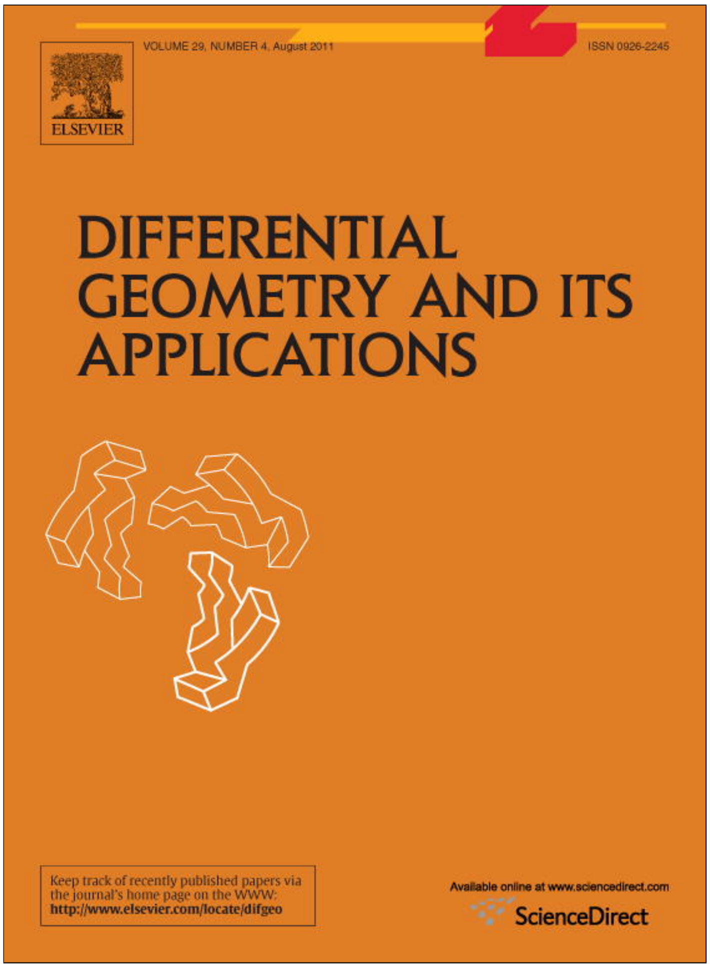
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Rigidity in the class of orientable compact surfaces of minimal total absolute curvature ☆

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ABSTRACT

Consider an orientable compact surface in three-dimensional Euclidean space with minimum total absolute curvature. If the Gaussian curvature changes sign to finite order and satisfies a nondegeneracy condition along closed asymptotic curves, we show that any other isometric surface differs by at most a Euclidean motion.

1. Introduction

A surface in $\mathbb{R}^3$ is said to be rigid, if any other isometric surface differs from it by at most a Euclidean motion. The well-known Cohn-Vossen Theorem asserts that any compact (closed without boundary) convex surface is rigid [2]. However, there are standard examples [8] to show that without the convexity assumption this theorem is false. It is then a natural question to ask, under what conditions is a compact nonconvex surface rigid? Intuition suggests that a partial answer to this question should be: a compact surface is rigid when it is as close to being convex as is possible. It will be our aim to verify this assertion under a nondegeneracy condition on the Gaussian curvature and the closed asymptotic curves.

In 1938 Alexandrov [1] introduced the class T, of compact surfaces (immersed in $\mathbb{R}^3$) characterized by the condition

$$\int_{S^+} K\,dA = 4\pi, \tag{1.1}$$

where K is the Gaussian curvature of $S \in T$, $S^+ \subset S$ is the set on which $K > 0$, and dA is the element of area. It is well-known that for an arbitrary compact surface in $\mathbb{R}^3$

$$\int_S |K|\,dA \geqslant 2\pi\big(4 - \chi(S)\big),$$

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where $\chi(S)$ denotes the Euler characteristic. The quantity on the left-hand side is referred to as the total absolute curvature. By the Gauss–Bonnet Theorem, if (1.1) occurs then the minimum of the total absolute curvature is attained. It is for this reason that elements of the class T are said to have minimal total absolute curvature, and are sometimes referred to as tight surfaces. The term tightness alludes to the following geometric consequence of (1.1).

Convexity Property. *(See [6].) If $S \in T$, then the tangent plane at any point of S^+ is a plane of support, that is, S lies entirely on one side of the tangent plane.*

Clearly, this convexity property describes the class T as those compact surfaces which are as close to being convex as is possible for surfaces of sign changing curvature. A standard example of such a surface is the torus of revolution.

The first rigidity result for tight surfaces was obtained by Alexandrov [1], who showed that if $S \in T$ is analytic, then it is rigid. In an attempt to remove this assumption of analyticity, Nirenberg [6] obtained the following partial result. Let $S^- \subset S \in T$ denote the set on which $K < 0$. If $\nabla K|_{\partial S^-} \neq 0$, and each component of S^- contains at most one closed asymptotic curve, then S is rigid. Here we shall improve Nirenberg's result by weakening the condition on the Gaussian curvature, and by allowing multiple closed asymptotic curves with a nondegeneracy assumption.

Theorem 1. *Let $S \in T$ be orientable of class C^{a+2}. If $|\nabla^b K|_{\partial S^-} \neq 0$ for some odd integer $b < a$, and $\int_\Gamma k_g k_n |K|^{-1/2}\,ds \neq 0$ for any closed asymptotic curve Γ where k_g is the geodesic curvature, k_n is the normal curvature in the direction perpendicular to Γ, and ds is the element of arclength, then S is rigid.*

At the expense of adding a small extrinsic condition on ∂S^-, we can allow the Gaussian curvature to vanish to infinite order there.

Theorem 2. *Let $S \in T$ be orientable of class C^4. If the mean curvature H satisfies $H|_{\partial S^-} \neq 0$ (that is, ∂S^- is not umbilic), K changes sign monotonically across ∂S^-, and $\int_\Gamma k_g k_n |K|^{-1/2}\,ds \neq 0$ for any closed asymptotic curve Γ where k_g is the geodesic curvature, k_n is the normal curvature in the direction perpendicular to Γ, and ds is the element of arclength, then S is rigid.*

Remark. At this time it is unknown whether closed asymptotic curves in tight surfaces must always satisfy the nondegeneracy condition of Theorems 1 and 2. If true, then Theorem 1 would yield a purely intrinsic condition for rigidity.

In order to show that a surface S is rigid, it is necessary and sufficient by the fundamental theorem of surface theory to show that the solution of the Gauss–Codazzi equations is unique (modulo trivial solutions). We will use the following strategy to verify Theorems 1 and 2. First, a well-known argument will show that the complement $S' = S - S^-$ is rigid. Then writing the Gauss–Codazzi equations as a quasi-linear weakly hyperbolic 2×2 system inside S^- near ∂S^-, we will use the nondegeneracy conditions placed on the Gaussian curvature to obtain a degenerate estimate which is sufficient to show uniqueness near ∂S^- for the Cauchy problem with data given on ∂S^-. An argument of Nirenberg [6] may then be applied to extend this local uniqueness result inside S^- up to a closed asymptotic curve. Local uniqueness for the Cauchy problem with data given on a closed asymptotic curve will then be obtained by showing that the Gauss–Codazzi equations form a symmetric positive system [3], from which we can find suitable estimates. Then repeating this procedure throughout S^- will yield its rigidity.

2. The geometry of S' and S^-

We begin by recalling a result concerning the geometry of S' which will lead to its rigidity. Let $\mathcal{M}$ denote a 2-dimensional compact Riemannian manifold satisfying condition (1.1), and possessing a C^2 isometric immersion $X : \mathcal{M} \hookrightarrow \mathbb{R}^3$. In [4,5], Kuiper showed that $\mathcal{M}$ may be decomposed into two disjoint open sets U and V with $\mathcal{M} = \overline{U} \cup \overline{V}$ (where $\overline{U}$, $\overline{V}$ denote the closures of U, V) such that the restriction of X to the set U is an embedding and is comprised of the boundary of the convex hull of $X(U)$ minus a finite (possibly zero) number of planar convex disks $D_1, \dots, D_k$. Furthermore $K(p) > 0$ for $p \in U$ and $K(p) < 0$ for $p \in V$, and the boundary of each disk D_i, $1 \leqslant i \leqslant k$, is the image of a nontrivial 1-cycle in $\mathcal{M}$. Now let $S_1, S_2 \in T$ be two isometric immersions of $\mathcal{M} = \overline{U} \cup \overline{V}$ which satisfy the conditions of Theorem 1 or 2, then we see that $S_i' = X_i(\overline{U})$ and $S_i^- = X_i(V)$, $i = 1, 2$, where $X_i : \mathcal{M} \hookrightarrow \mathbb{R}^3$ are the given immersions. Let γ_i denote a boundary curve of S_i', then since γ_i lies in a plane with the normal to the surface S_i normal to the plane, the geodesic curvature of γ_i is equal to the curvature of γ_i in the plane, which is nonnegative after appropriate orientation. It follows that γ_i is uniquely determined by the metric of S_i, up to a rigid body motion. Therefore, by filling in the empty disks in $X_1(\overline{U})$ and $X_2(\overline{U})$ we obtain two isometric convex surfaces Σ_1 and Σ_2. Since Σ_1 and Σ_2 may not be C^2 smooth, we cannot use the Cohn-Vossen Theorem to conclude that Σ_1 is congruent to Σ_2. However, we may apply Pogorelov's rigidity theorem [7] for nonsmooth convex surfaces to obtain the same conclusion. We now have

Lemma 2.1. *If $S_1, S_2 \in T$ satisfy the conditions of Theorem 1 or 2, then S_1' is congruent to S_2'.*

We will now investigate the geometry of S^- under the hypotheses of Theorems 1 and 2. Our goal will be to show that any component of S^- must be topologically equivalent to a cylinder. Let γ denote one of the planar convex curves which comprise the boundary of S^-. Fix a point $p \in \gamma$ and introduce local coordinates (u, v) near p, such that γ is the u-axis and the second fundamental form of S is given by

$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2.$$

We would like to eliminate the coefficient M of the second fundamental form. In the case of Theorem 2, p is not umbilic, and therefore this could be done by introducing the lines of curvature as local coordinates. However under the hypotheses of Theorem 1, p may be umbilic, so extra arguments are needed.

Lemma 2.2. *Let S be as in Theorem 1 or 2, then there exist C^1 local coordinates (x, t) near p, such that γ is the x-axis and $M \equiv 0$ in this new coordinate system.*

Proof. We assume that the hypotheses of Theorem 1 hold. Since γ is a plane curve it has zero normal curvature and therefore $L(u, 0) = 0$, this in turn implies (through the Gauss equation (3.1)) that $M(u, 0) = 0$. Then by successively differentiating the Codazzi equations (3.1) we find that

$$\partial_v^l N(u,0) = 0, \quad l \leqslant k, \quad \text{implies} \quad \partial_v^{l+1} L(u,0) = \partial_v^{l+1} M(u,0) = 0, \quad l \leqslant k. \tag{2.1}$$

In order to obtain the desired change of coordinates set

$$x = u, \qquad t = t(u, v),$$

where t solves

$$t_u - \frac{M}{N} t_v = 0, \qquad t(0, v) = v. \tag{2.2}$$

By what we have just shown and by the nondegeneracy assumption on the Gaussian curvature, the function $\frac{M}{N}$ is C^1 across the u-axis and satisfies $\frac{M}{N}(u, 0) = 0$. Therefore the v-axis is noncharacteristic for (2.2), and by the theory of first order partial differential equations it possesses a unique C^1 local solution. Furthermore since $t_u(u, 0) = 0$, the x-axis corresponds to the curve γ. $\square$

Using Lemma 2.2 we will be able to apply an argument of Nirenberg [6], to conclude that each component of S^- must be a cylinder under the hypotheses of Theorem 1 or 2. Let (x, t) be the coordinates of Lemma 2.2 around $p \in \gamma$, and let $t > 0$ denote the region lying inside S^-. Then near γ the asymptotic curves of S^-, the curves which have zero normal curvature at every point, are given by

$$\frac{dt}{dx} = \pm\sqrt{-\frac{L}{N}}.$$

According to (2.1) $\frac{L}{N}(x, 0) = 0$ (under the hypotheses of Theorem 2 this also holds, since in this case $N(x, 0) \neq 0$), and so it follows that the asymptotic curves are tangent to γ. We will now show

Lemma 2.3. *Let S be as in Theorem 1 or 2, then each component of S^- is topologically a cylinder.*

Proof. Each component of S^- is bounded by a finite number of planar convex curves. Choose a component and let $\gamma_1, \ldots, \gamma_k$ denote its boundary curves. S^- has two families of intersecting asymptotic curves which are distinguishable by how the surface rises above and below the tangent plane at each intersection. Take now one of these families on the component in question. It defines a line field without singularity on the components closure. By filling in the convex boundary curves $\gamma_1, \ldots, \gamma_k$ with disks $D_1, \ldots, D_k$, we obtain a closed surface on which this line field may be extended to have a single singularity in each disk D_i. This extension is possible since we have shown that the asymptotic curves are tangent to each boundary curve γ_i. The singularity in each disk clearly has index one. Therefore, since S is orientable we may apply Poincaré's Theorem on the indices of a line field to conclude that the sum of the indices is $2 - 2g$, that is $k = 2 - 2g$ where g denotes the genus of the closed surface obtained from our component by filling in the boundary curves with disks $D_1, \ldots, D_k$. Since $k > 0$ it follows that $g = 0$ and $k = 2$. $\square$

In order to better analyze the consequences of Lemma 2.3, we point out one consequence of the nondegeneracy condition concerning the closed asymptotic curves in Theorems 1 and 2.

Lemma 2.4. *Suppose that $\int_\Gamma k_g k_n |K|^{-1/2}\, ds \neq 0$ for a closed asymptotic curve Γ, where k_g is the geodesic curvature, k_n is the normal curvature in the direction perpendicular to Γ, and ds is the element of arclength. Then Γ cannot be a limit point of closed asymptotic curves.*

Proof. Let $(x,t) \subset [0,x_0) \times [0,t_0)$ be an arbitrary semi-global coordinate system near Γ, such that the curve $t=0$ corresponds to Γ and the curves $t = \text{const.}$ are all homotopic to Γ. We will show that

$$\int_0^{x_0} M^{-1}\partial_t L(x,0)\,dx = \pm\int_\Gamma k_g k_n |K|^{-1/2}\,ds \neq 0. \tag{2.3}$$

Using the Gauss–Codazzi equations ((3.1) below) and the formulae

$$k_g = \Gamma_{11}^2 E^{-3/2}\sqrt{\det I}, \qquad b := \Gamma_{11}^1 - \Gamma_{12}^2 = 2\Gamma_{11}^1 - \frac{1}{2}\partial_x \log\det I,$$

where I denotes the first fundamental form, at $t=0$ we calculate

$$\begin{aligned}
M^{-1}L_t &= M^{-1}\big(M_x + bM + \Gamma_{11}^2 N\big)\\
&= b + k_g M^{-1} N E^{3/2}(\det I)^{-1/2} + \partial_x \log|M|\\
&= 2\Gamma_{11}^1 \pm \frac{k_g N E^{3/2}}{\sqrt{-K}\det I} + \partial_x\log|M| - \frac{1}{2}\partial_x\log\det I\\
&= 2\left(\Gamma_{11}^1 \pm \frac{k_g F M\sqrt{E}}{\sqrt{-K}\det I}\right) \pm \frac{k_g}{\sqrt{-K}}\left(\frac{EN-2FM}{\det I}\right)\sqrt{E} + \partial_x\left(\log|M| - \frac{1}{2}\log\det I\right).
\end{aligned}$$

To discover the meaning of this expression note that

$$\Gamma_{11}^1 \pm \frac{k_g F M\sqrt{E}}{\sqrt{-K}\det I} = E^{-1}\big(E\Gamma_{11}^1 + F\Gamma_{11}^2\big) = \frac{1}{2}\partial_x \log E.$$

Furthermore the unit vector

$$Z = -\frac{F}{\sqrt{E\det I}}\partial_x + \sqrt{\frac{E}{\det I}}\partial_t$$

is normal to Γ and satisfies

$$II(Z,Z) = \frac{EN-2FM}{\det I}.$$

Thus

$$M^{-1}L_t = \pm\frac{k_g k_n}{\sqrt{-K}}\sqrt{E} + \partial_x \log\frac{E|M|}{\sqrt{\det I}},$$

so that integrating over Γ yields (2.3).

As a direct consequence of (2.3) we may confirm that Γ cannot be a limit point of closed asymptotic curves. Proceeding by contradiction, assume that this is the case. Then we may introduce coordinates (x,t), with the property that a sequence of closed asymptotic curves $\{\Gamma_i\}_{i=1}^\infty$ converging to Γ is given by $t = c_i$, for some constants $c_i \to 0$ as $i \to \infty$. It follows that $L_t(x,0) = 0$, in contradiction to (2.3). □

We now analyze the consequences of Lemma 2.3 with regards to the asymptotic curves of S^-, following Nirenberg. Let γ_1 and γ_2 be the boundary curves for a cylindrical component C of S^-. It is possible to orient each family of asymptotic curves on C with a suitable parameterization, so that the corresponding vector field of tangent vectors for each family has no singularity. Therefore we see that any asymptotic curve emanating from γ_1, say, cannot return to γ_1, since otherwise a singularity in the tangent vector field for this family must arise. Furthermore since C is topologically a planar annulus, we may apply the Poincaré–Bendixson Theorem to conclude that all the asymptotic curves of one family emanating from γ_1 either end on γ_2 or spiral towards a closed asymptotic curve $\widetilde{\gamma}_1$, wrapping themselves around C infinitely often. It is clear that $\widetilde{\gamma}_1$ must be homotopic to γ_1 and γ_2 since it can have no self-intersection and cannot be homotopically trivial without creating a singularity in the tangent vector field, and thus it divides C into two disjoint parts C_1 containing γ_1 and C_2 containing γ_2. Inside C_2 and near $\widetilde{\gamma}_1$, the curves of this family are all spiraling towards $\widetilde{\gamma}_1$. This follows from Lemma 2.4, since $\widetilde{\gamma}_1$ cannot be a limit point of closed asymptotic curves. By applying the Poincaré–Bendixson Theorem again, we find that these spiraling asymptotic curves will either end on γ_2 or will spiral towards another closed asymptotic curve $\widetilde{\gamma}_2$ homotopic to $\widetilde{\gamma}_1$, again dividing C into two disjoint parts $C_1' \cup C_2'$, where C_2' contains γ_2 and $C_1' = C_1 \cup (C_2 - C_2')$. Inside C_2' the same arguments may be applied. This procedure may be repeated up to γ_2. Since the closure of C is compact, Lemma 2.4 implies that there can only be a finite number of closed asymptotic curves, and so this procedure will terminate with a finite number of iterations. We have shown

Lemma 2.5. *Corresponding to each family of asymptotic curves in C, is a decomposition of C into a sequence $C_1, \ldots, C_m$ of cylindrical domains with $C = \bigcup_{i=1}^{m} C_i$, such that ∂C_i, $i \geqslant 3$, consists of two closed asymptotic curves of the same family homotopic to γ_1 and γ_2, and ∂C_1, ∂C_2 consist of one closed asymptotic curve each in addition to γ_1 and γ_2 respectively. Furthermore inside each C_i, $i \geqslant 3$, the asymptotic curves of this family are all spiraling towards ∂C_i. Inside C_1, C_2 the asymptotic curves of this family are tangent to γ_1, γ_2 and spiral towards the closed asymptotic curve which forms the other boundary component.*

3. The Gauss–Codazzi system

Suppose that we have two isometric surfaces $S, \bar{S} \in T$ satisfying the assumptions of Theorem 1 or 2, and let $\mathcal{M}$ denote the underlying Riemannian manifold. Then by Lemma 2.1, we know that S' is congruent to $\bar{S}'$. In this section we will prove two lemmas concerning uniqueness of the Gauss–Codazzi system which will be instrumental in showing that S^- is congruent to $\bar{S}^-$, that is we wish to show this for any component C of S^- and the corresponding component $\bar{C}$ for $\bar{S}^-$. We will denote the corresponding cylinder in $\mathcal{M}^-$ by $\mathcal{C}$. Since the boundaries of these cylinders lie in S' and $\bar{S}'$, we may assume that they coincide, after a Euclidean motion is applied. In order to show that C is congruent to $\bar{C}$, we must show that their second fundamental forms agree on $\mathcal{C}$. Noting that they are identical on the boundary, we will first prove a local (near $\partial\mathcal{C}$) uniqueness result for the Cauchy problem of the weakly hyperbolic system of Gauss–Codazzi equations. This local uniqueness will then be extended throughout $\mathcal{C}$ in the next section.

We will now put the Gauss–Codazzi system in a suitable form for obtaining estimates. Let I and K denote the metric and Gaussian curvature of $\mathcal{C}$, and let

$$II = L\,dx^2 + 2M\,dx\,dt + N\,dt^2, \qquad \bar{II} = \bar{L}\,dx^2 + 2\bar{M}\,dx\,dt + \bar{N}\,dt^2,$$

denote the second fundamental forms of C and $\bar{C}$ in some local coordinate system. Then both triples (L, M, N) and $(\bar{L}, \bar{M}, \bar{N})$ satisfy the Gauss–Codazzi equations:

$$\begin{aligned} &LN - M^2 = K \det I, \\ &L_t - M_x + aL + bM + cN = 0, \\ &M_t - N_x + \alpha L + \beta M + \gamma N = 0, \end{aligned} \tag{3.1}$$

where I denotes the first fundamental form and a, b, c, α, β, γ are given in terms of Christoffel symbols by

$$\begin{aligned} &a = -\Gamma_{12}^1, \qquad b = \Gamma_{11}^1 - \Gamma_{12}^2, \qquad c = \Gamma_{11}^2, \\ &\alpha = -\Gamma_{22}^1, \qquad \beta = \Gamma_{12}^1 - \Gamma_{22}^2, \qquad \gamma = \Gamma_{12}^2. \end{aligned}$$

Set $u = \bar{L} - L$, $v = \bar{M} - M$, and $w = \bar{N} - N$, then the triple (u, v, w) satisfies the last two (Codazzi) equations of (3.1), and in the first we may solve for u by

$$u = \frac{-Lw + 2Mv + v^2}{N + w}, \tag{3.2}$$

assuming for now that the expression on the right-hand side of (3.2) is smooth. Plugging this into the Codazzi equations, we have

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x + \begin{pmatrix} 1 & 0 \\ \frac{2M}{N+w} & -\frac{L}{N+w} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} B_{11} &= \beta + \frac{\alpha v}{N + w} + \frac{2\alpha M}{N + w}, \\ B_{12} &= \gamma - \frac{\alpha L}{N + w}, \\ B_{21} &= b + \frac{2v_t}{N + w} - \frac{(N + w)_t}{(N + w)^2} v + 2\left(\frac{M}{N + w}\right)_t + \frac{2aM}{N + w} + \frac{av}{N + w}, \\ B_{22} &= c - \left(\frac{L}{N + w}\right)_t - \frac{aL}{N + w}. \end{aligned}$$

In order to symmetrize the principal portion of the system, we multiply through by

$$\begin{pmatrix} N + w & 0 \\ -2M & N + w \end{pmatrix}$$

to obtain

$$\widetilde{A}^1 U_x + \widetilde{A}^2 U_t + \widetilde{B} U = 0,$$

where $U = (v, w)^*$ (the upper $*$ will denote the transpose operation) and

$$\widetilde{A}^1 = \begin{pmatrix} 0 & -(N+w) \\ -(N+w) & 2M \end{pmatrix}, \qquad \widetilde{A}^2 = \begin{pmatrix} N+w & 0 \\ 0 & -L \end{pmatrix},$$
$$\widetilde{B} = \begin{pmatrix} (N+w)B_{11} & (N+w)B_{12} \\ -2MB_{11} + (N+w)B_{21} & -2MB_{12} + (N+w)B_{22} \end{pmatrix}.$$

Lastly upon removing w from the principal part the system becomes

$$A^1 U_x + A^2 U_t + BU = 0, \tag{3.3}$$

where

$$A^1 = \begin{pmatrix} 0 & -N \\ -N & 2M \end{pmatrix}, \qquad A^2 = \begin{pmatrix} N & 0 \\ 0 & -L \end{pmatrix}, \qquad B = \widetilde{B} + \begin{pmatrix} 0 & v_t - w_x \\ 0 & -v_x \end{pmatrix}.$$

We will now obtain a local uniqueness result for system (3.3) near each of the boundary curves Γ_1 and Γ_2.

Lemma 3.1. *There exist neighborhoods of Γ_1 and Γ_2 inside $\mathcal{C}$ on which $II = \overline{II}$.*

Proof. Let $p \in \Gamma_1$, and introduce the local coordinate system (x, t) of Lemma 2.2, where $p = (0, 0)$ and $t > 0$ represents the region inside $\mathcal{C}$. Then $M \equiv 0$. Let us first assume that $\partial_t^k K(x, 0) \neq 0$, $k = \text{odd}$, as in the hypothesis of Theorem 1, so that $N = O(t^r)$ for some $0 \leqslant r < k$. By the arguments of Lemma 2.2, $L = O(t^{r+1})$. Furthermore since $(\overline{L}, \overline{M}, \overline{N})$ agrees (to all orders) with (L, M, N) on Γ_1, we also have $\overline{N} = O(t^r)$, $\overline{M} = O(t^{r+1})$, and $\overline{L} = O(t^{r+1})$. It follows that the expression on the right-hand side of (3.2) is regular, so that system (3.3) is valid near p. We may assume further that $N > 0$ and $L < 0$ inside $\mathcal{C}$, since $LN = K \det I$, that is

$$N = n(x)t^r + O\left(t^{r+1}\right), \qquad L = -l(x)t^{k-r} + O\left(t^{k-r+1}\right), \tag{3.4}$$

for some positive functions n and l.

Let $D \subset \mathcal{C}$ be a small characteristic triangle, bounded by the x-axis and two intersecting asymptotic curves given by the equations

$$\frac{dt}{dx} = \pm\sqrt{-\frac{L}{N}}. \tag{3.5}$$

Note that the asymptotic curves are characteristics for the system (3.3). Next set $\overline{U} = e^{-\lambda} U$ where $\lambda(x, t)$ is to be determined, so that (3.3) becomes

$$A^1 \overline{U}_x + A^2 \overline{U}_t + \left(B + \lambda_x A^1 + \lambda_t A^2\right)\overline{U} = 0. \tag{3.6}$$

Now multiply (3.6) by $\overline{U}^*$ and integrate by parts to obtain

$$\iint\limits_D \overline{U}^* \left(\frac{\mathcal{B} + \mathcal{B}^*}{2}\right)\overline{U} + \int\limits_{\partial D} \frac{1}{2}\overline{U}^*\left(A^1 \nu_1 + A^2 \nu_2\right)\overline{U} = 0, \tag{3.7}$$

where (ν_1, ν_2) denotes the unit outward normal to ∂D, and

$$\mathcal{B} = B + \lambda_x A^1 + \lambda_t A^2 - \frac{1}{2}A_x^1 - \frac{1}{2}A_t^2.$$

In order to show that $\frac{\mathcal{B}+\mathcal{B}^*}{2} := (\mathcal{B}_{ij})$ is positive definite, we use $v, w = O(t^{k+1})$ to calculate

$$\mathcal{B}_{11} = N\beta + \lambda_t N - \frac{1}{2}N_t + O\left(t^k\right),$$
$$\mathcal{B}_{22} = Nc - \frac{1}{2}L_t + \frac{N_t}{N}L - aL - \lambda_t L + O\left(t^k\right),$$
$$\mathcal{B}_{12} = \mathcal{B}_{21} = \frac{N}{2}(\gamma + b) - \frac{\alpha}{2}L - \lambda_x N + \frac{1}{2}N_x + O\left(t^k\right).$$

Moreover solving for γ and c in (3.1) yields

$$\gamma = \frac{N_x - \alpha L}{N}, \qquad c = -\frac{L_t + aL}{N},$$

so that

$$\mathcal{B}_{12} = N_x - \alpha L + \frac{b}{2}N - \lambda_x N + O\left(t^k\right).$$

If we set $\lambda = \log N + \frac{1}{2}\int_0^x b(\bar{x},t)\,d\bar{x} + \bar{\lambda}t$ for some positive constant $\bar{\lambda}$, then by (3.4)

$$\mathcal{B}_{11} = N\left(\bar{\lambda} + \beta + \int\limits_0^x b_t(\bar{x},t)\,d\bar{x}\right) + \frac{1}{2}N_t + O\left(t^k\right) = q_1 t^{r-1} + \bar{\lambda}q_2 t^r + O\left(t^{r+1}\right),$$

$$\mathcal{B}_{22} = -\frac{3}{2}L_t - \left(\bar{\lambda} + 2a + \frac{1}{2}\int\limits_0^x b_t(\bar{x},t)\,d\bar{x}\right)L + O\left(t^k\right) = q_3 t^{k-r-1} + \bar{\lambda}lt^{k-r} + O\left(t^{k-r+1}\right),$$

$$\mathcal{B}_{12} = O\left(t^{k-r}\right), \tag{3.8}$$

where q_1, q_2, and q_3 are strictly positive functions on D, and $q_1 \equiv 0$ if $r = 0$ (in which case $\bar{\lambda}$ will be used to ensure that $\mathcal{B}_{11} > 0$). It follows that $\frac{\mathcal{B}+\mathcal{B}^*}{2}$ is positive definite since $r < k/2$. More precisely we have

$$\iint\limits_D q t^{k-r-1}|\overline{U}|^2 \leqslant \iint\limits_D \overline{U}^*\left(\frac{\mathcal{B}+\mathcal{B}^*}{2}\right)\overline{U}, \tag{3.9}$$

for some positive constant q.

We now show that the boundary integral in (3.7) is nonnegative. Along the x-axis $\overline{U}|_{t=0} = 0$, since $U = O(t^{k+1})$ and $\overline{U} = e^{-\lambda}U = O(N^{-1}t^{k+1}) = O(t)$. Furthermore according to (3.5) we find that

$$\nu_1 = \pm\frac{\sqrt{-\frac{L}{N}}}{\sqrt{1-\frac{L}{N}}}, \qquad \nu_2 = \frac{1}{\sqrt{1-\frac{L}{N}}}.$$

Therefore

$$\int\limits_{\partial D-\{t=0\}} \frac{1}{2}\overline{U}^*\left(A^1\nu_1 + A^2\nu_2\right)\overline{U} = \int\limits_{\partial D-\{t=0\}} \frac{1}{2}\left(\bar{v} \pm \sqrt{-\frac{L}{N}}\bar{w}\right)^2 N\nu_2 \geqslant 0. \tag{3.10}$$

By combining (3.7), (3.9), and (3.10) we conclude that $U = 0$ for D sufficiently small.

Now assume that the hypotheses of Theorem 2 are valid. Then we will slightly modify the above procedure to obtain the same result. We may assume that $N \geqslant N_0 > 0$, $L \leqslant 0$, and $L_t|_{t>0} < 0$ for D small. As above the boundary integral of (3.7) will be nonnegative. Moreover by choosing $\bar{\lambda}$ sufficiently large in (3.8) we obtain

$$\iint\limits_D -L_t|\overline{U}|^2 \leqslant 0,$$

so that again $U = 0$. $\square$

We would now like to obtain a local uniqueness result for (3.3) in the neighborhood of a closed asymptotic curve, Γ. The first step will be to construct a special semi-global coordinate system near Γ.

Lemma 3.2. *Assume that $\int_\Gamma k_g k_n |K|^{-1/2}\,ds \neq 0$ where k_g is the geodesic curvature, k_n is the normal curvature in the direction perpendicular to Γ, and ds is the element of arclength. Then there exists a system of smooth local coordinates (x,t) near Γ with the following properties. The coordinate curves $t = \text{const.}$ are homotopic to Γ with $t = 0$ corresponding to Γ, and the coordinate curves $x = \text{const.}$ correspond to lines of curvature all having normal curvature of the same sign. Furthermore if*

$$II = L\,dx^2 + 2M\,dx\,dt + N\,dt^2$$

is the second fundamental form near Γ, then either

$$\begin{aligned} &L(x,0) = 0, \qquad L(x,t) < 0 \quad \textit{for } t > 0, \qquad \partial_t L(0,0) < 0,\\ &N(x,t) > 0, \qquad \left|M(x,t)\right| > 0, \end{aligned} \tag{3.11}$$

or

$$\begin{aligned} &L(x,0) = 0, \qquad L(x,t) > 0 \quad \textit{for } t > 0, \qquad \partial_t L(0,0) > 0,\\ &N(x,t) < 0, \qquad \left|M(x,t)\right| > 0. \end{aligned} \tag{3.12}$$

Proof. According to Lemma 2.4 Γ cannot be a limit point of closed asymptotic curves, and so it must be the case that the asymptotic curves belonging to the same family as Γ are spiraling towards Γ. Moreover since any asymptotic curve of the other family (not including Γ) must intersect Γ transversely, and principal directions bisect asymptotic directions, it follows that both lines of curvature intersect Γ transversely at every point. We may therefore choose a preliminary semi-global coordinate system $(\tilde{x},\tilde{t})$ in a sufficiently small neighborhood of Γ such that the curves $\tilde{t}=\text{const.}$ are closed curves homotopic to Γ with $\tilde{t}=0$ coinciding with Γ, and the curves $\tilde{x}=\text{const.}$ are lines of curvature corresponding to positive normal curvature. Then the components of the second fundamental form (in these coordinates) $(\widetilde{L},\widetilde{M},\widetilde{N})$ satisfy

$$\widetilde{N}(\tilde{x},\tilde{t})>0,\qquad \widetilde{M}(\tilde{x},\tilde{t})<0, \tag{3.13}$$

near Γ. The estimate for $\widetilde{N}$ follows immediately from the description of the coordinate system. Furthermore since $\widetilde{L}(\tilde{x},0)=0$, the Gauss equation shows that $\widetilde{M}(\tilde{x},0)$ cannot change sign. Thus (3.13) may be obtained by making the change of coordinates $\tilde{x}\to-\tilde{x}$ if necessary.

We now construct the desired coordinate system. The curves $x=\text{const.}$ remain the same, that is, they are the lines of curvature corresponding to positive normal curvature. Choose one of these to represent $x=0$. The remaining curves are labelled according to their distance along Γ from the point $(0,0)$. More precisely, choose an orientation for Γ and move along Γ away from $(0,0)$ in the positive direction stopping at a distance c. Then the (positive) line of curvature passing through Γ at this point will be labelled as $x=c$. The x-coordinates then lie in the range $0\leqslant x<l$, where l is the length of Γ, and all continuous functions depending on x will be periodic with period l.

Before constructing the curves $t=\text{const.}$, we make a few observations. In terms of the previous coordinates $(\tilde{x},\tilde{t})$, the following tangent vectors represent directions of nonpositive normal curvature:

$$Y_\sigma=\widetilde{N}\partial_{\tilde{x}}+\big(-\widetilde{M}+\sigma\sqrt{-K\det\widetilde{I}}\,\big)\partial_{\tilde{t}},\quad |\sigma|\leqslant 1.$$

In fact upon evaluating the second fundamental form in these directions we find that

$$II(Y_\sigma,Y_\sigma)=\big(1-\sigma^2\big)\widetilde{N}K\det\widetilde{I}. \tag{3.14}$$

Clearly then, $Y_{\pm1}$ are asymptotic directions and Y_0 is a principal direction. Furthermore since $\widetilde{M}<0$, Y_{-1} corresponds to the asymptotic direction belonging to the same family as Γ, since this family spirals towards Γ and thus requires the $\partial_{\tilde{t}}$-component to be small.

We now construct the curves $t=\text{const.}$ Starting from the point $(0,0)$, move along the curve $x=0$ (in the $t>0$ direction) a distance d; this point may then be labelled $(x,t)=(0,d)$. Now starting from $(0,d)$ follow the asymptotic curve which spirals towards Γ (in the positive x direction), that is, the one having direction Y_{-1}, until it again intersects $x=0$ at a point $(0,d')$. We assume for the time being that $d'<d$, and conclude that following a curve having a tangent vector at each point which is on the boundary of the region with nonpositive normal curvature, produces a curve which is not closed (since $d'<d$). It follows that a smooth closed curve, starting at $(0,d)$, may be constructed by following a curve with tangent vector at each point given by $Y_{\sigma(x,d)}$ for appropriately chosen $\sigma(x,d)>-1$. This curve will be labelled by $t=d$. Analogous closed curves may be constructed for all sufficiently small $d>0$ in such a way that the resulting coordinate system is smooth up to Γ $(t=0)$. Since the curves $t=\text{const.}$ have tangent vector $Y_{\sigma(x,\text{const.})}$ with $|\sigma(x,\text{const.})|<1$, (3.14) shows that $L(x,t)$ is proportional to

$$\big(1-\sigma^2\big)\widetilde{N}K\det\widetilde{I}(x,t)<0\quad\text{for } t>0.$$

It follows that $L_t(x,0)\leqslant 0$. Therefore (2.3) guarantees that there exists a point on Γ at which $L_t<0$, and we can always arrange that this point occurs at $x=0$. Lastly the conclusion concerning N is a direct consequence of the definition of the coordinate curves, and the conclusion concerning M is a consequence of the Gauss equation and the fact that $L(x,0)=0$.

Now assume that in the construction above, $d'>d$. Note that for all sufficiently small d either $d'>d$ or $d'<d$, according to the spiraling behavior of the asymptotic curves belonging to the same family as Γ. In this case we choose the curves $x=\text{const.}$ to be lines of curvature corresponding to negative normal curvature, and label them in the same manner as described above. To construct the curves $t=\text{const.}$, we also follow a similar procedure to that given above. Again, starting from the point $(x,t)=(0,d)$, follow the asymptotic curve (having direction Y_{-1}) which spirals towards Γ in the positive x direction until it intersects $x=0$ at $(0,d')$. Since $d'>d$, we conclude that a smooth closed curve may be constructed, starting at $(0,d)$, by following a curve with tangent vector at each point given by $Y_{\sigma(x,d)}$ for appropriately chosen $\sigma(x,d)<-1$. This curve will be labelled as $t=d$. Analogous closed curves may be constructed for all sufficiently small $d>0$ in such a way that the resulting coordinate system is smooth up to Γ $(t=0)$. Since the curves $t=\text{const.}$ now have tangent vector $Y_{\sigma(x,\text{const.})}$ with $\sigma(x,\text{const.})<-1$, (3.14) shows that $L(x,t)$ is proportional to

$$\big(1-\sigma^2\big)\widetilde{N}K\det\widetilde{I}(x,t)>0\quad\text{for } t>0.$$

It follows that $L_t(x,0)\geqslant 0$. Therefore (2.3) guarantees that there exists a point on Γ at which $L_t>0$, and we can always arrange that this point occurs at $x=0$. Lastly the conclusion concerning N is a direct consequence of the definition of the coordinate curves, and the conclusion concerning M is a consequence of the Gauss equation and the fact that $L(x,0)=0$. $\square$

We now use this special coordinate system to show that the Gauss–Codazzi system is of symmetric positive type near the closed asymptotic curve Γ. This leads to the following uniqueness result.

Lemma 3.3. *If II and $\overline{II}$ agree on Γ, then they agree in a neighborhood of Γ.*

Proof. Let (x,t) be the coordinates of Lemma 3.2 and consider the cylindrical domain $D_\delta = \{(x,t) \mid 0 < t < \delta\}$. We will show that (L, M, N) and $(\bar{L}, \overline{M}, \overline{N})$ coincide within D_δ for δ sufficiently small. In what follows we will assume that (3.11) is valid; if (3.12) holds then nearly identical arguments yield the desired result after multiplying system (3.3) through by -1. First observe that for sufficiently small δ the right-hand side of (3.2) is smooth since $N(x,t) > 0$ and $w(x,0) = 0$, therefore the system (3.3) is valid in D_δ. Set $U = e^{\lambda(x,t)+\lambda_0 t}\overline{U}$ for some constant $\lambda_0 > 0$ and a function $\lambda(x,t)$ to be determined, then (3.3) becomes

$$A^1\overline{U}_x + A^2\overline{U}_t + \overline{B}\,\overline{U} = 0,$$

where

$$\overline{B} = B + \lambda_x A^1 + (\lambda_0 + \lambda_t)A^2.$$

Multiplying by $\overline{U}^*$ and integrating by parts produces

$$\iint\limits_{D_\delta} \overline{U}^*\left(\frac{\mathcal{B}+\mathcal{B}^*}{2}\right)\overline{U} + \int\limits_{\partial D_\delta} \frac{1}{2}\overline{U}^*\big(A^1\nu_1 + A^2\nu_2\big)\overline{U} = 0, \tag{3.15}$$

where (ν_1,ν_2) denotes the unit outer normal to ∂D_δ, and

$$\mathcal{B} = \overline{B} - \frac{1}{2}A^1_x - \frac{1}{2}A^2_t.$$

We now show that $\frac{\mathcal{B}+\mathcal{B}^*}{2} := (\mathcal{B}_{ij})$ is positive definite for δ sufficiently small, if $\lambda(x,t)$ and λ_0 are chosen appropriately. A calculation yields

$$\begin{aligned}
\mathcal{B}_{11} &= (\lambda_0 + \lambda_t + \beta)N + \beta w - \frac{1}{2}N_t + \alpha v + 2\alpha M,\\
\mathcal{B}_{22} &= 2\left(\lambda_x - \gamma + \frac{\alpha L}{N+w}\right)M - M_x + \left(c - \left(\frac{L}{N+w}\right)_t\right)(N+w) - (\lambda_0 + \lambda_t + a)L + \frac{1}{2}L_t - v_x,\\
\mathcal{B}_{12} = \mathcal{B}_{21} &= \left(-\lambda_x + \frac{\gamma}{2} + \frac{b}{2}\right)N + \left(\frac{\gamma}{2} + \frac{b}{2}\right)w - \frac{\alpha}{2}L + 2v_t - w_x + \frac{1}{2}(N+w)_x - \frac{(N+w)_t}{2(N+w)}v\\
&\quad - \left(\beta - a + \frac{\alpha v}{N+w} + \frac{2\alpha M}{N+w} + \frac{(N+w)_t}{N+w}\right)M + M_t + \frac{av}{2}.
\end{aligned} \tag{3.16}$$

By applying the Codazzi equations we find that

$$\begin{aligned}
\mathcal{B}_{22} &= (2\lambda_x - 2\gamma)M - M_x + cN - \frac{1}{2}L_t - \big(\lambda_0 + \lambda_t + O(1)\big)L + O\big(|w| + |\nabla w| + |\nabla v|\big)\\
&= (2\lambda_x - 2\gamma - b)M - \frac{3}{2}L_t - \big(\lambda_0 + \lambda_t + O(1)\big)L + O\big(|w| + |\nabla w| + |\nabla v|\big)\\
&= \left(2\lambda_x - \frac{1}{2}\partial_x \log\det I\right)M - \frac{3}{2}L_t - \big(\lambda_0 + \lambda_t + O(1)\big)L + O\big(|w| + |\nabla w| + |\nabla v|\big),
\end{aligned} \tag{3.17}$$

where we have also used the identity

$$2\gamma + b = \Gamma^1_{11} + \Gamma^2_{12} = \frac{1}{2}\partial_x \log\det I.$$

This motivates the choice

$$\lambda := \lambda_1 + \varepsilon\lambda_2, \quad \lambda_1 = \frac{1}{4}\log\det I,$$

where $\varepsilon > 0$ is a small parameter and λ_2 is required to satisfy $\partial_x\lambda_2 > 0$ if $M(x,t) > 0$ ($\partial_x\lambda_2 < 0$ if $M(x,t) < 0$) except in a sufficiently small neighborhood of $(0,0)$ where $L_t < 0$. Note that since all functions, and in particular λ_2, must be periodic in x, it is not possible to choose λ_2 such that $\partial_x\lambda_2 > 0$ for all x. Therefore by choosing $\lambda_0 > 0$ sufficiently large and $\varepsilon, \delta > 0$ sufficiently small, the matrix $(\mathcal{B}_{ij})$ is positive definite.

Although (2.3) yields the nondegeneracy conditions for L_t in (3.11) and (3.12), which then yield the above uniqueness proof, we would like to show how (2.3) can be used directly to obtain this goal. According to (2.3) there exists a smooth function $f(x,t) < 0$ (assuming, as we may, that $M(x,t) < 0$) such that

$$\int_0^l \big(M^{-1}L_t(x,t) + f(x,t)\big)\,dx = 0,$$

where as in the proof of Lemma 3.2 the x-coordinate lies in the range $[0,l)$. Therefore the following equation admits a smooth solution λ_2:

$$2M\partial_x\lambda_2 - \frac{3}{2}L_t = \frac{3}{2}Mf.$$

Then by choosing

$$\lambda := \lambda_1 + \lambda_2, \quad \lambda_1 = \frac{1}{4}\log\det I,$$

taking $\lambda_0 > 0$ sufficiently large, and $\delta > 0$ sufficiently small, the calculations (3.16) and (3.17) show that $(\mathcal{B}_{ij})$ is positive definite in D_δ.

Remark. Without (2.3) it is unclear if $\mathcal{B}_{22}$ can be made positive. Thus we may view (2.3) as that which makes the system symmetric positive near a closed asymptotic curve.

Lastly, we show that the boundary integral in (3.15) is nonnegative. The boundary of D_δ consists of two curves, namely $t = 0$ and $t = \delta$. On $t = 0$ the function $\overline{U}$ vanishes identically, and on $t = \delta$ the unit normal has components $\nu_1 = 0$, $\nu_2 = 1$. Thus since $N > 0$ and $L \leqslant 0$, the desired conclusion follows. $\square$

4. Proof of Theorems 1 and 2

We would now like to use Lemmas 3.1 and 3.3 to extend the agreement of the second fundamental forms of S and $\overline{S}$ on $\partial\mathcal{C}$ to the whole of $\mathcal{C}$. Choose one family of asymptotic curves and decompose $\mathcal{C}$ into the cylindrical domains $\mathcal{C}_1, \ldots, \mathcal{C}_m$ of Lemma 2.5. We first show that the second fundamental forms II and $\overline{II}$ agree on $\mathcal{C}_1$, which contains the boundary curve γ_1 of $\mathcal{C}$. Let σ_1 be an asymptotic curve of this family which emanates from γ_1 and spirals toward $\partial\mathcal{C}_1$. Let σ_2 be an asymptotic curve of the other family which emanates from γ_1, and let p be a point at which they intersect. Let $R(p)$ denote the segment of σ_2 starting on γ_1 at p_0 and ending at p on σ_1. Suppose that II and $\overline{II}$ agree on γ_1. By Lemma 3.1 they agree also in a neighborhood of p_0. Therefore, starting at points on $R(p)$ near p_0, we may apply the Goursat uniqueness theorem for the hyperbolic system (3.3), in which one prescribes data on two intersecting characteristics and obtains a unique solution in a small characteristic rectangle. This shows that II and $\overline{II}$ agree in a small characteristic rectangle which contains a portion of $R(p)$. Then working our way up $R(p)$ we may repeatedly apply this procedure to show that II and $\overline{II}$ agree in a neighborhood of $R(p)$ all the way up to σ_1. Since this holds for any $p \in \sigma_1$ which has an asymptotic curve of the other family connecting it to γ_1, we conclude by the continuity method that II and $\overline{II}$ agree on all of σ_1 which lies below the first closed asymptotic curve of the other family. Since σ_1 was arbitrary, we now have that the second fundamental forms agree on all of the portion of $\mathcal{C}_1$ which is bounded by γ_1 and a closed asymptotic curve Γ of the other family.

In order to continue this uniqueness beyond Γ, we apply Lemma 3.3. Note that Γ cannot intersect $\partial\mathcal{C}_1$, since if it does then an asymptotic curve of the same family as Γ must be tangent to $\partial\mathcal{C}_1$ at some point, which is not possible. We may now continue the method of the previous paragraph to obtain uniqueness up until we hit another closed asymptotic curve of the same family as Γ. Eventually we will reach $\partial\mathcal{C}_1$. The same technique may be applied in $\mathcal{C}_2, \ldots, \mathcal{C}_m$. It follows that $II = \overline{II}$ on all of $\mathcal{C}$.

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