_{\bar{g}}$ is the unit outer normal to S_r in the $\bar{g}$ metric and $d\sigma_{\bar{g}}$, $d\sigma_g$ are area elements, to obtain

$$m(r) \geq 4\pi \int_0^r \rho_{,s}(\mu - J(w))\rho^2 ds - \frac{1}{8\pi} \int_{S_r} \phi \bar{g}(q, n_{\bar{g}}) d\sigma_{\bar{g}}$$

$$\geq \frac{4\pi}{3} \int_0^r (\rho^3)_{,r} dr \min_{B_r}(\mu - J(w)) + \frac{1}{4\pi} \int_{S_r} \frac{\rho_{,r} v}{\sqrt{g_{11}}} \left(\sqrt{g^{11}} \frac{\rho_{,r}}{\rho} v - k_b \right) d\sigma_g$$

$$= \frac{4\pi}{3} \rho^3(r) \min_{B_r}(\mu - J(w)) + \frac{\rho_{,r} v}{\sqrt{g_{11}}} \left(\sqrt{g^{11}} \frac{\rho_{,r}}{\rho} v - k_b \right) \rho^2(r). \tag{3.2}$$

However since

$$m(r) = \frac{1}{2}\rho(r) - \frac{1}{2}\left(\frac{1 - v^2}{g_{11}} \right) \rho_{,r}^2 \rho(r),$$

it follows that

$$\frac{1}{2}\rho(r) \geq \frac{4\pi}{3} \rho^3(r) \min_{B_r}(\mu - J(w)) + \frac{1}{2}(1 + v^2) g^{11} \rho_{,r}^2 \rho(r) - \frac{\rho_{,r}}{\sqrt{g_{11}}} k_b \rho^2 v(r)$$

$$= \frac{4\pi}{3} \rho^3(r) \min_{B_r}(\mu - J(w)) + \frac{1}{2}\rho^3 \left(g^{11} \frac{\rho_{,r}^2}{\rho^2} - k_b^2 \right) + \frac{1}{2}\rho^3 \left(k_b - \sqrt{g^{11}} \frac{\rho_{,r}}{\rho} v \right)^2$$

$$\geq \frac{4\pi}{3} \rho^3(r) \min_{B_r}(\mu - J(w)) + \frac{1}{8}\rho^3(r)(H_{S_r}^2 - (\mathrm{Tr}_{S_r}k)^2). \tag{3.3}$$

Lastly, because $\rho_{,r} > 0$ we have

$$\rho^2(r) \geq \frac{\int_0^r \sqrt{g_{11}} \rho^2}{\int_0^r \sqrt{g_{11}}} = \frac{1}{4\pi} \frac{\mathrm{Vol}(B_r)}{\mathrm{Rad}(B_r)},$$

and hence

$$\frac{3}{2} \frac{\mathrm{Rad}(B_r)}{\mathrm{Vol}(B_r)} \geq \min_{B_r}(\mu - J(w)) + \frac{3}{32\pi} \theta_+ \theta_-(r).$$

We conclude that if (1.1) holds, then B_r must contain an apparent horizon.

IV. PROPERTIES OF THE MISNER-SHARP ENERGY

The Misner-Sharp energy [22] is widely regarded as the correct measure of quasilocal energy contained in centered spacelike 2-spheres in spherically symmetric spacetimes. When evaluated on a sphere S_r it takes the form

$$E_r = \sqrt{\frac{A(S_r)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{S_r} \theta_+ \theta_- \right),$$

which also happens to be the expression for the Hawking energy [23] of a spacelike 2-surface in an arbitrary spacetime. Here we would merely like to point out that the arguments of the previous section immediately imply positivity and monotonicity properties for the Misner-Sharp energy. To see this, let $B_{r_1 r_2}$ denote the region between two concentric spheres S_{r_1} and S_{r_2} with $r_2 > r_1$. We will refer to this region as untrapped if $\theta_+ \theta_- > 0$ throughout. For definiteness let us assume that both $\theta_+ > 0$ and $\theta_- > 0$. Then $H_{S_{r_1}} \neq 0$ implies that $|H_{S_{r_1}}^{-1} \mathrm{Tr}_{S_{r_1}} k| \leq 1$, so in analogy with Theorem 2 there exists a regular solution of the

modified Jang equation with ϕ given by (2.3) and such that $v(r_1) = H_{S_{r_1}}^{-1} \mathrm{Tr}_{S_{r_1}} k$ (see [20]). Note that this does not exclude the possibility that S_{r_1} and/or S_{r_2} are apparent horizons, and if this is the case then we impose the restriction that they can be either future or past but not both, which ensures that $H_{S_{r_1}} \neq 0$. Therefore we may follow precisely the same arguments presented in (3.1), (3.2), and (3.3) while keeping the two middle terms of (2.2), to find that

$$E_{r_2} - E_{r_1} = \frac{1}{16\pi} \int_{B_{r_1 r_2}} \rho_{,s}(16\pi(\mu - J(w))$$

$$+ |h - K|_{\Sigma}^2|_{\bar{g}}^2 + 2|q|_{\bar{g}}^2) d\omega_{\bar{g}}$$

$$+ \frac{1}{8}\rho^3(r_2)(\mathrm{Tr}_{S_{r_2}} k - v(r_2)H_{S_{r_2}})^2$$

$$- \frac{1}{8}\rho^3(r_1)(\mathrm{Tr}_{S_{r_1}} k - v(r_1)H_{S_{r_1}})^2.$$

But since $v(r_1) = H_{S_{r_1}}^{-1} \mathrm{Tr}_{S_{r_1}} k$ we obtain

$$E_{r_2} \geq E_{r_1}.$$

Conversely, if both $\theta_+ < 0$ and $\theta_- < 0$ then the same arguments with $v(r_2) = H_{S_{r_2}}^{-1} \mathrm{Tr}_{S_{r_2}} k$ give

$$E_{r_2} \leq E_{r_1}.$$

We have thus found the following result.

Theorem 3.—Let (M, g, k) be a spherically symmetric initial data set satisfying the dominant energy condition $\mu \geq |J|$. Then the Misner-Sharp energy is always monotonic on untrapped regions. In particular, the Misner-Sharp energy of a centered 2-sphere not enclosing any apparent

horizon is nonnegative $E_r \geq 0$, and the Misner-Sharp energy of a centered two-sphere enclosing the outermost apparent horizon S_{r_0} satisfies the lower bound $E_r \geq \sqrt{A(S_{r_0})/16\pi}$. Furthermore if $E_r = 0$ or $E_r = \sqrt{A(S_{r_0})/16\pi}$, then (B_r, g, k) [respectively $(B_{r_0 r}, g, k)$] arises from a spacelike hypersurface in the Minkowski (respectively Schwarzschild) spacetime.

These observations concerning the Misner-Sharp energy have previously been established by Hayward in [24] (see also [25]) using different methods, although the rigidity result appears to be new (for details see [20]). The novelty of our method lies with the fact that it can easily be generalized to the nonspherically symmetric case, once a general existence theory for the modified Jang equation has been obtained. When this is done, an expanded version of Theorem 3 would give new positivity and monotonicity properties for the Hawking energy, and would lead to a proof of the Penrose inequality [18] for general initial data.

ACKNOWLEDGMENTS

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