Class. Quantum Grav. **26** (2009) 078001 (5pp) doi:10.1088/0264-9381/26/7/078001

COMMENTS, REPLIES AND NOTES

A note on the nonexistence of generalized apparent horizons in Minkowski space

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Received 14 November 2008, in final form 26 February 2009
Published 23 March 2009
Online at stacks.iop.org/CQG/26/078001

Abstract
We establish a positive mass theorem for initial data sets of the Einstein equations having generalized trapped surface boundary. In particular, we answer a question posed by R Wald concerning the existence of generalized apparent horizons in Minkowski space.

PACS numbers: 04.20.−q, 04.70.Bw

Let (M, g, k) be an initial data set for the Einstein equations, that is, M is a Riemannian 3-manifold with metric g, and k is a symmetric 2-tensor representing the extrinsic curvature of a spacelike slice; both are required to satisfy the constraint equations

$$16\pi\mu = R + (\mathrm{Tr}_g k)^2 - |k|^2, \qquad 8\pi J_i = \nabla^j(k_{ij} - (\mathrm{Tr}_g k)g_{ij}),$$

where R is scalar curvature and μ, J are respectively the energy and momentum densities of the matter fields. If all measured energy densities are nonnegative then $\mu \geqslant |J|$, which will be referred to as the dominant energy condition. We assume that the initial data are asymptotically flat (with one end), so that at spatial infinity the metric and extrinsic curvature satisfy the following fall-off conditions:

$$|\partial^l(g_{ij} - \delta_{ij})| = O(r^{-l-1}), \qquad |\partial^l k_{ij}| = O(r^{-l-2}), \qquad l = 0, 1, 2, \quad \text{as} \quad r \to \infty.$$

The ADM energy and momentum are then well defined by

$$E = \lim_{r\to\infty} \frac{1}{16\pi}\int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii})\nu^j, \qquad \vec{P}_i = \lim_{r\to\infty} \frac{1}{8\pi}\int_{S_r} (k_{ij} - (\mathrm{Tr}_g k)g_{ij})\nu^j,$$

where S_r are coordinate spheres in the asymptotic end with unit outward normal ν.

The strength of the gravitational field in the vicinity of a 2-surface $\Sigma \subset M$ may be measured by the null expansions,

$$\theta_\pm := H_\Sigma \pm \mathrm{Tr}_\Sigma k,$$

where H_Σ is the mean curvature with respect to the unit outward normal (pointing towards spatial infinity). The null expansions measure the rate of change of area for a shell of light

[1] The author is partially supported by NSF grant DMS-0707086 and a Sloan Research Fellowship.

emitted by the surface in the outward future direction (θ_+) and outward past direction (θ_-). Thus the gravitational field is interpreted as being strong near Σ if $\theta_+ \leqslant 0$ or $\theta_- \leqslant 0$, in which case Σ is referred to as a future (past) trapped surface. Future (past) apparent horizons arise as boundaries of future (past) trapped regions and satisfy the equation $\theta_+ = 0$ ($\theta_- = 0$).

In an attempt to find the most general conditions under which the Penrose inequality is valid, Bray and the author [4] have proposed the notion of a generalized apparent horizon, which we take to be any surface Σ satisfying the equation

$$H_\Sigma = |\mathrm{Tr}_\Sigma k|.$$

A very natural question, posed by Wald [10], is to ask whether such surfaces can exist inside Minkowski space. Our purpose here is to show that this is not possible. The strategy will be to follow Witten's proof of the positive mass theorem, and show that if such a surface exists in any initial data set satisfying the dominant energy condition then the ADM mass is strictly positive, which, of course, cannot occur for a slice of Minkowski space. In fact, this result will be a special case of the positive mass theorem for spacetimes containing a generalized trapped surface, that is a surface Σ satisfying the inequality

$$H_\Sigma \leqslant |\mathrm{Tr}_\Sigma k|. \tag{1}$$

It has been shown [5] that the existence of a compact generalized trapped surface in an asymptotically flat initial data set implies the existence of a generalized apparent horizon. This is analogous to the relationship between classical trapped surfaces and apparent horizons [1]. The following theorem exhibits another analogy between classical and generalized trapped surfaces.

Theorem. *Let (M, g, k) be an asymptotically flat initial data set for the Einstein equations satisfying the dominant energy condition $\mu \geqslant |J|$. If the boundary ∂M is nonempty and consists of finitely many compact components each of which is a generalized trapped surface, then the ADM mass is strictly positive, $E > |\overrightarrow{P}|$.*

Proof. Let $(\mathcal{M}, \gamma)$ be a portion of the spacetime arising from the initial data (M, g, k), and let $c : Cl(T\mathcal{M}) \to \mathrm{End}(S)$ be the usual representation of the Clifford algebra on the bundle of spinors S, so that

$$c(X)c(Y) + c(Y)c(X) = -2\gamma(X, Y)\,\mathrm{Id}.$$

We choose a local orthonormal frame $e_a, a = 0, 1, 2, 3$ such that e_0 is normal to M and $e_i, i = 1, 2, 3$, are tangent to M. Then the 'spacetime spin connection' on M is given by

$$\nabla_{e_i}\psi = \left(e_i(\psi^I) + \tfrac{1}{4}\psi^I\Gamma_{ij}^l c(e^j)c(e_l) + \tfrac{1}{2}\psi^I k_{ij} c(e^j)c(e_0)\right)\phi_I,$$

where $\psi = \psi^I\phi_I$ with $\phi_I, I = 1, 2, 3, 4$, being a choice of spin frame associated with the orhtonormal frame e_i, and Γ_{ij}^l are Christoffel symbols for the metric g. Note that we are using Dirac spinors ψ, which consist of a pair of $\mathrm{SL}(2, \mathbb{C})$ spinors, one left handed and one right handed. Consider the following chiral boundary value problem for the Dirac operator:

$$\mathcal{D}\psi = \sum_{i=1}^{3} c(e_i)\nabla_{e_i}\psi = 0 \quad \text{on} \quad M, \quad \psi = \psi_\infty + o\left(\frac{1}{|x|^{1-\delta}}\right) \quad \text{as} \quad |x| \to \infty,$$
$$\epsilon\psi - \psi = 0 \quad \text{on} \quad \partial M_+, \qquad \epsilon\psi + \psi = 0 \quad \text{on} \quad \partial M_-, \tag{2}$$

where $\epsilon = c(e_3)c(e_0)$ with e_3 normal to ∂M and pointing towards spatial infinity, ψ_∞ is a nonzero spinor which is constant in the asymptotic end (the components ψ_∞^I, with respect to a fixed frame at spatial infinity, are constant), and $\partial M_\pm$ denotes the portion of ∂M on which $\theta_\pm \leqslant 0$. Note that these boundary conditions are the usual ones used to establish the positive mass theorem with black holes in which ∂M is assumed to consist of classical future and past apparent horizons. Below, we will show that this boundary value problem is coercive. Moreover (2) falls into a class of elliptic boundary value problems treated in [3]. Therefore, we conclude that there exists a unique solution with $\psi - \psi_\infty \in W^{1,2}_{-1}(M) \cap W^{2,2}_{\mathrm{loc}}(M)$; here $W^{1,2}_{-1}(M)$ and $W^{2,2}_{\mathrm{loc}}(M)$ represent Sobolev spaces of square integrable derivatives up to orders one and two, respectively, with the subscript -1 indicating an appropriate weight to obtain the correct fall-off at spatial infinity.

Consider the following Lichnerowicz formula ([3, 7, 9]) for the solution of (2):

$$\mathcal{D}^*\mathcal{D}\psi = \nabla^*\nabla\psi + \mathcal{R}\psi = 0,$$

where

$$\mathcal{R}\psi = 4\pi(\mu + J^i c(e_0)c(e_i))\psi.$$

Then integrating by parts produces

$$\begin{aligned}\int_M (|\nabla\psi|^2 + 4\pi(\mu|\psi|^2 + J^i\langle\psi, c(e_0)c(e_i)\psi\rangle))\\ = 4\pi P^a\langle\psi_\infty, c(e_0)c(e_a)\psi_\infty\rangle - \int_{\partial M}\left\langle\psi, c(e_3)\sum_{i=1}^{2} c(e_i)\nabla_{e_i}\psi\right\rangle,\end{aligned} \tag{3}$$

where P^a is the ADM 4-momentum. In order to facilitate the calculation of the boundary term, we define the boundary covariant derivative by

$$\overline{\nabla}_{e_i}\psi = e_i(\psi) + \frac{1}{4}\sum_{j,l=1}^{2}\Gamma^l_{ij}c(e_j)c(e_l)\psi + \frac{1}{2}k_{i3}c(e_3)c(e_0)\psi, \qquad i = 1, 2,$$

and we define the boundary Dirac operator by

$$\mathcal{D}_{\partial M}\psi = c(e_3)\sum_{i=1}^{2} c(e_i)\overline{\nabla}_{e_i}\psi.$$

The boundary term may now be calculated by using properties of Clifford multiplication, symmetries of the Christoffel symbols and the special boundary conditions of (2), as follows:

$$\begin{aligned}c(e_3)\sum_{i=1}^{2} c(e_i)\nabla_{e_i}\psi &= \mathcal{D}_{\partial M}\psi + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{3}\Gamma^3_{ij}c(e_3)c(e_i)c(e_j)c(e_3)\psi\\ &\quad + \frac{1}{2}\sum_{i,j=1}^{2} k_{ij}c(e_3)c(e_i)c(e_j)c(e_0)\psi\\ &= \mathcal{D}_{\partial M}\psi - \frac{1}{2}H_{\partial M}\psi - \frac{1}{2}(\mathrm{Tr}_{\partial M}k)c(e_3)c(e_0)\psi\\ &= \mathcal{D}_{\partial M}\psi - \frac{1}{2}\theta_\pm\psi \qquad \text{on} \quad \partial M_\pm.\end{aligned} \tag{4}$$

Moreover, a similar calculation shows that $\mathcal{D}_{\partial M}\epsilon = -\epsilon\mathcal{D}_{\partial M}$, and therefore since ϵ is self-adjoint with respect to the (positive definite) inner product $\langle\cdot,\cdot\rangle$ on S we have

$$\begin{aligned}\langle\psi, \mathcal{D}_{\partial M}\psi\rangle &= \pm\langle\psi, \mathcal{D}_{\partial M}\epsilon\psi\rangle\\ &= \mp\langle\psi, \epsilon\mathcal{D}_{\partial M}\psi\rangle\\ &= \mp\langle\epsilon\psi, \mathcal{D}_{\partial M}\psi\rangle = -\langle\psi, \mathcal{D}_{\partial M}\psi\rangle.\end{aligned} \tag{5}$$

Then by combining (3), (4) and (5), choosing ψ_∞ so that

$$P^a \langle \psi_\infty, c(e_0)c(e_a)\psi_\infty \rangle = E - |\overrightarrow{P}|,$$

and applying the dominant energy condition, it follows that

$$\int_M |\nabla\psi|^2 \leqslant 4\pi(E - |\overrightarrow{P}|). \tag{6}$$

Note that the same arguments used to obtain this inequality also yield the coercivity of the boundary value problem (2), which is needed for establishing the existence and regularity of solutions.

We now proceed by contradiction and assume that $E \leqslant |\overrightarrow{P}|$. Then (6) shows that ψ is covariantly constant. First, consider the case in which at least one boundary component Σ is a true generalized trapped surface, that is Σ satisfies (1) and $\mathrm{Tr}_\Sigma k$ changes sign along Σ. Then according to the boundary conditions imposed on ψ, and the fact that ψ is continuous up to the boundary (with the help of a Sobolev embedding), there is a point $p \in \Sigma$ at which $\psi(p) = 0$. Now parallel transport ψ along any curve emanating from p to find that $\psi = 0$ along this curve (since ψ restricted to the curve is itself the solution of parallel transport). But this implies that $\psi \equiv 0$ on M, which is impossible as $\psi_\infty \neq 0$.

In the remaining case to consider, all boundary components are either pure future or past trapped surfaces. Then according to Andersson and Metzger [1] there exists a smooth compact outermost apparent horizon, each component of which either has spherical topology or is a flat torus [6]. First assume that at least one component Σ of the outermost apparent horizon has spherical topology, and we further assume that it is a future apparent horizon, that is $\theta_+ = 0$ (similar arguments will hold for a past apparent horizon). By writing the full Dirac operator in terms of the induced operator on the boundary with the help of calculation (4), and using the fact that ψ is covariantly constant, on Σ we have

$$0 = c(e_3)\mathcal{D}\psi = -\nabla_{e_3}\psi + \mathcal{D}_\Sigma\psi - \tfrac{1}{2}\theta_+\psi = \mathcal{D}_\Sigma\psi.$$

Since ψ cannot vanish on Σ (according to arguments above), this says that Σ admits a nontrivial harmonic spinor. However, since Σ is topologically a 2-sphere, this is impossible according to the Hijazi–Bär inequality [2, 8], which states that all eigenvalues of the Dirac operator on a 2-sphere must satisfy

$$|\lambda(\mathcal{D}_\Sigma)| \geqslant \sqrt{\frac{4\pi}{\mathrm{Area}(\Sigma)}} > 0.$$

The application of the Hijazi–Bär inequality was first suggested by Bartnik and Chruściel in [3].

If all components of the outermost apparent horizon are flat tori then we proceed as follows. Assume that ∂M coincides with the outermost apparent horizon. Then we may choose a spin structure on M for which the induced spin structure on one of the boundary components Σ is not the 'trivial' one (note that the 2-torus admits four distinct spin structures). By arguing as above we find that Σ admits a nontrivial harmonic spinor. However this is impossible, since only the trivial spin structure on a flat torus admits nontrivial harmonic spinors [2]. With this contradiction we conclude that the ADM mass must be strictly positive. □

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