Lower bounds for the area of black holes in terms of mass, charge, and angular momentum

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The most general formulation of Penrose's inequality yields a lower bound for Arnowitt-Deser-Misner mass in terms of the area, charge, and angular momentum of black holes. This inequality is in turn equivalent to an upper and lower bound for the area in terms of the remaining quantities. In this paper, we establish the lower bound for a single black hole in the setting of axisymmetric maximal initial data sets for the Einstein-Maxwell equations, when the non-electromagnetic matter fields are not charged and satisfy the dominant energy condition. It is shown that the inequality is saturated if and only if the initial data arise from the extreme Kerr-Newman spacetime. Further refinements are given when either charge or angular momentum vanish. Last, we discuss the validity of the lower bound in the presence of multiple black holes.

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I. INTRODUCTION

The standard model of gravitational collapse [1,2] consists of two main parts. Namely, gravitational collapse should always result in a black hole (weak cosmic censorship), and eventually the spacetime should settle down to a stationary electrovacuum final state. According to the black hole uniqueness theorem this final state will be a Kerr-Newman spacetime; however, it should be noted that many important technical aspects of black hole uniqueness remain open (see [3] for a recent review). Let (m_0, A_0, q_0, J_0) denote the mass, black hole area, charge, and angular momentum of the Kerr-Newman solution, then direct calculation yields an expression for the mass in terms of the remaining quantities

$$m_0^2 = \frac{A_0}{16\pi} + \frac{q_0^2}{2} + \frac{\pi(q_0^4 + 4J_0^2)}{A_0}. \tag{1.1}$$

In general, the expression on the right-hand side is the square of the so-called Christodoulou mass [4] of a black hole. Observe that, as a function of A_0 (keeping q_0 and J_0 fixed), the right-hand side is nondecreasing precisely when

$$A_0 \geq 4\pi\sqrt{q_0^4 + 4J_0^2}. \tag{1.2}$$

Consider now a Cauchy surface M in an asymptotically flat spacetime which undergoes gravitational collapse, and settles down to the Kerr-Newman solution above. Let (m, A, q, J) be the Arnowitt-Deser-Misner (ADM) mass, black hole area, total charge, and ADM angular momentum associated with this slice. Since gravitational waves

carry positive energy, the ADM (total) mass of the spacetime should not be smaller than the mass of the final state

$$m \geq m_0, \tag{1.3}$$

and $m - m_0$ should measure the amount of radiation emitted by the system. Moreover, the Hawking area theorem [5] (which relies on cosmic censorship) yields

$$A_0 \geq A. \tag{1.4}$$

Therefore, if conditions are imposed to ensure that the charge and angular momentum are conserved, that is $q = q_0$ and $J = J_0$, then we are lead to the following generalized version of the Penrose inequality [6]

$$m^2 \geq \frac{A}{16\pi} + \frac{q^2}{2} + \frac{\pi(q^4 + 4J^2)}{A} \tag{1.5}$$
$$\text{whenever } A \geq 4\pi\sqrt{q^4 + 4J^2}.$$

Typical assumptions which guarantee conserved charge and angular momentum are that the spacetime be electrovacuum and axially symmetric. Furthermore, it is expected that equality is achieved in the first inequality of (1.5), only if M arises from the Kerr-Newman spacetime.

In the case that the area-charge-angular momentum inequality of (1.2) is not satisfied, similar arguments motivate the inequality

$$m^2 \geq \frac{q^2 + \sqrt{q^4 + 4J^2}}{2}. \tag{1.6}$$

Notice that the right-hand side of (1.6) is the minimum value of the right-hand side of (1.5), as a function of A, for $A \geq 4\pi\sqrt{q^4 + 4J^2}$. Equality in (1.6) should hold only when M arises from the extreme Kerr-Newman spacetime. These two inequalities yield variational characterizations

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of the Kerr-Newman and extreme Kerr-Newman initial data, as those with minimal mass for fixed black hole area, total charge, and angular momentum, or fixed total charge and angular momentum. A violation of (1.5) or (1.6) would present a serious challenge to the standard picture of gravitational collapse mentioned above.

The area A appearing in (1.5) is that of the intersection of the event horizon with the Cauchy surface M. Unfortunately, from an initial data perspective this is not a useful quantity, since it requires the full evolution of spacetime in order to determine its value. Thus it is convenient to replace event horizon with apparent horizon, and to replace A with $A_{\min}$, the minimal area required to enclose the outermost apparent horizon or the minimal area required to enclose all but one asymptotic end. We may now give the Penrose inequality a purely initial data formulation

$$m^2 \geq \frac{A_{\min}}{16\pi} + \frac{q^2}{2} + \frac{\pi(q^4 + 4J^2)}{A_{\min}} \tag{1.7}$$

$$\text{whenever } A_{\min} \geq 4\pi\sqrt{q^4 + 4J^2}.$$

Cosmic censorship implies that the outermost apparent horizon is contained within the event horizon, so that $A \geq A_{\min}$ and hence (1.7) is implied by (1.5). It follows that a counterexample to (1.7) would be just as significant as a counterexample to (1.5) or (1.6).

In this paper we will prove "one half" of inequality (1.7) for a single component black hole, under the assumption of maximal initial data. In order to explain what is meant by one half, let us multiply the inequality by $A_{\min}$ and view it as a bound for a quadratic polynomial in $A_{\min}$. This is then equivalent to the following upper and lower bound for $A_{\min}$

$$m^2 - \frac{q^2}{2} - \sqrt{\left(m^2 - \frac{q^2}{2}\right)^2 - \frac{q^4}{4} - J^2}$$
$$\leq \frac{A_{\min}}{8\pi} \leq m^2 - \frac{q^2}{2} + \sqrt{\left(m^2 - \frac{q^2}{2}\right)^2 - \frac{q^4}{4} - J^2}. \tag{1.8}$$

Notice that the quantity inside the square root is nonnegative by (1.6). We may then state two conjectured inequalities which are motivated by cosmic censorship

$$\frac{A_{\min}}{8\pi} \geq m^2 - \frac{q^2}{2} - \sqrt{\left(m^2 - \frac{q^2}{2}\right)^2 - \frac{q^4}{4} - J^2} \tag{1.9}$$

$$\text{whenever } A_{\min} \geq 4\pi\sqrt{q^4 + 4J^2},$$

and

$$\frac{A_{\min}}{8\pi} \leq m^2 - \frac{q^2}{2} + \sqrt{\left(m^2 - \frac{q^2}{2}\right)^2 - \frac{q^4}{4} - J^2}. \tag{1.10}$$

The auxiliary area-charge-angular momentum inequality does not appear with the upper bound (1.10) for a reason. Namely, one may derive this inequality directly from the same type of heuristic arguments which lead to (1.7). In fact, inequality (1.10) is the usual form of the generalized Penrose inequality that appears in the literature [7,8]. For maximal initial data, this upper bound has been proven when $q = J = 0$ for a single black hole in [9] and for multiple black holes in [10]; it has also been established in the case $q \neq 0$, $J = 0$ for a single black hole [9,11] (see [12] for the case of equality). The case of equality in (1.9) and (1.10) should only be achieved by extreme Kerr-Newman and Kerr-Newman initial data, respectively. The "half" of (1.7) which will be established here is the lower bound (1.9), when the horizon is connected, and without the assumption of the auxiliary area-charge-angular momentum inequality. In fact when the horizon is connected, the area-charge-angular momentum inequality is a theorem itself, rather than a hypothesis. The case when $q \neq 0$ and $J = 0$, where the hypothesis of axial symmetry is not required, was established in [13].

II. STATEMENT AND PROOF OF THE MAIN RESULT

We begin with the appropriate definitions. Let (M, g, k, E, B) be an initial data set for the Einstein-Maxwell equations, consisting of a 3-manifold M, Riemannian metric g, symmetric 2-tensor k (representing the extrinsic curvature in spacetime), and vector fields E and B representing the electric and magnetic fields. It is assumed that there is no charged matter, that is

$$\text{div}E = 0, \qquad \text{div}B = 0. \tag{2.1}$$

Consider a manifold M that has at least two ends, with one being asymptotically flat, and the remainder being either asymptotically flat or asymptotically cylindrical. Recall that a domain $M_{\text{end}} \subset M$ is an asymptotically flat end if it is diffeomorphic to $\mathbb{R}^3 \setminus \text{Ball}$, and in the coordinates given by the asymptotic diffeomorphism the following falloff conditions hold

$$g_{ij} = \delta_{ij} + o_l(r^{-1/2}), \quad \partial g_{ij} \in L^2(M_{\text{end}}), \quad k_{ij} = O_{l-1}(r^{-3}), \tag{2.2}$$

$$E^i = O_{l-1}(r^{-2}), \qquad B^i = O_{l-1}(r^{-2}), \tag{2.3}$$

for some $l \geq 6$.[1] These asymptotics may be weakened, see for example [15–17]. If M is simply connected and the data are axially symmetric, it is shown in [14] that the analysis reduces to the study of manifolds of the form $M \simeq \mathbb{R}^3 \setminus \sum_{n=0}^{N} i_n$, where i_n are points in $\mathbb{R}^3$ and represent

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[1]The notation $f = o_l(r^{-\alpha})$ asserts that $\lim_{r\to\infty} r^{\alpha+n}\partial^n f = 0$ for all $n \leq l$, and $f = O_l(r^{-\alpha})$ asserts that $r^{\alpha+n}|\partial^n f| \leq C$ for all $n \leq l$. The assumption $l \geq 6$ is needed for the results in [14].

asymptotic ends (in total there are $N + 1$ ends). Moreover, there exists a global (cylindrical) Brill coordinate system on M, where the points i_n all lie on the z-axis, and in which the appropriate asymptotics for the metric coefficients near a cylindrical end are given in [18]. The falloff conditions in the asymptotically flat ends guarantee that the asymptotic limits defining the ADM mass, and total charge exist

$$m = \frac{1}{16\pi} \int_{S_\infty} (g_{ij,i} - g_{ii,j})\nu^j, \qquad (2.4)$$

$$q_e = \frac{1}{4\pi} \int_{S_\infty} E_i \nu^i, \qquad q_b = \frac{1}{4\pi} \int_{S_\infty} B_i \nu^i, \qquad (2.5)$$

where S_∞ indicates the limit as $r \rightarrow \infty$ of integrals over coordinate spheres S_r, with unit outer normal ν. Here q_e and q_b denote the total electric and magnetic charge, respectively, and we denote the square of the total charge by $q^2 = q_e^2 + q_b^2$.

We say that the initial data are axially symmetric if the group of isometries of the Riemannian manifold (M, g) has a subgroup isomorphic to $U(1)$, and that the remaining quantities defining the initial data are invariant under the $U(1)$ action. In particular, if η denotes the Killing field associated with this symmetry, then

$$\mathfrak{L}_\eta g = \mathfrak{L}_\eta k = \mathfrak{L}_\eta E = \mathfrak{L}_\eta B = 0, \qquad (2.6)$$

where $\mathfrak{L}_\eta$ denotes Lie differentiation. The (gravitational) angular momentum, in the direction of the axis of rotation, of a 2-surface $\Sigma \subset M$ whose tangent space includes η, is represented by a scalar [7]

$$J(\Sigma) = \frac{1}{8\pi} \int_\Sigma (k_{ij} - (Trk)g_{ij})\nu^i \eta^j. \qquad (2.7)$$

The ADM angular momentum is then given by

$$J = \lim_{r \rightarrow \infty} J(S_r), \qquad (2.8)$$

and note that the falloff conditions (2.2) guarantee that the limit exists, as $|\eta|$ grows like ρ, the radial coordinate in the (cylindrical) Brill coordinate system.

Unfortunately, the angular momentum (2.7) is not necessarily conserved. We are thus motivated to define an alternate angular momentum which has this property. In order to do this, we will make use of a vector potential $B = \nabla \times \vec{A}$. However, note that the topology of M does not allow for a globally defined (smooth) vector potential. The typical construction which avoids this difficulty involves removing a "Dirac string" associated with each point i_n. That is, removing from M either the portion of the z-axis below or above i_n, to obtain a ($U(1)$ invariant) potential $\vec{A}_\pm^n$, defined on the complement of the respective Dirac string. We then define

$$\vec{A} = \frac{1}{2N} \sum_{n=1}^{N} (\vec{A}_+^n + \vec{A}_-^n) \quad \text{on } \mathbb{R}^3 \setminus \{z\text{-axis}\}. \qquad (2.9)$$

The (total) angular momentum of a surface Σ, after the contribution of the electromagnetic field has been added, is given by

$$\tilde{J}(\Sigma) = \frac{1}{8\pi} \int_\Sigma (k_{ij} - (\text{Tr}k)g_{ij})\nu^i \eta^j + \frac{1}{4\pi} \int_\Sigma (E_i \nu^i)(\vec{A}_j \eta^j). \qquad (2.10)$$

Although $\vec{A}$ is discontinuous on the z-axis, the product $\vec{A}_j \eta^j$ remains well behaved since η vanishes on the z-axis. Below it will be shown that this angular momentum is gauge invariant with respect to gauge transformations which vanish in the black hole region, and is conserved under appropriate conditions on (linear) momentum density. Recall that the matter density and (linear) momentum density for the non-electromagnetic matter fields are given by the constraint equations

$$16\pi\mu = R + (\text{Tr}k)^2 - |k|^2 - 2(|E|^2 + |B|^2),$$
$$8\pi P = \text{div}(k - (\text{Tr}k)g) + 2E \times B, \qquad (2.11)$$

where R denotes the scalar curvature of g. The non-electromagnetic matter fields will be said to satisfy the dominant energy condition if

$$\mu \geq |P|. \qquad (2.12)$$

Lemma 2.1 Let (M, g, k, E, B) have the properties described above. If $P_i \eta^i = 0$, then $\tilde{J}$ is conserved. That is, if surfaces Σ_1 and Σ_2 are $U(1)$ invariant and bound a domain, then

$$\tilde{J}(\Sigma_1) = \tilde{J}(\Sigma_2). \qquad (2.13)$$

Moreover, $\tilde{J}$ is invariant under gauge transformations which vanish in a neighborhood of the points i_n, and

$$\tilde{J}(S_\infty) = J. \qquad (2.14)$$

Proof. Consider the linear momentum density of the electromagnetic field in the η-direction

$$(E \times B)_l \eta^l = \epsilon_{ijl} E^i \epsilon^{abj}(\nabla_a \vec{A}_b)\eta^l$$
$$= \nabla_a(\epsilon_{ijl} E^i \epsilon^{abj}\vec{A}_b \eta^l) - \epsilon_{ijl}\epsilon^{abj}\vec{A}_b(\nabla_a E^i)\eta^l$$
$$\quad - \epsilon_{ijl}\epsilon^{abj}\vec{A}_b E^i \nabla_a \eta^l$$
$$= \nabla_a(\epsilon_{ijl} E^i \epsilon^{abj}\vec{A}_b \eta^l) + 4\vec{A}_i \eta^i \nabla_j E^j. \qquad (2.15)$$

In this calculation we used the fact that g and E are invariant under the $U(1)$ action

$$0 = (\mathfrak{L}_\eta g)_{ij} = \nabla_i \eta_j + \nabla_j \eta_i,$$
$$0 = (\mathfrak{L}_\eta E)^i = \eta^j \nabla_j E^i - E^j \nabla_j \eta^i. \qquad (2.16)$$

If $D \subset M$ is the domain with boundary $\partial D = \Sigma_1 \cup \Sigma_2$ then

$$\int_D E \times B \cdot \eta = \int_D 4(\vec{A} \cdot \eta) \text{div} E + \int_{\partial D} (\vec{A} \cdot \eta)(E \cdot \nu), \tag{2.17}$$

where in the calculation of the boundary term $\eta \perp \nu$ is used. Therefore, since $\text{div}E = 0$

$$\begin{aligned}
0 &= \int_D P_i \eta^i \\
&= \frac{1}{8\pi} \int_D \text{div}(k - (\text{Tr}k)g)_i \eta^i + \frac{1}{4\pi} \int_D (E \times B)_i \eta^i \\
&= \frac{1}{8\pi} \int_{\partial D} (k_{ij} - (\text{Tr}k)g_{ij})\eta^i \nu^j + \frac{1}{4\pi} \int_{\partial D} (E_i \nu^i)(\vec{A}_j \eta^j) \\
&= \tilde{J}(\Sigma_1) - \tilde{J}(\Sigma_2). \tag{2.18}
\end{aligned}$$

To show that $\tilde{J}$ is gauge invariant, consider a gauge transformation $\vec{A} \mapsto \vec{A} + \nabla u$ in which u vanishes near the points i_n. According to (2.17) and $\text{div}E = 0$, the definition (2.10) may be expressed in terms of quantities independent of u, as the relevant boundary integral near i_n, and involving u, vanishes. Note also that the restriction to gauge transformations vanishing near i_n is physically relevant, since that restriction is confined within the black hole.

In order to prove (2.14), it suffices to show that

$$\lim_{r \to \infty} \int_{S_r} (E_i \nu^i)(\vec{A}_j \eta^j) = 0. \tag{2.19}$$

In light of (2.3), we find that $|\vec{A}| = O(r^{-1})$. Moreover $|\eta| \sim |x\partial_y - y\partial_x| = O(\rho)$, where $\rho \sim \sqrt{x^2 + y^2}$ is the radial coordinate in the (cylindrical) Brill coordinate system. Thus, the expansion

$$E = \frac{q_e}{r^2}\partial_r + O\left(\frac{1}{r^3}\right) \tag{2.20}$$

yields

$$\lim_{r \to \infty} \int_{S_r} (E_i \nu^i)(\vec{A}_j \eta^j) = \lim_{r \to \infty} \frac{q_e}{r^2} \int_{S_r} \vec{A}_j \eta^j. \tag{2.21}$$

Suppose that i_0 is situated at the origin. Then since

$$B = \frac{q_b}{r^2}\partial_r + O\left(\frac{1}{r^3}\right), \tag{2.22}$$

we have

$$\vec{A}_+^0 = \frac{q_b}{r(z+r)}(x\partial_y - y\partial_x) + O\left(\frac{1}{r^2}\right) \\
\text{on } \mathbb{R}^3 \setminus \{(0,0,z) \mid z \le 0\}, \tag{2.23}$$

and

$$\vec{A}_-^0 = \frac{q_b}{r(z-r)}(x\partial_y - y\partial_x) + O\left(\frac{1}{r^2}\right) \\
\text{on } \mathbb{R}^3 \setminus \{(0,0,z) \mid z \ge 0\}. \tag{2.24}$$

Thus

$$\vec{A}_+^0 \cdot \eta = \frac{q_b(x^2 + y^2)}{r(z+r)} + O\left(\frac{1}{r}\right) = q_b\left(1 - \frac{z}{r}\right) + O\left(\frac{1}{r}\right), \tag{2.25}$$

$$\vec{A}_-^0 \cdot \eta = \frac{q_b(x^2 + y^2)}{r(z-r)} + O\left(\frac{1}{r}\right) = -q_b\left(1 + \frac{z}{r}\right) + O\left(\frac{1}{r}\right), \tag{2.26}$$

and it follows that

$$\lim_{r \to \infty} \frac{1}{r^2} \int_{S_r} (\vec{A}_+^0 + \vec{A}_-^0) \cdot \eta = 0. \tag{2.27}$$

Similarly, this type of cancelation occurs for the pair of potentials associated with each i_n. The desired result (2.19) now follows from (2.21). $\square$

In order to establish (1.9), we require the global inequality (1.6), which relies on the existence of a twist potential ω:

$$\epsilon_{ijl}(\pi^{jn} + 2\chi^{jn})\eta^l \eta_n dx^i = d\omega, \tag{2.28}$$

where

$$\pi_{jn} = k_{jn} - (\text{Tr}k)g_{jn}, \qquad \chi_{jn} = \epsilon_{imn}E^i\epsilon_j^{lm}\vec{A}_l. \tag{2.29}$$

Such a potential exists, for example, in the electrovacuum setting [19]. Here we show that a weaker condition is sufficient.

Lemma 2.2 Let (M, g, k, E, B) have the properties described above. If $P_i\eta^i = 0$, then a twist potential exists.

Proof. For any 2-tensor t_{ij} (not necessarily symmetric), consider the expression

$$\mathcal{T}_i = \epsilon_{ijl}t^{jn}\eta^l\eta_n. \tag{2.30}$$

A direct calculation shows that

$$(d\mathcal{T})_{ij} = \nabla_i\mathcal{T}_j - \nabla_j\mathcal{T}_i = \nabla^a(t_{ab}\eta^b)\epsilon_{jil}\eta^l. \tag{2.31}$$

Thus if we choose

$$t_{ij} = \pi_{ij} + 2\chi_{ij}, \tag{2.32}$$

then

$$\begin{aligned}
(d\mathcal{T})_{ij} &= \nabla^a(\pi_{ab}\eta^b + 2\chi_{ab}\eta^b)\epsilon_{jil}\eta^l \\
&= [(\nabla^a\pi_{ab})\eta^b + 2\nabla^a(\chi_{ab}\eta^b)]\epsilon_{jil}\eta^l \\
&= [(\nabla^a\pi_{ab})\eta^b + 2(E \times B)_b\eta^b - 8\vec{A}_b\eta^b\nabla_aE^a]\epsilon_{jil}\eta^l \\
&= (8\pi P_b\eta^b - 8\vec{A}_b\eta^b\nabla_aE^a)\epsilon_{jil}\eta^l, \tag{2.33}
\end{aligned}$$

where we have used that π_{ij} is symmetric and η^i is a Killing field, as well as formula (2.15). Therefore since $P_i\eta^i = \text{div}E = 0$, $\mathcal{T}$ is a closed 1-form when t is given by (2.32). As M is simply connected, it follows that a twist potential exists. $\square$

The precise statement of conditions under which the inequality (1.6) is valid, will now be reviewed. Typically such a result is stated in the electrovacuum setting, since this guarantees the existence of a twist potential. However, with Lemma 2.2, we obtain a slight generalization by replacing the electrovacuum assumption with the dominant energy condition and $P_i \eta^i = 0$. Recall also that the initial data are said to be maximal if $\mathrm{Tr} k = 0$.

Theorem 2.3 ([15–17]). Let (M, g, k, E, B) be a simply connected, axially symmetric, maximal initial data set with two ends, one asymptotically flat and the other either asymptotically flat or asymptotically cylindrical. If there is no charged matter, the dominant energy condition is satisfied, and $P_i \eta^i = 0$ (all of which are satisfied in electrovacuum), then

$$m^2 \geq \frac{q^2 + \sqrt{q^4 + 4J^2}}{2}. \qquad (2.34)$$

Moreover, equality holds if and only if the initial data arise from an extreme Kerr-Newman spacetime.

We will also make use of the area-charge-angular momentum inequality. In the case where both ends of M are asymptotically flat, there exists an axisymmetric stable minimal surface $\Sigma_{\min} \subset M$, separating both ends. Here, stable means that the second variation of area is nonnegative. $\Sigma_{\min}$ minimizes area among all closed 2-surfaces enclosing either end, and thus $A(\Sigma_{\min}) = A_{\min}$, where $A_{\min}$ is the least area required to enclose an end. If one of the ends is cylindrical, the least area $A_{\min}$ required to enclose this end is realized either by a stable minimal surface $\Sigma_{\min} \subset M$, or by the surface Σ_0 obtained by taking the asymptotic limit of cross sections Σ_ρ of the end. These observations allow an application of the results in [20] to obtain the next theorem. For any 2-surface $\Sigma \subset M$, define $q(\Sigma)^2 = q_e(\Sigma)^2 + q_b(\Sigma)^2$ where $q_e(\Sigma)$ and $q_b(\Sigma)$ are defined analogously to (2.5) with the only difference being that the integrals are taken over Σ. Define also

$$q(\Sigma_0) = \lim_{\rho \to 0} q(\Sigma_\rho), \qquad \tilde{J}(\Sigma_0) = \lim_{\rho \to 0} \tilde{J}(\Sigma_\rho). \qquad (2.35)$$

Theorem 2.4 ([20]). Let (M, g, k, E, B) be an axially symmetric, maximal initial data set with two ends, one asymptotically flat and the other either asymptotically flat or asymptotically cylindrical. If the dominant energy condition is satisfied, then

$$A_{\min} \geq 4\pi \sqrt{q(\Sigma)^4 + 4\tilde{J}(\Sigma)^2}, \qquad (2.36)$$

where Σ denotes either $\Sigma_{\min}$ or Σ_0. Moreover, equality is achieved if and only if $\Sigma = \Sigma_0$ is the extreme Kerr-Newman sphere.

We may now state and prove our main result.

Theorem 2.5 Let (M, g, k, E, B) be a simply connected, axially symmetric, maximal initial data set with two ends, one (M^1_{end}) asymptotically flat and the other (M^2_{end}) either asymptotically flat or asymptotically cylindrical. If there is

no charged matter, the dominant energy condition is satisfied, and $P_i \eta^i = 0$ (all of which are satisfied in electrovacuum), then

$$\frac{A_{\min}}{8\pi} \geq m^2 - \frac{q^2}{2} - \sqrt{\left(m^2 - \frac{q^2}{2}\right)^2 - \frac{q^4}{4} - J^2}, \qquad (2.37)$$

where $A_{\min}$ is the minimum area required to enclose M^2_{end}. Moreover, equality holds if and only if the initial data arise from an extreme Kerr-Newman spacetime.

Proof. Apply Lemma 2.1 and Theorems 2.3 and 2.4 to find

$$m^2 - \frac{q^2}{2} = \sqrt{\left(m^2 - \frac{q^2}{2}\right)^2 - \frac{q^4}{4} - J^2 + \frac{q^4}{4} + J^2}$$

$$\leq \sqrt{\frac{q^4}{4} + J^2} + \sqrt{\left(m^2 - \frac{q^2}{2}\right)^2 - \frac{q^4}{4} - J^2}$$

$$= \sqrt{\frac{q(\Sigma)^4}{4} + \tilde{J}(\Sigma)^2} + \sqrt{\left(m^2 - \frac{q^2}{2}\right)^2 - \frac{q^4}{4} - J^2}$$

$$\leq \frac{A_{\min}}{8\pi} + \sqrt{\left(m^2 - \frac{q^2}{2}\right)^2 - \frac{q^4}{4} - J^2}, \qquad (2.38)$$

where Σ is as in Theorem 2.4.

In the case of equality, we must have either

$$\frac{q^4}{4} + J^2 = 0 \quad \text{or} \quad \left(m^2 - \frac{q^2}{2}\right)^2 - \frac{q^4}{4} - J^2 = 0. \qquad (2.39)$$

If the first equality is satisfied, then $q = J = 0$. From the equality in (2.38), we then obtain $A_{\min} = 0$, which contradicts the existence of two ends. Thus, the second equality must hold. This, however, is equivalent to the case of equality in (2.34). The desired result now follows from Theorem 2.3. $\qquad \square$

III. CHARGE AND ANGULAR MOMENTUM SEPARATELY

It is typically thought that charge and angular momentum behave in a somewhat similar manner with regard to such geometric inequalities. However, in the context of (2.37), their behavior is quite different when multiple horizons are present. Let us first consider the case of charge alone, that is $q \neq 0$ and $J = 0$. In this case the assumption of simple connectivity, axial symmetry, maximality, and the existence of a twist potential are not required.

Theorem 3.1 Let (M, g, k, E, B) be an initial data set without charged matter and satisfying the dominant energy condition.

(1) If the initial data are asymptotically flat with one end, and possess a single component boundary consisting of an outermost apparent horizon with area A, then

$$\sqrt{\frac{A}{4\pi}} \geq m - \sqrt{m^2 - q^2}. \qquad (3.1)$$

(2) If the initial data are maximal with two ends, one (M^1_{end}) symptotically flat and the other (M^2_{end}) either asymptotically flat or asymptotically cylindrical, then

$$\sqrt{\frac{A_{\min}}{4\pi}} \geq m - \sqrt{m^2 - q^2}, \qquad (3.2)$$

where $A_{\min}$ is the minimum area required to enclose M^2_{end}. Moreover, equality holds if and only if the initial data arise from an extreme Reissner-Nordström spacetime.

Proof. Statement (1) is proven in Corollary 2 of [13]. The inequality in statement (2) is equivalent to (2.37) in Theorem 2.5 when $J = 0$, and may be established in the same way, since the positive mass theorem with charge [21,22] as well as the area-charge inequality [20,23] are valid under the current hypotheses.

Ideally one would like to show that (3.1) holds when A is replaced by the minimum area required to enclose the outermost apparent horizon. This is of course a stronger result; however, the relevant area-charge inequality needed to establish it is currently not available. Moreover, the case of equality in (3.1) should also imply that the initial data arise from the extreme Reissner-Nordström spacetime. However, the relevant case of equality for the positive mass theorem with charge, needed to establish this result, is also currently not available, although a proposal for its proof has been put forth in [24].

Consider now the case of angular momentum alone, that is $q = 0$ and $J \neq 0$. The situation for a single black hole is very similar to that of charge alone. For instance, as a corollary of Theorem 2.5 we have the following statement.

Corollary 3.2. Let (M, g, k) be a simply connected, axially symmetric, maximal initial data set with two ends, one (M^1_{end}) asymptotically flat and the other (M^2_{end}) either asymptotically flat or asymptotically cylindrical. If the dominant energy condition is satisfied, and $P_i \eta^i = 0$ (all of which are satisfied in vacuum), then

$$\frac{A_{\min}}{8\pi} \geq m^2 - \sqrt{m^4 - J^2}, \qquad (3.3)$$

where $A_{\min}$ is the minimum area required to enclose M^2_{end}. Moreover, equality holds if and only if the initial data arise from an extreme Kerr spacetime.

When multiple black holes are present, similarity between the charged case and the angular momentum case break down. To see this, recall that the Majumdar-Papapetrou spacetime—the static extremal black hole solution to the electrovacuum equations—violates the area-charge inequality whenever two or more black holes are present. Based on this observation, Weinstein and Yamada were able to perturb Majumdar-Papapetrou initial data to find the following counterexample to the lower bound (3.2).

Theorem 3.3 ([25]). There exists a time symmetric ($k = 0$), asymptotically flat initial data set (M, g) for the

Einstein-Maxwell system, having outermost minimal surface boundary (with two components) and such that

$$\frac{A}{4\pi} < m - \sqrt{m^2 - q^2}, \qquad (3.4)$$

where A is the area of the boundary.

On the other hand, consider the case of multiple black holes with angular momentum alone. Let us label the areas of the stable minimal surfaces and angular momentums of each black hole by A_i and J_i, $i = 1, \ldots, I$ respectively. Under the assumption of maximal axisymmetric initial data, satisfying the dominant energy condition, the area-angular momentum inequality [20] for single black holes implies that

$$A = \sum_{i=1}^{I} A_i \geq \sum_{i=1}^{I} 8\pi |J_i| \geq 8\pi \left| \sum_{i=1}^{I} J_i \right| = 8\pi |J|. \quad (3.5)$$

Thus, the area-angular momentum inequality holds for multiple black holes, since the left-hand side is additive in A and subadditive in J. This leads to the following conjecture.

Conjecture 3.4. Let (M, g, k) be a simply connected, axially symmetric, maximal initial data set with multiple ends, one (M^1_{end}) asymptotically flat and the others (M^i_{end}), $i = 2, \ldots, I$ either asymptotically flat or asymptotically cylindrical. If the dominant energy condition is satisfied, and $P_i \eta^i = 0$ (all of which are satisfied in vacuum), then

$$\frac{A}{8\pi} \geq m^2 - \sqrt{m^4 - J^2}, \qquad (3.6)$$

where J is the sum of the angular momentums, and A is the sum of the areas of the stable minimal surfaces enclosing each end M^i_{end}, $i = 2, \ldots, I$. Moreover, equality holds if and only if the initial data arise from an extreme Kerr spacetime.

If the positive mass theorem with angular momentum for multiple black holes were known to be valid

$$m^2 \geq \left| \sum_{i=1}^{I} J_i \right|, \qquad (3.7)$$

then we could establish (3.6) with the help of (3.5)

$$\begin{aligned} m^2 &= \sqrt{m^4 - J^2 + J^2} \\ &\leq |J| + \sqrt{m^4 - J^2} \\ &\leq \frac{A}{8\pi} + \sqrt{m^4 - J^2}. \end{aligned} \qquad (3.8)$$

Furthermore, there is strong physical evidence in support of (3.7). Namely, the same heuristic arguments presented in Sec. I, and based on cosmic censorship, may be used to derive (3.7). It then appears likely that Conjecture 3.4 is true. Hence we find distinctly different behavior with regard to the lower bound (2.37) in the case of multiple black holes, as counterexamples exist when $q \neq 0$, $J = 0$

and counterexamples should not exist when $q = 0$, $J \neq 0$. The key reason for this difference seems to be that the area-angular momentum inequality is subadditive in J, whereas the area-charge inequality does not have this property for q. Moreover, there do not exist analogues of the Majumdar-Papapetrou solutions in the case of angular momentum alone, and somehow the inequality (2.37) seems to know this fact.

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