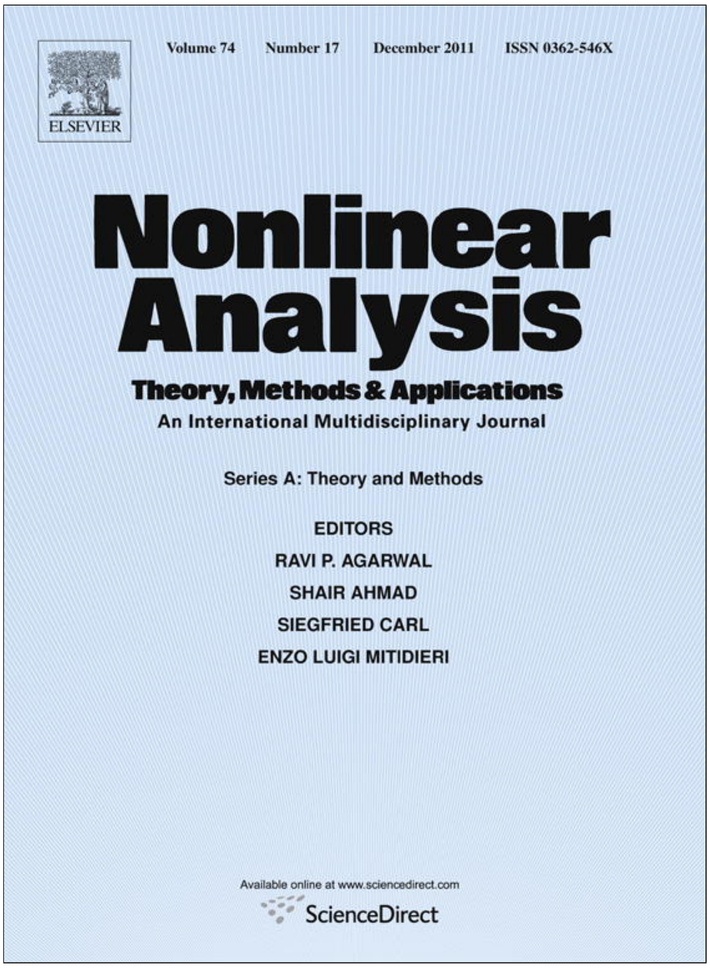
Volume 74 Number 17 December 2011 ISSN 0362-546X
ELSEVIER
Nonlinear Analysis
Theory, Methods & Applications
An International Multidisciplinary Journal
Series A: Theory and Methods
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Available online at www.sciencedirect.com
ScienceDirect

Nonlinear Analysis 74 (2011) 6405–6415

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Boundary value problems for mixed type equations and applications

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ARTICLE INFO

Article history:
Received 17 March 2011
Accepted 17 June 2011
Communicated by Enzo Mitidieri

Keywords:
Mixed type equations
Boundary value problems
Monge–Ampère equations
Isometric embeddings

ABSTRACT

In this paper we outline a general method for finding well-posed boundary value problems for linear equations of mixed elliptic and hyperbolic type, which extends previous techniques of Berezanskii, Didenko, and Friedrichs. This method is then used to study a particular class of fully nonlinear mixed type equations which arise in applications to differential geometry.

1. Introduction

An old classical problem from differential geometry asks, when can one realize a two-dimensional Riemannian manifold, locally, in a three-dimensional Euclidean space? In other words, when can one "see" an abstract surface, at least locally? As it turns out, this question is equivalent to finding local solutions $z(x, y)$ to a Monge–Ampère type equation, referred to as the Darboux equation:

$$\det \nabla_{ij} z = K(\det h)(1 - |\nabla_h z|^2). \tag{1.1}$$

Here h is the given Riemannian metric, ∇_{ij} are second covariant derivatives, and K is the Gaussian curvature of h. Another related problem is that of locally prescribing the Gaussian curvature of surfaces in three-dimensional Euclidean space. More precisely, given a function $K(x, y)$ defined in a neighborhood of the origin, does there exist a graph $z = z(x, y)$ having Gaussian curvature K? Note that every surface may be expressed locally as a graph. This problem is also equivalent to the local solvability of a Monge–Ampère equation, namely

$$\det \partial_{ij} z = K(1 + |\nabla z|^2)^2, \tag{1.2}$$

where ∂_{ij} are second partial derivatives. In both Eqs. (1.1) and (1.2), the sign of the Gaussian curvature completely determines the type of the equation. When K is positive the equation is elliptic, and when K is negative the equation is hyperbolic. Thus classical results may be used to analyze these problems in these two situations. However when K changes sign, the equation is of mixed type, and is very difficult to study. Nevertheless, it can be shown [1] that by a suitable application of a Nash–Moser iteration, these two problems reduce to the study of a linear equation having a particular form described below. More precisely, in order to successfully apply the Nash–Moser iteration, one must find a well-posed boundary value problem for the associated linearized equation, in a fixed domain about the origin, and establish certain a priori estimates. In previous work by Han, Hong, Lin, as well as the author, this has been accomplished in the case for which the Gaussian curvature changes sign to finite order along a single smooth curve (see [2–6]), and also in the case for which the Gaussian curvature vanishes to finite order and has a zero set consisting of two transversely intersecting curves (see [1,7]). Our goal

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doi:10.1016/j.na.2011.06.022

here is to extend these results by giving a general condition on the Gaussian curvature, which in particular allows for a change of sign to infinite order and a zero set for K which is more general than a finite number of intersecting curves. Counterexamples to the local solvability of mixed type Monge–Ampère equations, similar to, but not exactly of the form studied here, have been found [8] in the case of infinite order vanishing. Our main result is

Theorem 1.1. *Let $\varepsilon > 0$ be a small parameter. Suppose that the Gaussian curvature $K \in C^{\infty}$ satisfies the following condition in a neighborhood of a point,*

$$\nabla_V K \geq \varepsilon(|\nabla K| + |K|) \tag{1.3}$$

for some smooth vector field V. Then both Eqs. (1.1) and (1.2) admit sufficiently smooth local solutions.

By a *sufficiently smooth* local solution, we mean that for each sufficiently large integer m there exists a neighborhood Ω_m such that the solution $z \in C^m(\Omega_m)$. However, this does not necessarily imply that smooth local solutions exist, since the size of the domains Ω_m may become arbitrarily small as $m \to \infty$. Note that condition (1.3) will be satisfied for a wide variety of Gaussian curvatures. To see this, suppose that local coordinates x, y have been chosen near a point (corresponding to the origin in the xy-plane) such that the vector field V is given by ∂_y. Then we may take $K(x, y) = k(y)\phi(x, y)$ where $\phi > 0$, $k(y) = \exp(-|y|^{-1})$ for $y > 0$, $k(y) = -\exp(-|y|^{-1})$ for $y < 0$, and $k(0) = 0$. In this example K changes sign to infinite order across a single curve. However the zero set $K^{-1}(0)$ may be much more general. For instance $K^{-1}(0)$ may be given by the region $|y| \leq |x|$; if $K > 0$ for $y > |x|$ and $K < 0$ for $y < -|x|$ then condition (1.3) will be satisfied in a sufficiently small neighborhood of the origin.

Consider the following class of boundary value problems for linear second order partial differential equations of the form:

$$\begin{aligned} Lu &= Ku_{xx} + u_{yy} + Au_x + Bu_y = f \quad \text{in } \Omega, \\ \mathscr{B}u &= \alpha u_x + \beta u_y + \gamma u = g \quad \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

The coefficient functions of L and $\mathscr{B}$ are assumed to be smooth in the domain $\Omega \subset \mathbb{R}^2$ and on its (piecewise smooth) boundary $\partial\Omega$, respectively. Moreover the function K will be required to change sign in Ω, so that L is of mixed elliptic and hyperbolic type. In the case that $K = y$ and $A = B = 0$, L is the well-known Tricomi operator, which has been heavily studied in the context of transonic flows. The change from elliptic to hyperbolic type as one crosses the x-axis represents the passing from subsonic to supersonic speeds. In [9], Tricomi studied the homogeneous equation ($f = 0$) inside a domain bounded by a simple arc in the elliptic region $y > 0$, and two intersecting characteristic curves in the hyperbolic region $y < 0$, which emanate from the two points where the arc intersects $y = 0$. Dirichlet boundary data, that is $\mathscr{B}u = u$, were then prescribed on the simple arc and on one of the characteristic curves, leaving the other characteristic curve without any prescribed boundary conditions. He was able to show that this boundary value problem is well posed: it admits a unique regular solution, with continuous dependence on the given data. Such problems may be described as *open* boundary value problems, since the solution is not prescribed in any way along some portion of the boundary. Open boundary value problems arise in flows in nozzles and in other applications, and have received considerable attention. In contrast, *closed* boundary value problems, in which $\mathscr{B}u$ is prescribed on the whole boundary, are less well studied. This lack of attention is not due, however, to the absence of applications. For instance closed problems arise in constructing smooth flows about airfoils. Rather, closed problems turn out to be more difficult to study, since they are often overdetermined for regular solutions. In [10], Lupo, Morawetz, and Payne considered such closed problems for the Chaplygin equation, where $A = B = 0$ and $K = K(y)$ satisfies the condition

$$K(0) = 0, \quad \text{and} \quad yK(y) > 0, \quad \text{for } y \neq 0.$$

They showed the existence and uniqueness of weak solutions for the Dirichlet and mixed Dirichlet-conormal boundary value problems, with minimal restrictions on the boundary geometry of the domain. Previous results on closed problems, often required restrictions on the boundary geometry or on the way in which K changes sign that were too strong to be of much help in applications to transonic fluid flows.

In this paper, we will study the case of homogeneous boundary conditions for problem (1.4), with very weak restrictions on the possible ways in which K changes sign. Unlike in the Tricomi case, when $K = y$, one cannot ignore the lower order terms, and thus we will find appropriate conditions to impose on the functions A and B for which this problem is well posed. Our goal is then to find a natural closed boundary value problem which admits a unique, regular solution for each right-hand side f, and which admits appropriate a priori estimates to show a strong continuous dependence on the given data. The domain will be taken to be a rectangle

$$\Omega = \{(x, y) \mid |x| < 1, |y| < 1\}, \tag{1.5}$$

however the two sides $x = \pm 1$ will be identified so that Ω becomes a cylinder. Thus all the functions involved must be 2-periodic. Altogether this has the effect of greatly simplifying the problem by eliminating half of the boundary. On the remaining two portions of the boundary, conditions will be imposed as follows. On the top of the cylinder $y = 1$ (in the elliptic region), Dirichlet conditions $\mathscr{B}u = u$ will be fixed, while on the bottom $y = -1$ (in the hyperbolic region), an oblique derivative condition $\mathscr{B}u = \alpha u_x + u_y$ will be applied for some appropriately chosen constant α depending on K.

In the process of studying this problem, we will outline a general method for determining appropriate boundary value problems for mixed type equations of the form (1.4). The procedure is in fact just a reorganized version of the classical $a-b-c$ method of Friedrichs (also referred to as the multiplier method [11]) together with the techniques of Berezanskii [12] and Didenko [13], which involve global energy estimates and negative norm spaces.

In order to state our result for the linearized equation, let $\varepsilon > 0$ be a small parameter, and let Ω be given by (1.5). Consider the following boundary value problem

$$\begin{aligned} &L_\varepsilon u = \varepsilon K u_{xx} + u_{yy} + \varepsilon A u_x + \varepsilon B u_y = f \quad \text{in } \Omega, \\ &u(x,1) = 0, \qquad (\alpha u_x + u_y)(x,-1) = 0, \quad u \text{ is 2-periodic in } x. \end{aligned} \tag{1.6}$$

We would like to point out that similar boundary conditions were studied by Han in [2], in the setting of a first order system and where $K = y + O(\varepsilon)$.

The Sobolev space of square integrable derivatives up to and including order m, for functions 2-periodic in x, will be denoted by $H^m(\Omega)$, and its norm will be denoted by $\|\cdot\|_{H^m(\Omega)}$. We will prove the following theorem.

Theorem 1.2. *Let m be a nonnegative integer, $\varepsilon > 0$ a small parameter, and α a constant. Suppose that the coefficients K, A, and B are smooth, 2-periodic in x, and satisfy the following condition*

$$K_y - \alpha K_x + 2\alpha A \geq \varepsilon^{1/4}(|K_x| + |K| + |A|) \quad \text{in } \Omega. \tag{1.7}$$

If $\alpha^2 > -\varepsilon \min_{|x|\leq 1} K(x,-1)$, and ε is sufficiently small, depending on m, α as well as on the coefficients of L, then for each $f \in H^{m+1}(\Omega)$ there exists a unique solution $u \in H^m(\Omega)$ of boundary value problem (1.6). Moreover, there exists a constant C depending only on m and the coefficients of L and their derivatives up to and including order m, such that

$$\|u\|_{H^m(\Omega)} \leq C\|f\|_{H^{m+1}(\Omega)}. \tag{1.8}$$

We also remark that the solutions produced by Theorem 1.2 actually possess slightly better regularity than is stated here. This will become clear from the proof in Section 3.

This paper is organized as follows. In Section 2 we review the required functional analysis, and introduce the general procedure for ascertaining appropriate boundary conditions to obtain a well-posed problem. In Section 3 this procedure is used to treat (1.6), and to prove Theorem 1.2. Finally, the proof of our main result Theorem 1.1 is given in Section 4. The Appendix contains proofs of some functional analysis results.

2. Finding the appropriate boundary conditions

We begin by introducing the necessary functional analysis needed to apply the general procedure for ascertaining appropriate boundary conditions associated with a differential operator. Much of the discussion in this section is expository, and is reorganized here for our particular application.

Frequently when dealing with mixed type equations, regularity will occur at different levels for different directions, and it is then advantageous to have function spaces which can identify this difference. Thus we will be working with the anisotropic Sobolev spaces $H^{(m,l)}(\Omega)$, which consist of functions having square integrable derivatives up to and including order m in the x-direction and order l in the y-direction. Here Ω is a domain in the xy-plane, and the norm on these spaces is given by

$$\|u\|^2_{(m,l)} = \int_\Omega \sum_{\substack{0\leq s\leq m \\ 0\leq t\leq l}} (\partial_x^s \partial_y^t u)^2.$$

We will also have the need of the negative norm spaces of Lax [14]. For each $v \in L^2(\Omega)$ the negative norms are given by

$$\|v\|_{(-m,-l)} = \sup_{u\in H^{(m,l)}(\Omega)} \frac{|(u,v)|}{\|u\|_{(m,l)}}, \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product, and the spaces $H^{(-m,-l)}(\Omega)$ are defined to be the completion of $L^2(\Omega)$ in this norm. Clearly

$$\|v\|_{(-m,-l)} \leq \|v\| := \|v\|_{(0,0)} \leq \|v\|_{(m,l)},$$

and so the following inclusions hold

$$H^{(m,l)}(\Omega) \subset L^2(\Omega) \subset H^{(-m,-l)}(\Omega).$$

Moreover we have the generalized Schwarz inequality

$$|(u,v)| \leq \|u\|_{(m,l)}\|v\|_{(-m,-l)}, \quad u \in H^{(m,l)}(\Omega),\ v \in H^{(-m,-l)}(\Omega). \tag{2.2}$$

The negative norm spaces are important because they arise as the dual spaces to the Sobolev spaces.

Let L be a linear partial differential operator, and consider the boundary value problem

$$Lu = f \quad \text{in } \Omega, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

and the associated adjoint problem

$$L^*v = g \quad \text{in } \Omega, \qquad \mathcal{B}^*v = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

where $\mathcal{B}$ is as in (1.4), L^* is the formal adjoint of L, and the adjoint boundary conditions $\mathcal{B}^*v = 0$ are defined as follows. Let $C^\infty_{\mathcal{B}}(\overline{\Omega})$ denote the space of smooth functions (up to the boundary) on Ω satisfying the boundary condition in (2.3). Then a function $v \in C^1(\overline{\Omega})$ is said to satisfy the adjoint boundary conditions if $(Lu, v) = (u, L^*v)$ for all $u \in C^\infty_{\mathcal{B}}(\overline{\Omega})$. The space of smooth functions (up to the boundary) on Ω satisfying the boundary conditions of (2.4) will be denoted by $C^\infty_{\mathcal{B}^*}(\overline{\Omega})$. Our first task is to find an appropriate notion of weak solution for (2.3). We will say that $u \in H^{(m,l)}(\Omega)$ is a weak solution of (2.3), if

$$(u, L^*v) = (f, v) \quad \text{for all } v \in C^\infty_{\mathcal{B}^*}(\overline{\Omega}). \tag{2.5}$$

Clearly a weak solution in $C^2(\overline{\Omega})$ satisfies (2.3) in the classical sense.

Theorem 2.1. *Let $m, l, s, t \in \mathbb{Z}_{\geq 0}$. There exists a weak solution $u \in H^{(m,l)}(\Omega)$ of (2.3) for each $f \in H^{(s,t)}(\Omega)$, if and only if there exists a constant C such that*

$$\|v\|_{(-s,-t)} \leq C\|L^*v\|_{(-m,-l)} \quad \textit{for all } v \in C^\infty_{\mathcal{B}^*}(\overline{\Omega}). \tag{2.6}$$

This theorem generalizes a well-known result in the context of classical Sobolev spaces (see [12]) to the case of the anisotropic Sobolev spaces. The proof requires only slight modification of the original and is thus relegated to the Appendix. Moreover, this theorem shows that the problem of existence for (2.3) is reduced to establishing the inequality (2.6). We now outline the basic procedure for accomplishing this goal. This procedure will be implemented in the next section, for boundary value problem (1.6).

Let $v \in C^\infty_{\mathcal{B}^*}(\overline{\Omega})$, and consider an auxiliary boundary value problem

$$Mu = v \quad \text{in } \Omega, \qquad \widetilde{\mathcal{B}}u = 0 \quad \text{on } \partial\Omega,$$

where the differential operator M and boundary operator $\widetilde{\mathcal{B}}$ are to be determined. The use of auxiliary boundary value problems to study mixed type equations was first put forth by Didenko [13]. Note that upon integrating by parts we have

$$(L^*v, u) - (v, Lu) = \int_{\partial\Omega} I_1(u, v), \qquad (Mu, Lu) = \int_{\Omega} I_2(u, u) + \int_{\partial\Omega} I_3(u, u), \tag{2.7}$$

for some quadratic forms I_1, I_2, and I_3. The goal is then to choose M, $\widetilde{\mathcal{B}}$, and $\mathcal{B}^*$ appropriately so that

$$\int_{\Omega} I_2(u, u) \geq C^{-1}\|u\|^2_{(m,l)}, \tag{2.8}$$

$$\|v\|_{(-s,-t)} \leq C\|u\|_{(m,l)}, \tag{2.9}$$

and

$$\int_{\partial\Omega} (I_1(u, Mu) + I_3(u, u)) \geq 0, \tag{2.10}$$

where an additional integration by parts may be needed to obtain this last inequality. If this is successfully achieved, then by applying the generalized Schwarz inequality, (2.8) and (2.10), we have

$$\begin{aligned}
\|u\|_{(m,l)}\|L^*v\|_{(-m,-l)} &\geq (L^*v, u) \\
&= (v, Lu) + \int_{\partial\Omega} I_1(u, v) \\
&= (Mu, Lu) + \int_{\partial\Omega} I_1(u, v) \\
&= \int_{\Omega} I_2(u, u) + \int_{\partial\Omega} (I_1(u, Mu) + I_3(u, u)) \\
&\geq C^{-1}\|u\|^2_{(m,l)}.
\end{aligned}$$

The desired inequality (2.6) then follows from (2.9). In choosing the boundary conditions $\mathcal{B}^*$, we note that the stronger the condition, the easier it is to establish (2.10), and hence existence. However a strong condition $\mathcal{B}^*$ implies a weak condition

$\mathcal{B}$, which could then make proving uniqueness for (2.3) difficult. Conversely, if the condition $\mathcal{B}^*$ is weak, then the condition $\mathcal{B}$ will be strong, which is an advantageous situation for uniqueness but not existence. This just illustrates the intuitive fact, that a certain balance, between existence and uniqueness, is needed when choosing boundary conditions in order to achieve a well-posed problem.

Lastly we point out how this procedure differs from the standard techniques. The first difference is the use of the anisotropic Sobolev spaces, while the second difference concerns the use of inequality (2.10). Typically boundary conditions are chosen so that each of the boundary integrals involving $I_1(u, v)$ and $I_3(u, u)$, vanish. This is of course much more restrictive than the requirement (2.10). It is primarily this observation (that only (2.10) is needed) which allows us to establish the main theorems.

3. Proof of Theorem 1.2

In this section we will study the following boundary value problem

$$\begin{aligned} &L_\varepsilon u = \varepsilon K u_{xx} + u_{yy} + \varepsilon A u_x + \varepsilon B u_y = f \quad \text{in } \Omega, \\ &u(x,1) = 0, \qquad (\alpha u_x + u_y)(x,-1) = 0, \quad u \text{ is 2-periodic in } x, \end{aligned} \tag{3.1}$$

where

$$\Omega = \{(x,y) \mid |x| < 1, |y| < 1\},$$

and where α is a constant and all coefficients K, A, B, as well as the right-hand side f, are 2-periodic in x. The adjoint boundary value problem is given by

$$\begin{aligned} &L_\varepsilon^* v = \varepsilon K v_{xx} + v_{yy} + \varepsilon(2K_x - A)v_x - \varepsilon B v_y + \varepsilon(K_{xx} - A_x - B_y)v = g \quad \text{in } \Omega, \\ &v(x,1) = 0, \qquad (\alpha v_x - v_y)(x,-1) = 0, \quad v \text{ is 2-periodic in } x. \end{aligned} \tag{3.2}$$

We will first establish existence for (3.1) in the appropriate spaces, under the assumption (1.7). This will be accomplished by following the procedure from Section 2.

To begin, consider the auxiliary problem

$$\begin{aligned} &Mu = \sum_{s=0}^{m} (-1)^s \lambda^{-s} [\partial_x^s (a \partial_x^s u_x) + \partial_x^s (b \partial_x^s u_y) + \partial_x^s (c \partial_x^s u)] = v \quad \text{in } \Omega, \\ &u(x,1) = 0, \quad u \text{ is 2-periodic in } x, \end{aligned} \tag{3.3}$$

where $v \in C^\infty_{\mathcal{B}^*}(\overline{\Omega})$, and a, b, c are functions to be given below (which are 2-periodic in x); in fact b and c will be functions of y alone. We claim that a unique smooth solution always exists. To see this, let

$$w = \sum_{s=0}^{m} (-1)^s \lambda^{-s} \partial_x^{2s} u.$$

Clearly knowledge of w yields knowledge of u. Thus we may create an iteration scheme in the following way, to find u. Let $u_0 = 0$. Given u_i, solve

$$\begin{aligned} &a\partial_x w_{i+1} + b\partial_y w_{i+1} + c w_{i+1} = v - \sum_{s=0}^{m} (-1)^s \lambda^{-s} \sum_{l=1}^{s} \binom{s}{l} \partial_x^l a \partial_x^{2s-l} (u_i)_x, \quad \text{in } \Omega, \\ &w_{i+1}(x,1) = 0, \quad w_{i+1} \text{ is 2-periodic in } x, \end{aligned}$$

for w_{i+1} to obtain u_{i+1}. Note that this equation admits a unique smooth solution as long as $b \neq 0$ in Ω, according to the theory of first order partial differential equations. Moreover estimates are readily available and can be used to show that the sequence $\{u_i\}$, so obtained, converges to the unique smooth solution of (3.3).

Let (n_1, n_2) denote the unit outer normal to $\partial\Omega$. In order to find the quadratic forms I_1, I_2, and I_3 of (2.7), we integrate by parts and calculate

$$\begin{aligned} (au_x + bu_y + cu, L_\varepsilon u) = {} & \varepsilon \int_\Omega \frac{1}{2}[(bK)_y - 2cK - (aK)_x + 2aA]u_x^2 + [bA - (bK)_x - \varepsilon^{-1} a_y - aB]u_x u_y \\ & + \varepsilon \int_\Omega \frac{1}{2}[\varepsilon^{-1}(a_x - b_y - 2c) + 2bB]u_y^2 + \frac{1}{2}[(cK)_{xx} + \varepsilon^{-1} c_{yy} - (cA)_x - (cB)_y]u^2 \\ & + \varepsilon \int_{\partial\Omega} \frac{1}{2}[aKn_1 - bKn_2]u_x^2 + [bKn_1 + \varepsilon^{-1} a n_2]u_x u_y + \varepsilon^{-1} \frac{1}{2}[bn_2 - an_1]u_y^2 \\ & + \varepsilon \int_{\partial\Omega} [cKn_1]uu_x + [\varepsilon^{-1} c n_2]uu_y + \frac{1}{2}[cAn_1 + cBn_2 - (cK)_x n_1 - \varepsilon^{-1} c_y n_2]u^2, \end{aligned} \tag{3.4}$$

$$(Mu, L_\varepsilon u) = \sum_{s=0}^{m} \lambda^{-s}(a(\partial_x^s u)_x + b(\partial_x^s u)_y + c(\partial_x^s u), L_\varepsilon(\partial_x^s u)) + \sum_{s=0}^{m} \varepsilon\lambda^{-s}\left(a(\partial_x^s u)_x + b(\partial_x u)_y + c(\partial_x^s u), \sum_{l=1}^{s}\binom{s}{l}(\partial_x^l K \partial_x^{s-l+2}u + \partial_x^l A\partial_x^{s-l+1}u + \partial_x^l B\partial_x^{s-l}u_y)\right), \tag{3.5}$$

and also

$$(L_\varepsilon^* v, u) - (v, L_\varepsilon u) = \int_{\partial\Omega}[\varepsilon n_1 K v_x u - \varepsilon n_1 K v u_x - n_2 v u_y + n_2 v_y u + \varepsilon(n_1 K_x - n_1 A - n_2 B)uv]. \tag{3.6}$$

Note that no boundary terms appear in (3.5) due to periodicity in the x-direction. According to the choice of the domain Ω, we may disregard any boundary term with a factor of n_1. Moreover, we will choose b so that $b \neq 0$ in Ω, and thus it is clear from (3.2) and (3.3) that $u(x, 1) = u_y(x, 1) = 0$. These two facts help us to simplify the expressions in (3.4)–(3.6). Furthermore by using the boundary condition $\mathcal{B}^* v = 0$, and replacing v with Mu, in (3.6), we find that

$$\begin{aligned}
(L_\varepsilon^* v, u) - (v, L_\varepsilon u) &= \int_{y=-1}(-n_2 v u_y + n_2 v_y u - \varepsilon n_2 B u v) \\
&= \int_{y=-1} v(u_y + \alpha u_x + \varepsilon B u) \\
&= \int_{y=-1}\sum_{s=0}^{m}\lambda^{-s}(a\partial_x^s u_x + b\partial_x^s u_y + c\partial_x^s u)(\alpha\partial_x^s u_x + \partial_x^s u_y + \varepsilon B\partial_x^s u) \\
&\quad + \int_{y=-1}\varepsilon\sum_{s=0}^{m}\lambda^{-s}(a\partial_x^s u_x + b\partial_x^s u_y + c\partial_x^s u)\sum_{l=1}^{s}\binom{s}{l}\partial_x^l B\partial_x^{s-l}u \\
&= \int_{y=-1}\sum_{s=0}^{m}\lambda^{-s}\left[\alpha a(\partial_x^s u_x)^2 + (a+\alpha b)(\partial_x^s u_x)(\partial_x^s u_y) - b(\partial_x^s u_y)^2 - \frac{1}{2}\alpha c_x(\partial_x^s u)^2 + c(\partial_x^s u)(\partial_x^s u_y)\right] \\
&\quad - \int_{y=-1}\varepsilon\sum_{s=0}^{m}\lambda^{-s}\left[\frac{1}{2}(aB)_x(\partial_x^s u)^2 - bB(\partial_x^s u)(\partial_x^s u_y) - cB(\partial_x^s u)^2\right] \\
&\quad + \int_{y=-1}\varepsilon\sum_{s=0}^{m}\lambda^{-s}(a\partial_x^s u_x + b\partial_x^s u_y + c\partial_x^s u)\sum_{l=1}^{s}\binom{s}{l}\partial_x^l B\partial_x^{s-l}u.
\end{aligned} \tag{3.7}$$

Therefore by combining Eqs. (3.4) (with various derivatives of u), (3.5) and (3.7) we obtain

$$\begin{aligned}
(L_\varepsilon^* v, u) &= \sum_{s=0}^{m}\lambda^{-s}\varepsilon\int_\Omega \frac{1}{2}[(bK)_y - 2cK - (aK)_x + 2aA](\partial_x^s u_x)^2 + [bA - (bK)_x - \varepsilon^{-1}a_y - aB] \\
&\quad \times (\partial_x^s u_x)(\partial_x^s u_y) + \sum_{s=0}^{m}\lambda^{-s}\varepsilon\int_\Omega \frac{1}{2}[\varepsilon^{-1}(a_x - b_y - 2c) + 2bB](\partial_x^s u_y)^2 + \frac{1}{2}[(cK)_{xx} \\
&\quad + \varepsilon^{-1}c_{yy} - (cA)_x - (cB)_y](\partial_x^s u)^2 + \sum_{s=0}^{m}\varepsilon\lambda^{-s}\Bigg(a(\partial_x^s u)_x + b(\partial_x^s u)_y + c(\partial_x^s u), \\
&\quad \sum_{l=1}^{s}\binom{s}{l}(\partial_x^l K\partial_x^{s-l+2}u + \partial_x^l A\partial_x^{s-l+1}u + \partial_x^l B\partial_x^{s-l}u_y)\Bigg) \\
&\quad + \sum_{s=0}^{m}\lambda^{-s}\int_{y=-1}\left[\alpha a + \frac{1}{2}\varepsilon bK\right](\partial_x^s u_x)^2 + \frac{1}{2}b(\partial_x^s u_y)^2 + \alpha b(\partial_x^s u_x)(\partial_x^s u_y) \\
&\quad + \sum_{s=0}^{m}\lambda^{-s}\int_{y=-1}\varepsilon bB(\partial_x^s u)(\partial_x^s u_y) + \frac{1}{2}[c_y + \varepsilon(cB - (aB)_x)](\partial_x^s u)^2 \\
&\quad + \sum_{s=0}^{m}\lambda^{-s}\int_{y=-1}\varepsilon\sum_{l=1}^{s}\binom{s}{l}(a\partial_x^s u_x + b\partial_x^s u_y + c\partial_x^s u)\partial_x^l B\partial_x^{s-l}u.
\end{aligned} \tag{3.8}$$

We are now ready to choose the functions a, b, and c. Let ϕ solve the following ODE

$$\alpha\phi_y + \varepsilon\alpha B\phi = \varepsilon(A - K_x) \quad \text{in } \Omega, \qquad \phi(x, -1) = 1.$$

Note that according to the definition of α in Theorem 1.2, $\phi = 1 + O(\varepsilon/|\alpha|)$. Now set

$$a = \alpha\phi, \qquad b = 1, \qquad c = -\varepsilon^{1/2} + \varepsilon^{3/4}(3y + y^2). \tag{3.9}$$

We immediately have

$$(cK)_{xx} + \varepsilon^{-1}c_{yy} - (cA)_x - (cB)_y \geq \varepsilon^{-1/4} + O(\varepsilon^{1/2}),$$
$$\varepsilon^{-1}(a_x - b_y - 2c) + 2bB \geq \varepsilon^{-1/2} + O(\varepsilon^{-1/4}),$$

and by the hypothesis (1.7) we also have

$$(bK)_y - 2cK - (aK)_x + 2aA \geq 0.$$

By construction of ϕ it follows that the coefficient of the mixed derivative term $(\partial_x^s u_x)(\partial_x^s u_y)$, in the second line of (3.8), is zero. There is however another mixed derivative term in the third line of this same equation, however for ε sufficiently small this is dominated by the sum of the two terms involving $(\partial_x^s u_x)^2$ and $(\partial_x^s u_y)^2$. The remaining interior terms may be treated by one more integration by parts, and by taking λ sufficiently large. As for the boundary terms, we have that the quadratic form

$$\left[\alpha a + \frac{1}{2}\varepsilon bK\right](\partial_x^s u_x)^2 + \alpha b(\partial_x^s u_x)(\partial_x^s u_y) + \frac{1}{2}b(\partial_x^s u_y)^2$$

is positive, since

$$\frac{1}{2}\left(\alpha^2 + \frac{1}{2}\varepsilon K(x,-1)\right) - \frac{1}{4}\alpha^2 > 0$$

by the definition of α. Moreover

$$[c_y + \varepsilon(cB - (aB)_x)](x,-1) = \varepsilon^{3/4} + O(\varepsilon),$$

and so the remaining boundary terms may be absorbed into those that are positive by taking ε small, and λ large, after performing the appropriate integration by parts. Therefore there exists a constant $C > 0$ such that

$$(L_\varepsilon^* v, u) \geq C^{-1}\|u\|_{(m,1)}^2.$$

The generalized Schwarz inequality then yields

$$\|L_\varepsilon^* v\|_{(-m,-1)} \geq C^{-1}\|u\|_{(m,1)}.$$

Furthermore, an integration by parts shows that

$$\|v\|_{(-m-1,0)} \leq C\|u\|_{(m,1)},$$

and hence

$$\|v\|_{(-m-1,0)} \leq C^2\|L_\varepsilon^* v\|_{(-m,-1)}.$$

Theorem 2.1 may now be applied to boundary value problem (3.1), to obtain the existence of a weak solution $u \in H^{(m,1)}(\Omega)$ for each $f \in H^{(m+1,0)}(\Omega)$.

We claim that this weak solution is in fact unique. This follows almost immediately from the calculations above. Consider (3.4) with the same choices for a, b, and c as in (3.9). The interior terms, all together, are nonnegative, with the coefficient of u^2 positive, as we have shown. As for the boundary integral, we may apply the boundary conditions $\mathcal{B}u = 0$ (we are assuming here that $f \in C^\infty(\overline{\Omega})$ and hence $u \in C^1(\overline{\Omega})$, according to the additional regularity established below) and integrate by parts to obtain

$$\int_{y=1}\frac{1}{2}bu_y^2 + \int_{y=-1}\frac{1}{2}[\varepsilon bK + 2\alpha a - \alpha^2 b]u_x^2 + \frac{1}{2}[c_y - \alpha c_x - \varepsilon cB]u^2.$$

This is clearly nonnegative. Therefore if $f = 0$, we find that the only possible solution is $u = 0$, and hence uniqueness follows.

In order to obtain higher regularity for the solution given by Theorem 2.1, we will utilize the following standard lemma concerning the difference quotient

$$u^q(x,y) := \frac{u(x, y+q) - u(x,y)}{q}.$$

Lemma 3.1. (i) *Let $u \in H^{(0,1)}(\Omega)$ and $\Omega' \subset\subset \Omega$ (that is, Ω' is compactly contained in Ω). Then*

$$\|u^q\|_{L^2(\Omega')} \leq \|u_y\|_{L^2(\Omega)}$$

for all $0 < |q| < \frac{1}{2}\mathrm{dist}(\Omega', \partial\Omega)$.

(ii) *If* $u \in L^2(\Omega)$ *and* $\|u^q\|_{L^2(\Omega')} \le C$ *for all* $0 < |q| < \frac{1}{2}\text{dist}(\Omega', \partial\Omega)$, *then* $u \in H^{(0,1)}(\Omega')$.

Let $u \in H^{(m,1)}(\Omega)$ be the weak solution given by Theorem 2.1, for $f \in H^{m+1}(\Omega)$. We will show that in fact $u \in H^m(\Omega)$. If $m \le 1$ then this statement follows trivially, so assume that $m \ge 2$. We may integrate by parts to obtain

$$-(u_y + \varepsilon Bu, v_y) = (f - \varepsilon K u_{xx} - \varepsilon A u_x + \varepsilon B_y u, v) + \int_{\partial\Omega} (\varepsilon A v u n_1 - v_y u n_2 - \varepsilon K v_x u n_1 - \varepsilon K_x v u n_1 + \varepsilon K v u_x n_1),$$

for all $v \in C^\infty_{\mathcal{B}^*}(\overline{\Omega})$. Note that since $u \in H^1(\Omega)$ we have that $u|_{\partial\Omega}$ is meaningful in $L^2(\partial\Omega)$, and in particular, as $\mathcal{B}u = 0$, we have that $u(x, 1) = 0$ in the L^2-sense. Moreover $u_x \in H^1(\Omega)$ and so $u_x|_{\partial\Omega} \in L^2(\partial\Omega)$. Thus we may integrate by parts and use that $\mathcal{B}^* v = 0$ in order to show that

$$\int_{\partial\Omega} (\varepsilon A v u n_1 - v_y u n_2 - \varepsilon K v_x u n_1 - \varepsilon K_x v u n_1 + \varepsilon K v u_x n_1) = 0.$$

We may then write

$$(\bar{u}, v_y) = (\bar{f}, v) \quad \text{for all } v \in C^\infty_{\mathcal{B}^*}(\overline{\Omega}),$$

where

$$\bar{u} = -u_y - \varepsilon Bu, \qquad \bar{f} = f - \varepsilon K u_{xx} - \varepsilon A u_x + \varepsilon B_y u.$$

Furthermore

$$(\bar{u}^q, v_y) = (\bar{f}^q, v) \quad \text{for all } v \in C^\infty_c(\Omega),$$

so that choosing a sequence $v_i \in C^\infty_c(\Omega)$ with $v_i \to -\eta u^q$ in $H^{(0,1)}(\Omega)$ for some nonnegative $\eta \in C^\infty_c(\Omega)$, implies that

$$\|\sqrt{\eta}\bar{u}^q\|^2 \le |(\bar{f}^q, \eta u^q)| + |(\bar{u}^q, \eta_y u^q)| + |(\bar{u}^q, \eta(\varepsilon Bu)^q)| \le \|\sqrt{\eta}\bar{f}^q\|\|\sqrt{\eta}u^q\| + \|\sqrt{\eta}\bar{u}^q\|\|\frac{\eta_y}{\sqrt{\eta}}u^q\| + \|\sqrt{\eta}\bar{u}^q\|\|\sqrt{\eta}(\varepsilon Bu)^q\|,$$

where $\|\cdot\|$ denotes the $L^2(\Omega)$ norm. Then since $u, \bar{f} \in H^{(0,1)}(\Omega)$ and $|\nabla\eta|^2 \le C\eta$, Lemma 3.1(i) yields $\|\sqrt{\eta}\bar{u}^q\| \le C$ for some constant C independent of q, if $|q|$ is sufficiently small. Now Lemma 3.1(ii) shows that $\bar{u} \in H^{(0,1)}_{\text{loc}}(\Omega)$, as η was arbitrary. Hence $u_{yy} \in L^2_{\text{loc}}(\Omega)$. It follows that the equation $L_\varepsilon u = f$ holds in $L^2_{\text{loc}}(\Omega)$, and since we can solve for u_{yy}, we may boot-strap in the usual way to obtain $u \in H^m(\Omega)$.

Lastly, to show that the solution u satisfies the estimate (1.8), we recall the proof of uniqueness above. This proof immediately gives

$$(au_x + bu_y + cu, f) \ge C\|u\|^2_{(0,1)}.$$

Upon integrating by parts

$$(au_x, f) = -(u, af_x + a_x f),$$

and thus we have

$$\|f\|_{(1,0)} \ge C\|u\|_{(0,1)}.$$

By differentiating Eq. (3.1) with respect to x, and applying a similar procedure, we find that

$$\|f\|_{(m+1,0)} \ge C\|u\|_{(m,1)}.$$

By solving for u_{yy} in Eq. (3.1), we may then estimate all remaining derivatives to obtain the desired estimate (1.8). This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.1

Theorem 1.1 follows almost immediately from Theorem 1.2 and previous work. More precisely, as is shown in [1], the nonlinear problems (1.1) and (1.2) can be reduced to a study of the linearized equation via an application of the Nash–Moser implicit function theorem. By an appropriate choice of coordinates (see [1]) the following may be arranged. First, the linearized equation will have the form (1.6) where $A = K_x + \psi K$ for some smooth function ψ, and second, the vector field V from (1.3) will be given by $\varepsilon^{7/8}(\partial_y + O(\varepsilon))$, where the parameter ε represents a rescaling of the original coordinates and thus determines the size of the domain of existence for the nonlinear equations. Moreover, since we are only concerned with local solutions for Eqs. (1.1) and (1.2), we may suitably modify the coefficients of the linearized equation away from the origin so that they are 2-periodic in x. Now also, (1.3) implies that (1.7) holds with $\alpha = O(\varepsilon^{1/2})$, for ε sufficiently small. Therefore upon applying Theorem 1.2 we obtain a unique solution satisfying an a priori estimate. Lastly, in order to carry out the Nash–Moser iteration, a more precise a priori estimate, referred to as the Moser-estimate, is needed. The Moser-estimate elucidates the dependence of the solution on the coefficients of the linearization, and is easily derived from the energy method of the previous section (see [1]). This completes the proof of Theorem 1.1.

Acknowledgments

The author is partially supported by NSF Grant DMS-1007156 and a Sloan Research Fellowship.

Appendix

In this section we include a proof of Theorem 2.1 for convenience of the reader. To begin recall that the negative norm spaces arise as the dual spaces of Sobolev spaces.

Lemma A.1. $H^{(-m,-l)}(\Omega) = H^{(m,l)}(\Omega)^*$.

Proof. For each $v \in L^2(\Omega)$ define a bounded linear function $F_v(u) = (u, v)$ on $H^{(m,l)}(\Omega)$. We first show that the set

$$\Lambda_{(m,l)} = \{F_v \in H^{(m,l)}(\Omega)^* \mid v \in L^2(\Omega)\}$$

is dense in $H^{(m,l)}(\Omega)^*$. To see this, observe that if $\Lambda_{(m,l)}$ is not dense, then there exists $F \in H^{(m,l)}(\Omega)^* - \overline{\Lambda_{(m,l)}}$; here $\overline{\Lambda_{(m,l)}}$ denotes the closure of $\Lambda_{(m,l)}$. According to a standard corollary of the Hahn–Banach theorem, there then exists $\mathfrak{L} \in H^{(m,l)}(\Omega)^{**}$ such that $\mathfrak{L}(F) \neq 0$ and $\mathfrak{L} = 0$ on $\overline{\Lambda_{(m,l)}}$. However by reflexivity of Hilbert spaces there exists a nonzero $f \in H^{(m,l)}(\Omega)$ such that $\mathfrak{L}(\widetilde{F}) = \widetilde{F}(f)$ for all $\widetilde{F} \in H^{(m,l)}(\Omega)^*$. Thus $F_v(f) = 0$ for all $F_v \in \Lambda_{(m,l)}$, which implies that $(f, v) = 0$ for all $v \in L^2(\Omega)$, so that $f = 0$, a contradiction. This shows that $\Lambda_{(m,l)}$ is dense.

Now consider the map

$$\mathfrak{I} : H^{(-m,-l)}(\Omega) \to H^{(m,l)}(\Omega)^*$$

defined in the following way. Each $v \in H^{(-m,-l)}(\Omega)$ arises as a limit $v = \lim_{n\to\infty} v_n$, for some $v_n \in L^2(\Omega)$. We may then set $\mathfrak{I}(v) = \lim_{n\to\infty} F_{v_n}$, where convergence is with respect to the operator norm. To see that this is well defined, let $v = \lim_{n\to\infty} v_n = \lim_{n\to\infty} \overline{v}_n$, and observe that since $\|v\|_{(-m,-l)} = \|F_v\|$ we have

$$\|F_{v_n} - F_{\overline{v}_n}\| = \|F_{v_n - \overline{v}_n}\| = \|v_n - \overline{v}_n\| \to 0.$$

To see that this map is one-to-one, suppose that $\mathfrak{I}(v) = \mathfrak{I}(w)$ then

$$0 = \lim_{n\to\infty} \|F_{v_n} - F_{w_n}\| = \lim_{n\to\infty} \|v_n - w_n\|_{(-m,-l)} = \|v - w\|_{(-m,-l)},$$

so that $v = w$. Also by the density property proved above, $\mathfrak{I}$ is onto. Lastly

$$\|\mathfrak{I}(v)\| = \|F_v\| = \|v\|_{(-m,-l)}$$

so that $\mathfrak{I}$ is an isometric isomorphism. □

We may now construct an inner product on $H^{(-m,-l)}(\Omega)$. Let

$$\mathfrak{F} : H^{(m,l)}(\Omega)^* \to H^{(m,l)}(\Omega)$$

be the isometric isomorphism given by the Riesz representation theorem. Then set

$$(u, v)_{(-m,-l)} = (\mathfrak{F} \circ \mathfrak{I}(u), \mathfrak{F} \circ \mathfrak{I}(v))_{(m,l)},$$

where $(\cdot,\cdot)_{(m,l)}$ is the usual inner product on $H^{(m,l)}(\Omega)$. Note that if $v_n \to v$ in $H^{(-m,-l)}(\Omega)$ then $F_{v_n} \to F_v$ with respect to the operator norm, since for any $u \in H^{(m,l)}(\Omega)$,

$$|(u, v - v_n)| \leq \|u\|_{(m,l)} \|v - v_n\|_{(-m,-l)} \to 0.$$

This shows that every bounded linear functional on $H^{(m,l)}(\Omega)$ can be represented by F_v for some $v \in H^{(m,l)}(\Omega)$, and may be used to find that

$$(v, v)_{(-m,-l)} = (\mathfrak{F} \circ \mathfrak{I}(v), \mathfrak{F} \circ \mathfrak{I}(v))_{(m,l)} = \|\mathfrak{I}(v)\| = \|F_v\| = \sup_{u \in H^{(m,l)}(\Omega)} \frac{|(u, v)|}{\|u\|_{(m,l)}}.$$

Therefore the inner product $(\cdot,\cdot)_{(-m,-l)}$ correctly generates the norm $\|\cdot\|_{(-m,-l)}$ given by (2.1). We also note that since Hilbert spaces are reflexive, we could conclude from Lemma A.1 that $H^{(-m,-l)}(\Omega)^* = H^{(m,l)}(\Omega)^{**} = H^{(m,l)}(\Omega)$, however we would like a specific form of this result.

Lemma A.2. *Any $G \in H^{(-m,-l)}(\Omega)^*$ may be represented by a unique $u \in H^{(m,l)}(\Omega)$, such that $G(v) = (u, v)$ for all $v \in H^{(-m,-l)}(\Omega)$. In particular $H^{(-m,-l)}(\Omega)^* = H^{(m,l)}(\Omega)$.*

Proof. Given $u \in H^{(m,l)}(\Omega)$ set $G_u(v) = (u, v)$, $v \in H^{(-m,-l)}(\Omega)$. By the generalized Schwarz inequality (2.2), $\|G_u\| \leq \|u\|_{(m,l)}$ so that $G_u \in H^{(-m,-l)}(\Omega)^*$. Moreover

$$\|G_u\| = \sup_{v\in H^{(-m,-l)}(\Omega)} \frac{|(u,v)|}{\|v\|_{(-m,-l)}} \geq \frac{|(u,v_0)|}{\|v_0\|_{(-m,-l)}} = \frac{|F_{v_0}(u)|}{\|F_{v_0}\|},$$

where v_0 is chosen such that $F_{v_0}(u) = \|u\|_{(m,l)}$ and $\|F_{v_0}\| = 1$. This yields $\|G_u\| \geq \|u\|_{(m,l)}$, so we have $\|G_u\| = \|u\|_{(m,l)}$.

Consider the set

$$\Lambda_{(-m,-l)} = \{G_u \in H^{(-m,-l)}(\Omega)^* \mid u \in H^{(m,l)}(\Omega)\}.$$

Then $\Lambda_{(-m,-l)}$ is dense in $H^{(-m,-l)}(\Omega)^*$. If not, then there exists $G \in H^{(-m,-l)}(\Omega)^* - \overline{\Lambda_{(-m,-l)}}$. By a standard corollary of the Hahn–Banach theorem there exists $\mathfrak{L} \in H^{(-m,-l)}(\Omega)^{**}$ such that $\mathfrak{L}(G) \neq 0$ and $\mathfrak{L} = 0$ on $\overline{\Lambda_{(-m,-l)}}$. By reflexivity there is a nonzero $f \in H^{(-m,-l)}(\Omega)$ with $\mathfrak{L}(\widetilde{G}) = \widetilde{G}(f)$ for all $\widetilde{G} \in H^{(-m,-l)}(\Omega)^{**}$. Thus $G_u(f) = 0$ for all $G_u \in \Lambda_{(-m,-l)}$, which implies that $(u, f) = 0$ for all $u \in H^{(m,l)}(\Omega)$, and hence $f = 0$, a contradiction.

Define a map

$$\mathfrak{G} : H^{(m,l)}(\Omega) \to H^{(-m,-l)}(\Omega)^*$$

by $\mathfrak{G}(u) = G_u$. By the density property proved above, each $G \in H^{(-m,-l)}(\Omega)^*$ may be given by a limit $G = \lim_{n\to\infty} G_{u_n}$, for some $u_n \in H^{(m,l)}(\Omega)$. Because G_{u_n} converges and $\|G_{u_n}\| = \|u_n\|_{(m,l)}$, we have that $u_n \to u$, and thus $G(v) = (u, v)$ for all $v \in H^{(-m,-l)}(\Omega)$. That is, $\mathfrak{G}$ is onto. It is also clear that $\mathfrak{G}$ is one-to-one, and $\|\mathfrak{G}(u)\| = \|G_u\| = \|u\|_{(m,l)}$, so that $\mathfrak{G}$ is an isometric isomorphism. □

We now restate and give a proof of Theorem 2.1.

Theorem A.3. *Let $m, l, s, t \in \mathbb{Z}_{\geq 0}$. There exists a weak solution $u \in H^{(m,l)}(\Omega)$ of (2.3) for each $f \in H^{(s,t)}(\Omega)$, if and only if there exists a constant C such that*

$$\|v\|_{(-s,-t)} \leq C\|L^*v\|_{(-m,-l)} \quad \textit{for all } v \in C^\infty_{\mathcal{B}^*}(\overline{\Omega}). \tag{A.1}$$

Proof. Suppose that the inequality (A.1) holds, and consider the linear functional

$$F : L^*C^\infty_{\mathcal{B}^*}(\overline{\Omega}) =: X \to \mathbb{R}$$

given by

$$F(L^*v) = (f, v),$$

for some fixed $f \in H^{(s,t)}(\Omega)$. Note that by the generalized Schwarz inequality and (A.1),

$$|F(L^*v)| \leq \|f\|_{(s,t)}\|v\|_{(-s,-t)} \leq C\|f\|_{(s,t)}\|L^*v\|_{(-m,-l)},$$

and therefore F is a bounded linear functional on the subspace $X \subset H^{(-m,-l)}(\Omega)$. The Hahn–Banach theorem then yields an extension $\widetilde{F}$ of F to a bounded linear functional on all of $H^{(-m,-l)}(\Omega)$. According to Lemma A.2, there then exists $u \in H^{(m,l)}(\Omega)$ such that

$$\widetilde{F}(w) = (u, w) \quad \text{for all } w \in H^{(-m,-l)}(\Omega).$$

Upon restricting w back to X, we obtain

$$(u, L^*v) = \widetilde{F}(L^*v) = F(L^*v) = (f, v) \quad \text{for all } v \in C^\infty_{\mathcal{B}^*}(\overline{\Omega}).$$

Conversely, assume that for any $f \in H^{(s,t)}(\Omega)$ there exists a weak solution $u_f \in H^{(m,l)}(\Omega)$, then

$$|(f, v)| \leq |(u_f, L^*v)| \leq \|u_f\|_{(m,l)}\|L^*v\|_{(-m,-l)} = C_f\|L^*v\|_{(-m,-l)}.$$

Consider the linear functional $G_f(v) = (f, v)$ on $H^{(-s,-t)}(\Omega)$. By the Riesz representation theorem we may write $G_f(v) = (\mathfrak{F}^{-1} \circ \mathfrak{I}^{-1}(f), v)_{(-s,-t)}$ for some $\mathfrak{F}^{-1} \circ \mathfrak{I}^{-1}(f) \in H^{(-s,-t)}(\Omega)$. From the proof of Lemmas A.1 and A.2, we know that $\mathfrak{F}^{-1} \circ \mathfrak{I}^{-1} : H^{(s,t)}(\Omega) \to H^{(-s,-t)}(\Omega)$ is an isometry. Thus

$$|(\mathfrak{F}^{-1} \circ \mathfrak{I}^{-1}(f), v\|L^*v\|^{-1}_{(-m,-l)})_{(-s,-t)}| = |(f, v\|L^*v\|^{-1}_{(-m,-l)})| \leq C_f.$$

We now have a family of bounded linear functionals $J_v \in H^{(-s,-t)}(\Omega)^*$ given by

$$J_v(u) = (u, v\|L^*v\|^{-1}_{(-m,-l)})_{(-s,-t)},$$

where $u = \mathfrak{F}^{-1} \circ \mathfrak{I}^{-1}(f)$ for some $f \in H^{(s,t)}(\Omega)$. This family is pointwise bounded for all $v \in C^\infty_{\mathcal{B}^*}(\overline{\Omega})$, and therefore the Banach–Steinhaus theorem asserts that this family is uniformly bounded, that is, $\|J_v\| \leq C$ for all $v \in C^\infty_{\mathcal{B}^*}(\overline{\Omega})$. However

$$|J_v(u)| \leq \|\mathfrak{F}^{-1} \circ \mathfrak{I}^{-1}(f)\|_{(-s,-t)}\|v\|L^*v\|^{-1}_{(-m,-l)}\|_{(-s,-t)},$$

so that

$$\|J_v\| \leq \|v\|L^*v\|^{-1}_{(-m,-l)}\|_{(-s,-t)}.$$

Also by choosing

$$\mathfrak{F}^{-1} \circ \mathfrak{I}^{-1}(f) = v\|L^*v\|^{-1}_{(-m,-l)}$$

we obtain

$$|J_v(v\|L^*v\|^{-1}_{(-m,-l)})| = \|v\|L^*v\|^{-1}_{(-m,-l)}\|^2_{(-s,-t)},$$

so that

$$\|J_v\| \geq \|v\|L^*v\|^{-1}_{(-m,-l)}\|_{(-s,-t)}.$$

Hence

$$\|J_v\| = \|v\|L^*v\|^{-1}_{(-m,-l)}\|_{(-s,-t)}.$$

Therefore the uniform bound yields

$$\|v\|_{(-s,-t)} \leq C\|L^*v\|_{(-m,-l)}. \quad \square$$

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