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A JANG EQUATION APPROACH TO THE PENROSE INEQUALITY

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Abstract. We introduce a generalized version of the Jang equation, designed for the general case of the Penrose Inequality in the setting of an asymptotically flat space-like hypersurface of a spacetime satisfying the dominant energy condition. The appropriate existence and regularity results are established in the special case of spherically symmetric Cauchy data, and are applied to give a new proof of the general Penrose Inequality for these data sets. When appropriately coupled with an inverse mean curvature flow, analogous existence and regularity results for the associated system of equations in the nonspherical setting would yield a proof of the full Penrose Conjecture. Thus it remains as an important and challenging open problem to determine whether this system does indeed admit the desired solutions.

1. **Introduction.** In 1978 P. S. Jang introduced a quasilinear elliptic equation [9], which Schoen and Yau [15] successfully employed to reduce the positive mass theorem for general Cauchy data to the case of time symmetry. For this reason it has been widely suggested that the Jang equation could be used in a similar way to reduce the general Penrose Inequality to the time symmetric case. However as pointed out by Malec and Ó Murchadha [12], serious issues arise when one tries to apply the steps taken by Schoen and Yau in [15] without modification (other issues with this process will be pointed out below). Therefore a new idea is needed, and in this paper is provided in the form of a generalized Jang equation specifically designed to treat the Penrose Inequality (PI).

In order to motivate the modification to the Jang equation, let us recall the precise statement of the PI as well as the suggested method of proof via the classical Jang equation. An initial data set for the Einstein equations is a triple (M, g, k) consisting of a 3-manifold M (for our purposes with boundary) on which a positive definite metric g and a symmetric 2-tensor (the extrinsic curvature) k are defined,

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which satisfy the constraint equations

$$\begin{aligned} 16\pi\mu &= R - k^{ij}k_{ij} + (g^{ij}k_{ij})^2, \\ 8\pi J_i &= \nabla^j(k_{ij} - (g^{ab}k_{ab})g_{ij}), \end{aligned}$$

where R is the scalar curvature, ∇^j denotes covariant differentiation, and μ and J_i are the local matter and momentum densities respectively. If the initial data are asymptotically flat, satisfy the dominant energy condition $\mu \geq |J|_g$, and contain an apparent horizon boundary ∂M, then the PI relates the total ADM mass (of a chosen end) M_{ADM} to the area A of its outermost minimal area enclosure (for ∂M) by the inequality

$$M_{\text{ADM}} \geq \sqrt{\frac{A}{16\pi}}. \tag{1}$$

Furthermore it asserts that if equality holds and the outermost minimal area enclosure is the boundary of an open bounded domain $U \subset M$, then $(M - U, g)$ admits an isometric embedding into the Schwarzschild spacetime with second fundamental form given by k. The suggested approach for confirming this statement is as follows. Look for a surface Σ in the product manifold $(M \times \mathbb{R}, g + dt^2)$ given by the graph of a function $t = f(x)$, where f is a solution of Jang's equation. Then the induced metric $\overline{g} = g + df^2$ on Σ has a certain positivity property for its scalar curvature $\overline{R}$, and the ADM mass remains unchanged. One then uses this positivity property to solve the scalar curvature equation

$$\Delta_{\overline{g}} u - \frac{1}{8}\overline{R}u = 0 \tag{2}$$

on Σ, to obtain a new metric $u^4\overline{g}$ with zero scalar curvature, and smaller mass. Thus the hope is that the area of $\partial\Sigma$ in the new metric $u^4\overline{g}$ is greater than or equal to the area of ∂M in the original metric g, so that an application of the Riemannian PI would give the desired result. However, as in [15] it is expected that the correct boundary behavior for Σ is to blow-up and approximate a cylinder over ∂M, but this implies that the solution u of (2) must vanish exponentially fast at $\partial\Sigma$ (as observed in [12] and [15]) so that we obtain no contribution from the area of ∂M and hence little hope of establishing (1). Another failure of this method is that it has no chance of working in the case of equality, where we wish to embed the initial data into the Schwarzschild spacetime. The problem here is that the Schwarzschild spacetime is given by a warped product metric, and the classical Jang approach only gives an embedding into a pure product metric.

So we see that there are several problems with the classical approach to the Penrose Inequality. The biggest of these problems is the fact that when the classical Jang surface blows-up inside the product metric $g + dt^2$, it blows-up like a cylinder. In other words, the boundary of the Jang surface is infinitely far away from every point in the surface; this is what causes the conformal factor to have zero Dirichlet boundary data. A natural (and probably first) idea that comes to mind in order to overcome both difficulties (this one, and the case of equality) is to consider the warped product metric $g + \phi^2 dt^2$ instead of the product metric, and require the warping factor ϕ to vanish on ∂M. Note that this is compatible with the case of equality since the warping factor for the Schwarzschild metric also satisfies this property. We would also like to point out that although the classical Jang equation has virtually no chance of establishing the full PI, it has been shown to yield a Penrose-Like Inequality [10] for general initial data.

Lastly we encourage those who are interested in the current paper, to compare the motivations and perspective presented here with the equivalent spacetime formulation presented in [2]. More precisely, this paper generalizes the Schoen/Yau approach ([13]) to the Positive Mass Theorem in a way which is suitable for the Penrose Inequality, whereas [2] derives its stimulus from the dual Lorentzian setting. We also recommend the excellent recent survey paper [13], as well as the forthcoming article [3] which yields a counterexample to a generalized version of the PI proposed in [2].

2. **The generalized Jang equation.** At this point we see that because of several considerations it is natural to make the first modification of the Jang approach, by looking for the Jang surface inside the warped product space $(M\times\mathbb{R}, g+\phi^2dt^2)$. In order to have any chance of obtaining a positivity property for the scalar curvature here, we would like the Jang surface Σ to satisfy an equation with the same structure, namely

$$H_\Sigma - \mathrm{Tr}_\Sigma K = 0, \tag{3}$$

where H_Σ denotes the mean curvature, the tensor K on $M\times\mathbb{R}$ is an extended version of k from the initial data, and $\mathrm{Tr}_\Sigma K$ denotes the trace of K over Σ. Of course we are free to extend k as we wish. Note that Schoen and Yau chose to extend k trivially, however as we will see this extension will not be appropriate for our problem. The first consideration when looking for a choice of extension, is that we would like the solutions of Jang's equation to blow-up at the horizon just as in the classical case, because this gives zero mean curvature and preserves the area of the horizon inside the Jang surface. However, it is easily seen that the trivial extension will not allow this in the warped product metric.

Let us consider the extension:

$$\begin{aligned}
K(\partial_{x^i},\partial_{x^j}) = K(\partial_{x^j},\partial_{x^i}) &= k(\partial_{x^i},\partial_{x^j}) \quad \text{for} \quad 1\le i,j\le 3,\\
K(\partial_{x^i},\partial_{x^4}) = K(\partial_{x^4},\partial_{x^i}) &= 0 \qquad\qquad\quad\ \ \text{for} \quad 1\le i\le 3,\\
K(\partial_{x^4},\partial_{x^4}) &= k_{44},
\end{aligned}$$

where x^i, $i=1,2,3$, are local coordinates on M, $x^4=t$ is the coordinate on $\mathbb{R}$, and k_{44} is to be determined. If the Jang surface blows-up at the horizon appropriately then it will still approximate a cylinder over the horizon, but a calculation shows that

$$H_{\partial M\times\mathbb{R}} = H_{\partial M} + \phi^{-1}\langle n_g, \nabla_g\phi\rangle_{g_\phi} \quad \text{and} \quad \mathrm{Tr}_{\partial M\times\mathbb{R}}K = \mathrm{Tr}_{\partial M}k + \phi^{-2}k_{44},$$

where n_g is the unit inner normal to ∂M inside (M,g) and g_ϕ is the warped product metric. Therefore since the unit normal to Σ, given by

$$N = \frac{\nabla_g f - \phi^{-2}\partial_{x^4}}{\sqrt{\phi^{-2}+|\nabla_g f|^2}}$$

where $x^4 = f(x^1,x^2,x^3)$ expresses Σ as a graph, converges to $\mp n_g$ in the process of blowing up to $\pm\infty$, it is natural to choose

$$k_{44} = \langle N, \phi\nabla_g\phi\rangle_{g_\phi} = \frac{\langle\nabla_g f, \phi\nabla_g\phi\rangle_g}{\sqrt{\phi^{-2}+|\nabla_g f|^2}} \tag{4}$$

since ∂M is an apparent horizon and thus satisfies

$$H_{\partial M} \pm \mathrm{Tr}_{\partial M}k = 0.$$

The $\pm$ indicates a future (past) horizon respectively, and the same expression for k_{44} is valid for both since the Jang surface will blow-up to $+\infty$ at a future horizon and down to $-\infty$ at a past horizon. In other words when k_{44} is chosen in this way, it is possible for the Jang surface to have the desired blow-up behavior at horizons.

When the tensor k is extended according to (4), we will refer to equation (3) as the generalized Jang equation. It is important to note that this particular extension has a natural interpretation in the dual Lorentzian setting, that is in the setting of the static spacetime $(M \times \mathbb{R}, g - \phi^2 dt^2)$. More precisely, if we consider the Jang surface Σ inside this spacetime then the generalized Jang equation (3) expresses the fact that the second fundamental form of Σ in the Lorentzian setting and the data k, when both are pulled back to the $t = 0$ slice, have the same trace over the metric on the $t = 0$ slice (see Appendix B for the relevant calculations). Thus, the extension given by (4) can be interpreted as the trivial extension in the dual Lorentzian setting, a fact which is of paramount importance when proving the rigidity statement in the case of equality for (1).

It turns out that this choice of extension given by (4) actually solves three problems. Namely, as we have seen it allows the modified Jang equation to have solutions which blow-up at horizons, second as we will see later (and eluded to in the previous paragraph) it is precisely what is needed for the case of equality, and third it is used to establish a positivity property for the scalar curvature of the Jang surface Σ in the warped product metric. The following formula for the Jang surface Σ in the warped product metric is one of the most important observations of this paper, as it is fundamental for any approach taken towards the general PI. As the proof is heavy with calculation, it is placed in Appendix A.

Theorem 1. *Let μ and J denote the local energy density and current density associated with the initial data, respectively. If the surface Σ satisfies the generalized Jang equation (3) and is given by a graph $t = f(x)$, then its scalar curvature $\overline{R}$ is given by*

$$\overline{R} = 16\pi(\mu - J(w)) + |h - K|_\Sigma|_g^2 + 2|q|_g^2 - 2\phi^{-1}\mathrm{div}_{\overline{g}}(\phi q), \tag{5}$$

where h is the second fundamental form, $K|_\Sigma$ is the restriction to Σ of the extended tensor K, q is a 1-form and w is a vector with $|w|_g \leq 1$ given by

$$w = \frac{f^i \partial_{x^i}}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}, \qquad q_i = \frac{f^j}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}(h_{ij} - (K|_\Sigma)_{ij}),$$

with $f^j = g^{ij} f_{,i}$.

Remark. The full formula for $\overline{R}$, when Σ does not satisfy any equation, is given in [2], and is referred to as the generalized Schoen-Yau identity.

Note that (5) reduces to the formula obtained by Schoen and Yau in [15] when $\phi \equiv 1$, which of course corresponds to the case of the classical Jang equation. Furthermore the dominant energy condition ensures that $\mu \geq |J|_g$, so that only the $\mathrm{div}_{\overline{g}}$ term prevents $\overline{R}$ from being nonnegative. However, as we shall see, with an appropriate choice of the warping factor ϕ this difficulty can be overcome to yield the PI, once a full existence theory for the generalized Jang equation coupled to an inverse mean curvature flow has been established.

3. **Existence for the generalized Jang equation in spherical symmetry.**

In this section we prove the necessary existence and regularity result needed for the generalized Jang equation, if it is to be applied to the PI. We will restrict ourselves to spherically symmetric initial data. Therefore the metric g and extrinsic curvature k have the form

$$g = g_{11}(r)dr^2 + \rho^2(r)d\Omega^2, \qquad k_{ij} = n_i n_j k_a + (g_{ij} - n_i n_j)k_b,$$

for some functions g_{11}, ρ, k_a, k_b with the appropriate fall-off conditions at infinity (to be specified below), where

$$n = n^1\partial_r + n^2\partial_{\psi^2} + n^3\partial_{\psi^3} = \sqrt{g^{11}}\partial_r$$

is the unit normal to spheres centered at the origin which will be denoted by S_r, and

$$d\Omega^2 = (d\psi^2)^2 + \sin^2\psi^2(d\psi^3)^2$$

is the round metric on $\mathbb{S}^2$. We assume that $M = \mathbb{R}^3 - B_0$ (B_0 is the ball with boundary S_0) so that $\partial M = S_0$, with S_0 an apparent horizon. Furthermore, we assume that no other apparent horizons exist in M. This means that the (outgoing) null expansions satisfy

$$\theta_\pm = 2\left(\sqrt{g^{11}}\frac{\rho_{,r}}{\rho} \pm k_b\right)(r) > 0, \qquad r > 0, \tag{6}$$

where $\rho_{,r} = \frac{d\rho}{dr}$, and that either $\theta_+(0) = 0$, $\theta_-(0) = 0$, or $\theta_+(0) = \theta_-(0) = 0$, depending on whether S_0 is a future horizon, past horizon, or both, respectively.

We now derive the generalized Jang equation. Let the Jang surface Σ be given as the graph of a function $t = f(r)$, then the unit normal to Σ in the warped product metric $g_\phi = g + \phi^2 dt^2$ is

$$N = \frac{g^{11}f_{,r}\partial_r - \phi^{-2}\partial_t}{\sqrt{\phi^{-2} + g^{11}f_{,r}^2}} := N^1\partial_r + N^2\partial_{\psi^2} + N^3\partial_{\psi^3} + N^4\partial_t.$$

The mean curvature of Σ with respect to N is

$$H_\Sigma = \sum_{i=1}^{4} N^i_{;i} = \sum_{i=1}^{4}\frac{1}{\sqrt{|g_\phi|}}\partial_i\left(\sqrt{|g_\phi|}N^i\right) = \frac{\sqrt{g^{11}}}{\phi}\partial_r(\phi v) + 2\sqrt{g^{11}}\frac{\rho_{,r}}{\rho}v$$

where $|g_\phi| = \det g_\phi$, $N^i_{;i}$ denotes covariant differentiation with respect to g_ϕ, and

$$v = \frac{\phi\sqrt{g^{11}}f_{,r}}{\sqrt{1 + \phi^2 g^{11}f_{,r}^2}}.$$

Furthermore the extension K of the extrinsic curvature given by (4) requires that

$$k_{44} = \frac{g^{11}\phi\phi_{,r}f_{,r}}{\sqrt{\phi^{-2} + g^{11}f_{,r}^2}},$$

so that if $\overline{g} = g + \phi^2 df^2$ denotes the induced metric on Σ we have

$$\begin{aligned} \mathrm{Tr}_\Sigma K &= \overline{g}^{ij}k_{ij} + \overline{g}^{ij}f_{,i}f_{,j}k_{44} \\ &= k_a + 2k_b - \frac{g^{11}f_{,r}^2}{\phi^{-2} + g^{11}f_{,r}^2}k_a + \left(\frac{g^{11}f_{,r}^2}{1 + \phi^2 g^{11}f_{,r}^2}\right)\left(\frac{g^{11}\phi^2\phi_{,r}f_{,r}}{\sqrt{1 + \phi^2 g^{11}f_{,r}^2}}\right) \\ &= (1 - v^2)k_a + 2k_b + \sqrt{g^{11}}\frac{\phi_{,r}}{\phi}v^3. \end{aligned}$$

Thus the generalized Jang equation (3) takes the form

$$\sqrt{g^{11}}v_{,r} + 2\left(\sqrt{g^{11}}\frac{\rho_{,r}}{\rho}v - k_b\right) + (v^2-1)k_a + \sqrt{g^{11}}v\frac{\phi_{,r}}{\phi}(1-v^2) = 0. \tag{7}$$

For the proof of the PI in the next section we will need to set

$$\phi = \rho_{,s} = \frac{\sqrt{1-v^2}}{\sqrt{g_{11}}}\rho_{,r} \tag{8}$$

where s is the radial arc length parameter in the $\overline{g}$ metric, that is

$$s = \int_0^r \sqrt{g_{11} + \phi^2 f_{,r}^2} = \int_0^r \frac{\sqrt{g_{11}}}{\sqrt{1-v^2}}.$$

Thus our existence results shall only concern the case in which ϕ is given by (8). First note that $\rho_{,r}(r) > 0$, $r > 0$, since the condition (6) shows that

$$2\sqrt{g^{11}}\frac{\rho_{,r}}{\rho}(r) = H_{S_r,g} = \frac{1}{2}(\theta_+ + \theta_-)(r) > 0, \qquad r > 0,$$

so ϕ is well-defined. Secondly, when ϕ is given by (8) we have

$$\sqrt{g^{11}}v\frac{\phi_{,r}}{\phi}(1-v^2) = v\frac{\phi_{,s}}{\phi}\sqrt{1-v^2} = v\frac{\rho_{,ss}}{\rho_{,s}}\sqrt{1-v^2},$$

and

$$\rho_{,ss} = \rho_{,rr}\left(\frac{1-v^2}{g_{11}}\right) - \frac{vv_{,r}}{g_{11}}\rho_{,r} - \frac{1}{2}\frac{g_{11,r}}{g_{11}^2}(1-v^2)\rho_{,r}.$$

Therefore, the generalized Jang equation (7) becomes

$$\sqrt{g^{11}}(1-v^2)v_{,r} + (1-v^2)F_{\mp}(r,v) \pm \theta_{\mp} = 0 \tag{9}$$

where

$$F_{\mp}(r,v) = \mp 2\sqrt{g^{11}}\frac{\rho_{,r}}{\rho}\frac{1}{1\pm v} - k_a + \frac{v}{\sqrt{g^{11}}}\frac{\rho_{,rr}}{\rho_{,r}} - \frac{v}{2}\frac{g_{11,r}}{g_{11}^2}.$$

It is interesting to note the role of the null expansions in equation (9). It turns out that the outermost apparent horizon hypothesis (6) is the primary reason that we are able to obtain an existence and regularity result in M. This is analogous with the theory developed by Schoen and Yau in [15] for the classical Jang equation, in that the absence of horizons leads to regularity. Furthermore observe that according to the definition of v, $v = \pm 1$ corresponds to blow-up of the Jang surface Σ. Thus for us, by regularity of a solution to (9) we mean not only that the solution possesses a large number of continuous derivatives, but that it satisfies $-1 < v < 1$ as well. The following existence result is what we require for the PI in the next section. Assume that the initial data satisfy the following fall-off conditions as $r \to \infty$:

$$\begin{aligned} |k(r)|_g &\leq Cr^{-2}, \qquad |\mathrm{Tr}_g k(r)| \leq Cr^{-3}, \\ |(g_{11}-1)(r)| + r|g_{11,r}(r)| &\leq Cr^{-1}, \qquad |\rho(r) - r| + r|\rho_{,r}(r) - 1| + r^2|\rho_{,rr}(r)| \leq C, \end{aligned} \tag{10}$$

for some constant C.

Theorem 2. *Assume that the initial data are smooth, satisfy the outermost apparent horizon condition (6), and the asymptotics (10). Then given $\alpha \in (-1,1)$ there exists a unique solution $v \in C^\infty((0,\infty)) \cap C^1([0,\infty))$ of (9) such that $-1 < v(r) < 1$, $r > 0$, and $v(0) = \alpha$. If S_0 is a past (future) horizon then the same* *conclusion holds with* $v(0) = \pm 1$*, respectively. Furthermore the solution* v *has the following asymptotics*

$$|v(r)| + r|v_{,r}(r)| \leq Cr^{-2}, \quad as \quad r \to \infty,$$

for a constant C *depending only on* $|g|_{C^1((0,\infty))}$ *and* $|k|_{C^0((0,\infty))}$.

Remark. The generalized Jang equation addressed here corresponds to a specific choice of ϕ, namely that given by (8).

Proof. We first establish the fundamental a priori estimate

$$-1 < v(r) < 1, \qquad r > 0, \tag{11}$$

as a consequence of the outermost apparent horizon condition (6). First consider the case when $|v(0)| < 1$. Then arguing by contradiction there must exist a smallest value $r_0 > 0$ such that $v(r_0) = 1$. It follows that there is an $\varepsilon > 0$ such that

$$v(r) < 1, \qquad v_{,r}(r) \geq 0, \qquad r_0 - r < \varepsilon.$$

However from equation (9) and (6) we have

$$\sqrt{g^{11}}(1 - v^2)v_{,r}(\overline{r}) = -\theta_-(\overline{r}) - (1 - v^2)F_-(\overline{r}, v) < 0 \quad \text{for some} \quad r_0 - \overline{r} < \varepsilon.$$

A similar argument can be used if $v(r_0) = -1$ by replacing θ_- with θ_+. This establishes (11) if $|v(0)| < 1$. If $v(0) = 1$ and S_0 is a past horizon then these same arguments yield (11) as long as $v_{,r}(0) < 0$ (similarly if $v(0) = -1$ and S_0 is a future horizon then we need $v_{,r}(0) > 0$). To confirm this it suffices to write out partial Taylor expansions at $r = 0$ for all functions appearing in (9). It follows that

$$\sqrt{g^{11}}(0)v_{,r}(0)^2 + F_-(0, v)v_{,r}(0) - \frac{1}{2}\theta_{-,r}(0) = 0. \tag{12}$$

We can assume without loss of generality that $\theta_{-,r}(0) > 0$ (which is of course consistent with (6)), by slightly perturbing the initial data. Therefore we find that $v_{,r}(0) < 0$ as desired. This establishes (11).

In order to prove existence, we need to obtain a priori estimates for the derivatives of v. The first task in this direction is to improve (11). Let (r_0, r_1) be an interval on which $v(r) \geq 0$ and $v_{,r}(r) \leq 0$, then

$$v(r) \leq v(r_0), \qquad r \in (r_0, r_1). \tag{13}$$

If (r_0, r_1) is an interval on which $v(r) \geq 0$ and $v_{,r}(r) \geq 0$ then from equation (9) and (11) we have

$$0 \leq -\theta_-(r) + \overline{F}_-(r)(1 - v^2), \qquad r \in (r_0, r_1),$$

where

$$|F_-(r)| \leq \overline{F}_-(r) \quad \text{with} \quad Cr^{-1} \leq \overline{F}_-(r) \quad \text{and} \quad \frac{\theta_-(r)}{\overline{F}_-(r)} \to C^{-1} \quad \text{as} \quad r \to \infty,$$

for some universal constant $C > 0$. Then according to (6) and the fall-off conditions (10),

$$(1 - v^2) \geq \frac{\theta_-(r)}{\overline{F}_-(r)} \geq \delta > 0 \tag{14}$$

for some $0 < \delta < 1$ independent of the interval (r_0, r_1) if $r_0 \geq \varepsilon > 0$. Similar estimates can be obtained when $v(r) \leq 0$. Thus by combining (13) and (14) we conclude that there exists $0 < \delta(\alpha) < 1$ for each $|\alpha = v(0)| < 1$ such that

$$-1 + \delta(\alpha) \leq v(r) \leq 1 - \delta(\alpha), \qquad r \in [0, \infty). \tag{15}$$

If $v(0) = \pm 1$ and S_0 is a past (future) horizon then $v_{,r}(0) < (>) 0$ from (12), so the same arguments provide $0 < \delta(\varepsilon) < 1$ for each $\varepsilon > 0$ such that

$$-1 + \delta(\varepsilon) \leq v(r) \leq 1 - \delta(\varepsilon), \qquad r \in [\varepsilon, \infty). \tag{16}$$

With the aid of (15) and (16) we can now simply differentiate equation (9) to inductively show that there exist constants $C(l,\alpha)$, $|\alpha = v(0)| < 1$, $l \in \mathbb{Z}_+$ such that

$$|v|_{C^l([0,\infty))} \leq C(l,\alpha),$$

and constants $C(l,\varepsilon)$, $\varepsilon > 0$, if $v(0) = \pm 1$ and S_0 is a past (future) horizon, such that

$$|v|_{C^l([\varepsilon,\infty))} \leq C(l,\varepsilon).$$

Moreover because we can solve for $v_{,r}(0)$ from (12), we also obtain the global C^1 estimate

$$|v|_{C^1([0,\infty))} \leq C.$$

At this point we can then make a standard application of the Leray-Schauder fixed point theorem (or alternatively the method of continuity) to obtain a global solution $v \in C^\infty((0,\infty)) \cap C^1([0,\infty))$ with prescribed $v(0) \in [-1,1]$. Of course if $v(0) = \pm 1$ then we require S_0 to be a past (future) horizon respectively.

Lastly we show that v has the correct asymptotics at infinity. By (15), (16), and the fall-off conditions (10) we can write equation (9) as

$$v_{,r} + \frac{2r^{-1}}{1-v^2} v = O(r^{-3} + r^{-2}v), \qquad 0 < r_0 < r < \infty, \tag{17}$$

noting that

$$\mathrm{Tr}_g k = k_a + 2k_b.$$

Thus the solution on the interval (r_0, ∞) can be represented by

$$v(r) = \exp\left(-\int_{r_0}^{r} \frac{2r^{-1}}{1-v^2}\right)\left[\int_{r_0}^{r} O(r^{-3} + r^{-2}v)\exp\left(\int_{r_0}^{r} \frac{2r^{-1}}{1-v^2}\right) + v(r_0)\right].$$

It follows that

$$|v(r)| \leq Cr^{-1} \quad \text{as} \quad r \to \infty.$$

Plugging this back into the above representation produces

$$|v(r)| \leq Cr^{-2} \quad \text{as} \quad r \to \infty. \tag{18}$$

Therefore from (17) and (18) we have

$$|v_{,r}(r)| \leq C(r^{-3} + r^{-1}|v(r)|) \leq Cr^{-3} \quad \text{as} \quad r \to \infty.$$

□

4. **Proof of the Penrose inequality in the case of spherical symmetry.** In this section we show how to apply the generalized Jang equation to treat the PI as stated in section §1. The proof presented here is restricted to the case of spherically symmetric initial data. However as illustrated in the next section, this method could be generalized to cover arbitrary data if an analogous existence result for the generalized Jang equation is established. A significant difference in the general case is that it is necessary to solve a system of equations (see [2]), whereas in the case of spherical symmetry only the generalized Jang equation need be solved, as the system actually decouples. Note that several different proofs of the PI for spherically symmetric initial data have been put forward (eg. [5], [6], [8], [11]). The

proof presented below is new, and appears to be novel in that it has the potential to generalize.

Using notation already established in the previous section, let $\overline{g} = g + \phi^2 df^2$ denote the induced metric on the Jang surface and write

$$\overline{g} = ds^2 + \rho^2(s)d\Omega^2,$$

where

$$s = \int_0^r \sqrt{g_{11} + \phi^2 f_{,r}^2} = \int_0^r \frac{\sqrt{g_{11}}}{\sqrt{1-v^2}}$$

is the radial arclength parameter in the $\overline{g}$ metric. We first derive the Hawking mass. By trying to transform $\overline{g}$ into a Schwarzschild metric we have

$$\overline{g} = \frac{1}{\rho_{,s}^2}d\rho^2 + \rho^2 d\Omega^2 = \left(1 - \frac{2m(s)}{\rho}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2$$

for some function $m(s)$. Solving for $m(s)$ produces

$$2m = \rho(1 - \rho_{,s}^2),$$

where

$$\rho_{,s} = \frac{\sqrt{1-v^2}}{\sqrt{g_{11}}}\rho_{,r}.$$

As in the previous section let S_r denote a sphere of radius r, then

$$A_{\overline{g}}(S_r) = A_g(S_r) = 4\pi\rho^2, \qquad H_{S_r,\overline{g}} = 2\frac{\rho_{,s}}{\rho} = 2\frac{\sqrt{1-v^2}}{\sqrt{g_{11}}}\frac{\rho_{,r}}{\rho}, \tag{19}$$

where $A_{\overline{g}}(S_r)$ and $H_{S_r,\overline{g}}$ denote area and mean curvature in the $\overline{g}$ metric respectively. It follows that

$$m(r) = \frac{1}{2}\rho(1-\rho_{,s}^2) = \sqrt{\frac{A_{\overline{g}}(S_r)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{S_r} H_{S_r,\overline{g}}^2 d\sigma_{\overline{g}}\right)$$

is precisely the Hawking mass with $d\sigma_{\overline{g}}$ representing the area form in the $\overline{g}$ metric. Furthermore a calculation shows that the scalar curvature of $\overline{g}$ is given by

$$\overline{R} = 2\rho^{-2}(1 - 2\rho\rho_{ss} - \rho_s^2),$$

and therefore

$$2m_{,s} = \rho_{,s} - \rho_{,s}^3 - 2\rho\rho_{,ss}\rho_{,s} = \frac{1}{2}\rho_{,s}\rho^2\overline{R}. \tag{20}$$

We will now use these formulas to obtain the PI. Set the warping factor by $\phi = \rho_{,s}$ so that Theorem 2 guarantees a unique solution of the generalized Jang equation. We also assume that S_0 is a past horizon so that we can take $v(0) = 1$ in Theorem 2 (the same arguments below will work if S_0 is a future horizon). Note that the outermost apparent horizon condition (6) guarantees that

$$H_{S_r,g} = \frac{1}{2}(\theta_+ + \theta_-)(r) > 0, \qquad r > 0,$$

which implies

$$\phi(r) = \rho_{,s}(r) = \frac{\sqrt{1-v^2}}{\sqrt{g_{11}}}\rho_{,r}(r) = \frac{\sqrt{1-v^2}}{2}\rho H_{S_r,g} > 0, \quad r > 0, \quad \phi(0) = 0, \tag{21}$$

where we have also used the estimate (11) and $v(0) = 1$. Now in order to obtain the PI just integrate equation (20):

$$m(\infty) - m(0) = \int_0^\infty m_{,s} ds = \int_0^\infty \frac{1}{4}\rho_{,s}\rho^2 \overline{R} ds = \frac{1}{16\pi}\int_\Sigma \rho_{,s}\overline{R} d\omega_{\overline{g}}$$

by (19), where $d\omega_{\overline{g}}$ is the volume form on the Jang surface Σ. Then applying the formula for $\overline{R}$ from Theorem 1, the dominant energy condition, the definition of ϕ, as well as the divergence theorem and (21), we have

$$\begin{aligned} m(\infty) - m(0) &= \frac{1}{16\pi}\int_\Sigma \rho_{,s}(16\pi(\mu - J(w)) + |h - K|_\Sigma|^2_{\overline{g}} + 2|q|^2_{\overline{g}}) d\omega_{\overline{g}} \\ &\quad -\frac{1}{8\pi}\int_\Sigma \mathrm{div}_{\overline{g}}(\phi q) d\omega_{\overline{g}} \qquad (22) \\ &\geq -\frac{1}{8\pi}\int_{\partial\Sigma\cup\partial\infty} \phi\overline{g}(q, n_{\overline{g}}) d\sigma_{\overline{g}}, \end{aligned}$$

where $K|_\Sigma$ is the restriction to Σ of the extended (by (4)) tensor K, $n_{\overline{g}}$ is the unit outer normal (as a 1-form), and q, w are given in Theorem 1. According to a calculation relegated to Appendix C

$$\phi\overline{g}(q, n_{\overline{g}}) d\sigma_{\overline{g}} = \pm\frac{2\rho_{,r}v}{\sqrt{g_{11}}}\left(\sqrt{g^{11}}\frac{\rho_{,r}}{\rho}v - k_b\right)\rho^2 d\sigma,$$

where $d\sigma$ is the Euclidean area element. Therefore the boundary integral of (22) taken over $\partial\Sigma$ is zero, since $v(0) = 1$ and S_0 is a past horizon. Also the boundary integral over $\partial\infty$ is zero as well according to the asymptotics for $v(r)$ given in Theorem 2 and the fall-off conditions (10). It follows that

$$M_{\mathrm{ADM}} - \sqrt{\frac{A_g(S_r)}{16\pi}} = m(\infty) - m(0) \geq 0,$$

since (19) and $v(0) = 1$ give $H_{S_0,\overline{g}} = 0$.

Lastly we prove the rigidity statement in the case of equality. The same arguments can be used to deal with the $\mathrm{div}_{\overline{g}}$ term in (22), thus

$$0 = M_{\mathrm{ADM}} - \sqrt{\frac{A_g(S_r)}{16\pi}} \geq \frac{1}{16\pi}\int_\Sigma \rho_{,s}(16\pi(\mu - |J|_g) + |h - K|_\Sigma|^2_{\overline{g}} + 2|q|^2_{\overline{g}}) d\omega_{\overline{g}}.$$

Hence

$$\mu - |J|_g \equiv 0, \quad h - K|_\Sigma \equiv 0, \quad q \equiv 0.$$

It then follows from Theorem 1 that $\overline{R} \equiv 0$. We can now apply the time symmetric PI to the Jang surface Σ to obtain $\overline{g} \cong g_{\mathrm{SC}}$, that is $\overline{g}$ is isometric to the standard slice of the Schwarzschild spacetime

$$g_{\mathrm{SC}} = \left(1 - \frac{2M_{\mathrm{ADM}}}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

Hence

$$\rho = r, \quad \overline{g}_{11} = \left(1 - \frac{2M_{\mathrm{ADM}}}{r}\right)^{-1},$$

so that

$$\phi = \rho_{,s} = \frac{1}{\sqrt{\overline{g}_{11}}}\rho_{,r} = \left(1 - \frac{2M_{\mathrm{ADM}}}{r}\right)^{1/2}.$$

This says that ϕ is the correct warping factor for the Schwarzschild spacetime, and furthermore since

$$g = \overline{g} - \phi^2 df^2 = g_{\mathrm{SC}} - \phi^2 df^2,$$

the graph map $G : M \to \mathbb{SC}^4$ provides an isometric embedding of the initial data (M, g) into the Schwarzschild spacetime $(\mathbb{SC}^4, g_{\mathrm{SC}} - \phi^2 dt^2)$. Finally a calculation (Appendix B) shows that $h - K|_\Sigma \equiv 0$ implies that the second fundamental form of $G(M) \subset \mathbb{SC}^4$ is precisely given by the initial data k.

5. **Approach to the general case.** Here we discuss how the approach of the previous section may be generalized to the case of arbitrary initial data (without spherical symmetry), whenever appropriate solutions exist to a canonical system of equations constructed from the generalized Jang equation and the inverse mean curvature flow. Whether or not such solutions do indeed exist is thus a very important and challenging open problem. To proceed, we assume without loss of generality that ∂M is an outermost apparent horizon. More precisely, none of the components of ∂M are separated from spatial infinity by another horizon. The idea is that if ∂M is not outermost, then we should replace M with the submanifold $\widetilde{M}$ such that $\partial\widetilde{M}$ is an outermost horizon. Then the PI for $\widetilde{M}$ implies the PI for M. This assumption is made in order to facilitate the existence of a smooth Jang surface Σ on the interior of M, which also blows-up at the boundary (this boundary behavior guarantees that $\partial\Sigma$ is minimal). Such solutions have been shown to exist in [14], at least for the classical Jang equation. Furthermore, we assume the existence of a smooth Inverse Mean Curvature Flow (IMCF) inside the Jang surface, starting from any one of the components of the outermost minimal surface enclosing ∂M (it is customary to take the one with largest area). As the name suggests, IMCF refers to the flow of 2-surfaces in Σ in which the surfaces flow in the outward normal direction at a rate equal to the inverse of their mean curvatures at each point. Originally introduced by Geroch [4], this flow has been generalized and used successfully by Huisken and Ilmanen [7] to prove the PI in the time symmetric case. Note that since $\partial\Sigma$ is minimal, the existence of an outermost minimal surface is guaranteed ([7], Lemma 4.1), and moreover the region between the outermost minimal surface and spatial infinity, denoted by $\widetilde{\Sigma}$, contains no other compact minimal surfaces, and each component of $\widetilde{\Sigma}$ has spherical topology. This observation is required so that the weak formulation of IMCF given by Huisken and Ilmanen has a smooth start at the boundary. Although use of the weak formulation is necessary, since it is not difficult to find examples where the flow develops singularities, for the sake of simplicity of exposition, in this paper we assume the existence of a smooth flow. This means that in $\widetilde{\Sigma}$, the induced metric $\overline{g}$ may be written as

$$\overline{g} = H_{S_r,\overline{g}}^{-2} dr^2 + \sum_{i,j=1}^{2} \widehat{g}_{ij} d\theta^i d\theta^j,$$

where the surfaces $r =$ const. are the flow surfaces denoted by S_r (each having spherical topology), and θ^i are local coordinates on S_r. We also set S_0 to be a component of the outermost minimal surface $\partial\widetilde{\Sigma}$.

Let $m(r)$ again be the Hawking mass, then a well-known formula (due to Geroch [4]) gives

$$\frac{dm}{dr}(r) = \sqrt{\frac{A_{\overline{g}}(S_r)}{16\pi}} \left[\frac{1}{2} + \frac{1}{16\pi}\int_{S_r}\left(2\frac{|\nabla_{S_r} H_{S_r,\overline{g}}|^2}{H_{S_r,\overline{g}}^2} + \overline{R} - 2K_{S_r} + \frac{1}{2}(\lambda_1 - \lambda_2)^2\right) d\sigma_{\overline{g}}\right]$$

where K_{S_r} is the Gaussian curvature of S_r and λ_1, λ_2 are its principal curvatures. Since each S_r has spherical topology (as the flow is assumed to be smooth), the Gauss-Bonnet Theorem shows that

$$\int_{S_r} K_{S_r} d\sigma_{\overline{g}} = 4\pi.$$

We then have

$$\begin{aligned} m(\infty) - m(0) &= \int_0^\infty \frac{dm}{dr} dr \qquad (23)\\ &\geq \frac{1}{(16\pi)^{3/2}} \int_0^\infty \left(\int_{S_r} \sqrt{A_{\overline{g}}(S_r)} \overline{R} d\sigma_{\overline{g}} \right) dr \\ &= \frac{1}{(16\pi)^{3/2}} \int_{\widetilde{\Sigma}} \sqrt{A_{\overline{g}}(S_r)} H_{S_r,\overline{g}} \overline{R} d\omega_{\overline{g}} \\ &\geq -\frac{2}{(16\pi)^{3/2}} \int_{\widetilde{\Sigma}} \frac{\sqrt{A_{\overline{g}}(S_r)} H_{S_r,\overline{g}}}{\phi} \mathrm{div}_{\overline{g}}(\phi q) d\omega_{\overline{g}} \end{aligned}$$

according to (5), the coarea formula, and the fact that $H_{S_r,\overline{g}} > 0$ under smooth IMCF. This motivates the choice

$$\phi = \sqrt{A_{\overline{g}}(S_r)} H_{S_r,\overline{g}}, \qquad (24)$$

since an application of the divergence theorem then yields

$$m(\infty) - m(0) \geq -\frac{2}{(16\pi)^{3/2}} \int_{S_0 \cup \partial\infty} \phi \overline{g}(q, n_{\overline{g}}) d\sigma_{\overline{g}}, \qquad (25)$$

where $n_{\overline{g}}$ is the unit outer normal with respect to $\overline{g}$.

In order to obtain the PI from (25), we note that since the solution of the generalized Jang equation vanishes very fast at spatial infinity, $m(\infty)$ is the original ADM mass of M and the integral at $\partial\infty$ vanishes (as ϕ remains bounded). Furthermore because S_0 is a minimal surface $m(0) = \sqrt{A_{\overline{g}}(S_0)/16\pi}$. If S_0 does not intersect $\partial\Sigma$ then the solution of the generalized Jang equation remains bounded on S_0, so $\phi|_{S_0} = 0$ implies that the boundary integral vanishes in this case. On the other hand, if a portion of S_0 coincides with $\partial\Sigma$ then we calculate the integrand on level sets of f approaching $\partial\Sigma$ as follows. Let Λ be a level set, N the unit normal to Σ, ν the unit inner normal to Λ in the horizontal space (that is, in a $t =$ const. slice of $M \times \mathbb{R}$), and τ a unit tangent to Σ which is normal to Λ (pointing inside Σ), then a calculation [2] shows that

$$H_{\Lambda,\overline{g}} = g_\phi(\tau,\nu) H_{\Lambda,g},$$

$$\overline{g}(q, n_{\overline{g}})|_\Lambda = g_\phi(\tau,\nu)^{-1}(H_{\Lambda,g} - g_\phi(N,\nu)\mathrm{Tr}_{\Lambda,g} k) - H_{\Lambda,\overline{g}},$$

where $g_\phi = g + \phi^2 dt^2$ is the metric on $M \times \mathbb{R}$ and $H_{\Lambda,g}$, $H_{\Lambda,\overline{g}}$ are the mean curvatures of Λ with respect to g and $\overline{g}$. Thus since the Jang surface Σ blows-up to $\pm\infty$ at horizons

$$g_\phi(\tau,\nu) \to 0, \quad g_\phi(N,\nu) \to \pm 1, \quad \text{as} \quad \Lambda \to \partial\Sigma.$$

The apparent horizon equations

$$H_{\partial\Sigma,g} \pm \mathrm{Tr}_{\partial\Sigma,g} k = 0$$

then imply that the boundary integral at S_0 vanishes. From (25) we now have

$$M_{\mathrm{ADM}} \geq \sqrt{\frac{A_{\overline{g}}(S_0)}{16\pi}} \geq \sqrt{\frac{A_g(S_0)}{16\pi}} \geq \sqrt{\frac{A}{16\pi}}, \qquad (26)$$

after observing that $\overline{g}$ measures areas to be at least as large as does g.

Lastly we treat the case of equality. By appealing to the above arguments including (23), and using the full expression for $\overline{R}$ in (5), we find that

$$0 \geq \int_{\widetilde{\Sigma}} \phi \left(16\pi(\mu - |J|_g) + |h - K|_\Sigma|_{\overline{g}}^2 + 2|q|_{\overline{g}}^2\right) d\omega_{\overline{g}}.$$

As $\phi > 0$ away from $\partial\widetilde{\Sigma}$ this implies that

$$\mu - |J|_g \equiv 0, \qquad h - K|_\Sigma \equiv 0, \qquad q \equiv 0,$$

from which we obtain $\overline{R} \equiv 0$. Therefore $(\widetilde{\Sigma}, \overline{g})$ is isometric to $(\mathbb{SC}^3, g_{\mathrm{SC}})$ the exterior region of the $t = 0$ slice of the Schwarzschild spacetime, by the time symmetric version of the PI. Hence we may write

$$\overline{g} = \left(1 - \frac{2M_{\mathrm{ADM}}}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

so that

$$\phi = \sqrt{A_{\overline{g}}(S_r)}H_{S_r,\overline{g}} = \sqrt{4\pi r^2}\sqrt{\overline{g}^{11}}\frac{2}{r} = 4\sqrt{\pi}\left(1 - \frac{2M_{\mathrm{ADM}}}{r}\right)^{1/2}.$$

Moreover as $g = \overline{g} - \phi^2 df^2$, it follows that the graph map $G : M \to \mathbb{SC}^4$ given by $G(x) = (x, f(x))$ provides an isometric embedding of $(M - \widetilde{U}, g)$ into the Schwarzschild spacetime $(\mathbb{SC}^4, g_{\mathrm{SC}} - \phi^2 dt^2)$, where $\partial\widetilde{U}$ is the image of S_0 in M. By (26) (with all inequalities replaced by equalities) $A_g(\partial\widetilde{U}) = A$. Then if ∂U is the outermost minimal area enclosure, we must have $\widetilde{U} \subset U$. Lastly a calculation (see [2]) shows that $h - K|_\Sigma \equiv 0$ implies that the extrinsic curvature of $G(M - \widetilde{U}) \subset \mathbb{SC}^4$ is given by k.

Now some comments concerning the above methods. The definition of ϕ in (24) is not as simple as it appears, in that ϕ is also present on the right-hand side due to the definition of $\overline{g}$. Furthermore the IMCF in Σ depends on f and ϕ for the same reason. Thus we are not only concerned with solving a single equation, namely the generalized Jang equation (3), rather we must solve a system of equations. We may write this system down in the following way. If we take the level set formulation of IMCF (as in [7]), so that the flow surfaces S_r are given by the level sets $r = u(x)$ for some function u on M, then u must satisfy the equation

$$\mathrm{div}_{\overline{g}}\left(\frac{\nabla_{\overline{g}}u}{|\nabla_{\overline{g}}u|}\right) = |\nabla_{\overline{g}}u| \tag{27}$$

in which the left-hand side represents the mean curvature of S_r and the right-hand side is the inverse speed of the flow. It then follows that ([7])

$$A_{\overline{g}}(S_r) = e^r A_{\overline{g}}(S_0) = e^u A_{\overline{g}}(S_0),$$

so by definition of ϕ and (27) we have

$$\phi^2 = A_{\overline{g}}(S_0)e^u|\nabla_{\overline{g}}u|^2. \tag{28}$$

Equations (3), (27), and (28) now form a 3×3 degenerate elliptic system for the unknowns u, ϕ, and f. In section §3 we have successfully solved this system for the special case of spherically symmetric initial data, and have found that the solution has the same behavior as conjectured in this paper for the general case. As each of the equations (3) and (27) already have full existence theories in the classical case when $\phi \equiv 1$ ([7], [15]), it is possible that similar techniques will yield the corresponding theory for this generalized Jang/IMCF system.

Finally we mention that the method proposed here, which in a nut shell can be thought of as simply integrating away the "bad" term from the expression of $\overline{R}$ in (5), can also be modified to generalize the other known proof of the time symmetric PI, namely the conformal flow proof of Bray [1]. When this is done a new modified Jang/conformal flow system is generated. While this method of proof would yield a stronger result (since it applies to multiple black holes), the system obtained is less tractable at the moment [2].

6. **Appendix A.** In this appendix we confirm Theorem 1. The following notation will be used. Suppose that Σ is a smooth hypersurface inside the warped product space $(M\times\mathbb{R}, g_\phi = g+\phi^2dt^2)$, and let e_1, e_2, e_3, e_4 be a local orthonormal frame for Σ with $e_4 = N$ normal and e_1, e_2, e_3 tangent to Σ. The Levi-Civita connection for g_ϕ will be denoted by $\nabla_a = \nabla_{e_a}$ and that for $\overline{g}$, the induced metric on Σ, by $\overline{\nabla}_a = \overline{\nabla}_{e_a}$. Furthermore x^1, x^2, x^3 will be local coordinates on M with $x^4 = t$, and the second fundamental form of Σ will be denoted

$$h_{ij} = h(e_i, e_j) = \langle \nabla_{e_j} N, e_i\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the inner product of g_ϕ.

The preliminary calculations will generalize those of [15]. We have

$$\overline{\nabla}_j\langle\partial_{x^4}, N\rangle = \langle\nabla_j\partial_{x^4}, N\rangle + \langle\partial_{x^4}, e_i\rangle h_{ij},$$

where the repeated index i is summed from 1 to 3. Next

$$\begin{aligned}\overline{\nabla}_l\overline{\nabla}_j\langle\partial_{x^4}, N\rangle \quad = \quad & \langle\nabla_l\nabla_j\partial_{x^4}, N\rangle + \langle\nabla_j\partial_{x^4}, e_i\rangle h_{il} + \langle\partial_{x^4}, e_i\rangle\overline{\nabla}_l h_{ij} \\ & +\langle\nabla_l\partial_{x^4}, e_i\rangle h_{ij} + \langle\partial_{x^4}, \nabla_l e_i\rangle h_{ij} - \langle\partial_{x^4}, e_p\rangle\overline{\Gamma}^p_{li}h_{ij},\end{aligned}$$

where $\overline{\Gamma}^p_{li}$ are Christoffel symbols for $\overline{g}$. However the Codazzi equations give

$$\overline{\nabla}_l h_{ij} - \overline{\nabla}_i h_{lj} = R_{Njil}$$

where R_{Njil} are components of the Riemann tensor for g_ϕ, and we also have

$$\begin{aligned}\langle\partial_{x^4}, \nabla_l e_i\rangle &= \langle\langle\partial_{x^4}, N\rangle N + \langle\partial_{x^4}, e_p\rangle e_p, \nabla_l e_i\rangle \\ &= -\langle\partial_{x^4}, N\rangle h_{il} + \langle\partial_{x^4}, e_p\rangle\overline{\Gamma}^p_{li}.\end{aligned}$$

Therefore, adopting the convention that indices i, j, l, p run from 1 to 3 and a, b (appearing later) run from 1 to 4, it follows that

$$\begin{aligned}\Delta_{\overline{g}}\langle\partial_{x^4}, N\rangle &= \sum_i\overline{\nabla}_i\overline{\nabla}_i\langle\partial_{x^4}, N\rangle \\ &= \sum_i\langle\nabla_i\nabla_i\partial_{x^4}, N\rangle + 2\sum_{i,j}\langle\nabla_j\partial_{x^4}, e_i\rangle h_{ij} \\ &\quad + \sum_i\overline{\nabla}_i H\langle\partial_{x^4}, e_i\rangle + \sum_{i,j}R_{Njij}\langle\partial_{x^4}, e_i\rangle - |h|^2\langle\partial_{x^4}, N\rangle,\end{aligned}$$

where

$$H = \sum_i h_{ii}, \qquad |h|^2 = \sum_{i,j}h_{ij}h_{ij}.$$

Moreover

$$\overline{\nabla}_i H\langle\partial_{x^4}, e_i\rangle = \nabla_{\partial_{x^4}}H - \nabla_N H\langle\partial_{x^4}, N\rangle = -N(H)\langle\partial_{x^4}, N\rangle,$$

and

$$R_{Njil}\langle \partial_{x^4}, e_i\rangle = -R_{NjNl}\langle \partial_{x^4}, N\rangle + \mathrm{Riem}(N, e_j, \partial_{x^4}, e_l).$$

Hence

$$\begin{aligned}\Delta_{\overline{g}}\langle \partial_{x^4}, N\rangle \;=\; & -(|h|^2 + N(H) + \sum_i R_{NiNi})\langle \partial_{x^4}, N\rangle + \sum_i \langle \nabla_i\nabla_i\partial_{x^4}, N\rangle \\ & + \sum_i \mathrm{Riem}(N, e_i, \partial_{x^4}, e_i) + 2\sum_{i,j}\langle \nabla_j \partial_{x^4}, e_i\rangle h_{ij}. \end{aligned} \tag{29}$$

Let K be the extended version (by (4)) of the initial data. We then define the extended versions of the local energy and current densities by

$$2\mu^{\mathrm{ext}} = R_{g_\phi} - \sum_{a,b} K_{ab}^2 + (\sum_a K_{aa})^2, \qquad J^{\mathrm{ext}}(e_b) = \sum_a (\nabla_a K_{ab} - \nabla_b K_{aa}),$$

where R_{g_ϕ} is the scalar curvature of g_ϕ and $K_{ab} = K(e_a, e_b)$. Notice that

$$R_{g_\phi} = 2\sum_i R_{NiNi} + \sum_{i,j} R_{ijij}$$

and by the Gauss equations

$$\overline{R}_{ijpl} = R_{ijpl} + h_{ip}h_{jl} - h_{il}h_{jp},$$

(here $\overline{R}_{ijpl}$ denotes the Riemann tensor of $\overline{g}$) which implies

$$R_{g_\phi} = 2\sum_i R_{NiNi} + \overline{R} - H^2 + |h|^2,$$

where $\overline{R}$ is the scalar curvature of $\overline{g}$. So by definition of μ^{ext},

$$\sum_i R_{NiNi} = \mu^{\mathrm{ext}} + \frac{1}{2}\left(-\overline{R} + \sum_{a,b} K_{ab}^2 - (\sum_a K_{aa})^2 - |h|^2 + H^2\right).$$

Thus (29) becomes

$$\begin{aligned} & \Delta_{\overline{g}}\langle \partial_{x^4}, N\rangle \\ = \; & -\left(\mu^{\mathrm{ext}} + \frac{1}{2}|h|^2 + \frac{1}{2}H^2 + N(H) - \frac{1}{2}\overline{R} + \frac{1}{2}\sum_{a,b} K_{ab}^2 - \frac{1}{2}(\sum_a K_{aa})^2\right)\langle \partial_{x^4}, N\rangle \\ & + \sum_i \mathrm{Riem}(N, e_i, \partial_{x^4}, e_i) + \sum_i \langle \nabla_i\nabla_i\partial_{x^4}, N\rangle + 2\sum_{i,j}\langle \nabla_j\partial_{x^4}, e_i\rangle h_{ij}. \end{aligned} \tag{30}$$

We now obtain another expression for $\Delta_{\overline{g}}\langle \partial_{x^4}, N\rangle$. First extend the second fundamental form tensor h to all of $M \times \mathbb{R}$ by

$$h(X, Y) = \langle \nabla_Y N, X\rangle, \qquad X, Y \in T_{x_0}(M \times \mathbb{R}),$$

so that

$$h_{iN} = h(e_i, N), \qquad h_{Ni} = h_{NN} = 0, \qquad i = 1, 2, 3.$$

Observe that

$$\nabla_{\partial_{x^4}} N = \sum_{i,j}\langle \partial_{x^4}, e_i\rangle h_{ij}e_j + \sum_j \langle \partial_{x^4}, N\rangle h_{jN}e_j$$

so that

$$h_{jN} = -\overline{\nabla}_j \log\langle \partial_{x^4}, N\rangle + \langle \partial_{x^4}, N\rangle^{-1}(\langle \nabla_j \partial_{x^4}, N\rangle + \langle \nabla_{\partial_{x^4}} N, e_j\rangle), \tag{31}$$

which implies

$$\begin{aligned}
&\Delta_{\overline{g}} \log\langle\partial_{x^4}, N\rangle \\
= \ & -\langle\partial_{x^4}, N\rangle^{-2}\sum_j(\overline{\nabla}_j\langle\partial_{x^4}, N\rangle)^2 + \langle\partial_{x^4}, N\rangle^{-1}\Delta_{\overline{g}}\langle\partial_{x^4}, N\rangle \\
= \ & -\sum_j[h_{jN} - \langle\partial_{x^4}, N\rangle^{-1}(\langle\nabla_j\partial_{x^4}, N\rangle + \langle\nabla_{\partial_{x^4}}N, e_j\rangle)]^2 \\
& +\langle\partial_{x^4}, N\rangle^{-1}\Delta_{\overline{g}}\langle\partial_{x^4}, N\rangle.
\end{aligned}$$

With the help of (31) we have

$$\begin{aligned}
&\langle\partial_{x^4}, N\rangle^{-1}\Delta_{\overline{g}}\langle\partial_{x^4}, N\rangle \\
= \ & \sum_j[h_{jN} - \langle\partial_{x^4}, N\rangle^{-1}(\langle\nabla_j\partial_{x^4}, N\rangle + \langle\nabla_{\partial_{x^4}}N, e_j\rangle)]^2 \\
& -\sum_j\overline{\nabla}_j h_{jN} + \sum_j\overline{\nabla}_j\left(\frac{\langle\nabla_j\partial_{x^4}, N\rangle}{\langle\partial_{x^4}, N\rangle} + \frac{\langle\nabla_{\partial_{x^4}}N, e_i\rangle}{\langle\partial_{x^4}, N\rangle}\right) \\
= \ & \sum_j(h_{jN}^2 - 2\langle\partial_{x^4}, N\rangle^{-1}\langle\nabla_j\partial_{x^4}, N\rangle h_{jN} + \langle\partial_{x^4}, N\rangle^{-2}\langle\nabla_{\partial_{x^4}}N, e_j\rangle^2) \\
& -\sum_j 2\langle\partial_{x^4}, N\rangle^{-1}\langle\nabla_{\partial_{x^4}}N, e_j\rangle(h_{jN} - \langle\partial_{x^4}, N\rangle^{-1}\langle\nabla_j\partial_{x^4}, N\rangle) \\
& +\sum_j[\langle\partial_{x^4}, N\rangle^{-1}(\langle\nabla_j\nabla_j\partial_{x^4}, N\rangle + \sum_i\langle\nabla_j\partial_{x^4}, e_i\rangle h_{ji}) - \overline{\nabla}_j h_{jN}] \\
& +\sum_j\left(\overline{\nabla}_j\frac{\langle\nabla_{\partial_{x^4}}N, e_j\rangle}{\langle\partial_{x^4}, N\rangle} - \sum_i\langle\partial_{x^4}, N\rangle^{-2}\langle\partial_{x^4}, e_i\rangle h_{ji}\langle\nabla_j\partial_{x^4}, N\rangle\right).
\end{aligned}$$

However

$$\sum_i\langle\partial_{x^4}, e_i\rangle h_{ji} = -\langle\partial_{x^4}, N\rangle h_{jN} + \langle\nabla_{\partial_{x^4}}N, e_j\rangle$$

and

$$\begin{aligned}
&\sum_j\overline{\nabla}_j\frac{\langle\partial_{x^4}N, e_j\rangle}{\langle\partial_{x^4}, N\rangle} \\
= \ & \sum_j\langle\partial_{x^4}, N\rangle^{-1}\overline{\nabla}_j\langle\nabla_{\partial_{x^4}}N, e_j\rangle \\
& -\sum_j\langle\partial_{x^4}, N\rangle^{-2}\langle\nabla_{\partial_{x^4}}N, e_j\rangle(\langle\nabla_j\partial_{x^4}, N\rangle - \langle\partial_{x^4}, N\rangle h_{jN} + \langle\nabla_{\partial_{x^4}}N, e_j\rangle),
\end{aligned}$$

therefore

$$\begin{aligned}
&\langle\partial_{x^4}, N\rangle^{-1}\Delta_{\overline{g}}\langle\partial_{x^4}, N\rangle \\
= \ & \sum_j(h_{jN}^2 - \langle\partial_{x^4}, N\rangle^{-1}\langle\nabla_j\partial_{x^4}, N\rangle h_{jN} - \overline{\nabla}_j h_{jN}) \\
& +\sum_j\langle\partial_{x^4}, N\rangle^{-1}(\langle\nabla_j\nabla_j\partial_{x^4}, N\rangle + \sum_i\langle\nabla_j\partial_{x^4}, e_i\rangle h_{ji}) \\
& +\langle\partial_{x^4}, N\rangle^{-1}\sum_j(\overline{\nabla}_j\langle\nabla_{\partial_{x^4}}N, e_j\rangle - \langle\nabla_{\partial_{x^4}}N, e_j\rangle h_{jN}).
\end{aligned} \tag{32}$$

In order to compare quantities appearing in (30) and (32) to the local current density, we employ a formula on page 239 of [15]:

$$J^{\mathrm{ext}}(N) = \sum_i \overline{\nabla}_i K_{i4} - N(\sum_i K_{ii}) + K_{NN}H - \sum_{i,j} K_{ij}h_{ij} - 2\sum_i K_{iN}h_{iN}. \quad (33)$$

This formula still remains valid in our situation. To see this observe that

$$J^{\mathrm{ext}}(N) = J^{\mathrm{ext}}(e_4) = \sum_i (\nabla_i K_{iN} - \nabla_N K_{ii}).$$

Moreover if $(\overline{\delta}^{ij}) = (\overline{g}(e_i, e_j))^{-1}$ then

$$\begin{aligned}
\sum_i \nabla_N K_{ii} &= \overline{\delta}^{ij}(N(K_{ij}) - 2\Gamma^a_{Ni}K_{ja}) \\
&= N(\sum_i K_{ii}) - N(\overline{\delta}^{ij})K_{ij} - 2\sum_{i,j}\Gamma^j_{iN}K_{ij} - 2\sum_i \Gamma^N_{Ni}K_{iN} \\
&= N(\sum_i K_{ii}) + \sum_i 2h_{iN}K_{iN}
\end{aligned}$$

since

$$\Gamma^N_{Ni} = \langle N, \nabla_N e_i\rangle = -\langle \nabla_N N, e_i\rangle = -h_{iN},$$
$$\Gamma^j_{iN} = \langle e_j, \nabla_i N\rangle = h_{ij}, \qquad N(\overline{\delta}^{ij}) = -2h_{ij},$$

and

$$\begin{aligned}
\sum_i \nabla_i K_{iN} &= \sum_i (e_i(K_{iN}) - K(\nabla_i e_i, N) - K(e_i, \nabla_i N)) \\
&= \sum_i \overline{\nabla}_i K_{iN} + HK_{NN} - \sum_{i,j} h_{ij}K_{ij}
\end{aligned}$$

since

$$\nabla_i e_i = \Gamma^N_{ii}N + \sum_j \Gamma^j_{ii}e_j = -h_{ii}N + \sum_j \overline{\Gamma}^j_{ii}e_j, \qquad \nabla_i N = \sum_j h_{ij}e_j.$$

The desired formula now follows.

Equations (30) and (32) yield an expression for μ^{ext}. Then by combining this expression with (33) we arrive at

$$\begin{aligned}
&2(\mu^{\mathrm{ext}} - J^{\mathrm{ext}}(N)) \qquad (34)\\
= {}& \overline{R} - \sum_{i,j}(h_{ij} - K_{ij})^2 - 2\sum_i (h_{iN} - K_{iN})^2 + 2\sum_i \overline{\nabla}_i(h_{iN} - K_{iN}) \\
&+ (\sum_i K_{ii})^2 - H^2 + 2K_{NN}(\sum_i K_{ii} - H) + 2N(\sum_i K_{ii} - H) \\
&+ 2\langle \partial_{x^4}, N\rangle^{-1}\sum_i (\mathrm{Riem}(N, e_i, \partial_{x^4}, e_i) + \langle \nabla_i \partial_{x^4}, \sum_j h_{ij}e_j + h_{iN}N\rangle \\
&+ \langle \nabla_{\partial_{x^4}} N, e_i\rangle h_{iN} - \overline{\nabla}_i\langle \nabla_{\partial_{x^4}} N, e_i\rangle),
\end{aligned}$$

where the repeated indices i, j are summed from 1 to 3.

The remainder of the proof will consist of evaluating certain terms from (34) in local coordinates. We assume from now on that Σ satisfies the generalized Jang equation (3), so that the 5th, 6th, and 7th terms on the right-hand side of (34) vanish. We also assume that Σ is given by the graph of a function $x^4 = f(x^1, x^2, x^3)$, and we will write $f_{,i} = \partial f/\partial x^i$, $f^i = g^{ij} f_{,j}$. Let

$$X_i = \partial_{x^i} + f_{,i}\partial_{x^4}, \qquad i = 1, 2, 3,$$

be tangent vectors to Σ and

$$N = \frac{f^i\partial_{x^i} - \phi^{-2}\partial_{x^4}}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}$$

be the unit normal to Σ. Also we will write

$$\overline{g}_{ij} = \overline{g}(X_i, X_j) = g_{ij} + \phi^2 f_{,i} f_{,j}, \qquad \overline{g}^{ij} = g^{ij} - \frac{f^i f^j}{\phi^{-2} + |\nabla_g f|^2}.$$

The next three claims will simplify (34).

Claim 1.

$$\langle \partial_{x^4}, N\rangle^{-1} \sum_i \mathrm{Riem}(N, e_i, \partial_{x^4}, e_i) = -\phi^{-1}\Delta_g \phi$$

Proof. Christoffel symbols for the metric g_ϕ in the above local coordinates are given by

$$\widehat{\Gamma}_{44}^4 = \widehat{\Gamma}_{ij}^4 = \widehat{\Gamma}_{i4}^j = 0, \qquad 1 \leq i, j \leq 3, \tag{35}$$
$$\widehat{\Gamma}_{i4}^4 = (\log\phi)_{,i}, \qquad \widehat{\Gamma}_{44}^i = -\phi\phi^i.$$

Note that $\widehat{\Gamma}_{ij}^k$ are the Christoffel symbols for the initial data metric g when $1 \leq i, j, k \leq 3$. The Riemann tensor is then given by

$$\begin{aligned} \widehat{R}_{4ijk} &= (g_\phi)_{4a}\widehat{R}_{ijk}^a = \phi^2 \widehat{R}_{ijk}^4 \\ &= \phi^2(\widehat{\Gamma}_{ik,j}^4 - \widehat{\Gamma}_{ij,k}^4 + \widehat{\Gamma}_{ik}^b\widehat{\Gamma}_{bj}^4 - \widehat{\Gamma}_{ij}^b\widehat{\Gamma}_{bk}^4) \\ &= 0, \end{aligned} \tag{36}$$

$$\begin{aligned} \widehat{R}_{4i4j} &= (g_\phi)_{4a}\widehat{R}_{i4j}^a = \phi^2 \widehat{R}_{i4j}^4 \\ &= \phi^2(\widehat{\Gamma}_{ij,4}^4 - \widehat{\Gamma}_{i4,j}^4 + \widehat{\Gamma}_{ij}^b\widehat{\Gamma}_{b4}^4 - \widehat{\Gamma}_{i4}^b\widehat{\Gamma}_{bj}^4) \\ &= -\phi\phi_{;ij}, \end{aligned} \tag{37}$$

where the semicolon denotes covariant differentiation with respect to g. Moreover if $1 \leq i, j, k, l \leq 3$ then $\widehat{R}_{ijkl}$ are just the components of the Riemann tensor for g. Therefore with the help of (36) and (37) we have

$$\begin{aligned} \sum_i \mathrm{Riem}(N, e_i, \partial_{x^4}, e_i) &= \overline{g}^{ij}\mathrm{Riem}\left(\frac{f^l\partial_{x^l} - \phi^{-2}\partial_{x^4}}{\langle\partial_{x^4}, N\rangle^{-1}}, X_i, \partial_{x^4}, X_j\right) \\ &= \langle\partial_{x^4}, N\rangle\overline{g}^{ij}(\phi^{-2}\widehat{R}_{4i4j} - f^l\widehat{R}_{li4j} - f^l f_{,i}\widehat{R}_{l44j}) \\ &= -\langle\partial_{x^4}, N\rangle\phi^{-1}\Delta_g\phi. \end{aligned}$$

□

Claim 2.

$$\langle\partial_{x^4}, N\rangle^{-1}\sum_i\langle\nabla_i\partial_{x^4}, \sum_j h_{ij}e_j + h_{iN}N\rangle$$
$$= -|\nabla_{\overline{g}}\log\phi + \phi\phi^l f_{,l}\nabla_{\overline{g}} f|^2 - \langle\partial_{x^4}, N\rangle^{-1}\overline{g}^{ij} f_{,j}\phi\phi^l h(X_i, X_l)$$

Proof. First observe that

$$\langle \nabla_i \partial_{x^4}, \sum_j h_{ij} e_j + h_{iN} N \rangle = h(e_i, \nabla_{e_i} \partial_{x^4}),$$

and therefore

$$\sum_i \langle \nabla_i \partial_{x^4}, \sum_j h_{ij} e_j + h_{iN} N \rangle = \overline{g}^{ij} h(X_i, \nabla_{X_j} \partial_{x^4}).$$

Now compute with the help of (35):

$$\begin{aligned} h(X_i, \nabla_{X_j} \partial_{x^4}) &= h(X_i, \widehat{\Gamma}^a_{j4} \partial_{x^a} + f_{,j} \widehat{\Gamma}^a_{44} \partial_{x^a}) \\ &= ((\log \phi)_{,j} + f_{,j} f_{,k} \phi \phi^k) h(X_i, \partial_{x^4}) - f_{,j} \phi \phi^k h(X_i, X_k). \end{aligned}$$

Moreover

$$\begin{aligned} h(X_i, \partial_{x^4}) &= -\langle N, \nabla_{\partial_{x^4}} X_i \rangle \\ &= -\langle N, \widehat{\Gamma}^a_{4i} \partial_{x^a} + f_{,i} \widehat{\Gamma}^a_{44} \partial_{x^a} \rangle \\ &= -\langle \partial_{x^4}, N \rangle ((\log \phi)_{,i} + f_{,i} f_{,k} \phi \phi^k). \end{aligned} \tag{38}$$

□

Claim 3.

$$\begin{aligned} &\langle \partial_{x^4}, N \rangle^{-1} \sum_i (\langle \nabla_{\partial_{x^4}} N, e_i \rangle h_{iN} - \overline{\nabla}_i \langle \nabla_{\partial_{x^4}} N, e_i \rangle) \\ &= \phi^{-1} \Delta_g \phi + \langle \partial_{x^4}, N \rangle^{-1} \overline{g}^{ij} f_{,j} \phi \phi^l h(X_i, X_l) + |\nabla_{\overline{g}} \log \phi + \phi \phi^l f_{,l} \nabla_{\overline{g}} f|^2 \end{aligned}$$

Proof. First note that

$$\begin{aligned} \nabla_{\partial_{x^4}} N &= \frac{1}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} (f^i \widehat{\Gamma}^a_{i4} \partial_{x^a} - \phi^{-2} \widehat{\Gamma}^a_{44} \partial_{x^a}) \\ &= -\langle \partial_{x^4}, N \rangle (\log \phi)^i X_i, \end{aligned}$$

and therefore

$$\begin{aligned} &\langle \partial_{x^4}, N \rangle^{-1} \sum_i \langle \nabla_{\partial_{x^4}} N, e_i \rangle h_{iN} \\ =\ &\langle \partial_{x^4}, N \rangle^{-1} \overline{g}^{ij} \langle \nabla_{\partial_{x^4}} N, X_j \rangle h(X_i, N) \\ =\ &-\overline{g}^{ij} (\log \phi)^k \overline{g}_{kj} h \left(X_i, \frac{f^l \partial_{x^l} - \phi^{-2} \partial_{x^4}}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} \right) \\ =\ &-\frac{1}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} (\log \phi)^i f^j h(X_i, X_j) + \sqrt{\phi^{-2} + |\nabla_g f|^2} (\log \phi)^i h(X_i, \partial_{x^4}). \end{aligned} \tag{39}$$

For the other term we have

$$\begin{aligned} &-\langle \partial_{x^4}, N \rangle^{-1} \sum_i \overline{\nabla}_i \langle \partial_{x^4} N, e_i \rangle \\ =\ &-\langle \partial_{x^4}, N \rangle^{-1} \overline{g}^{ij} \overline{\nabla}_{X_i} \langle \partial_{x^4} N, X_j \rangle \\ =\ &\sqrt{\phi^{-2} + |\nabla_g f|^2}\, \overline{g}^{ij} \overline{\nabla}_{X_i} \left(\frac{(\log \phi)^k \overline{g}_{kj}}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} \right) \\ =\ &\overline{\nabla}_{X_i} (\log \phi)^i + \frac{\phi^{-2} |\nabla_g \log \phi|^2}{\phi^{-2} + |\nabla_g f|^2} - \frac{(\log \phi)^i f^j f_{;ij}}{\phi^{-2} + |\nabla_g f|^2}. \end{aligned} \tag{40}$$

Also

$$\begin{aligned}\overline{\nabla}_{X_i}(\log\phi)^i &= X_i[(\log\phi)^i] + \overline{\Gamma}^i_{ik}(\log\phi)^k \\ &= \partial_{x^i}[g^{il}(\log\phi)_{,l}] + \widehat{\Gamma}^i_{ik}(\log\phi)^k + (\overline{\Gamma}^i_{ik} - \widehat{\Gamma}^i_{ik})(\log\phi)^k \\ &= \Delta_g \log\phi + (\overline{\Gamma}^i_{ik} - \widehat{\Gamma}^i_{ik})(\log\phi)^k\end{aligned} \tag{41}$$

since

$$\partial_{x^i} g^{il} = -g^{ij}\widehat{\Gamma}^k_{jk} - g^{jk}\widehat{\Gamma}^i_{jk},$$

where the overline indicates Christoffel symbols for the induced metric $\overline{g}$. In order to calculate the difference of Christoffel symbols appearing above, notice that

$$\begin{aligned}\overline{\Gamma}^l_{jk}X_l = \overline{\nabla}_{X_j}X_k &= \nabla_{X_j}X_k + h(X_k, X_j)N \\ &= \nabla_{\partial_{x^j}}(\partial_{x^k} + f_{,k}\partial_{x^4}) + f_{,j}\nabla_{\partial_{x^4}}(\partial_{x^k} + f_k\partial_{x^4}) + h(X_k, X_j)N \\ &= \left(\widehat{\Gamma}^l_{jk} - \phi\phi^l f_{,j}f_{,k} + \frac{f^l h(X_k, X_j)}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}\right)\partial_{x^l} \\ &\quad + \left(f_{,jk} + (\log\phi)_{,j}f_{,k} + (\log\phi)_{,k}f_{,j} - \frac{\phi^{-2}h(X_k, X_j)}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}\right)\partial_{x^4} \\ &= \left(\widehat{\Gamma}^l_{jk} - \phi\phi^l f_{,j}f_{,k} + \frac{f^l h(X_k, X_j)}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}\right)X_l,\end{aligned}$$

where we have used the formula

$$\begin{aligned}h(X_i, X_j) &= \langle\nabla_{X_j}N, X_i\rangle \\ &= \frac{1}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}(f_{;ij} + (\log\phi)_{,i}f_{,j} + (\log\phi)_{,j}f_{,i} + \phi\phi^l f_{,l}f_{,i}f_{,j})\end{aligned} \tag{42}$$

which is easily established from (35). Hence

$$\overline{\Gamma}^l_{jk} = \widehat{\Gamma}^l_{jk} - \phi\phi^l f_{,j}f_{,k} + \frac{f^l h(X_k, X_j)}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}. \tag{43}$$

By combining (40)-(43) we arrive at

$$-\langle\partial_{x^4}, N\rangle^{-1}\sum_i \overline{\nabla}_i\langle\partial_{x^4}N, e_i\rangle = \phi^{-1}\Delta_g\phi. \tag{44}$$

Lastly, with (39) and (44) the desired result is obtained after making a short calculation (using (38)) to show that

$$\sqrt{\phi^{-2} + |\nabla_g f|^2}(\log\phi)^i h(X_i, \partial_{x^4}) = |\nabla_{\overline{g}}\log\phi + \phi\phi^l f_{,l}\nabla_{\overline{g}}f|^2,$$

and also

$$\overline{g}^{ij}f_{,j} = \frac{\phi^{-2}f^i}{\phi^{-2} + |\nabla_g f|^2}.$$

□

Claims 1, 2, and 3 show that the last four terms of (34) cancel to yield

$$\begin{aligned}&2(\mu^{\text{ext}} - J^{\text{ext}}(N)) \\ = \;& \overline{R} - \sum_{i,j}(h_{ij} - K_{ij})^2 - 2\sum_i(h_{iN} - K_{iN})^2 + 2\sum_i \overline{\nabla}_i(h_{iN} - K_{iN}).\end{aligned} \tag{45}$$

We continue by writing the "extended" energy and current densities in terms of the original densities.

Claim 4.

$$\mu^{\text{ext}} - J^{\text{ext}}(N) = 8\pi(\mu - J(w)) - \phi^{-1}\Delta_g\phi + Q(k,\phi,f)$$

where

$$w = \frac{f^i\partial_{x^i}}{\sqrt{\phi^{-2}+|\nabla_g f|^2}},$$

and

$$Q(k,\phi,f) = \phi^{-2}(\mathrm{Tr}_g k)k_{44} + \frac{f^i}{\sqrt{\phi^{-2}+|\nabla_g f|^2}}(\phi^{-2}k_{44,i} - \phi^{-2}(\log\phi)_{,i}k_{44} - (\log\phi)^j k_{ij})$$

with the extension term k_{44} given by (4) and $k_{ij} = k(\partial_{x^i}, \partial_{x^j})$.

Proof. We first treat the energy density

$$\begin{aligned} 2\mu^{\text{ext}} &= R_{g_\phi} - g_\phi^{ac}g_\phi^{bd}K_{ab}K_{cd} + (g_\phi^{ab}K_{ab})^2 \\ &= R_{g_\phi} - R + 16\pi\mu + 2\phi^{-2}(\mathrm{Tr}_g k)k_{44}, \end{aligned}$$

where R is the scalar curvature of g. Moreover by (35)

$$\begin{aligned} R_{g_\phi} &= g_\phi^{jl}\widehat{\Gamma}^k_{jl,k} - g_\phi^{ik}\widehat{\Gamma}^j_{ij,k} + g_\phi^{jl}\widehat{\Gamma}^k_{ik}\widehat{\Gamma}^i_{jl} - g_\phi^{ik}\widehat{\Gamma}^j_{kl}\widehat{\Gamma}^l_{ij} \\ &= R + g_\phi^{44}\widehat{\Gamma}^k_{44,k} - g_\phi^{ik}\widehat{\Gamma}^4_{i4,k} + g_\phi^{jl}\widehat{\Gamma}^i_{jl}\widehat{\Gamma}^4_{i4} + g_\phi^{jl}\widehat{\Gamma}^4_{jl}\widehat{\Gamma}^k_{4k} \\ &\quad + g_\phi^{44}\widehat{\Gamma}^i_{44}\widehat{\Gamma}^k_{ik} - g_\phi^{ik}\widehat{\Gamma}^4_{k4}\widehat{\Gamma}^4_{i4} - g_\phi^{44}\widehat{\Gamma}^j_{4l}\widehat{\Gamma}^l_{4j} \\ &= R - 2\phi^{-1}\Delta_g\phi, \end{aligned}$$

and therefore

$$\mu^{\text{ext}} = 8\pi\mu - \phi^{-1}\Delta_g\phi + \phi^{-2}(\mathrm{Tr}_g k)k_{44}.$$

Now consider the current density. If $1 \le i \le 3$ then

$$\begin{aligned} J^{\text{ext}}(\partial_{x^i}) &= g_\phi^{ab}K_{bi;a} - g_\phi^{ab}K_{ab;i} \\ &= g_\phi^{ab}(\partial_{x^a}K_{bi} - \widehat{\Gamma}^c_{ab}K_{ci} - \widehat{\Gamma}^c_{ia}K_{cb}) \\ &\quad - g_\phi^{ab}(\partial_{x^i}K_{ab} - \widehat{\Gamma}^c_{bi}K_{ca} - \widehat{\Gamma}^c_{ai}K_{cb}) \\ &= 8\pi J(\partial_{x^i}) - \phi^{-2}k_{44,i} + \phi^{-2}(\log\phi)_{,i}k_{44} + (\log\phi)^j k_{ij}. \end{aligned}$$

In addition

$$\begin{aligned} J^{\text{ext}}(\partial_{x^4}) &= g_\phi^{ab}K_{b4;a} - g_\phi^{ab}K_{ab;4} \\ &= g_\phi^{ab}(\partial_{x^a}K_{b4} - \widehat{\Gamma}^c_{ab}K_{c4} - \widehat{\Gamma}^c_{4a}K_{cb}) \\ &\quad - g_\phi^{ab}(\partial_{x^4}K_{ab} - \widehat{\Gamma}^c_{b4}K_{ca} - \widehat{\Gamma}^c_{a4}K_{cb}) \\ &= 0, \end{aligned}$$

so that

$$\begin{aligned} J^{\text{ext}}(N) &= J^{\text{ext}}\left(\frac{f^i\partial_{x^i} - \phi^{-2}\partial_{x^4}}{\sqrt{\phi^{-2}+|\nabla_g f|^2}}\right) \\ &= 8\pi J(w) - \frac{f^i}{\sqrt{\phi^{-2}+|\nabla_g f|^2}}(\phi^{-2}k_{44,i} - \phi^{-2}(\log\phi)_{,i}k_{44} - (\log\phi)^j k_{ij}). \end{aligned}$$

□

Our next goal will be to simplify the expression for $Q(k,\phi,f)$. To this end we will need the following

Claim 5.

$$\sum_i (h_{iN}-K_{iN})^2 = |q|^2_{\overline{g}} + |\nabla_{\overline{g}}\log\phi|^2 - \overline{g}^{ij}q_i(\log\phi)_{,j},$$

where

$$q_i = \frac{f^j}{\sqrt{\phi^{-2}+|\nabla_g f|^2}}(h(X_i,X_j)-K(X_i,X_j)).$$

Proof. Employ (38) and (4) to find

$$\begin{aligned}
& h(X_i,N)-K(X_i,N) \qquad (46)\\
= \;& \frac{1}{\sqrt{\phi^{-2}+|\nabla_g f|^2}}(h(X_i,f^j\partial_{x^j}-\phi^{-2}\partial_{x^4})-K(X_i,f^j\partial_{x^j}-\phi^{-2}\partial_{x^4}))\\
= \;& \frac{f^j}{\sqrt{\phi^{-2}+|\nabla_g f|^2}}(h(X_i,X_j)-K(X_i,X_j))\\
& +\sqrt{\phi^{-2}+|\nabla_g f|^2}(f_{,i}k_{44}-h(X_i,\partial_{x^4}))\\
= \;& q_i-(\log\phi)_{,i}.
\end{aligned}$$

□

As an immediate corollary of (46) we also have

Claim 6.

$$\sum_i \overline{\nabla}_i(h_{iN}-K_{iN}) = \operatorname{div}_{\overline{g}}q - \Delta_{\overline{g}}\log\phi$$

We now come to the simplification of $Q(k,\phi,f)$.

Claim 7.

$$\begin{aligned}
& \sum_i (h_{i4}-K_{i4})^2 + \sum_{i,j}(h_{ij}-K_{ij})^2 + 2Q(k,\phi,f)\\
= \;& |h-K|_\Sigma|^2_{\overline{g}} + |q|^2_{\overline{g}} - 2\Delta_{\overline{g}}\log\phi - |\nabla_{\overline{g}}\log\phi|^2 + 2\phi^{-1}\Delta_g\phi
\end{aligned}$$

Proof. Using (4), the following term of $Q(k,\phi,f)$ (from Claim 4) may be calculated by

$$\frac{f^i(\log\phi)^j k_{ij}}{\sqrt{\phi^{-2}+|\nabla_g f|^2}} = \frac{\phi^{-2}(\phi^l f_{,l})^2}{\phi^{-2}+|\nabla_g f|^2}. \qquad (47)$$

To see this observe that

$$\begin{aligned}
f^j k_{ij} &= K(\partial_{x^i}+f_{,i}\partial_{x^4}, f^j\partial_{x^j})\\
&= \sqrt{\phi^{-2}+|\nabla_g f|^2}K(X_i,N)+\phi^{-2}f_{,i}k_{44},
\end{aligned}$$

and

$$\begin{aligned}
2(\log\phi)^i K(X_i,N) &= -2\langle\partial_{x^4},N\rangle^{-1}K(\nabla_{\partial_{x^4}}N,N)\\
&= -\langle\partial_{x^4},N\rangle^{-1}(\partial_{x^4}K(N,N)-(\nabla_{\partial_{x^4}}K)(N,N))\\
&= 0.
\end{aligned}$$

Therefore by calculating the remaining terms of $Q(k,\phi,f)$ in a straight forward way, we have

$$\begin{aligned} Q(k,\phi,f) &= \frac{(\mathrm{Tr}_g k)\phi^{-1}\phi^l f_{,l}}{\sqrt{\phi^{-2}+|\nabla_g f|^2}} - \frac{\phi^{-2}(\phi^l f_{,l})^2}{\phi^{-2}+|\nabla_g f|^2} + \frac{\phi^{-1}f^i f^j \phi_{;ij}}{\phi^{-2}+|\nabla_g f|^2} \\ &\quad + \frac{(\log\phi)^i f^j f_{;ij}}{\phi^{-2}+|\nabla_g f|^2} + \frac{\phi^{-4}(\phi^l f_{,l})^2}{(\phi^{-2}+|\nabla_g f|^2)^2} - \frac{\phi^{-1}(\phi^l f_{,l}) f^i f^j f_{;ij}}{(\phi^{-2}+|\nabla_g f|^2)^2}. \end{aligned}$$

Moreover with the help of (43)

$$\begin{aligned} \frac{\phi^{-1}f^i f^j \phi_{;ij}}{\phi^{-2}+|\nabla_g f|^2} &= \phi^{-1}\Delta_g\phi - \phi^{-1}\overline{g}^{ij}\phi_{;ij} \\ &= \phi^{-1}(\Delta_g\phi - \Delta_{\overline{g}}\phi) + \phi^{-1}(\widehat{\Gamma}^k_{ij} - \overline{\Gamma}^k_{ij})\phi_{,k} \\ &= \phi^{-1}(\Delta_g\phi - \Delta_{\overline{g}}\phi) + |\nabla_g\phi|^2\overline{g}^{ij}f_{,i}f_{,j} - \frac{\phi^{-1}(\phi^l f_{,l})H}{\sqrt{\phi^{-2}+|\nabla_g f|^2}}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{Tr}_g k &= g^{ij}K(X_i,X_j) - |\nabla_g f|^2 k_{44} \\ &= \overline{g}^{ij}K(X_i,X_j) - |\nabla_g f|^2 k_{44} \\ &\quad + \frac{f^i f^j}{\phi^{-2}+|\nabla_g f|^2}(K(X_i,X_j) - h(X_i,X_j)) + \frac{f^i f^j}{\phi^{-2}+|\nabla_g f|^2}h(X_i,X_j). \end{aligned}$$

It follows that with (42)

$$\begin{aligned} Q(k,\phi,f) &= -\frac{\phi^{-1}(\phi^l f_{,l})}{\sqrt{\phi^{-2}+|\nabla_g f|^2}}(H - \sum_i K_{ii}) + \Delta_g\log\phi - \Delta_{\overline{g}}\log\phi \\ &\quad - \frac{\phi^{-1}(\phi^l f_{,l})f^i f^j}{(\phi^{-2}+|\nabla_g f|^2)^{3/2}}(h(X_i,X_j) - K(X_i,X_j)) + \frac{\phi^{-2}(\phi^l f_{,l})^2}{\phi^{-2}+|\nabla_g f|^2} \\ &\quad + \frac{(\log\phi)^i f^j f_{;ij}}{\phi^{-2}+|\nabla_g f|^2} + \frac{|\nabla_g f|^2|\nabla_g\log\phi|^2}{\phi^{-2}+|\nabla_g f|^2}. \end{aligned} \tag{48}$$

Now set

$$p_{ij} = h(X_i,X_j) - K(X_i,X_j).$$

Many of the terms appearing in (48) are similar to those appearing in the following expression, which is derived from (46)

$$\begin{aligned} &\sum_{i,j}(h_{ij}-K_{ij})^2 + \sum_i (h_{iN}-K_{iN})^2 \\ &= g^{il}g^{jk}p_{ij}p_{lk} - \frac{g^{il}f^j f^k}{\phi^{-2}+|\nabla_g f|^2}p_{ij}p_{lk} - \frac{2f^l(\log\phi)^i p_{il}}{\sqrt{\phi^{-2}+|\nabla_g f|^2}} \\ &\quad + |\nabla_g\log\phi|^2 + \frac{2f^l f^i p_{il} f^j(\log\phi)_{,j}}{(\phi^{-2}+|\nabla_g f|^2)^{3/2}} - \frac{\phi^{-2}(\phi^l f_{,l})^2}{\phi^{-2}+|\nabla_g f|^2}. \end{aligned} \tag{49}$$

The two expressions (48) and (49) may now be combined to obtain

$$\begin{aligned}
&\sum_i (h_{i4} - K_{i4})^2 + \sum_{i,j}(h_{ij} - K_{ij})^2 + 2Q(k,\phi,f) \\
= \quad & -\frac{2\phi^{-1}(\phi^l f_{,l})}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}(H - \sum_i K_{ii}) + g^{il}\overline{g}^{jk}p_{ij}p_{lk} \\
& -2\Delta_{\overline{g}}\log\phi - |\nabla_{\overline{g}}\log\phi|^2 + 2\phi^{-1}\Delta_g\phi.
\end{aligned}$$

For this last calculation it is necessary to use (47) in addition to

$$\begin{aligned}
p_{ij} \quad &= \quad h(X_i, X_j) - k_{ij} - f_{,i}f_{,j}k_{44} \\
&= \quad \frac{1}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}(f_{;ij} + (\log\phi)_{,i}f_{,j} + (\log\phi)_{,j}f_{,i}) - k_{ij}.
\end{aligned}$$

The Claim now follows from the generalized Jang equation. □

Theorem 1 is now a consequence of (45) as well as Claims 4, 5, 6, and 7.

7. **Appendix B.** Suppose that the graph map $G(x) = (x, f(x))$ provides an isometric embedding of (M, g) into the Schwarzschild spacetime $(\mathbb{SC}^4, g_{\mathrm{SC}} - \phi^2 dt^2)$. We will show that $h = K|_\Sigma$ implies that the second fundamental form π of the embedding $G(M) \subset \mathbb{SC}^4$ is given by the initial data k, where h is the second fundamental form of the graph $t = f(x)$ (denoted by Σ) in the warped product space $(M \times \mathbb{R}, g + \phi^2 dt^2)$ and $K|_\Sigma$ is the restriction to Σ of the extended (by (4)) version of k.

Let

$$X_i = \partial_{x^i} + f_{,i}\partial_{x^4}, \qquad i = 1, 2, 3,$$

be tangent vectors to Σ. Then according to (42) the second fundamental form of $\Sigma \subset (M \times \mathbb{R}, g + \phi^2 dt^2)$ is given by

$$h_{ij} := h(X_i, X_j) = \frac{\nabla_{ij} f + (\log\phi)_{,i}f_{,j} + (\log\phi)_{,j}f_{,i} + g^{lp}\phi\phi_{,l}f_{,p}f_{,i}f_{,j}}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}$$

where ∇_{ij} denotes covariant differentiation with respect to g, and the second fundamental form of $G(M) \subset \mathbb{SC}^4$ is given by

$$\pi_{ij} := \pi(X_i, X_j) = \frac{\nabla^{\mathrm{SC}}_{ij} f + (\log\phi)_{,i}f_{,j} + (\log\phi)_{,j}f_{,i} - g^{lp}_{\mathrm{SC}}\phi\phi_{,l}f_{,p}f_{,i}f_{,j}}{\sqrt{\phi^{-2} - |\nabla_{g_{\mathrm{SC}}} f|^2}}$$

where $\nabla^{\mathrm{SC}}_{ij}$ denotes covariant differentiation with respect to g_{SC}. Utilizing the isometry we can write $g_{ij} = (g_{\mathrm{SC}})_{ij} - \phi^2 f_{,i}f_{,j}$. Then direct calculation shows that

$$\begin{aligned}
& \nabla_{ij} f + (\log\phi)_{,i}f_{,j} + (\log\phi)_{,j}f_{,i} + g^{lp}\phi\phi_{,l}f_{,p}f_{,i}f_{,j} \\
= \quad & \frac{\phi^{-2}}{\phi^{-2} - |\nabla_{g_{\mathrm{SC}}} f|^2}(\nabla^{\mathrm{SC}}_{ij} f + (\log\phi)_{,i}f_{,j} + (\log\phi)_{,j}f_{,i}).
\end{aligned}$$

Furthermore

$$\frac{1}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} = \phi^2\sqrt{\phi^{-2} - |\nabla_{g_{\mathrm{SC}}} f|^2},$$

and so

$$h_{ij} = \pi_{ij} + \frac{\langle \phi\nabla_g\phi, \nabla_g f\rangle_g}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} f_{,i}f_{,j}.$$

However

$$(K|_\Sigma)_{ij} = K(X_i, X_j) = k_{ij} + \frac{\langle \phi\nabla_g \phi, \nabla_g f\rangle_g}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} f_{,i} f_{,j}.$$

Therefore if $h = K|_\Sigma$, it follows that $k = \pi$ as desired.

8. Appendix C.

Here we calculate the boundary term in (22). Observe that

$$\overline{g}(q, n_{\overline{g}})d\sigma_{\overline{g}} = \overline{g}^{ij} q_i n_j \sqrt{g_{11} + \phi^2 f_{,r}^2}\rho^2 d\sigma, \tag{50}$$

where (n_1, n_2, n_3) is the Euclidean unit outer normal and $d\sigma$ is the Euclidean area element. Since the metric $\overline{g}$ is diagonal and $f = f(r)$ we have $q_2 = q_3 = 0$. Next we calculate

$$\begin{aligned}
q_1 &= \frac{f^r}{\sqrt{\phi^{-2} + |\nabla_g f|^2}}(h_{rr} - (K|_\Sigma)_{rr}) \\
&= \frac{\phi g^{11} f_{,r}}{\sqrt{1 + \phi^2 g^{11} f_{,r}^2}} \left(\frac{f_{;rr} + 2(\log\phi)_{,r} f_{,r}}{\sqrt{\phi^{-2} + |\nabla_g f|^2}} - k_{rr} \right) \\
&= (g_{11} + \phi^2 f_{,r}^2)^{-1} \phi f_{,r} (\phi f_{,rr} - \frac{1}{2}\phi g^{11} g_{11,r} f_{,r} + 2\phi_{,r} f_{,r}) - \sqrt{g^{11}} v k_{rr} \\
&= (g_{11} + \phi^2 f_{,r}^2)^{-1} \left[\frac{1}{2} g_{11} (\phi^2 g^{11} f_{,r}^2)_{,r} + \phi\phi_{,r} f_{,r}^2 \right] - \sqrt{g_{11}} v k_a \qquad (51) \\
&= g^{11}(1 - v^2) \left[\frac{1}{2} g_{11} \left(\frac{v^2}{1 - v^2} \right)_{,r} + g_{11} \frac{\phi_{,r}}{\phi} \frac{v^2}{1 - v^2} \right] - \sqrt{g_{11}} v k_a \\
&= \frac{v v_{,r}}{1 - v^2} + \frac{\phi_{,r}}{\phi} v^2 - \sqrt{g_{11}} v K_a \\
&= \sqrt{g_{11}} \frac{v}{1 - v^2} \left[-2 \left(\sqrt{g^{11}} \frac{\rho_{,r}}{\rho} v - k_b \right) + (1 - v^2) k_a - \sqrt{g^{11}} v \frac{\phi_{,r}}{\phi}(1 - v^2) \right] \\
&\quad + \frac{\phi_{,r}}{\phi} v^2 - \sqrt{g_{11}} v k_a \\
&= -2\sqrt{g_{11}} \frac{v}{1 - v^2} \left(\sqrt{g^{11}} \frac{\rho_{,r}}{\rho} v - k_b \right),
\end{aligned}$$

where $f_{;rr}$ denotes covariant differentiation with respect to g and we have used equation (7) as well as $k_{rr} = k(\partial_r, \partial_r) = g_{11} k_a$. Furthermore

$$\phi = \rho_{,s} = \frac{\sqrt{1 - v^2}}{\sqrt{g_{11}}} \rho_{,r}, \qquad \sqrt{g_{11} + \phi^2 f_{,r}^2} = \frac{\sqrt{g_{11}}}{\sqrt{1 - v^2}}. \tag{52}$$

Thus combining (50), (51), and (52) we have

$$\phi\overline{g}(q, n_{\overline{g}})d\sigma_{\overline{g}} = \pm\frac{2\rho_{,r} v}{\sqrt{g_{11}}} \left(\sqrt{g^{11}} \frac{\rho_{,r}}{\rho} v - k_b \right) \rho^2 d\sigma.$$

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