POSITIVE MASS THEOREMS FOR ASYMPTOTICALLY FLAT AND ASYMPTOTICALLY LOCALLY FLAT MANIFOLDS

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ABSTRACT. In dimensions $n \leq 7$, the mass of AF manifolds having nonnegative scalar curvature is shown to be nonnegative if a sphere in the asymptotic end is trivial in $H_{n-2}(M)$, with zero mass achieved only for the product $\mathbb{R}^{n-1} \times \mathbb{S}^1$. The same conclusions are obtained in dimension 4 for ALF manifolds admitting an almost free $\mathbb{S}^1$-action. Moreover, the mass is shown to be bounded below by a multiple of the degree of the $\mathbb{S}^1$-bundle at infinity.

1. INTRODUCTION

A landmark result in the study of scalar curvature and in mathematical relativity is the positive mass theorem, proved originally for asymptotically Euclidean (AE) manifolds of nonnegative scalar curvature by Schoen-Yau [29] and Witten [32]. This theorem has been successfully extended in the asymptotically hyperbolic regime [9, 31, 33], as well as in the asymptotically locally hyperbolic [2] and complex hyperbolic settings [6, 20]. Motivated by considerations in quantum gravity, the result was conjectured to hold for asymptotically locally Euclidean (ALE) manifolds, however counterexamples where found by LeBrun [21]. Versions of the positive mass theorem in the ALE case for Kähler manifolds have been established by Hein-LeBrun [19], and under a spin structure matching condition by Dahl [11]. More recently, with inspiration coming from the study of gravitational instantons, Minerbe [27] has obtained versions of the theorem pertaining to the AF and ALF settings, and Liu-Shi-Zhu [25] as well as Chen-Liu-Shi-Zhu [7] have proved incarnations for the AF and other cases. See also the related results of Dai [12], and Barzegar-Chruściel-Hörzinger [4]. However, the AF and ALF settings remain the least well understood.

The Euclidean Reissner-Nordstrom metrics on $\mathbb{R}^2 \times \mathbb{S}^2$ are AF and scalar flat, but they can have negative mass for certain choices of parameters. In [27, Theorem 0.1] Minerbe sought to explain this phenomena by asserting that mass is nonnegative for ALF manifolds under the assumption of nonnegative Ricci curvature, which Reissner-Nordstrom does not satisfy; note that Minerbe's definition of mass in this result is different than the standard one. In a different direction, Liu-Shi-Zhu [25, Theorem 1.2] show that for AF manifolds of dimensions less than 8 the positive mass theorem holds under an incompressible condition for the circle at infinity.

Definition 1.1. A complete Riemannian manifold (M^n, g) is called asymptotically flat (AF) if there is a compact set K such that each component of $M \setminus K$ is homeomorphic to $(\mathbb{R}^{n-1} \setminus B^{n-1}(0,1)) \times \mathbb{S}^1$ and on each end

$$|g - g_0| + |x||\nabla(g - g_0)| + |x|^2|\nabla^2(g - g_0)| = O(|x|^{-\mu}),$$

where g_0 is the flat metric on $\mathbb{R}^{n-1} \times \mathbb{S}^1$ and $\mu > (n-3)/2$. On the end $\mathcal{E}$, the sphere $\Gamma_{\theta,r} := \partial B^{n-1}(0,r) \times \{\theta\}$ in $\mathcal{E}$ is called a sphere at $\mathcal{E}$, where $\theta \in \mathbb{S}^1$, and r is large enough.

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Our first result is related to that of Liu-Shi-Zhu [25, Theorem 1.2], in that topological aspects of the asymptotic end leads to positivity of mass. The proof is based on stable minimal hypersurfaces.

Theorem 1.2. *Let (M^n, g) be an AF manifold and with non-negative scalar curvature, where $4 \leq n \leq 7$. If some sphere $\Gamma_{\theta,r}$ at $\mathcal{E}$ is trivial in $H_{n-2}(M^n)$, then the mass is nonnegative. Moreover, the mass vanishes if and only if (M^n, g) is isometric to $\mathbb{R}^{n-1} \times \mathbb{S}^1$.*

In another direction, Minerbe [27, Theorem 0.2] considered AF manifolds with nonnegative scalar curvature and a matching condition for the spin structure at infinity, to establish a positive mass theorem. The restriction to AF asymptotics does not allow for an application to examples such as the (multi)-Taub-NUT geometries, which have nontrivial fibrations in the asymptotic end. To the author's knowledge, there are no known positive mass theorems in the literature for general ALF manifolds with nonnegative scalar curvature. The two theorems below aim to fill this gap.

Let $\pi_0 : \mathcal{X} \to \mathbb{R}^3 \setminus B(0,1)$ be a principle $\mathbb{S}^1$ bundle, and let T_0 be the infinitesimal generator of the $\mathbb{S}^1$ action on $\mathcal{X}$. The model metric h on $\mathcal{X}$ is given by

$$h = \pi_0^*(g_{\mathbb{R}^3}) + \eta_0^2,$$

where η_0 is a 1-form with $\eta_0(T_0) = 1$. We further define

$$|\mathrm{Deg}(\mathcal{X})| = \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|} \int_{\mathcal{X}|_{\partial B(0,R)}} \eta_0 \wedge d\eta_0,$$

and observe that this is the modulus of the degree of the $\mathbb{S}^1$-bundle, and is independent of R.

Definition 1.3. Let $(M^4, g, \mathcal{E})$ be a complete 4-manifold with asymptotic end $\mathcal{E}$. The manifold is called asymptotically locally flat (ALF) if $\mathcal{E}$ is homeomorphic to $\mathcal{X}$ and admits the following asymptotics

$$|g - h| + |x||\nabla(g - h)| + |x|^2|\nabla^2(g - h)| = O(|x|^{-\mu}),$$

where $\mu > 1/2$. Moreover we set $|\mathrm{Deg}(\mathcal{E})| := |\mathrm{Deg}(\mathcal{X})|$.

Toric and $\mathbb{S}^1$-symmetries play an important role in the study of gravitational instantons [1]. Therefore, it is natural to consider positive mass theorems in the ALF setting which assume such symmetries. We will restrict attention to the case in which the circle action has a finite number of fixed points.

Definition 1.4. A $\mathbb{S}^1$-action on a complete ALF manifold $(M^4, g, \mathcal{E})$ is called *almost free* if it satisfies the following.

1. $\mathbb{S}^1$ acts on M^4 by isometrics.
2. The $\mathbb{S}^1$ action preserves the principle $\mathbb{S}^1$ bundle $\mathcal{X}$.
3. There are finitely many points $\{p_1, \cdots, p_k\}$ such that $\mathbb{S}^1$ acts on $M^4 \setminus \{p_1, \cdots, p_k\}$ freely.

O'Neill's formula for the scalar curvature of Riemannian submersions shows that if the fibers are flat, then the difference of the base and ambient scalar curvatures is weakly nonnegative. Based on this observation, we are able to perform a reduction argument to the 3-dimensional asymptotically Euclidean case in order to obtain mass positivity in the presence of an almost free action.

Theorem 1.5. *Let $(M^4, g, \mathcal{E})$ be a complete ALF manifold with nonnegative scalar curvature. If it admits an almost free $\mathbb{S}^1$ action, then the mass is nonnegative. Moreover, the mass vanishes if and only if (M^4, g) is isometric to $\mathbb{R}^3 \times \mathbb{S}^1$.*

Generalizations that include positive lower bounds for the mass in terms of horizon area, charge, and angular momentum are well-known in the AE setting, and are referred to as Penrose-type inequalities. In the ALF regime we are able to prove a mass lower bound in terms of the degree of the bundle at infinity. Although the constant obtained in the next result is not sharp, we anticipate that with an optimal constant rigidity should be obtained only for the (multi)-Taub-NUT manifolds.

Theorem 1.6. *Let $(M^4, g, \mathcal{E})$ be a complete ALF manifold with nonnegative scalar curvature and mass m. If it admits an almost free $\mathbb{S}^1$ action then*

$$m \geq \frac{1}{8}|Deg(\mathcal{E})|.$$

2. Preliminary: $\mathbb{S}^1$-invariant functions and ADM mass

2.1. **ALF 4-manifolds with an isometric $\mathbb{S}^1$-action.** Let $\pi_0 : \mathcal{X} \to \mathbb{R}^3 \setminus B(0,1)$ be a principle $\mathbb{S}^1$ bundle and T_0 be the infinitesimal generator of the $\mathbb{S}^1$ action on $\mathcal{X}$. The model metric h on $\mathcal{X}$ is defined as follows:

$$h = \pi_0^*(g_{\mathbb{R}^3}) + \eta_0^2,$$

where η_0 is a connection 1-form with $\eta_0(T_0) = 1$. The curvature of the principal $\mathbb{S}^1$-bundle is

$$d\eta_0 = (\pi_0')^*(\omega) \tag{2.1}$$

where ω is a closed 2-form on $\mathbb{S}^2$ and $\pi_0' := pr \circ \pi_0 : \mathcal{X} \xrightarrow{\pi_0} \mathbb{R}^3 \setminus B(0,1) \xrightarrow{pr} \mathbb{S}^2$. Then, for any $R > 1$, one obtains that

$$\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|} \int_{\mathcal{X}|_{\partial B(0,R)}} \eta_0 \wedge d\eta_0 = \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{S}^2} \omega \tag{2.2}$$

It is a topological invariant, independent of R, called the degree of the bundle $\mathcal{X}$.

Remark 2.1. The scalar curvature R_h does not vanish. It enjoys the following decay:

$$R_h = -\frac{|d\eta_0|^2}{4} = -\frac{(\mathrm{Deg}(\mathcal{X}))^2}{4r^4} \tag{2.3}$$

Example 2.2. Consider the Hopf fibration $\pi_0 : \mathbb{C}^2 \setminus B(0,1)$ and the $\mathbb{S}^1$-action induced by complex number's multiplication. The 1-form η_0 is the standard contact form on $\mathbb{S}^3 \subset \mathbb{R}^4$. The degree of the bundle is equal to 1.

The metric $h = \pi^*(g_{\mathbb{R}^3}) + \eta_0^2$ is the model at infinity of Multi-Taub-NUT metrics.

In the following, we suppose that $(M^4, g, \mathcal{E})$ is a complete ALF manifold with an almost free $\mathbb{S}^1$-action. Consider the infinitesimal generator T and the $\mathbb{S}^1$-invariant 1-form η with $\eta(T) = 1$. The metric g can be written as follows:

$$g = \pi^*(\bar{g}) + \frac{1}{|\eta|^2}\eta \otimes \eta$$

where $\pi : M \to M/\mathbb{S}^1$ is the quotient map and $(\bar{M}, \bar{g})$ is the quotient space.

Proposition 2.3. *Let $(M^4, g, \mathcal{E})$ be a complete ALF 4-manifold with almost free $\mathbb{S}^1$-action and T, η defined as above. Then, one has that*

$$\lim_{R\to\infty} \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|} \int_{\partial(B(p,R))} \eta \wedge d\eta \text{ exists}$$

and it is equal to $|Deg(\mathcal{E})|$.

Proof. Recall that $d\eta_0 \in C^{2,\alpha}_2$ and $\eta - \eta_0 \in C^{2,\alpha}_\mu$ and $d\eta - d\eta_0 \in C^{2,\alpha}_{1+\mu}$ where $\mu > 1/2$. Then, one has that

$$\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_{\partial B(p,R))} \eta \wedge d\eta_0 = \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_{\partial B(p,R)} \eta_0 \wedge d\eta_0 + o(1)$$

Notice that $\int_{\partial B(p,R)} \eta_0 \wedge d\eta = -\int_{\partial B(p,R)} d(\eta_0 \wedge \eta) + \int_{\partial B(p,R)} \eta \wedge d\eta_0 = \int_{\partial B(p,R)} \eta \wedge d\eta_0$. The second equality follows form Stokes' Theorem. Namely, one has that

$$\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_{\partial B(p,R))} \eta_0 \wedge d\eta = \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_{\partial B(p,R)} \eta_0 \wedge d\eta_0 + o(1)$$

The asymptotic behavior of η implies that $\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_{\partial B(p,R))} (\eta - \eta_0) \wedge d(\eta - \eta_0) = o(1)$. Then, one obtains that

$$\begin{aligned}\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_{\partial B(p,R))} \eta \wedge d\eta &= \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}(\int_{\partial B(p,R)} \eta_0 \wedge d\eta + \int_{\partial B(p,R))} \eta \wedge d\eta_0 - \int_{\partial B(p,R)} \eta_0 \wedge d\eta_0) + o(1)\\ &= \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_{\partial B(p,R)} \eta_0 \wedge d\eta_0 + o(1)\end{aligned}$$

$\square$

2.2. $\mathbb{S}^1$**-invariant functions.** Let $(M^4, \mathcal{E})$ be a complete ALF manifold with an almost free $\mathbb{S}^1$-action and T, η defined as above. The metric g is in the following form:

$$g = \pi^*(\bar{g}) + \frac{1}{|\eta|^2}\eta \otimes \eta \tag{2.4}$$

where $\pi : M \to M/\mathbb{S}^1$ is the quotient map and $(\bar{M}, \bar{g})$ is the quotient space.

In the following, we may assume that the $\mathbb{S}^1$-act on $(\mathcal{E}, g|_{\mathcal{E}})$ is free. That is to say, the quotient space $(\bar{\mathcal{E}}, \bar{g}|_{\bar{\mathcal{E}}})$ is a smooth manifold and asymptotically Euclidean, where $\bar{\mathcal{E}} := \mathcal{E}/\mathbb{S}^1$. Any $\mathbb{S}^1$-invariant function f (i.e. $f(x) = f(\gamma \cdot x)$ for any $\gamma \in \mathbb{S}^1$) on M can be expressed as follows:

$$f = \pi^*(\bar{f})$$

where $\bar{f}$ is a function on the quotient space $\bar{M}$.

Proposition 2.4. *For any $\mathbb{S}^1$-invariant function $f = \pi^*(\bar{f})$ on $\mathcal{E}$, one has that*

$$\nabla_g(f) = \pi^*(\nabla_{\bar{g}}\bar{f}) \tag{2.5}$$

$$\Delta_g(f) = \pi^*(\Delta_{\bar{g}}\bar{f} + \frac{1}{2}\nabla_{\bar{g}}\log(||\eta||^2) \cdot \nabla_{\bar{g}}\bar{f}) \tag{2.6}$$

$$\int_M f dvol_g = |\mathbb{S}^1| \int_{\bar{M}} \bar{f}|\eta| dvol_{\bar{g}} \tag{2.7}$$

The proof follows from (2.4).

Remark 2.5. Definition 1.3 implies that $\nabla_{\bar{g}}\log(||\eta||^2) \in C^{1,\alpha}_{1+\mu}$. Moreover, if $|||T|| - 1| < \delta$ on $\mathcal{E}$, where δ is a small constant, one obtains

$$||\nabla_{\bar{g}}\bar{f}||^{\bar{\mathcal{E}}}_{L^2} \approx ||\nabla f||^{\mathcal{E}}_{L^2} \quad ||\bar{f}||^{\bar{\mathcal{E}}}_{L^p} \approx ||f||^{\mathcal{E}}_{L^p} \tag{2.8}$$

Let's recall the weighted Schauder estimate on an asymptotic Euclidean end.

Theorem 2.6. *(Meyers [26] or Lee [23]) Let (X^3, g) be an asymptotically Euclidean end. Suppose $Lu = f$, where u is bounded smooth function and $Lu = a_{i,j}\partial^{i,j}u + b_i\partial_i u + cu$ is a linear elliptic operator. If L satisfies that for some $\epsilon \in (0,1)$.*

$$a_{i,j} - \delta_{i,j} \in C^{0,\alpha}_{\epsilon},\ b^i \in C^{0,\alpha}_{1+\epsilon},\ c \in C^{0,\alpha}_{3+\epsilon} \text{ and } f \in C^{0,\alpha}_{3+\epsilon}$$

Then, there exist constants a_∞, c_∞ such that

$$u - (a_\infty + \frac{c_\infty}{r}) \in C^{2,\alpha}_{1+\epsilon}$$

We use Theorem 2.6 to get a Schaduder estimate for ALF manifolds.

Lemma 2.7. *Let $(M^4, g, \mathcal{E})$ be a complete ALF manifold. Suppose that an isometric $\mathbb{S}^1$-action $(\mathcal{E}, g|_{\mathcal{E}})$ is free and three $\mathbb{S}^1$-invarinat functions u, f and c satisfy that*

$$\Delta_g u + c(x)u = f(x).$$

If u is bounded, $c, f \in C^{0,\alpha}_{3+\epsilon}(\mathcal{E})$, for some $\epsilon \in (0,1)$ then there are two constants a_∞ and b_∞ such that

$$u - (a_\infty + \frac{b_\infty}{r}) \in C^{2,\alpha}_{1+\epsilon'}$$

where $\epsilon' = \min\{\mu, \epsilon\}$ and $\mu > 1/2$.

Proof. The quotient space $\bar{\mathcal{E}} := \mathcal{E}/\mathbb{S}^1$ is asymptotically Euclidean, where $\bar{g}|_{\bar{\mathcal{E}}}$ satisfies that

$$\bar{g}|_{\bar{\mathcal{E}}} - g_{\mathbb{R}^3} \in C^{2,\alpha}_{\mu}(\bar{\mathcal{E}}).$$

One has that $u = \pi^*(\bar{u}), f = \pi^*(\bar{f}), c = \pi^*(\bar{c})$ where $\bar{f}, \bar{c} \in C^{0,\alpha}_{\epsilon+3}(\bar{\mathcal{E}})$. Using (2.6) the differential equation can be expressed as follows:

$$\Delta_{\bar{g}}\bar{u} + \bar{g}(\overrightarrow{b}, \nabla_{\bar{g}}\bar{u}) + \bar{c}\bar{u} = \bar{f}$$

where $\overrightarrow{b} := \frac{1}{2}\nabla_{\bar{g}}\log(||\eta||^2) \in C^{1,\alpha}_{1+\mu} \in C^{1,\alpha}$. We apply Theorem 2.6 to find two constants, a_∞ and b_∞ such that

$$\bar{u} - (a_\infty + \frac{b_\infty}{r}) \in C^{2,\alpha}_{1+\epsilon'}(\bar{\mathcal{E}})$$

Such an asymptotic estimate also holds for u. □

2.3. **ADM mass.** The ADM mass for an asymptotically Euclidean end is a geometric invariant, independent of the choice of the coordinate at infinity and the model metric (See [3]). In this subsection, our facus is on the ADM mass for ALF 4-manifolds.

Proposition 2.8. *Let $(M^4, g, \mathcal{E})$ be a complete ALF 4-manifold. Then the ADM mass*

$$m_{ADM}(M^4, g, \mathcal{E}) = \lim_{R\to\infty} \frac{1}{|\mathbb{S}^2||\mathbb{S}^1|}\int_{\partial B_R} *_h(div_h(g) - dtr_h(g)) \tag{2.9}$$

is a geometric invariant, independent of h.

Remark that if $(M^n, g, \mathcal{E})$ is a complete AF manifold, the ADM mass is a geometric invariant, independent of choice of h (See Theorem in [7]).

Proof. We first show that the limit exists. Then, we prove that the limit is independent of the choice of the model metric.

Step 1: The existence of the limit.

Let $h = \pi_0^*(g_{\mathbb{R}^3}) + \eta_0^2$ be the model metric on $\mathcal{E}$ and T_0 the infinitesimal as above. There is an orthogonal frame (X_1, X_2, X_3, T_0) for h. We can express the scalar curvature R_g under this frame:

$$R_g = \frac{1}{|g|^{\frac{1}{2}}}\partial_a(g_{ab,b} - g_{bb,a}) + O(r^{-2-2\mu}) \tag{2.10}$$

where a, b runs over all index $\{X_1, X_2, X_3, T_0\}$ Then, one has that for any $R > R'$

$$\begin{aligned}\int_{\partial B_R} *_h(\mathrm{div}_h(g) - d\mathrm{tr}_h(g)) - \int_{\partial B_{R'}} *_h(\mathrm{div}_h(g) - d\mathrm{tr}_h(g)) &= \int_{B_R\setminus B_{R'}} \partial_a(g_{ab,b} - g_{bb,a})d\mathrm{vol}_h \\ &= \int_{B_R\setminus B_{R'}} \frac{1}{|g|^{\frac{1}{2}}}\partial_a(g_{ab,b} - g_{bb,a})d\mathrm{vol}_g \\ &= \int_{B_R\setminus B_{R'}} (R_g + O(r^{-2-2\mu}))d\mathrm{vol}_g\end{aligned} \tag{2.11}$$

The last equation follows from Equation (2.11). Then, we can conclude that

$$\lim_{R\to\infty} \tfrac{1}{|\mathbb{S}^2||\mathbb{S}^1|} \textstyle\int_{\partial B_R} *_h(\mathrm{div}_h(g) - d\mathrm{tr}_h(g)) \text{ exists,}$$

because $R_g \in L^1(\mathcal{E})$ and $\mu > 1/2$.

Step 2: The limit is independent of the choice of the model metric h.

Consider two metrics $h = dx^2 + \eta_0^2$ and $h' = dx'^2 + \eta_0'^2$ on $\mathcal{X}$ with different connections. From the proof of Proposition 3.6 in [27], we have that the h-orthonormal frame field (X_1, X_2, X_3, T_0) defined in Setp 1, is r^{-1}-close to the h'-orthonormal frame field (X_1', X_2', X_3', T_0'). Further, we obtain that

$$*_h(\mathrm{div}_h(g) - d\mathrm{tr}_h(g)) - *_{h'}(\mathrm{div}_{h'}(g) - d\mathrm{tr}_{h'}(g)) = O(r^{-3}).$$

Hence, the limit is independent of the choice of the model metric h. $\square$

Remark. Let (X_1, X_2, X_3, T_0) be the h-orthonormal frame field on $\mathcal{X}$ defined in Step 1. Use the same argument in the proof of Lemma 4.8 in [7] to get that

$$m(M^4, g, \mathcal{E}) = \lim_{R\to\infty} \frac{1}{|\mathbb{S}^2||\mathbb{S}^1|} \int_{\partial B_R} (g_{ij,j} - g_{jj,i}) *_h dx^i - \lim_{R\to\infty} \frac{1}{|\mathbb{S}^2||\mathbb{S}^1|} \int_{\partial B_R} *_h dg(T_0, T_0) \tag{2.12}$$

where $i, j \in \{1, 2, 3\}$ and $X_i = \frac{\partial}{\partial x^i}$, $g_{ij} = g(X_i, X_j)$.

Corollary 2.9. *If $(M^4, g, \mathcal{E})$ admits an almost free $\mathbb{S}^1$-action, then*

$$m_{ADM}(M^4, g, \mathcal{E}) = m_{ADM}(\bar{M}, \bar{g}, \bar{\mathcal{E}}) + \lim_{R\to\infty} \frac{2}{|\mathbb{S}^2|} \int_{\partial B(R)} \pi_*(N) \cdot \nu \tag{2.13}$$

where the quotient map $\pi : (M^4, g) \to (M/\mathbb{S}^1, \bar{g})$ is a Riemannian immersion and N is the mean curvature of each fiber $\pi^{-1}(\bar{x})$ for $\bar{x} \in \bar{M}$.

Notice that $(\bar{\mathcal{E}}, \bar{g}|_{\bar{\mathcal{E}}})$ is an asymptotically Euclidean end and the ADM mass is

$$m(\bar{M}, \bar{g}, \bar{\mathcal{E}}) = \lim_{R\to\infty} \frac{1}{|\mathbb{S}|^2} \int_{\partial B(\bar{p}, R)} *_{g_{\mathbb{R}^3}}(\mathrm{div}_{g_{\mathbb{R}^3}}\bar{g} - d\mathrm{tr}_{g_{\mathbb{R}^3}}\bar{g})$$

Proof. Let T and η be defined in Section 2.2. The metric g can be expressed as follows:

$$g = \pi^*(\bar{g}) + \frac{1}{|\eta|^2}\eta^2$$

From Proposition 2.8, $m(M^4, g, \mathcal{E})$ is independent of the choice of h. We may assume that

$$h = \pi^*(g_{\mathbb{R}^3}) + \eta_0^2.$$

Let $(\bar{X_1}, \bar{X_2}, \bar{X_3})$ be a $g_{\mathbb{R}^3}$-orthonormal frame field on $\bar{\mathcal{E}}$. Then, $\{X_1, X_2, X_3, T_0\}$ is a h-orthonormal frame field on $\mathcal{E}$, where $\bar{X}_i = \pi_*(X_i)$. Notice that $g(X_i, X_j) = \bar{g}(\bar{X}_i, \bar{X}_j)$.

The first part of Equation (2.12) can rewritten in the following form:

$$\begin{aligned}
\lim_{R\to\infty} \frac{1}{|\mathbb{S}^2||\mathbb{S}^1|}\int_{\partial B(p,R)} (g_{ij,j} - g_{jj,i}) *_h dx^i &= \lim_{R\to\infty} \frac{1}{|\mathbb{S}^2||\mathbb{S}^1|}\int_{\partial B(p,R)} (\bar{g}_{ij,j} - \bar{g}_{jj,i}) *_{g_{\mathbb{R}^3}} (d\bar{x}^i) \wedge \eta_0 \\
&= \lim_{R\to\infty} \frac{1}{|\mathbb{S}^2|}\int_{\partial B(\bar{p},R)} (\bar{g}_{ij,j} - \bar{g}_{jj,i}) *_{g_{\mathbb{R}^3}} (d\bar{x}^i) \\
&= \lim_{R\to\infty} \frac{1}{|\mathbb{S}|^2}\int_{\partial B(\bar{p},R)} *_{g_{\mathbb{R}^3}}(\mathrm{div}_{g_{\mathbb{R}^3}}\bar{g} - d\mathrm{tr}_{g_{\mathbb{R}^3}}\bar{g}) \\
&= m(\bar{M}, \bar{g}, \bar{\mathcal{E}}).
\end{aligned}$$

We directly compute that

$$g(T,T) - g(T_0,T_0) \in C^{2,\alpha}_{2\mu} \text{ and } N = -\frac{1}{2} d\log g(T,T)$$

where $\mu > 1/2$. The second part of Equation (2.12) is

$$\begin{aligned}
-\lim_{R\to\infty} \frac{1}{|\mathbb{S}^2||\mathbb{S}^1|}\int_{\partial B_R} *_h dg(T_0,T_0) &= -\lim_{R\to\infty} \frac{1}{|\mathbb{S}^2||\mathbb{S}^1|}\int_{\partial B_R} *_h dg(T,T) \\
&= \lim_{R\to\infty} \frac{2}{|\mathbb{S}^2||\mathbb{S}^1|}\int_{\partial B_R} *_h N \\
&= \lim_{R\to\infty} \frac{2}{|\mathbb{S}^2|}\int_{\partial B(R)} \pi_*(N)\cdot\nu
\end{aligned}$$

□

3. Almost free action and Local structure of singularities of $M^4/\mathbb{S}^1$

In this section, we consider a complete ALF manifold $(M^4, g, \mathcal{E})$ admitting an almost free $\mathbb{S}^1$-action and study the local singular point of the quotient space $(M^4/\mathbb{S}^1, \bar{g})$.

Theorem 3.1. *Let (M^4, g) be a complete manifold with an isometric $\mathbb{S}^1$-action. If $\mathbb{S}^1$ acts on (M, g) freely, expect finitely many points $\{p_1, \cdots, p_k\}$, then $(M/\mathbb{S}^1, \bar{g})$ is a smooth manifold outside finitely many points, $\bar{p}_1, \bar{p}_2, \cdots \bar{p}_k$. Moreover, for each $\bar{p}_i$, there is a neighborhood $\bar{U}_i$ such that*

$$\bar{g}|_{\bar{U}_i} = (dr)^2 + r^2 g_{\mathbb{P}^1} + o(r^2),$$

where $g_{\mathbb{P}^1} = \frac{1}{4}g_{\mathbb{S}^2}$ is the Fubini-Study metric on $\mathbb{P}^1$.

We first study the behavior of the $\mathbb{S}^1$-action near the fixed points and its relationship with the quotient metric near singular points. Precisely, near any singular point, $\bar{g}$ can be expressed as follows:

$$\bar{g}|_{\bar{U}_i} = (dr)^2 + r^2 g_{\Sigma^2} + o(r^2)$$

where (Σ^2, g_{Σ^2}) is the quotient space of $(\mathbb{S}^3, g_{\mathbb{S}^3})$ by an isometric and free $\mathbb{S}^1$-action $A(t)$ (see Proposition 3.3 and Remark 3.4). Then we show that the action $A(t)$ induces the so-called Hopf-fibration.

3.1. **Metric structure near fixed points.** Without loss of generalization, we assume that

$$p \text{ is the unique fixed point of the } \mathbb{S}^1\text{-action}$$

That is to say, the quotient space $M/\mathbb{S}^1$ is a smooth manifold outside $\bar{p}$. The isometric $\mathbb{S}^1$-action is associated with a group homomorphism $S(t) : \mathbb{S}^1 \to \mathrm{Isom}(M, g)$.

Proposition 3.2. *If p is an isolated fixed point of the isometric $\mathbb{S}^1$-action on (M^4, g), then one has that*

- *the tangential map $A(t) := dS(t)|_p : T_pM \to T_pM$ is an isometry for each $t \in \mathbb{S}^1$;*
- *The action $S(t)$ is a free $\mathbb{S}^1$-action on $\partial B(p, r)$, for $r > 0$ small enough.*

Notice that $A(t) : \mathbb{S}^3 \subset T_pM \to \mathbb{S}^3 \subset T_pM$ is also an isometric $\mathbb{S}^1$-action on $\mathbb{S}^3$.

Proposition 3.3. *Let (M, g), $S(t)$ and p be assumed as in Proposition 3.2. Then, one has that*

- *the space $(\partial B(\bar{p}, r), \frac{1}{r^2}\bar{g}|_{\partial B(\bar{p},r)}) \subset (M^4/\mathbb{S}^1, \frac{1}{r^2}\bar{g})$ converges to a metric space (Σ^2, g_{Σ^2}) in the Gromov-Hausdorff topology;*
- *the space (Σ^2, g_{Σ^2}) is isometric to the quotient space of $(\mathbb{S}^3, g_{\mathbb{S}^3})$ by the action $A(t)$*

Proof. For $r_0 < \frac{1}{2}\mathrm{Inj}_g(p)$, the induced $\mathbb{S}^1$-action $\tilde{S}(t)$ on $B(0, r_0) \subset T_pM$ is defined as follows: for any $v \in B(0, r_0)$

$$\exp_p(\tilde{S}(t)v) = S(t)(\exp_p(v))$$

It is an isometric $\mathbb{S}^1$-action, with respect to the metric $exp_p^*(g)$ on $B(0, r_0)$.

In the following, we choose a sufficiently small constant r_0 such that 0 is the unique fixed point of the action $\tilde{S}(t)$ on $B(0, r_0) \subset T_pM$. Then, one has that for any $r < r_0$,

- $\tilde{S}(t) : \partial B(0, r) \subset T_pM \longrightarrow \partial B(0, r) \subset T_pM$ is a free action;
- it is an isometric action, with respect to the metric $g_{p,r} := \exp_p^*(g)|_{\partial B(0,r)}$. (because the metric $\exp_p^*(g)$ on $B(0, r_0)$ is $\tilde{S}(t)$-invariant.)

For any $r < r_0$, define a $\mathbb{S}^1$-action $\tilde{S}_r(t)$ on the space $(\mathbb{S}^3, \frac{1}{r^2}g_{p,r})$ as follows:

$$\begin{aligned} \tilde{S}_r(t) :(\mathbb{S}^3, \frac{1}{r^2}g_{r,p}) &\longrightarrow (\mathbb{S}^3, \frac{1}{r^2}g_{r,p}) \\ v \quad &\longrightarrow \qquad \frac{1}{r}\tilde{S}(t)(rv) \end{aligned}$$

From the construction, this action enjoys the following properties:

(A) It is an isometric and free $\mathbb{S}^1$-action;
(B) $\lim\limits_{r\to 0} \tilde{S}_r(t) = dS(t)|_p = A(t) \in \mathrm{Isom}(T_pM)$, since $\tilde{S}(0) = 0$.
(C) for $r < r_0$, $(B(\bar{p}, r), \bar{g}|_{B(\bar{p},r)}) \subset (M/\mathbb{S}^1, \bar{g})$ is the quotient space of $(B(0, r), \exp_p^*(g)|_{B(0,r)})$ by the action $\tilde{S}(t)$.
(D) for $r < r_0$, $(\partial B(\bar{p}, r), \frac{1}{r^2}\bar{g}|_{\partial B(\bar{p},r)})$ is the quotient space of $(\mathbb{S}^3, \frac{1}{r^2}g_{r,p})$ by the action $\tilde{S}_r(t)$.

The item (D) follows from the item (C) and the definition of $\tilde{S}_r(t)$.

The space $(\mathbb{S}^3, \frac{1}{r^2}g_{r,p})$ converges to $(\mathbb{S}^3, g_{\mathbb{S}^3})$ as $r \to 0$ in C^∞-topology. Moreover, from the item (B), one has that $(\mathbb{S}^3, \frac{1}{r^2}g_{r,p}, \tilde{S}_r(t))$ converges to $(\mathbb{S}^3, g_{\mathbb{S}^3}, A(t))$ in equivariant Gromov-Hausdorff topology (see Definition 3.3 in [15]).

$$\begin{array}{ccc} (\mathbb{S}^3, \frac{1}{r^2} g_{p,r}) & \xrightarrow{r\to 0} & (\mathbb{S}^3, g_{\mathbb{S}^3}) \\ \Big\downarrow{\scriptstyle \tilde{S}_r(t)-\text{action}} & & \Big\downarrow{\scriptstyle A(t)-\text{action}} \\ (\partial \bar{B}(\bar{p}, r), \frac{1}{r^2}\bar{g}|_{\partial B(\bar{p},r)}) & \xrightarrow{r\to 0} & (\Sigma, g_\Sigma) \end{array}$$

From Lemma 3.4 in [15], the second line is the Gromov-Hausdorff convergence in the diagram. Then, (Σ^2, g_{Σ^2}) is isometric to the quotient space of $\mathbb{S}^3$ by the action $A(t)$. $\square$

Remark 3.4. If $A(t)$ acts on $(\mathbb{S}^3, g_{\mathbb{S}^3})$ freely, then the space (Σ^2, g_{Σ^2}) is a smooth manifold. Namely, $(\partial \bar{B}(\bar{p}, r), \frac{1}{r^2}\bar{g}|_{\partial B(\bar{p},r)})$ converges (Σ, g_Σ) in the $C^{1,\alpha}$-topology. The quotient metric $\bar{g}$ can be expressed as follows:

$$\bar{g}|_{\bar{U}_i} = (dr)^2 + r^2 g_{\Sigma^2} + o(r^2) \tag{3.1}$$

In the following, it is sufficient to prove that $A(t)$ is a free $\mathbb{S}^1$-action on $(\mathbb{S}^3, g_{\mathbb{S}^3})$. Since $A(t) : T_pM \to T_pM$ is an isometric $\mathbb{S}^1$-action, it can be written as follows:

$$A(t) = e^{Et}$$

where $E \in so(4)$. Use the normal coordinate and the Jordan decomposition for $so(4)$ to rewrite E as follows:

$$E = \left[\begin{array}{cc|cc} 0 & n_1 & & \\ -n_1 & 0 & & \\ \hline & & 0 & n_2 \\ & & -n_2 & 0 \end{array}\right]$$

The complex coordinate on $T_pM = \mathbb{C}^2$ allows us to rewrite A and E

$$E = \begin{bmatrix} in_1 & \\ & in_k \end{bmatrix} \quad A(t) = \begin{bmatrix} e^{in_1 t} & \\ & e^{in_k t} \end{bmatrix}$$

Because the metric $A(t)$ is of period 2π, one has that $n_i \in \mathbb{Z}$ for $i = 1, 2$.

Theorem 3.5. *Let (M, g), $S(t)$, $A(t)$ and k_i be assumed as above. Then $A(t)$ is a free action on $(\mathbb{S}^3, g_{\mathbb{S}^3})$. Moreover, $k_i = \pm 1$ for $i = 1, 2$.*

Remark 3.6. From Theorem 3.5 and Remark 3.4, (Σ^2, g_Σ) is a smooth manifold. Moreover, one has two cases

- if $k_1k_2 = 1$, then $\mathbb{S}^3 \to \Sigma$ is the circle bundle of $O(1)$-bundle over $\mathbb{P}^1$.
- if $k_1k_2 = -1$, then $\mathbb{S}^3 \to \Sigma$ is the circle bundle of $O(-1)$-bundle over $\mathbb{P}^1$.

To sum up, (Σ, g_Σ) is isometric to $(\mathbb{P}^1, g_{FS})$, where g_{FS} is the Fubini-Study metric on $\mathbb{P}^1$. That's to say,

$$\bar{g} = (dr)^2 + r^2 g_{\mathbb{P}^1} + o(r^2).$$

For proving Theorem 3.1, it is sufficient to complete the proof of Theorem 3.5

3.2. **Link number.** In order to prove Theorem 3.5, we begin with the link number and then talk about its application for the $\mathbb{S}^1$ action on (M^4, g).

Definition 3.7. Let $L = L_1 \amalg L_2 \subset \mathbb{S}^3$ be a link with two components. The link number $lk(L_1, L_2)$ is defined as follows:

$$[L_2] = lk(L_1, L_2)[m_1] \in H_1(\mathbb{S}^3 \setminus L_1),$$

where m_1 is a generator of the kernel of $\pi_1(\partial B(L_1, \epsilon)) \to \pi_1(B(L_1, \epsilon))$ and $B(L_1, \epsilon)$ is the tubular neighborhood of L_1 in $\mathbb{S}^3 \setminus L_2$ with radius ϵ.

Lemma 3.8. *Let $L = L_1 \amalg L_2 \subset \mathbb{S}^3$ be a link with two components and $L' = L'_1 \amalg L'_2$ be another link, where L'_i is a subset of $B(L_i, \epsilon_i)$. Then one has that*

$$lk(L'_1, L'_2) = \deg(Pr_{L_1}|_{L'_1}) \cdot \deg(Pr_{L_2}|_{L'_2}) \cdot lk(L_1, L_2),$$

where $Pr_{L_i} : B(L_i, \epsilon_i) \to L_i$ is the projection.

Proof. From the loop lemma (see Theorem 3.1 in [17]), we find an embedded disc D_1 bounded by a meridian m_1 of L_1. We may assume that L'_1 intersects D_1 transversally and each component of $B(L'_1, \epsilon'_1) \cap D_1$ is a disc, where $B(L'_1, \epsilon'_1) \subset B(L_1, \epsilon_1)$ is the tubular neighborhood of L'_1 . Definition of the degree of a map shows that

$$[m_1] = \deg(Pr_{L_1}|_{L'_1})[m'_1] \text{ in } H_1(B(L_1, \epsilon) \setminus \mathrm{Int}B(L'_1, \epsilon'_1)), \tag{3.2}$$

where m'_1 is a meridian of L'_1. The equality also holds in $H_1(\mathbb{S}^3 \setminus L'_1)$.

Definition 3.7 gives that

$$[L_2] = lk(L_1, L_2)[m_1] \in H_1(\mathbb{S}^3 \setminus \text{ Int } B(L_1, \epsilon_1)). \tag{3.3}$$

It also holds in $H_1(\mathbb{S}^3 \setminus L'_1)$. Hence, one has that in $H_1(\mathbb{S}^3 \setminus L'_1)$

$$\begin{aligned} [L'_2] &= \deg(Pr_{L_2}|_{L'_2})[L_2] \\ &= \deg(Pr_{L_2}|_{L'_2}) lk(L_1, L_2)[m_1] && \text{from Equation (3.2)} \\ &= \deg(Pr_{L_2}|_{L'_2}) lk(L_1, L_2) \deg(Pr_{L_1}|_{L'_1})[m'_1] && \text{from Equation (3.3).} \end{aligned}$$

Therefore, we can conclude that $lk(L'_1, L'_2) = \deg(Pr_{L_1}|_{L'_1}) \cdot \deg(Pr_{L_2}|_{L'_2}) \cdot lk(L_1, L_2)$. □

Lemma 3.9. *Let $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ be a principal $\mathbb{S}^1$ bundle. Then it is the Hopf fibration. Moreover, for any two distinct points $p, q \in \mathbb{S}^2$, the pre-image $\pi^{-1}(\{p, q\})$ is a link and its link number is ± 1.*

Proof. From [8], the isomorphism classes of principal $\mathbb{S}^1$-bundles over $\mathbb{S}^2$ are in one-one correspondence with $\pi_1(SO(2))$.

$$\begin{array}{ccc} \mathbb{Z} \cong \pi_1(SO(2)) & \longleftrightarrow & \{\text{principle } \mathbb{S}^1\text{-bundle over } \mathbb{S}^2\} \\ & & \\ n & & P_n \end{array}$$

Namely, for any $n \in \mathbb{Z}$, there is a principle $\mathbb{S}^1$-bundle P_n over $\mathbb{S}^2$. Using the clutching construction, the total space of P_n is homeomorphic to the lens space $L(n, 1)$.

There is a $n_0 \in \mathbb{Z}$ such that the principal bundle P_{n_0} is isomorphic to $\pi : \mathbb{S}^3 \to \mathbb{S}^1$. Namely, $L(n_0, 1) \cong \mathbb{S}^3$, which implies that $n_0 = \pm 1$. The principal bundle $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ is the so-called Hopf fibration. Then, the link $\pi^{-1}(\{p, q\})$ is a Hopf Link whose link number is ± 1. □

Corollary 3.10. *Let $\tilde{S}_r(t) : \mathbb{S}^3 \subset T_pM \to \mathbb{S}^3 \subset T_pM$ be a free $\mathbb{S}^1$-action defined in the proof of Proposition 3.3. Then, one has that*

- *the quotient space of* $\mathbb{S}^3$ *by the action* $\tilde{S}_r(t)$ *is homeomorphic to* $\mathbb{S}^2$
- *the quotient map* $\mathbb{S}^3 \to \partial B(\bar{p}, r)$ *by the action*$\tilde{S}_r(t)$ *is a Hopf fibration.*

Proof. Since $\tilde{S}_r(t) : \mathbb{S}^3 \to \mathbb{S}^3$ is a free $\mathbb{S}^1$-action, the quotient map $\mathbb{S}^3 \to \partial B(\bar{p}, r)$ by the action $\tilde{S}_r(t)$ is a principal $\mathbb{S}^1$-bundle. The long exact sequence for fibration (see Theorem 4.41 in [18]) implies that $\pi_2(\partial B(\bar{p}, r)) \cong \mathbb{Z}$ and $\pi_1(\partial B(\bar{p}, r)) = \{1\}$. That is to say, the quotient space $\partial B(\bar{p}, r)$ is homeomorphic to $\mathbb{S}^2$.

From Lemma 3.9, the quotient map is a Hopf fibration.

□

3.3. **Proof Theorem 3.5.** We now use the link number to complete the proof of Theorem 3.5.

Proof. As in Section 3.1, the matrix $A(t)$ can be written as follows:

$$A(t) = \begin{bmatrix} e^{in_1 t} & 0 \\ 0 & e^{in_2 t} \end{bmatrix}$$

Choose two points $v_1 = (1, 0), v_2 = (0, 1) \in \mathbb{S}^3 := \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$ and consider two circles $C_1 := \{(z_1, 0) \mid |z_1| = 1\}$ and $C_2 := \{(0, z_2) \mid |z_2| = 1\}$. Then, one has that

- for any $t \in \mathbb{S}^1$ and $i \in \{1, 2\}$, $A(t)C_i = C_i$
- The union $C_1 \amalg C_2$ is a hopf link with $lk(C_1, C_2) = 1$.

Let $\tilde{S}_r : (\mathbb{S}^3, \frac{1}{r^2} g_{r,p}) \to (\mathbb{S}^3, \frac{1}{r^2} g_{r,p})$ and $A(t)$ be defined in Section 3.1. Since $(\mathbb{S}^3, \frac{1}{r^2} g_{p,r}, \tilde{S}_r(t))$ converges to $(\mathbb{S}^3, g_{\mathbb{S}^3}, A(t))$ in equivariant Gromov-Hausdorff sense, then for any $\epsilon > 0$, there is a positive constant $r_1 0$ such that for any $r < r_1$ and $v \in \mathbb{S}^3$,

$$|A(t)(v) - \tilde{S}_r(v)|_{g_{\mathbb{S}^3}} \leq \epsilon$$

That is to say, the orbit of $\tilde{S}_r(t)$ including v_i satisfies that

$$C_i' := \{\tilde{S}_r(t) v_i \mid t \in \mathbb{S}^1\} \subset B(C_i, \epsilon)$$

where $B(C_i, \epsilon)$ is the tubular neighborhood of C_i in $(\mathbb{S}^3, g_{\mathbb{S}^3})$ with radius ϵ. For $i = 1, 2$, the degree of $Pr_{C_i}|_{C_i'} : C_i' \to C_i$ is equal to n_i, where $Pr_{C_i} : B(C_i, \epsilon) \to C_i$ is the projection map.

From Corollary 3.10, the union $C_1' \amalg C_2'$ is a Hopf link in $\mathbb{S}^3$, with $lk(C_1', C_2') = \pm 1$. Lemma 3.8 implies that

$$lk(C_1', C_2') = \deg(\pi_{C_1}|_{C_1'}) \cdot \deg(\pi_{C_2}|_{C_2'}) lk(C_1, C_2) = n_1 n_2$$

From Lemma 3.9, $lk(C_1', C_2') = \pm 1$, which implies that $n_i = \pm 1$. □

4. The Density of ADM mass for ALF 4-manifolds

Let $\mathcal{X}$ be the total space of a principal $\mathbb{S}^1$ bundle $\pi_0 : \mathcal{X} \to \mathbb{R}^3 \setminus B(0, 1)$. The model metric h on $\mathcal{X}$ is defined as follows

$$h = \pi_0^*(g_{\mathbb{R}^3}) + \eta_0^2$$

where η_0 is a $\mathbb{S}^1$ invariant one form.

Remark 4.1. The model metric h is not scalar-flat and its scalar curvature enjoy the decay $R_h = O(r^{-4})$. For any function f on $\mathbb{R}^3 \setminus B(0, 1)$, we have that

$$\Delta_h \pi_0^*(f) = \pi_0^*(\Delta_{\mathbb{R}^3} f)$$

Then, $\Delta_h(\pi_0^*(1+\frac{m}{6r}))=0$. The scalar curvature of

$$(1+\frac{m}{6r})^2h$$

belongs to $O(r^{-4})$ and its ADM mass is equal to m.

Definition 4.2. Let $(M^4,g,\mathcal{E})$ be a complete ALF 4-manifold. The end $\mathcal{E}$ is called *Schwarzschild-like ALF* if it satisfies that

$$\begin{aligned} g|_{\mathcal{E}} &= (1+\frac{m}{r})^2h+w \\ |w|+r|\partial w|&+r^2|\partial^2 w|\in O(r^{-q}) \end{aligned}$$

where $q>1$

We now focus on the following mass density result.

Theorem 4.3. *Let $(M^4,g,\mathcal{E})$ be a complete ALF 4-manifold with almost free $\mathbb{S}^1$-action and with non-negative scalar curvature. For any $\epsilon>0$, there is a complete ALF metric g' satisfying that*

- *its scalar curvature is non-negative;*
- *it admits an almost free $\mathbb{S}^1$-action;*
- $|m(M^4,g,\mathcal{E})-m(M^4,g',\mathcal{E})|\le\epsilon$.

4.1. **Finding the conformal factor.** Assume that $(M^4,g,\mathcal{E})$ is a complete ALF manifold with almost free $\mathbb{S}^1$-action. the metric g on $\mathcal{E}$ is

$$g|_{\mathcal{E}}=\pi^*(g_{\bar{\mathcal{E}}})+\frac{1}{|\eta|^2}\eta\otimes\eta$$

where T is the infinitesimal generator of the isometric $\mathbb{S}^1$-action and η is a 1-form with $\eta(X)=g(T,X)$.

We may assume that the $\mathbb{S}^1$-action is free on $(\mathcal{E},g|_{\mathcal{E}})$ and $|T|$ is sufficiently close to 1 on $\mathcal{E}$. The quotient space $\bar{\mathcal{E}}:=\mathcal{E}/\mathbb{S}^1$ is an asymptotically Euclidean end. One has that

Lemma 4.4. *Let $u:\mathcal{E}\to\mathbb{R}$ be a $\mathbb{S}^1$-invariant function. Then, there is a universal constant C such that*

$$(\int_{\mathcal{E}}u^6dx)^{1/3}\le C\int_{\mathcal{E}}|\nabla u|^2$$

The proof is the combination of the sobolev inequality on $\bar{\mathcal{E}}=\mathcal{E}/\mathbb{S}^1$ and the norm inequalities (2.8).

In the following, we study the existence of the elliptic equation on a complete ALF manifold (M^4,g) with an almost free $\mathbb{S}^1$-action. Without the loss of the generalization, we may assume that (M^4,g) has a unique end $\mathcal{E}$ and p is the unique fixed point of the $\mathbb{S}^1$-action.

Theorem 4.5. *Let $(M^4,g,\mathcal{E})$ be a complete ALF manifold with almost free $\mathbb{S}^1$-action. Assume that the isometric $\mathbb{S}^1$-action is free on $\mathcal{E}$. If f is a $\mathbb{S}^1$-invariant function on M satisfying that*

- *f is supported in $\mathcal{E}$;*
- $(\int_{\mathcal{E}}|f|^{\frac{3}{2}}dx)^{\frac{2}{3}}\le\frac{1}{2C}$,
- *$f\in L^{\frac{6}{5}}$ and $f\in L^\infty$*
- $f\in O(r^{-4})$.

Then there is a positive $\mathbb{S}^1$-invariant function u satisfying that

$$\Delta_g u - fu = 0 \tag{4.1}$$

where C is a constant in Lemma 4.4 Moreover, u enjoys an asymptotic expression as follows:

$$u = 1 + \frac{A}{r} + O(r^2),$$

where $A = -\frac{1}{8\pi^2}\int_M fu$

Proof. Assume that there is an exhaustion $\{U_k\}$ of M such that

- each U_i is $\mathbb{S}^1$-invariant;
- $\partial U_i \subset \mathcal{E}$.

Step 1: On U_i, the kernel of $\Delta_g - f$ with Dirichlet boundary is trivial.

Assume that a function w satisfies that

$$\begin{cases} \Delta_g w - fw = 0 \\ w(x) = 0 \text{ on } \partial U_i \end{cases} \tag{4.2}$$

The integration by part implies that

$$\begin{aligned}\int_{U_i} |\nabla w|^2 &= -\int_{U_i} fw^2 dx = -\int_{U_i\cap\mathcal{E}} fw^2 dx \\ &\leq (\int_{\mathcal{E}} |f|^{\frac{3}{2}} dx)^{\frac{2}{3}} (\int_{\mathcal{E}} w^6 dx)^{\frac{1}{3}} \\ &\leq \frac{1}{2C}(\int_{\mathcal{E}} w^6 dx)^{\frac{1}{3}}\end{aligned}$$

Lemma 4.4 implies that $\frac{1}{C}(\int_{\mathcal{E}} w^6 dx)^{1/3} \leq \int_{\mathcal{E}} |\nabla w|^2$. Then, $w = 0$ on $\mathcal{E}$ and $\nabla w = 0$ in U_i. Namely, $w = 0$.

Remark that w is a function on $\mathcal{E}$ with zero extension on $\mathcal{E} \setminus U_i$.

Step 2 Existence of u.

From the Fredholm alternative (see Theorem 5.3 in [16]), the following PDE has a unique smooth solution w_i on U_i

$$\begin{cases} \Delta_g w_i - fw_i = f \text{ in } U_i \\ w_i = 0, \text{ on } \partial U_i \end{cases} \tag{4.3}$$

Since g, U_i and f are $\mathbb{S}^1$-invariant , the uniqueness of the solution deduces that w_i is also $\mathbb{S}^1$-invariant.

We now study the uniform estimate for w_i. Using the integration by part give that

$$\begin{aligned}\int_M |\nabla w_i|^2 &= -\int_{\mathcal{E}} w_i^2 f dx - \int_{\mathcal{E}} w_i f dx \qquad \text{since } f \text{ is supported in } \mathcal{E} \\ &\leq (\int_{\mathcal{E}} |f|^{\frac{3}{2}} dx)^{\frac{2}{3}} (\int_{\mathcal{E}} w_i^6 dx)^{\frac{1}{3}} + (\int_{\mathcal{E}} |f|^{\frac{6}{5}} dx)^{\frac{5}{6}} (\int_{\mathcal{E}} w_i^6 dx)^{\frac{1}{6}} \\ &\leq \frac{1}{2C}(\int_{\mathcal{E}} w_i^6 dx)^{\frac{1}{3}} + (\int_{\mathcal{E}} |f|^{\frac{6}{5}} dx)^{\frac{5}{6}} (\int_{\mathcal{E}} w_i^6 dx)^{\frac{1}{6}}\end{aligned}$$

From Lemma 4.4, it follows that

$$(\int_{\mathcal{E}} w_i^6 dx)^{\frac{1}{6}} \leq 2C(\int_{\mathcal{E}} |f|^{\frac{6}{5}} dx)^{\frac{5}{6}}. \tag{4.4}$$

Moreover, Nash-Moser iteration allows us to find a uniform L^∞-estimate (See Appendix I):

$$||w_i||_{L^\infty} \le C_2(||f||_{L^2}, ||f||_{L^{\frac{6}{5}}}, C)$$

From Schauder estimate (see Theorem 6.2 in [16]), all differentials of w_k enjoy a uniform estimate on a compact set K: for any non-negative integer k,

$$\max_{x\in K} |\nabla^k w_i| \le C(K, k)$$

We conclude that there is a $\mathbb{S}^1$-invariant function $w = \lim\limits_{i\to\infty} w_i$ satisfying that

$$\begin{cases} \Delta_g w - fw = f \\ w \text{ is bounded} \end{cases} \tag{4.5}$$

The function $u = w + 1$ is the required one.

Step 3: Asymptotic estimate.

Since $f = O(r^{-4}) \in C^{0,\alpha}_4$, we directly apply Lemma 2.7 to (4.5) and get that

$$w - (c_\infty + \frac{A}{r}) \in C^{2,\alpha}_{1+\mu}$$

The bound of L^6-norm of w (from (4.4)) implies that $c_\infty = 0$. That to say,

$$u - (1 + \frac{A}{r}) \in C^{2,\alpha}_{1+\mu}$$

The constant A can be expressed as follows:

$$-8\pi^2 A = \lim_{R\to\infty} \int_{\partial B(0,R)} \partial_v u = \int_M \Delta u = \int_M fu$$

where the last equation follows from the differential equation $\Delta u = fu$. Then,

$$A = -\frac{1}{8\pi^2} \int_M fu$$

Step 4: u is non-negative.

We argue by contradiction. Assume that $S = \{x \in M \mid u(x) < 0\}$ is non-empty. Since $\lim\limits_{|x|\to\infty} u(x) = 1$ on $\mathcal{E}$ and M has a unique end, the closure of S is compact. Then, we have that

$$\begin{cases} \Delta u - fu = 0 \text{ in } S \\ u = 0 \text{ on } \partial S \end{cases}$$

The integration by part implies that $\int_S |\nabla u| = -\int_S fu^2$. We have two cases, $S \cap \text{Int } \mathcal{E} = \emptyset$ and $S \cap \text{Int}\mathcal{E} \neq \emptyset$.

Case (I) If $S \cap \text{Int}\mathcal{E} = \emptyset$, then $f = 0$ on S. Namely, $\nabla u = 0$ in S. It implies that $u = 0$ in S, a contradiction.

Case (II) If $S \cap \text{Int}\mathcal{E} \neq \emptyset$, we have that $\int_{\mathcal{E}} |\nabla \tilde{u}|^2 = -\int_{\mathcal{E}} f\tilde{u}^2$, where $\tilde{u}$ is a function on $\mathcal{E}$ with zero-extension of $u|_S$ on $\mathcal{E} \setminus S$. Using the argue in Step 1 of the existence, we have that $\tilde{u} = 0$. That is to say, $u = 0$ on $S \cap \mathcal{E}$, a contradiction.

We can conclude that S is empty. That's to say, $u \ge 0$. We complete the proof of the claim.

If there is a point $x_0 \in M$ with $u(x_0) = 0$, Harnack's inequality and the non-negativity of u deduces that $u(x) = 0$, which leads to a contradiction with the asymptotic behavior of u.

We can conclude that u is a positive function. $\square$

4.2. **Proof for Theorem 4.3.** Let $(M^4, g, \mathcal{E})$ be a complete ALF manifold with almost free $\mathbb{S}^1$-action and with non-negative scalar curvature. We may assume that the isometric $\mathbb{S}^1$-action on $\mathcal{E}$ is free. On the end $\mathcal{E}$, the metric g can be written as

$$g|_{\mathcal{E}} = (1 + \frac{m}{6r})^2 h + w,$$

where $w = o(r^{-1})$ is $\mathbb{S}^1$-invariant tensor and $m = m_{ADM}(M^4, g, \mathcal{E})$. Proposition 2.8 shows that

$$\int_{\partial B(R,0)} *\{\mathrm{div}_h(w) - d\mathrm{Tr}_h(w)\} = o(1)$$

Step 1 Construct a family of metrics on M^4 with Schwarzschild-like ALF end

For any $s > s_0$, there is a $\mathbb{S}^1$-invariant cut-off function ϕ_s on $\mathcal{E}$ satisfying that

- $\phi_s = 0$ in $\{x \mid r(x) \geq 4s\}$
- $\phi_s = 1$ for $\{x \mid r(x) \leq 2s\} \cup M^4 \setminus \mathcal{E}$
- $|\nabla^k \phi_s| \leq \frac{10 \cdot 4^k}{r^k}$ for $k = 0, 1, 2, 3$

Such a cut-off function can be constructed from the cut-off function on the asymptotically Euclidean end $\bar{\mathcal{E}} := \mathcal{E}/\mathbb{S}^1$ via the pull-back of the Riemannian submersion $\pi : \mathcal{E} \to \bar{\mathcal{E}}$.

Consider a family of new metrics g_s on M with almost free $\mathbb{S}^1$-actions.

$$g_s = (1 + \frac{m}{6r})^2 h + \phi_s w$$

For any $s \geq s_0$, it has Schwarzschild-like ALF end its ADM mass is always equal to m. The scalar curvature R_{g_s} may be negative on $\{x \mid r(x) \geq 2s\}$.

Remark 4.6. The metric g_s enjoys the following properties.

$$g_s = \begin{cases} g & \text{in } M \setminus \mathcal{E} \cup \{x \mid r(x) \leq 2s\} \\ (1 + \frac{m}{6r})^2 h & \text{in } r(x) \geq 4s \end{cases}$$

The scalar curvature R_{g_s} has that

- $|R_{g_s}| \lesssim r^{-2-\mu}$ on $2s \leq r(x) \leq 4s$
- $R_{g_s} \lesssim \frac{1}{r^4}$ on $r(x) \geq 4s$
- R_{g_s} is negative in $\{x \mid r(x) \geq 2s\}$ and R_{g_s} is non-negative in $M \setminus \mathcal{E} \cup \{x \mid r(x) \leq 2s\}$
- $R_{g_s} \in L^\infty$, $R_{g_s} \in L^{\frac{6}{5}}$.

Step 2 Conformal deformation.

Let ψ_s be a $\mathbb{S}^1$-invariant function on (M, g_s) satisfying that

- $0 \leq \psi \leq 1$;
- ψ_s is supported in $\{x \mid r(x) \geq s\}$
- $\psi_s = 1$ in $\{x \mid r(x) \geq 2s\}$.

In the following, we first use Theorem 4.5 to solve the following PDE and then get the asymptotic behavior of the solution.

$$\begin{cases} \Delta_{g_s} u_s - \frac{1}{6}\psi_s R_{g_s} u_s = 0 \\ u(x) \to 1 \text{ as } r(x) \to \infty \end{cases}$$

Choose s large enough. Then, we have that

$$\int_M |\psi_s R_{g_s}|^{\frac{3}{2}} \leq \int_{r(x)\geq 4s} \frac{C_1}{r^6} + \int_{s\leq r(x)\leq 4s} \frac{C_2}{r^{3+\frac{3}{2}\mu}} \lesssim C_1 s^{-3} + C_2 s^{-\frac{3}{2}\mu} \leq (\frac{1}{2C})^{\frac{3}{2}}$$

Moreover, $\psi_s R_{g_s}$ is a $\mathbb{S}^1$-invariant function supported in $\mathcal{E}$ and it belongs to both L^∞ and $L^{\frac{6}{5}}$. Theorem 4.5 allows us to find a positive $\mathbb{S}^1$-invariant solution with the asymptotic behavior at $\mathcal{E}$

$$u_s - (1 + \frac{A}{r}) \in C^{2,\alpha}_{1+\mu} \tag{4.6}$$

where $A = -\frac{1}{8\pi}\int_{\mathcal{E}} \psi_s R_{g_s} u_s$

Remark 4.7. From (4.4), we have that $||u-1||_{L^6(\mathcal{E})} \leq C||\psi_s R_{g_s}||_{L^{\frac{6}{5}}(\mathcal{E})}$. The $L^{\frac{6}{5}}$ estimate follows that

$$\begin{aligned}\int |\psi_s R_{g_s}|^{\frac{6}{5}} &= \int_{s\leq r(x)\leq 4s} |\psi_s R_{g_s}|^{\frac{6}{5}} + \int_{r(x)\geq 4s} |\psi_s R_{g_s}|^{\frac{6}{5}} \\ &\leq \int_{s\leq r(x)\leq 4s} \frac{C_1^{\frac{6}{5}}}{r^{(2+\mu)\cdot\frac{6}{5}}} + \int_{r(x)\geq 4s} \frac{C_2^{\frac{6}{5}}}{r^{4\cdot\frac{6}{5}}} \\ &\lesssim s^{\frac{1}{2}-\mu} + s^{-\frac{9}{5}} = o(1)\end{aligned} \tag{4.7}$$

where $\mu > \frac{1}{2}$.

Lemma 4.8. $A = -\frac{1}{8\pi}\int_{\mathcal{E}} \psi_s R_{g_s} u_s = o(1)$ *as* $s \to \infty$

Remark. Proposition 2.8 shows that that for $k = 1, 2$ and s large enough,

$$\int_{r(x)=ks} *(\mathrm{div}_h(g_s) - d\mathrm{tr}_h(g_s)) = \int_{r(x)=ks} *(\mathrm{div}_h(g) - d\mathrm{tr}_h(g)) = m + o(1) \tag{4.8}$$

$$\lim_{R\to\infty} \int_{r(x)=R} *(\mathrm{div}_h(g_s) - d\mathrm{tr}_h(g_s)) = m \tag{4.9}$$

Proof. From Remark 4.7, we have that

$$\begin{aligned}|\int_M \psi_s R_{g_s} u - \int_{r(x)\geq s} \psi_s R_{g_s}| &= |\int_{r(x)\geq s} (u-1)\psi_s R_{g_s}| \\ &\leq ||u-1||^{\frac{1}{6}}_{L^6(\mathcal{E})} ||\psi_s R_{g_s}||^{\frac{5}{6}}_{L^{\frac{6}{5}}(\mathcal{E})} \\ &\lesssim ||\psi_s R_{g_s}||^{\frac{5}{6}}_{L^{\frac{6}{5}}(\mathcal{E})} = o(1)\end{aligned}$$

It is sufficient to show that $|\int_{r\geq s} \psi R_{g_s}| = o(1)$ as $s \to \infty$.

Notice that $R_{g_s} \geq 0$ in $\{s \leq r(x) \leq 2s\}$ and $\psi_s = 1$ in $\{r(x) \geq 2s\}$. We have two different cases:

Case I: If $\int_{r(x)\geq s}\psi_s R_{g_s}$ is positive, we have that

$$\begin{aligned}
|\int_{r(x)\geq s}\psi_s R_{g_s}| &= \int_{r(x)\geq s}\psi_s R_{g_s} \\
&= \int_{s\leq r(x)\leq 2s}\psi_s R_{g_s} + \int_{r(x)\geq 2s} R_{g_s} \quad \text{since } \psi_s = 1 \text{ on } r(x)\geq 2s \\
&\leq \int_{s\leq r(x)\leq 2s} R_{g_s} + \int_{r(x)\geq 2s} R_{g_s} \quad \text{because } R_{g_s}\geq 0 \text{ on } s\leq r(x)\leq 2s \\
&\leq \int_{r(x)\geq s} R_{g_s} \\
&= \int_{r(x)=s} *(\mathrm{div}_h(g_s) - d\mathrm{tr}_h(g_s)) - m + \int_{r(x)\geq s} O(r^{-2-2\mu}) \\
&= m + o(1) - m + O(s^{1-2\mu}) \\
&= O(s^{1-2\mu}) + o(1) \to 0.
\end{aligned}$$

where $\mu > 1/2$

Case II: If $\int_{r(x)\geq s}\psi_s R_{g_s}$ is negative, we have that

$$\begin{aligned}
|\int_{r(x)\geq s}\psi_s R_{g_s}| &= -\int_{r(x)\geq s}\psi_s R_{g_s} \\
&= -\int_{s\leq r(x)\leq 2s}\psi_s R_{g_s} - \int_{r(x)\geq 2s} R_{g_s} \quad \text{since } \psi_s = 1 \text{ on } r(x)\geq 2s \\
&\leq |\int_{r(x)\geq 2s} R_{g_s}| \quad \text{because } R_{g_s}\geq 0 \text{ on } s\leq r(x)\leq 2s \\
&= |\int_{r(x)=2s} *(\mathrm{div}_h(g_s) - d\mathrm{tr}_h(g_s)) - m| + \int_{r(x)\geq 2s} O(r^{-2-2\mu}) \\
&= m + o(1) - m + O(s^{1-2\mu}) \\
&= O(s^{1-2\mu}) + o(1) \to 0.
\end{aligned}$$

where $\mu > 1/2$

Step 3 Complete the proof. For sufficiently large s, the metric g' is defined as follows:

$$g' = u_s^2 g_s.$$

Then, from Lemma 4.8, we have that $m(M^4, g, \mathcal{E}) - m(M^4, g', \mathcal{E}) = A/6 = o(1)$.

The scalar curvature $R_{g'}$ is

$$R_{g'} = u_s^{-2}(R_{g_s} - \frac{6\Delta_{g_s}u_s}{u_s}) = u_s^{-2}R_{g_s}(1-\phi_s) \geq 0$$

Because $1-\psi_s$ is supported in $M\setminus\mathcal{E}\cup\{r(x)\leq 2s\}$ and $R_{g_s}\geq 0$ on $M\setminus\mathcal{E}\cup\{r(x)\leq 2s\}$, we have that $R_{g'}$ is non-negative. □

5. Proof for Theorem 1.5

Let $(M^4, g, \mathcal{E})$ be a complete ALF 4-manifold with an almost free $\mathbb{S}^1$-action, T be the infinitesimal generator of the $\mathbb{S}^1$-action and η be a 1-form with $\eta(X) = g(T, X)$. The metric g can expressed as follows:

$$g = \pi^*(\bar{g}) + \frac{1}{|\eta|^2}\eta^2 \tag{5.1}$$

where the quotient map $\pi : (M^4, g) \to (M/\mathbb{S}^1, \bar{g})$ is a Riemannian submersion. Moreover, the space $(M/\mathbb{S}^1, \bar{g}, \bar{\mathcal{E}})$ is a smooth asymptotically Euclidean manifold outside finitely many points.

We begin with O'Neil formula for Riemannian submersion and then use it to study the positive mass theorem on the quotient space $(M/\mathbb{S}^1, \bar{g})$.

5.1. **Riemannian Submersion and O'Neill formula.** Let $\pi : (X, g) \to (B, \bar{g})$ be a Riemannian submersion. Assume that it is also a principal $\mathbb{S}^1$-bundle (i.e. for any $b \in B$, the fiber $\pi^{-1}(b)$ is an embedded circle). For any $x \in X$ with $b = \pi(x)$, We denote by

$\mathcal{V}_x$	the tangent subspace to $\pi^{-1}(b)$ in Tx X
$\mathcal{H}_x$	the orthogonal complement to $\mathcal{V}_x$ in T_xX
$\mathcal{V} := \cup_x \mathcal{V}_x$,	the corresponding distribution of subspace $\mathcal{V}_x$
$\mathcal{H} := \cup_x \mathcal{H}_x$,	the corresponding distribution of subspace $\mathcal{H}_x$

Throughout this paper, U, V and W will be vertical vector field and X, Y, Z will be horizontal vector fields.

Definition 5.1. Let $\pi : X \to B$, $\mathcal{H}$, $\mathcal{V}$ be assumed as above. We denote two operators:

$$\begin{array}{rccc} A: & \mathcal{H} \times \mathcal{H} & \longrightarrow & \mathcal{V} \\ & (X, Y) & \longrightarrow & \nabla_X Y \end{array}$$

$$\begin{array}{rccc} \mathcal{T}: & \mathcal{V} \times \mathcal{V} & \longrightarrow & \mathcal{H} \\ & (U, V) & \longrightarrow & \nabla_U V \end{array}$$

At the point $x \in X$, T is the second fundamental form of the fiber $\pi^{-1}(\pi(x))$. The mean curvature of the fiber $\pi^{-1}(\pi(x))$ is

$$N = \mathrm{Tr}_{\mathcal{V}}(\mathcal{T}) \in \mathcal{H}$$

Then, $|\pi_*(N)|^2 = |N|^2$. Moreover, if $A = 0$, then $\mathcal{H}$ is completely integrable.

Lemma 5.2 (see Corollary 9.37 in [5])**.** *Let $\pi : (X, g) \to (B, \bar{g})$ be a Riemannian submersion and A, T and N be assumed as above. Then, the scalar curvature $R_{\bar{g}}$ is*

$$R_{\bar{g}}(b) = R_g(x) + |A|^2 + |\mathcal{T}|^2| + |N|^2 - 2div_{\bar{g}}(\pi_*(N)) \tag{5.2}$$

Since $\mathcal{V}$ is 1-dimensional, then $|N|^2 = |\mathcal{T}|^2$.

We now apply O'Neill's formula to the Riemannian submersion.

Proposition 5.3. *Let $(M^4, g, \mathcal{E})$ and $(\bar{M}, \bar{g}, \bar{\mathcal{E}})$ be assumed as in the begining of Section 5, where $\bar{M} = M/\mathbb{S}^1$. Assume that p is a fixed point of the $\mathbb{S}^1$. Then, one has that*

$$\begin{aligned} &N(x) = \frac{1}{r} + O(1) \ \textit{near} \ \ p \\ &N(x) \in C^{1,\alpha}_{1+\mu} \ \textit{in} \ \ \bar{\mathcal{E}} \\ &\bar{R}(\bar{x}) = R(x) + |A|^2 + |\mathcal{T}|^2| + |N|^2 - 2div_{\bar{g}}(\pi_*(N)) \end{aligned}$$

where $\mu > 1/2$, $r(x) = d(p, x)$, R and $\bar{R}$ are the scalar curvature of (M^4, g) and $(\bar{M}, \bar{g})$, respectively.

Remark. the expression in Equation (5.1) gives that

$$N = -\frac{d(|\eta|^2)}{2|\eta|^2}\bigg|_{\mathcal{H}}. \tag{5.3}$$

The form η enjoys that (1) $|\eta|^2 \sim r^2$ near p; (2) $|\eta|^2 - 1 \in C^{2,\alpha}_{\mu}$

5.2. **Conformal Laplacian on the quotient space.** Let $(M^4, g, \mathcal{E})$ be an complete ALF manifold which admits an almost free $\mathbb{S}^1$-action.In the following, for simplifying the proof of Theorem 1.5, we may assume that there is a fixed point p. The quotient space $(\bar{M}, \bar{g})$ is smooth outside $\bar{p}$. From Proposition 3.1, $\bar{g}$ can be written near $\bar{p}$, as follows:

$$\bar{g} = (dr)^2 + \frac{1}{4}r^2 g_{\mathbb{S}^2} + o(r^2). \tag{5.4}$$

Consider a a complete ALF Schwarzschild-like 4-manifold $(M^4, g, \mathcal{E})$. The metric g on $\mathcal{E}$ can be expressed as follows:

$$\begin{aligned} g|_{\mathcal{E}} &= (1 + \frac{m}{6r})^2 h + w \\ |w| + r|\partial w| &+ r^2|\partial^2 w| = O(r^{-q}) \end{aligned}$$

where $q > 1$. The scalar curvature R_g and the mean curvature N of the fiber satisfies that

$$\begin{aligned} &N = \frac{m}{6r^2}dr + O(r^{-3}) \\ &\frac{1}{|\mathbb{S}^2|}\int_{\partial B(R,\bar{p})} N \cdot \nu = \frac{m}{6} + o(1) \\ &R \in C^1_{2+q} \end{aligned} \tag{5.5}$$

The quotient space $(\bar{M}, \bar{g})$ is an asymptotically Euclidean 3-manifold outside the singular point $\bar{p}$. The metric $\bar{g}$ on $\bar{\mathcal{E}}$ is expressed as follows:

$$\begin{aligned} \bar{g}|_{\bar{\mathcal{E}}} &= (1 + \frac{m}{6r})^2 g_{\mathbb{R}^3} + \bar{w} \\ |\bar{w}| + r|\partial \bar{w}| &+ r^2|\partial^2 \bar{w}| \in O(r^{-q}) \end{aligned}$$

Its AMD mass $m(\bar{M}, \bar{g}, \mathcal{E})$ is $\frac{2m}{3}$ and its scalar curvature $\bar{R}$ belongs to $C^{1,\alpha}_{2+q}$.

Theorem 5.4. *Let $(M^4, g, \mathcal{E})$ be a complete ALF Schwarzschild-like 4-manifold and $(\bar{M}, \bar{g}, \bar{\mathcal{E}})$ be assumed as above. If the scalar curvature R of (M, g) is non-negative, then there is a function function u on $(\bar{M}, \bar{g}, \bar{E})$ satisfying that*

$$\begin{cases} \Delta_{\bar{g}} u - \frac{1}{8}\bar{R}u = 0 \\ u(x) \sim Cr^{1/2} \text{ near } \bar{p} \end{cases} \tag{5.6}$$

Moreover, u enjoys the following asymptotic behavior:

$$u - (1 + \frac{A}{r}) \in C^{2,\alpha}_q$$

where $A = -\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int |\nabla_{\bar{g}} u|^2 + \frac{1}{8}\bar{R}u^2 dvol_{\bar{g}}$.

Notice that the metric $u^4\bar{g}$ belongs to $W^{1,\infty}_{loc} \cap C^\infty(\bar{M} \setminus \bar{p}\})$

In the following, we first study the positivity of the conformal Laplacian on $(\bar{M}, \bar{g})$ and then combine with a metric deformation to prove Theorem 5.4.

5.2.1. *The positivity of the conformal Laplacian.* In the following, we abuse the notion and write N for $\pi_*(N)$.

Proposition 5.5. *Let $(M^4, g, \mathcal{E})$ and $(\bar{M}, \bar{g}, \bar{\mathcal{E}})$ be assumed as in Theorem 5.4. If the scalar curvature of $R(x)$ is non-negative, then for any bounded Lipschitz function ϕ on $\bar{M}$ which tends to zeros at ∞, one has that*

$$\int_{\bar{M}} |\nabla_{\bar{g}}\phi|^2 + \frac{1}{2}\bar{R}\phi^2 dvol_{\bar{g}} \geq 0. \tag{5.7}$$

Proof. For any $R > 0$ and $\epsilon > 0$, we use (5.2) to obtain that

$$\begin{aligned}
\int_{B(\bar{p},R)\setminus B(\bar{p},\epsilon)} |\nabla_{\bar{g}}\phi|^2 + \frac{1}{2}\bar{R}\phi^2 &\geq \int_{B(\bar{p},R)\setminus B(\bar{p},\epsilon)} |\nabla_{\bar{g}}\phi|^2 + |N|^2\phi^2 - \operatorname{div}(N)\phi^2 \\
&= \int_{B(\bar{p},R)\setminus B(\bar{p},\epsilon)} |\nabla_{\bar{g}}\phi|^2 + |N|^2\phi + 2\phi g(N, \nabla_{\bar{g}}\phi) \\
&\quad - \int_{\partial B(\bar{p},R)} (N \cdot v)\phi^2 + \int_{\partial B(\bar{p},\epsilon)} (N \cdot v)\phi^2 \\
&= \int |\nabla_{\bar{g}}\phi + N\phi|^2 - \int_{\partial B(\bar{p},R)} (N \cdot v)\phi^2 + \int_{\partial B(\bar{p},\epsilon)} (N \cdot v)\phi^2 \\
&\geq \int_{\partial B(\bar{p},R)} (N \cdot v)\phi^2 - \int_{\partial B(\bar{p},\epsilon)} (N \cdot v)\phi^2
\end{aligned}$$

The asymptotic behavior of N shows the boundary estimates as follows:

$$\begin{aligned}
&|\int_{\partial B(\bar{p},R)} (N \cdot \mu)\phi^2| \leq \max_{\partial B(\bar{p},R)} |\phi|^2 \int_{\partial B(\bar{p},R)} |N| \leq \max_{\partial B(\bar{p},R)} |\phi|^2 \cdot O(1) \to 0 \text{ as } R \to \infty \\
&|\int_{\partial B(\bar{p},\epsilon)} (N \cdot \mu)\phi^2| \leq C \int_{\partial B(\bar{p},\epsilon)} |N| \leq C\epsilon \to 0 \text{ as } \epsilon \to 0
\end{aligned}$$

where the first inequality follows from (5.5) Taking the limit as $R \to \infty$ and $\epsilon \to 0$, we get the inequality (5.7). □

5.2.2. *Conformal Deformation.* We need only (5.7) to finish the proof and the modification of $\bar{g}$ will afford us technical convenience. We use a conformal change to remove the singular point $\bar{p}$. Such a conformal deformation also appears in the article of Scheon-Yau [30]

Let ψ be a positive smooth function on $\bar{M}$ satisfying the following

$$\psi(\bar{x}) = \begin{cases} 1 & \text{on} \quad \bar{\mathcal{E}} \\ r^2 & \text{near} \quad \bar{p} \end{cases}$$

where $r(\bar{x}) = d(\bar{x}, \bar{p})$. Define a new metric on $\bar{M}$

$$g' = \psi^4 \bar{g}.$$

Near $\bar{p}$, g' is expressed as follows:

$$g' = r^2((dr)^2 + \frac{1}{4}r^4 g_{\mathbb{S}^2} + O(r^2)) \tag{5.8}$$

If we set $\rho = \frac{1}{2}r^2$, we then have that

$$g' = (d\rho)^2 + \rho^2 g_{\mathbb{S}^2} + O(\rho^2).$$

for ρ near zero. It follows from our construction that the metric g' is uniformly equivalent to a smooth metric in a neighborhood of $\bar{p}$. The scalar curvature $R_{g'}$ is

$$R_{g'} = \psi^{-4}(\bar{R} - \frac{8\Delta_{\bar{g}}\psi}{\psi}) \tag{5.9}$$

Lemma 5.6. *For any bounded Lipschitz function ϕ on $\bar{M}$ which tends to zero at ∞, then one has that*

$$\int_{\bar{M}} |\nabla_{g'}\phi|^2_{g'} + \frac{1}{8}R_{g'}\phi^2 dvol_{g'} \geq 0$$

The inequality is equivalent to the following inequality;

$$\int_{\bar{M}} |\nabla_{\bar{g}}(\phi\psi)|^2_{g'} + \frac{1}{8}\bar{R}(\phi\psi)^2 dvol_{\bar{g}} \geq 0$$

Proof. For any Lipschitz function ϕ on $\bar{M}$, one has that

$$|\nabla_{g'}\phi|^2_{g'} = \psi^{-4}|\nabla_{\bar{g}}\phi|^2_{\bar{g}} \text{ and } dvol_{g'} = \psi^2 dvol_{\bar{g}}$$

where $g' = \psi^4\bar{g}$. We directly compute that

$$\begin{aligned}
\int_{\bar{M}} |\nabla_{g'}\phi|^2_{g'} + \frac{1}{8}R_{g'}\phi^2 dvol_{g'} &= \int_{\bar{M}} |\nabla_{\bar{g}}\phi|^2_{\bar{g}}\psi^2 + \frac{1}{8}\bar{R}(\phi\psi)^2 - \psi\phi^2\Delta_{\bar{g}}\psi dvol_{\bar{g}} \\
&= \int_{\bar{M}} |\nabla_{\bar{g}}\phi|^2_{\bar{g}}\psi^2 + \frac{1}{8}\bar{R}(\phi\psi)^2 + |\nabla_{\bar{g}}\psi|^2_{\bar{g}}\phi^2 + 2\psi\phi\bar{g}(\nabla_{\bar{g}}\phi, \nabla_{\bar{g}}\psi) dvol_{\bar{g}} \\
&\quad - \lim_{R\to\infty}\int_{\partial B(\bar{p},R)} \psi^2\phi\frac{\partial\phi}{\partial\nu} - \lim_{\epsilon\to 0}\int_{\partial B(\bar{p},\epsilon)} \psi^2\phi\frac{\partial\phi}{\partial\nu} \\
&= \int_{\bar{M}} |\nabla_{\bar{g}}\phi|^2_{\bar{g}}\psi^2 + \frac{1}{8}\bar{R}(\phi\psi)^2 + |\nabla_{\bar{g}}\psi|^2_{\bar{g}}\phi^2 + 2\psi\phi\bar{g}(\nabla_{\bar{g}}\phi, \nabla_{\bar{g}}\psi) dvol_{\bar{g}} \\
&= \int_{\bar{M}} |\nabla_{\bar{g}}(\phi\psi)|^2_{\bar{g}} + \frac{1}{8}\bar{R}(\phi\psi)^2 dvol_{\bar{g}} \\
&\geq 0
\end{aligned}$$

The third equality follows from the behavior of ψ and ϕ ($\psi = r^{1/2}$ near $\bar{p}$ and ϕ tends to zero at ∞). The last inequality comes from Lemma 5.5. ☐

We use the same argument in [29] and Lemma 5.6 to prove Theorem 5.4.

Proof. Let g' be defined as above. We use Lemma 5.6 to find a positive function ϕ on M satisfying that

- ϕ is a bounded Lipschitz function;
- $\Delta_{g'}\phi - \frac{1}{8}R_{g'}\phi = 0$
- ϕ enjoys the following asymptotic behavior:

$$\phi - (1 + \frac{A}{r}) \in C^{2,\alpha}_q$$

where $A = -\frac{1}{|\mathbb{S}^2|}\int |\nabla_{g'}\phi|^2 + \frac{1}{8}R_{g'}\phi^2 dvol_{g'}$.

Notice that the asymptotic estimate follows from Lemma 2.6 and $R_{g'} \in C^{1,\alpha}_{q+2}$ (see (5.5)).

Choose the positive function $u = \phi\psi$ and use (5.9) to obtain that

$$\Delta_{\bar{g}}u - \frac{1}{8}\bar{R}u = 0$$

It has the asymptotic behavior on $\bar{E}$ (i.e. $u - (1+\frac{A}{r}) \in C^{2,\alpha}_{1+\mu}$). Use the same argument of the proof of Lemma 5.8 to have that

$$\begin{aligned} A &= -\frac{1}{|\mathbb{S}^2|}\int |\nabla_{g'}\phi|^2 + \frac{1}{8}R_{g'}\phi^2 dvol_{g'} \\ &= -\frac{1}{|\mathbb{S}^2|}\int |\nabla_{\bar{g}}u|^2 + \frac{1}{8}\bar{R}u^2 dvol_{\bar{g}} \end{aligned}$$

Moreover, since ϕ is a bounded function and $\psi = r^{1/2}$ near $\bar{p}$, then $u \sim Cr^{1/2}$ near $\bar{p}$ for some C. □

5.3. **Completing the proof of Theorem 1.5.** We first prove Theorem 1.5 for Schwarzschild-like ALF manifolds and then give a complete proof for the general cases.

Proposition 5.7. *Let $(M^4, g, \mathcal{E})$ be a complete ALF Schwarzschild-like manifold with an almost free $\mathbb{S}^1$-action. If the scalar curvature R is non-negative, then $m \geq 0$.*

Proof. From Theorem 5.4, there is a Lipchitz function u on $\bar{M}$ such hat

$$\begin{cases} \Delta_{\bar{g}}u - \frac{1}{8}\bar{R}u = 0 \\ u \to 1 \text{ at } \ \infty \text{ and } u \sim Cr^{1/2} \text{ near } \bar{p}. \end{cases}$$

The function u has the asymptotic behavior.

$$u = 1 + \frac{A}{r} + O(\frac{1}{r^q})$$

where $A = -\frac{1}{|\mathbb{S}^2|}\int_{\bar{M}} |\nabla_{\bar{g}}u|^2 + \frac{1}{8}\bar{R}u^2$ and $q > 1$. Moreover, the new metric $u^4\bar{g}$ belongs to $W^{1,\infty} \cap C^\infty(M \setminus \{\bar{p}\})$.

Claim: $A \leq \frac{m}{24}$
Proof of the claim: Use (5.2) to compute that

$$\begin{aligned} -A =&\frac{1}{|\mathbb{S}^2|}\int |\nabla_{\bar{g}}u|^2 + \frac{1}{8}\bar{R}u^2 dvol_{\bar{g}} \\ =&\frac{1}{|\mathbb{S}^2|}\int |\nabla_{\bar{g}}u|^2 + \frac{1}{8}(R(x) + |A|^2 + |\mathcal{T}|^2| + |N|^2 - 2\text{div}(N))u^2 dvol_{\bar{g}} \\ =&\frac{1}{|\mathbb{S}^2|}\int |\nabla_{\bar{g}}u|^2 + \frac{1}{8}u^2|N|^2 + \frac{1}{2}u\bar{g}(N, \nabla_{\bar{g}}u) + \frac{1}{8}(R(x) + |A|^2 + |\mathcal{T}|^2|)u^2 \text{dvol}_{\bar{g}} \\ &- \frac{2}{8|\mathbb{S}^2|}\lim_{R\to\infty}\int_{\partial B(\bar{p},R)} (N\cdot\nu)u^2 + \frac{2}{8|\mathbb{S}^2|}\lim_{\epsilon\to 0}\int_{\partial B(\bar{p},\epsilon)} (N\cdot\nu)u^2 \\ =&\frac{1}{|\mathbb{S}^2|}\int_{\bar{M}} \frac{1}{2}|\nabla_{\bar{g}}u|^2 + \frac{1}{2}|\nabla_{\bar{g}}u + \frac{1}{2}Nu|^2 + \frac{1}{8|\mathbb{S}^2|}\int_{\bar{M}} (R(x) + |A|^2 + |\mathcal{T}|^2|)u^2 \\ &- \frac{2}{8|\mathbb{S}^2|}\lim_{R\to\infty}\int_{\partial B(\bar{p},R)} N\cdot\nu \\ \geq& -\frac{2}{8|\mathbb{S}^2|}\lim_{R\to\infty}\int_{\partial B(\bar{p},R)} N\cdot\nu = -\frac{m}{24} \end{aligned}$$

We finish the proof of the claim.

We now use the claim to complete the proof.

Consider the new metric $u^4\bar{g}$ on $\bar{M}$ and its scalar curvature is zero outside $\bar{p}$. The metric belongs to $W^{1,\infty}\cap C^\infty(M\setminus\{\bar{p}\})$. A singular positive mass theorem (see [Theorem 1.3 [10]] or [22]) implies that $m_{ADM}(\bar{M},u^4\bar{g},\bar{\mathcal{E}})\geq 0$. From the above claim, one has that

$$0\leq m_{ADM}(\bar{M},u^4\bar{g},\bar{\mathcal{E}})=m_{ADM}(\bar{M},\bar{g},\bar{\mathcal{E}})+8A\leq\frac{2m}{3}+\frac{m}{3}=m$$

□

Remark 5.8. Precisely, the AMD mass is bounded below by $-8A$. That is to say,

$$\frac{2m}{3}\geq\frac{1}{|\mathbb{S}^2|}\int_{\bar{M}}8|\nabla_{\bar{g}}u|^2+\bar{R}u^2 \tag{5.10}$$

We now prove Theorem 1.5 for the general case.

Proof. Let $(M^4,g,\mathcal{E})$ be a complete ALF manifold with non-negative scalar curvature and with an almost free $\mathbb{S}^1$-action. We will use the proof by contradiction.

Suppose that the ADM mass $m(M^4,g,\mathcal{E})$ is negative. Theorem 9.6 allows us to find a complete ALF metric g' on M^4 satisfying that
(1) the scalar curvature $R_{g'}$ is non-negative;
(2) there is an almost free $\mathbb{S}^1$-action;
(3) $(\mathcal{E},g|_{\mathcal{E}})$ is a Schwarzschild-like ALF end;
(4)its ADM mass $m(M^4,g',\mathcal{E})$ is negative.
It leads to contradiction with Proposition 5.7 □

6. Rigidity for ALF manifold

In this section, we focus on the equality case of theorem 1.5. Due to a density argument in proving Theorem 1.5 (see Theorem 9.6), the rigidity aspect could not be established in the same way as in the asymptotically Euclidean case. We will study Harmonic functions from some coordinate functions and use them to get the rigidity statement.

6.1. Harmonic functions on ALF 4-manifolds.

We begin with a classical result for the equality case and then use them to study harmonic functions.

Lemma 6.1. *Let (M^4,g) be assumed as in Theorem 1.5. If $m(M^4,g,\mathcal{E})=0$, then (M^4,g) is Ricci-flat.*

The proof is the same as the argument in [29].

In the following of this section, we suppose that $(M^4,g,\mathcal{E})$ is assumed as in Theorem 1.5 and Ricci-flat. The quotient map $\pi:\mathcal{E}\to\bar{\mathcal{E}}$ is a submersion, where $\bar{\mathcal{E}}$ is asymptotically Euclidean. Choose a coordinate $(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ on $\bar{\mathcal{E}}$ and a function χ which is supported in $\bar{\mathcal{E}}$ and tends to 1 at ∞.

For $i=1,2,3$, there is a $\mathbb{S}^1$-invariant function u_i satisfying that

$$\Delta_g(u_i)=\Delta_g(\pi^*(\chi\bar{x}_i))$$

The reason is as follows: From (2.6), one has that $\Delta_g(\pi^*(\chi x_i))=O(r^{-1-\mu})\in L^2_{\delta'-2}$, where $\delta'>\frac{5}{2}-\mu$ and $\mu>1/2$. The uniqueness and solvability of the above PDE is ensured by Corollary 2 in [27]. Moreover, $u_i\in H^2_{\delta'}$. Since g and $\pi^*(\chi\bar{x}_i)$ are $\mathbb{S}^1$-invariant, the uniqueness of the solution shows that u_i is also $\mathbb{S}^1$-invariant.

Remark 6.2. Choose $\delta' = 5/2 - \mu + \epsilon_0$, where ϵ_0 is sufficient small. The function u_i also satisfies that

$$u_i = O(r^{1-\mu'}) \text{ and } \sum_{k=1}^{3} r^k |\partial^k u_i| = O(r^{1-\mu'}) \tag{6.1}$$

where $\mu' = \mu - \epsilon_0 > 1/2$. See Lemma 6 in [27] and the proof in the proof of Proposition 4.14 in [7]

The function $y_i = \pi^*(\chi \bar{x}_i) - u_i$ is a $\mathbb{S}^1$-invariant function satisfying that

$$\Delta_g y_i = 0 \tag{6.2}$$

We use these harmonic functions to show the following:

Lemma 6.3. *Let $(M, g, \mathcal{E})$ be assumed as in Theorem 1.5 and y_i defined as above. If it is Ricci-flat, then one has that*

$$m(M, g, \mathcal{E}) = 2/3 \sum_{i=1}^{3} \frac{1}{|\mathbb{S}^2||\mathbb{S}^1|} \int_M |\nabla \alpha_i|^2$$

where $\alpha_i = dy_i$

The proof is the same as the proof of Proposition 4.12 in [7]

Proof. Let T be the killing field generated by the $\mathbb{S}^1$-action and η the 1-form with $\eta(X) = g(T, X)$. The metric g can be written as follows

$$g = \pi^*(\bar{g}) + \frac{1}{|\eta|^2} \eta \otimes \eta$$

For simplifying the computation, we introduce some new notations

- $g_{i,j} = g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})$ and $|\eta|^2 = g(T, T)$
- $g_{i,T} = g(\frac{\partial}{\partial y_i}, T)$

Remark. One has that

- $\mathcal{H}|_{\mathcal{E}} = span\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\}$ and $\mathcal{V}|_{\mathcal{E}} = span\{T\}$
- $g_{i,T} = 0$
- The metric g is rewritten in the following form.

$$g = g_{i,j} dy_i \otimes dy_j + \frac{1}{|\eta|^2} \eta \otimes \eta$$

Step 1: show $g_{i,j} - (\delta_{i,j} + \frac{c_{i,j}}{r}) \in C^{2,\alpha}_{1+\epsilon}$, where $\epsilon > 0$ is some positive constant.

Apply Bochner's formula to the harmonic 1-form $\alpha_i = dy_i$ and obtain that

$$\begin{aligned} &\frac{1}{2}\Delta |dy_i|^2 = |\nabla dy_i|^2 + Ric(dy_i, dy_i) = |\nabla dy_i|^2 \\ &\frac{1}{2}\Delta g^{i,j} = g(\nabla dy_i, \nabla dy_j) \end{aligned} \tag{6.3}$$

The asymptotic behaviors of y_i and g (see (6.1)) implies that $\Delta g^{i,j} \in C^{0,\alpha}_{2+2\mu'}$, where $\mu' > 1/2$.

Since $g^{i,j}$ is $\mathbb{S}^1$-invariant, we apply Lemma 2.7 to the second equation in (6.3) and obtain that

$$g^{i,j} - (1 - \frac{c_{i,j}}{r}) \in C^{2,\alpha}_{1+\epsilon_1}$$

where $\epsilon_1 = \min\{\mu, 2\mu' - 1\}$ is some positive constant, $c_{i,j}$ is a constant. That is to say,

$$g_{i,j} - (1 + \frac{c_{i,j}}{r}) \in C^{2,\alpha}_{1+\epsilon_1}$$

Step 2: complete the proof

Without the loss generalization, we may assume that $c_{i,j} = \delta_{i,j} c_i$. Namely,

$$g = \sum_{i=1}^{3} (1 + \frac{c_i}{r}) dy_i \otimes dy_i + \eta \otimes \eta + C^{1,\alpha}_{1+\epsilon_1}, \tag{6.4}$$

From Corollary 2.9, the ADM mass $m(M^4, g, \mathcal{E})$ can be expressed as follows

$$m = \frac{2}{3} \sum_{i=1}^{3} c_i - \lim_{R \to \infty} \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|} \int_{\partial B(p,R)} *_g dg(T, T).$$

Use the expression of g (see (6.4)) and $T(y_i) = 0$ (i.e. y_i is $\mathbb{S}^1$-invariant) to obtain that

$$0 = \Delta y_i = (c_i - \frac{\sum_{j=1}^{3} c_j}{2}) \frac{y_i}{r^3} + \frac{1}{2} \frac{\partial g(T, T)}{\partial y_i} + O(r^{-2-\epsilon_2})$$

where $\epsilon_2 = \min\{\epsilon_1, \mu + \mu' - 1\}$. The, we obtain that

$$\begin{aligned}
-\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|} \int_{\partial B(p,R)} *_g (dg(T, T)) &= -\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|} \int_{\partial B(p,R)} \frac{\partial g(T, T)}{\partial y_i} \frac{y_i}{r} \\
&= \frac{2}{|\mathbb{S}^1||\mathbb{S}^2|} \int_{\partial B(p,R)} \sum_{i=1}^{3} (c_i - \frac{\sum_{j=1}^{3} c_j}{2}) \frac{y_i^2}{r^4} + o(1) \\
&= -\frac{1}{3} (\sum_{i=1}^{3} c_i) + o(1)
\end{aligned}$$

Then, $m = 1/3 \sum_{i=1}^{3} c_i$. We use Bochner's formula (6.3)

$$m = \frac{1}{3} \sum_{i=1}^{3} c_i = \frac{1}{3} \lim_{R \to} \sum_{i=1}^{3} \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|} \int_{\partial B(x,R)} \frac{\partial (g^{ii})}{\partial \nu} = \frac{2}{3} \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|} \int_M \sum_{i=1}^{3} \frac{1}{2} \Delta |\alpha_i|^2 = \frac{2}{3} \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|} \sum_{i=1}^{3} \int_M |\nabla \alpha_i|^2$$

The last equation follows from Bochner's formula for 1-$form$. □

6.2. Rigidity part of Theorem 1.5.

Proof. Let $(M^4, g, \mathcal{E})$ be assumed as in Theorem 1.5. If $m(M^4, g, \mathcal{E}) = 0$, then g is Ricci-flat (see Lemma 6.1). From Lemma 6.3, one has that

$$\sum_{i=1}^{3} \int_M |\nabla dy_i|^2 = 0$$

That's to say, dy_i is parallel for each i. Then, $\alpha_4 := *(dy_1 \wedge dy_2 \wedge dy_3)$ is also parallel. The spanned space, formed by four parallel forms $\{\alpha_i\}_{i=1}^4$, is equal to $\bigwedge^1 TM$. Then, the metric g is flat. We use the same argument in the proof of Theorem 3 in [27] to show that the manifold (M^4, g) is isometric to $\mathbb{S}^1 \times \mathbb{R}^3$. □

7. Degree of bundle and the proof of Theorem 1.6

7.1. **Killing form on ALF manifolds.** Let $(M^4, g, \mathcal{E})$ be a ALF manifold with an almost free $\mathbb{S}^1$-action and η be assumed in Sections 2.1 and 2.2. The metric g can be expressed as follows

$$g = \pi^*(\bar{g}) + \frac{1}{|\eta|^2}\eta^2$$

where $\bar{g}$ is the quotient metric on the quotient space $\bar{M}$. From Proposition 2.3, one has that

$$\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_{\partial B(x,R)} \eta \wedge d\eta = |\deg(\mathcal{E})| + o(1) \tag{7.1}$$

The infinitesimal generator T of the $\mathbb{S}^1$-action is a killing field We use the expression of g and the killing field T to obtain that

Proposition 7.1. *Let T and η be assumed in Section 2.1 and $\mathcal{H}$, $\mathcal{V}$, A, $\mathcal{T}$ and N be defined in Section 5.1. Then, if x is not a fixed point of the $\mathbb{S}^1$-action, then one has that at x*

$$A = -\frac{d\eta}{2|\eta|^2}\Bigg|_{\mathcal{H}} \otimes T \ \textit{and}\ N = -\frac{d|\eta|^2}{2|\eta|^2} = -\frac{d|\eta|}{|\eta|} \tag{7.2}$$

$$|A|^2 + |\mathcal{T}|^2 \geq \frac{|d\eta|^2}{4|\eta|^2} \ \textit{and}\ |N|^2 = |\mathcal{T}|^2 = \frac{\big|d|\eta|\big|^2}{|\eta|^2} \tag{7.3}$$

$$\bar{R} \geq R + \frac{|d\eta|^2}{4|\eta|^2} + \frac{|d|\eta||^2}{|\eta|^2} + 2div_{\bar{g}}(\frac{d|\eta|}{|\eta|}) \tag{7.4}$$

Equations (7.2) (7.3) follows from Definition 5.1 and the fact that T is the killing field. Equation (7.4) comes from Equation (5.2).

Lemma 7.2. *Let (X, g) be a complete 4-manifold and w a smooth 2-form on M with $\int_X |w|^2 < \infty$. Then, one has*

$$\int_M |w|^2 \geq \int_M w \wedge w$$

In the case of equality, w is self-dual.

Proof. Any 2-form on M can be decomposed into the self-dual part and the anti-self-dual part. Namely, w can be written as follows:

$$w = w_+ + w_- \in \Lambda^+ \oplus \Lambda^-.$$

Moreover, $\int_M w_+ \wedge w_- = 0$. Then, we directly obtain that

$$\begin{aligned} \int_M |w|^2 &= \int_M |w_+|^2 + |w_-|^2 \\ \int_M w \wedge w &= \int_M w_+ \wedge w_+ + w_- \wedge w_- = \int_M |w_+|^2 - |w_-|^2. \end{aligned} \tag{7.5}$$

We have that $\int_M |w|^2 \geq \int_M w \wedge w$. If the equality holds, then $w_- = 0$ (i.e. w is self-dual). $\square$

7.2. **Proof of Theorem 1.6.** Using Theorem 4.3, it is sufficient to prove Theorem 1.6 for Schwarzschild-like ALF 4-manifolds.

We assume that $(M^4, g, \mathcal{E})$ is a complete Schwarzschild-like ALF manifold with an almost free $\mathbb{S}^1$-action with non-negative scalar curvature $R \geq 0$. That is to say,

$$g|_{\mathcal{E}} = (1+\frac{m}{6r})^2 h + w$$
$$|w| + r|\partial w| + r^2|\partial^2 w| = O(r^{-q})$$

where $q > 1$.

For simplifying the proof, we may assume $\mathcal{E}$ is the unique end and p is the unique fixed point. From Theorem 5.4, there is a function function u on $(\bar{M}, \bar{g}, \bar{E})$ satisfying that

$$\begin{cases} \Delta_{\bar{g}} u - \frac{1}{8}\bar{R}u = 0 \\ u(x) \sim Cr^{1/2} \text{ near } \bar{p} \end{cases}$$

From (5.10) and (7.4), one has that

$$\begin{aligned}
\frac{2m}{3} &\geq \frac{1}{|\mathbb{S}^2|}\int_{\bar{M}} 8|\nabla_{\bar{g}} u|^2 + [R + \frac{|d\eta|^2}{4|\eta|^2} + \frac{|d|\eta||^2}{|\eta|^2} + 2\mathrm{div}_{\bar{g}}(\frac{d|\eta|}{|\eta|})]u^2 \\
&= \frac{1}{|\mathbb{S}^2|}\int_{\bar{M}} 8|\nabla_{\bar{g}} u|^2 + \frac{|d\eta|^2}{4|\eta|^2}u^2 + \frac{|d|\eta||^2}{|\eta|^2}u^2 - 4\frac{u}{|\eta|}\bar{g}(\nabla_{\bar{g}}|\eta|, \nabla_{\bar{g}} u) \\
&+ \frac{2}{|\mathbb{S}^2|}\lim_{R\to\infty}\int_{\partial B(\bar{p},R)} \frac{\partial|\eta|}{\partial\nu}\cdot\frac{u^2}{|\eta|} - \frac{2}{|\mathbb{S}^2|}\lim_{\epsilon\to 0}\int_{\partial B(\bar{p},\epsilon)} \frac{\partial|\eta|}{\partial\nu}\cdot\frac{u^2}{|\eta|} \\
&= \frac{1}{|\mathbb{S}^2|}\int_{\bar{M}} 8|\nabla_{\bar{g}} u|^2 + \frac{|d\eta|^2}{4|\eta|^2}u^2 + \frac{|d|\eta||^2}{|\eta|^2}u^2 - 4\frac{u}{|\eta|}\bar{g}(\nabla_{\bar{g}}|\eta|, \nabla_{\bar{g}} u) - \frac{m}{3}
\end{aligned}$$

The last equality follows from the asymptotic behavior at infinity and near $\bar{p}$(i.e. Near $\bar{p}$, $|\eta| \approx r$ and $u^2 \approx Cr$; On $\mathcal{E}$, $|\eta| = 1 + \frac{m}{6r} + O(r^{-1})$ and $u \to 1$.

In the following, we abuse the notation and write u for the function $\pi^*(u)$ on M. We combine with $\nabla|\eta| \cdot \nabla u = \bar{g}(\nabla_{\bar{g}}|\eta|, \nabla_{\bar{g}} u)$ and Equation (2.7) to obtain that

$$m \geq \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_M \frac{8|\nabla u|^2}{|\eta|} + \frac{|d\eta|^2}{4|\eta|^3}u^2 + \frac{|\nabla|\eta||^2 u^2}{|\eta|^3} - \frac{4(\nabla u \cdot \nabla|\eta|)u}{|\eta|^2} \tag{7.6}$$

Lemma 7.3. *Let η and u be assume as above. Then, one has that*

$$\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_M \frac{|d\eta|^2}{4|\eta|^3}u^2 \geq \frac{1}{8}\deg(\mathcal{E}) - \frac{1}{8}(\int_M |\eta|^2|d(\frac{u}{|\eta|^{3/2}})|^2$$

Proof. We see that $d(\eta\wedge d\eta)=d\eta\wedge d\eta$. Then we use Lemma 7.2 and the integration by part to estimate the term as follows:

$$\begin{aligned}
\int_M \frac{|d\eta|^3}{4|\eta|^2}u^2 &= \frac{1}{8}\int_M \frac{|d\eta|^2u^2}{|\eta|^3}+\frac{1}{8}\int_M \frac{|d\eta|^2u^2}{|\eta|^3}\\
&\geq \frac{1}{8}\int_M \frac{|d\eta|^2u^2}{|\eta|^3}+\frac{1}{8}\int_M \frac{(d(\eta\wedge d\eta))u^2}{|\eta|^3}\\
&= \frac{1}{8}\int_M \frac{|d\eta|^2u^2}{|\eta|^3}-\frac{1}{8}\{\int_M \eta\wedge d\eta\wedge d(\frac{u^2}{|\eta|^3})\\
&+\lim_{\partial R\to\infty}\int_{B(p,R)}\frac{u^2}{|\eta|^3}\cdot\eta\wedge d\eta-\lim_{\epsilon\to 0}\int_{\partial B(p,\epsilon)}\frac{u^2}{|\eta|^3}\cdot\eta\wedge d\eta\}\\
&= \frac{1}{8}|\deg(\mathcal{E})|+\frac{1}{8}\int_M \frac{|d\eta|^2u^2}{|\eta|^3}-\frac{1}{16}\int_M \eta\wedge d\eta\wedge d(\frac{u^2}{|\eta|^3})
\end{aligned}$$

The last equality follows from (7.1) and the asymptotic behavior of u and η (i.e. Near $\bar{p}$, $u^2\sim Cr$ and $|\eta|\sim r$; on $\mathcal{E}$, $u\to 1$ and $|\eta|\to 1$).

We observe that $d(\frac{u^2}{|\eta|^3})=d(\frac{u}{|\eta|^{3/2}})^2=2\frac{u}{|\eta|^{3/2}}d(\frac{u}{|\eta|^{3/2}})$. Then, we obtain that

$$\begin{aligned}
\int_M \frac{|d\eta|^3}{4|\eta|^2}u^2 &\geq \frac{1}{8}|\deg(\mathcal{E})|+\frac{1}{8}\int_M \frac{|d\eta|^2u^2}{|\eta|^3}-\frac{1}{16}\int_M \eta\wedge d\eta\wedge d(\frac{u^2}{|\eta|^3})\\
&= \frac{1}{8}|\deg(\mathcal{E})|+\frac{3}{16}\int_M \frac{|d\eta|^2u^2}{|\eta|^3}-\frac{2}{8}\int_M \frac{u}{|\eta|^{3/2}}\cdot\eta\wedge d\eta\wedge d(\frac{u}{|\eta|^{3/2}})\\
&\geq \frac{1}{8}|\mathbb{S}^1||\mathbb{S}^2||\deg(\mathcal{E})|+\frac{3}{16}\int_M \frac{|d\eta|^2u^2}{|\eta|^3}-\frac{1}{8}(\int_M \frac{|d\eta|^2u^2}{|\eta|^3}+|\eta|^2|d(\frac{u}{|\eta|^{3/2}})|^2)\\
&\geq \frac{1}{8}|\mathbb{S}^1||\mathbb{S}^2||\deg(\mathcal{E})|-\frac{1}{8}(\int_M |\eta|^2|d(\frac{u}{|\eta|^{3/2}})|^2)
\end{aligned}$$

□

We now use Lemma 7.3 to complete the proof of Theorem 1.6 as follows:

$$\begin{aligned}
m &\geq \frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_M \frac{8|\nabla u|^2}{|\eta|}+\frac{|d\eta|^2}{4|\eta|^3}u^2+\frac{|\nabla|\eta||^2u^2}{|\eta|^3}-\frac{4(\nabla u\cdot\nabla|\eta|)u}{|\eta|^2}\\
&\geq \frac{1}{8}|\deg\mathcal{E}|+\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_M \frac{8|\nabla u|^2}{|\eta|}+\frac{|\nabla|\eta||^2u^2}{|\eta|^3}-\frac{4(\nabla u\cdot\nabla|\eta|)u}{|\eta|^2}-\frac{1}{8}\int_M |\eta|^2|d(\frac{u}{|\eta|^{3/2}})|^2\\
&\geq \frac{1}{8}|\deg\mathcal{E}|+\frac{1}{|\mathbb{S}^1||\mathbb{S}^2|}\int_M (8-1/8)\frac{|\nabla u|^2}{|\eta|}+(1-9/32)\frac{|\nabla|\eta||^2u^2}{|\eta|^3}-(4-3/8)\frac{(\nabla u\cdot\nabla|\eta|)u}{|\eta|^2}\geq\frac{1}{8}|\deg\mathcal{E}|
\end{aligned}$$

The last inequality comes from the fact that $(4-3/8)^2-4(8-1/8)(1-9/32)<0$. We can conclude that

$$m\geq\frac{1}{8}\deg(\mathcal{E}).$$

8. Stable minimal hypersurfaces on AF manifolds

In this section, we assume that $(M^n.g)$ is a complete AF manifold and some sphere at infinity on some end $\mathcal{E}$ vanishes in $H_{n-2}(M)$. We split the coordinates on the AF end $\mathcal{E}$ as (x,θ) and define

- $\Gamma_{\alpha,R}=\{(x,\theta)\mid |x|=R,\theta=\alpha\}$

- $H^\alpha = \{(x,\theta) \mid \theta = \alpha\}$ and $H^\alpha_{r,R} = \{(x,\theta) \mid r \leq |x| \leq R, \theta = \alpha\} \subset \mathcal{E}$
- $\Pi_{\geq r} = \{(x,\theta) \mid |x| \geq r\}$ and $\Pi_{r_1,r_2} = \{(x,\theta) \, |r_1 \leq |x| \leq r_2\} \subset \mathcal{E}$
- $C_r = \{(x,\theta) \mid |x| = r\}$ is homeomorphic to $\mathbb{S}^{n-2} \times \mathbb{S}^1$
- $Z_r = \{(x,\theta) \mid |x| \leq r\}$ and $Z_{p,r} = \{(x,\theta) \mid |x-p| \leq r\}$.

Since $\Gamma_{\alpha,R}$ vanishes in $H_{n-2}(M)$ for any $R \geq 1$, $\Gamma_{\alpha,1}$ bounds a compact hypersurface $\Sigma_{\alpha,1} \subset M$.

8.1. **Construction of minimal hypersurfaces.** Assume that (M^n, g) is a complete AF manifold, where $4 \leq n \leq 7$ and some sphere at infinity, on the end $\mathcal{E} \subset M$, vanishes in $H_{n-2}(M)$. That is to say, for any $\alpha \in \mathbb{S}^1$, the sphere $\Gamma_{\alpha,R} \subset \mathcal{E}$ vanishes in $H_{n-2}(S)$, where R is sufficiently large. Federer-Fleming theory [14] allows us to find a volume-minimizing hypersurface $\Sigma_{\alpha,R}$ with boundary $\Gamma_{\alpha,R}$

$$\mathrm{Vol}(\Sigma_{\alpha,R}) = \min_{\partial\Sigma = \Gamma_{\alpha,R}} \mathrm{Vol}(\Sigma).$$

Proposition 8.1. *There is two positive constants C and r_0 satisfying that for $r \geq r_0$ and $R \geq r$,*

$$\begin{aligned} &Vol(\Sigma_{\alpha,R} \cap (M \setminus \mathcal{E})) \leq C, \\ &Vol(\Sigma_{\alpha,R} \cap \Pi_{1,r}) \leq C r^{n-1} \end{aligned} \tag{8.1}$$

Proof. We may assume that $\Sigma_{\alpha,R}$ intersects C_r transversally. Namely, $S^\alpha_{r,R} := \Sigma_{\alpha,R} \cap C_r$ is a $(n-2)$-manifold.

Step 1: show that $S^\alpha_{r,R}$ is homological to $\Gamma_{\alpha,r}$ in $H_{n-2}(C_r)$.

The hypersurface C_r cuts $\Sigma_{\alpha,R}$ into two parts, $\Sigma'_{\alpha,R} = \Sigma_{\alpha,R} \cap \Pi_{\geq r}$ and $\Sigma''_{\alpha,R} = \Sigma_{\alpha,R} \setminus \Sigma'_{\alpha,R}$, where $\partial\Sigma'_{\alpha,R} = \Gamma_{\alpha,R} \amalg S^\alpha_r$. Consider a $(n-1)$-current $\Sigma'_{\alpha,R} - H^\alpha_{r,R}$ on $\Pi_{\geq r}$, with boundary

$$\partial(\Sigma'_{\alpha,R} - H^\alpha_{r,R}) = S^\alpha_r - \Gamma_{\alpha,r}$$

The element $[\partial(\Sigma'_{\alpha,R} - H^\alpha_{r,R}]$ belongs to $H_{n-1}(\Pi_{\geq r}, \partial C_r)$. The boundary map in the long exact sequence for relative homology is given as follows:

$$\begin{array}{ccc} \{0\} \cong H_{n-1}(\Pi_{\geq r}, C_r) & \longrightarrow & H_{n-2}(C_r) \\ [\Sigma'_{\alpha,R} - H^\alpha_{r,R}] & \longrightarrow & [S^\alpha_r - \Gamma_{\alpha,r}] \end{array}$$

The excision property [18] gives the following isomorphism

$$H_{n-1}(\Pi_{\geq r}, C_r) \cong H_{n-1}(\mathbb{R}^{n-1} \setminus B(0,r) \times \mathbb{S}^1, \partial B(0,r) \times \mathbb{S}^1) \cong H_{n-1}(\mathbb{R}^{n-1} \times \mathbb{S}^1, B(0,1) \times \mathbb{S}^1) = \{0\}.$$

That is to say, one has

$$[S^\alpha_r - \Gamma_{\alpha,r}] = 0 \text{ in } H_{n-2}(C_r). \tag{8.2}$$

Step 2: Get the volume estimate.

From (8.2), there is a $(n-1)$ current W on C_r with $\partial W = -S^\alpha_r + \Gamma_{\alpha,r}$ and with $m(W) \leq m(C_r)$. Consider a $(n-1)$ current

$$\hat{\Sigma}_{\alpha,R} := \Sigma'_{\alpha,R} + W + H^\alpha_{1,r} + \Sigma_{\alpha,1}$$

with boundary $\Gamma_{\alpha,R}$, where $\Sigma_{\alpha,1}$ is a given (n-1)-current with boundary $\Gamma_{\alpha,1}$ The mass-minimizing property shows that

$$m(\Sigma''_{\alpha,R}) \leq m(W) + \mathrm{vol}(H^\alpha_{1,r}) + m(\Sigma_{\alpha,1})$$

In addition, we have the following mass estimates

$$\begin{aligned}
&m(W) \le m(C_r) = C' r^{n-2},\\
&m(H^{\alpha}_{1,r}) \le C'' r^{n-1}\\
&m(\Sigma_{\alpha,1}) \le \max_{\alpha\in\mathbb{S}^1} m(\Sigma_{\alpha,1}) \le C'''.
\end{aligned}$$

We complete the proof of this proposition. □

Corollary 8.2. *Let $\Sigma_{\alpha,R}$ be assumed as above. There is two positive constants C and r_0 such that for any $r \ge r_0$ and $R \ge r$, one has*

$$vol(\Sigma_{\alpha,R} \cap Z_{p,r}) \le C r^{n-1},$$

where $\frac{1}{2}\min\{d(p,K), d(p,\Gamma_{\alpha,R})\} \ge r$.

8.2. **Convergence of minimal hypersurface.**

Proposition 8.3. *Let (M^n, g) be a complete AF manifold, where $4 \le n \le 7$. If some sphere at infinity, on some end $\mathcal{E}$, vanishes in $H_{n-2}(M)$, then there is a complete stable minimal hypersurface Σ_∞. Furthermore, one has that for $r \ge r_0$*

$$\begin{aligned}
&vol(\Sigma_\infty \setminus \mathcal{E}) \le C\\
&vol(\Sigma_\infty \cap Z_{p,r}) \le C r^{n-1},
\end{aligned} \tag{8.3}$$

where r_0, C only depends on (M^n, g) and $\frac{1}{2}\min\{d(p,K), d(p,\Gamma_{\alpha,R})\} \ge r$

Proof. Let $\Sigma_{\alpha,R}$ be constructed in Section 2.1. First, we can conclude that $\Sigma_{\alpha,R} \cap K$ is non-empty.

If not, we have that $\Sigma_{\alpha,R} \subset \mathcal{E} \cong \mathbb{R}^3 \setminus B(0,1) \times \mathbb{S}^1$. Then, we have that $[\Gamma_{\alpha,R}]$ vanishes in $H_{n-2}(\mathbb{R}^3 \setminus B(0,1) \times \mathbb{S}^1)$, from $\partial\Sigma_{\alpha,R} = \Gamma_{\alpha,R}$. However, $[\Gamma_{\alpha,R}]$ is a generator of $H_2(\mathbb{R}^3 \setminus B(0,1) \times \mathbb{S}^1) \cong \mathbb{Z}$, a contradiction.

Choose a sequence $\{R_j\}$ going to infinity and choose α_j as follows:

$$\mathrm{Vol}(\Sigma_{\alpha_j,R_j}) = \min_{\alpha\in S^1} \mathrm{Vol}(\Sigma_{\alpha,R_j}) \tag{8.4}$$

The volume estimate in Proposition and the above intersection property implies that after passing to a subsequence, Σ_{α_j,R_j} converges to Σ_∞ in the sense of current (or pointed C^k topology). □

Remark 8.4. The choice of Σ_{α_j,R_j} implies a stronger stability inequality as follows: if $\frac{\partial}{\partial\theta} = \phi_j\nu_j + \hat{X}$ on the end $\mathcal{E}$, one has that

$$\begin{aligned}
&\int_{\Sigma_{\alpha_j,R_j}} a_j + \int_{\partial\Sigma_{\alpha_j,R_j}} G_j(\hat{X}) \ge 0\\
&a_j = |\nabla\phi_j|^2 + \frac{1}{2}R_{\Sigma_\infty}\phi_j^2 - \frac{1}{2}(R_M + |A|^2)\phi_j^2\\
&d_j(\hat{X}) =< (div_{\Sigma_{\alpha_j R_j}}\hat{X})\hat{X} - 2\phi S_j(\hat{X}) - \nabla_{\hat{X}}\hat{X} + \hat{X}, \eta >,
\end{aligned} \tag{8.5}$$

where ν_j is the unit normal vector of Σ_{α_j,R_j} and S_j is the shape operator and η_j is the conormal in Σ

8.3. **Projection map and non-zero degree map.** We can clearly define the orthogonal projection

$$p : \Pi_{\geq r} \subset \mathcal{E} \to H_{\geq r} := \{x \in \mathbb{R}^{n-1} : |x| \geq r\}$$

Lemma 8.5. *The degree of the map $p : \Sigma_\infty \cap \Pi_{\geq r} \to H_{\geq r}$ is ∓ 1.*

Remark 8.6. The degree of p is equal to the intersection number of Σ_∞ and $\{x'\} \times \mathbb{S}^1$, for any $x \in H_{\geq r}$.

Proof. Let $\Sigma_{\alpha,R}$ be a $(n-1)$-current with boundary $\Gamma_{\alpha,R}$ for $R > r$. As in Step 1 of the proof for Proposition 8.1, consider the $(n-1)$-current

$$(\Sigma_{\alpha,R} \cap \Pi_{\geq r}) - H^\alpha_{r,R}$$

whose boundary is a $(n-2)$-current on C_r.The element $[\Sigma_{\alpha,R} \cap \Pi_{\geq r}) - H^\alpha_{r,R}]$ belongs to $H_{n-1}(\Pi_{\geq r}, C_r)$. The excision property shows that $H_{n-1}(\Pi_{\geq r}, C_r) \cong H_{n-1}(\mathbb{R}^{n-1} \times \mathbb{S}^1; B^{n-1}(0,r) \times \mathbb{S}^1) = \{0\}$. That is to say,

$$[(\Sigma_{\alpha,R} \cap \Pi_{\geq r}) - H^\alpha_{r,R}] = 0 \text{ in } H_{n-1}(\Pi_{\geq r}, C_r) \tag{8.6}$$

Choose a point$x' \in H_{r,R}$. The intersection number of $C_{x'}$ and $[(\Sigma_{\alpha,R} \cap \Pi_{\geq r}) - H^\alpha_{r,R}]$ is equal zero, where $C_{x'} = p^{-1}(x')$. It follows from the vanishing property (See (8.6)).

Additionally, $C_{x'}$ intersects $H^\alpha_{r,R}$ transversally at a unique point and their intersection number is equal to ± 1. From the last paragraph, the intersection number of $C_{x'}$ and $\Sigma_{\alpha,R} \cap \Pi_{\geq r}$ is ∓ 1.

Since Σ_∞ is the limit of Σ_{α_j,R_j} in the sense of current as R_j goes to infinity, then the intersection number of $C_{x'}$ and $\Sigma_\infty \cap \Pi_{\geq r}$ is ∓ 1. □

Proposition 8.7. *The map $\pi_1(\Sigma_\infty \cap \Pi_{\geq r}) \to \pi_1(\Pi_{\geq r})$ is trivial for $r \geq 1$.*

Proof. We argue it by contradiction. Suppose that the map $\pi_1(\Sigma_\infty \cap \Pi_{\geq r}) \to \pi_1(\Pi_r)$ is non-trivial. Namely, there is a non-zero integer k and a circle $C \subset \Sigma_\infty \cap \Pi_{\geq r}$ such that $[C] = k[C_{x'}]$ in $\pi_1(\Pi_{\geq r})$, where $C_{x'} = p^{-1}(x')$.Lemma 8.5 implies that the intersection number of C and $\Sigma_\infty \cap \Pi_{\geq r}$ is $\mp k \neq 0$.

It is sufficient to show that the intersection number of C and $\Sigma_\infty \cap \Pi_{\geq r}$ is equal to zero, which leads to a contradiction.

Let $N(\Sigma_\infty \cap \Pi_{\geq r}, \epsilon)$ be the tubular neighborhood of $\Sigma_\infty \cap \Pi_{\geq r}$ with radius ϵ, with ϵ small enough. It is homeomorphic to $(\Sigma_\infty \cap \Pi_{\geq r}) \times [-\epsilon, \epsilon]$. Choose a circle $C' \subset (\Sigma_\infty \cap \Pi_{\geq r}) \times \{\epsilon\}$ such that C' is homotopic to C in $\Pi_{\geq r}$.

However, $(\Sigma_\infty \cap \Pi_{\geq r}) \times \{\epsilon\} \cap (\Sigma_\infty \cap \Pi_{\geq r}) \times \{0\}$ is empty. Namely, $(\Sigma_\infty \cap \Pi_{\geq r}) \cap C'$ is also empty. Then, their intersection number is equal to zero. From the homotopy property in last paragraph, we can conclude that the intersection number of C and Σ_∞ is also equal to zero, which leads to a contradiction. □

Corollary 8.8. *Let $(\hat{\mathcal{E}}, \hat{g}|_{\hat{\mathcal{E}}})$ be the universal cover of $(\mathcal{E}, g|_{\mathcal{E}})$, $\hat{\Sigma}_{\infty,r}$ the lifting of $\Sigma_\infty \cap \Pi_{\geq r}$ and $\pi : \hat{\Pi}_{\geq r} \to \Pi_{\geq r}$ the cover map. Then, the restriction of π on $\hat{\Sigma}_{\infty,r}$*

$$\pi : \hat{\Sigma}_{\infty,r} \to \Sigma_\infty \cap \Pi_{\geq r}$$

is an isometry. Moreover, we have that

$$vol(\pi^{-1}(Z_{p,R}) \cap \hat{\Sigma}_{\infty,r}) \leq CR^{n-1}$$

9. Asymptotic behavior of minimal hypersurfaces Σ_∞

In this section, we assume that $(M^n, g, \mathcal{E})$ be a complete AF 4-manifold and some sphere $\Gamma_{\alpha,R} \subset \mathcal{E}$ vanishes in $H_{n-2}(M)$. Proposition 8.3 allows us to find a complete stable minimal hypersurfacce Σ_∞ with volume estimates as in Corollary 8.2. In this section, we assume that g is Schwarzschild-like metric on $\mathcal{E}$ as follows:

$$g = (1 + \frac{m}{2r^{n-3}})^{\frac{4}{n-2}}(\sum_{i=1}^{n-1}(dx_i)^2 + (d\theta)^2) + O(r^{-\tau})$$

where $\tau > n-2$

Then, we will show that each Σ_∞ has an asymptotically Euclidean end $\mathcal{E}'$

Theorem 9.1. *Let (M^n, g) be a complete AF n-manifold and Σ_∞ be a complete stable minimal hypersurface in Proposition 8.3. If g is a Schwarzschild-like metric, then Σ_∞ is a Schwarzschild metric with one end $\mathcal{E}'$ and $m_{ADM}(\Sigma_\infty, \mathcal{E}') = m_{ADM}(M^n, g, \mathcal{E})$.*

9.1. **the universal cover of $\Pi_{\geq r}$ or $\mathcal{E}$.** Consider the universal cover $(\hat{\mathcal{E}}, \hat{g}|_{\hat{\mathcal{E}}})$ of $(\mathcal{E}, g|_{\mathcal{E}})$ as follows:

$$\hat{g}(x, \hat{\theta}) = (1 + \frac{m}{2r^{n-3}})^{\frac{4}{n-2}}(\sum_{i=1}^{n-1}(dx_i)^2 + (d\hat{\theta})^2) + O(r^{-\tau})$$

where $\tau > n-2$, $(x, \hat{\theta}) \in \mathbb{R}^{n-1} \times \mathbb{R}$ are the coordinate functions on $\hat{\mathcal{E}}$ and $r^2 = \sum_{i=1}^{n-1}(x_i)^2$.

Consider the universal cover map $\pi : \hat{\mathcal{E}} \to \mathcal{E}$ and the lifting $\hat{\Sigma}_\infty$ of some end $\Sigma_\infty \cap \mathcal{E}$. The hypersurface $\hat{\Sigma}_\infty$ is stable minimal with respect to the metric $\hat{g}$. Corollary 8.8 shows that $\pi : \hat{\Sigma}_\infty \to \Sigma_\infty \cap \mathcal{E}$ is an isometry. We also have the volume estimate as following:

$$\mathrm{vol}(\pi^{-1}(Z_{p,R}) \cap \hat{\Sigma}_\infty) \leq CR^{n-1}.$$

We use similar arguments in [24] to get that

Theorem 9.2. *Let $\hat{\Sigma}^{n-1}_\infty$ and $\hat{\mathcal{E}}$ be defined as above. Then, there is a positive constant R such that $\hat{\Sigma}_\infty \cap \pi^{-1}(\Pi_{\geq R})$ is a graph of function u over $\mathbb{R}^{n-1} \setminus B^{n-1}(0, R)$ and the function u satisfies that*

$$u(x) - a_\infty - \frac{c_\infty}{|x|^{n-4}} \in C^{3,\alpha}_{n-4+\epsilon} \text{ for } n > 4,$$
$$u(x) - a_\infty \in C^{3,\alpha}_{\epsilon} \text{ for } n = 4. \tag{9.1}$$

Sketch of the proof:

Step 1: Since $\hat{\Sigma}$ is stable minimal and it enjoys the volume estimate, the end $\hat{\Sigma}_\infty \subset \hat{\mathcal{E}}$ has the curvature decay

$$|A_{\hat{g}}| = O(|x|^{-1}) \tag{9.2}$$

with respect to the metric $\hat{g}_{\hat{\mathcal{E}}}$, where $(x, \hat{\theta}) \in \mathbb{R}^{n-1} \setminus B^{n-1}(0,1) \times \mathbb{R}$ are the coordinates on $\mathcal{E}$. Moreover, Σ_∞ also have the same curvature decay with respect to the metric g.

Step 2: Let $\hat{\delta}$ be the flat metric on $\hat{\mathcal{E}}$. The decay of $\hat{g} - \hat{\delta}$ is $|x|^{-(n-3)}$. As in [[24], Lemma 4.10], the difference of the norms defined by $\hat{g}$ and $\hat{\delta}$ enjoys the same decay. That is to say, for any X is any tangent vector, we have that

$$||X|_{\hat{g}} - |X|_{\hat{\delta}}| \lesssim |x|^{-(n-3)}|X|_{\hat{\delta}} \lesssim |x|^{-(n-3)}|X|_{\hat{g}}. \tag{9.3}$$

We use the formulas for the hypersurface $\hat{\Sigma}_\infty$ defined as a level set $\xi = 0$ for some smooth function ξ with $d\xi \neq 0$. It is standard [28] that the normal vector, the second fundamental form and the mean curvature respect to a metric h on $\hat{\mathcal{E}}$ is given as follows:

$$\begin{aligned} \mathbf{n}_h &= \frac{\nabla_h \xi}{|\nabla_h \xi|} \\ A_h(X,Y) &= \frac{\nabla_h^2 \xi(X,Y)}{|d\xi|_h} \text{ for } X, Y \in T\hat{\Sigma}_\infty \\ H_h &= \Pi_h^{i,j} \frac{(\nabla_h^2 \xi)_{i,j}}{|d\xi|_h} \end{aligned} \tag{9.4}$$

where $\Pi_h = h^{-1} - \mathbf{n}_h \otimes \mathbf{n}_h$. We now compute the difference of $A_{\hat{g}}$ and $A_{\hat{\delta}}$. For any vectors X and Y in $T\hat{\Sigma}_\infty$, we have that

$$(A_{\hat{g}} - A_{\hat{\delta}})(X,Y) = (\frac{1}{|d\xi|_{\hat{g}}} - \frac{1}{|d\xi|_{\hat{\delta}}})\partial_{i,j}\xi X^i Y^j - \hat{\Gamma}_{i,j}^k \frac{\partial_k \xi}{|d\xi|_{\hat{g}}} X^i Y^j,$$

where $\hat{\Gamma}^k_{i,j} = O(|x|^{-(n-2)})$ is Ch symbol of $\hat{g}$. Equation (9.3) shows that

$$|\frac{1}{|d\xi|_{\hat{g}}} - \frac{1}{|d\xi|_{\hat{\delta}}}| = \frac{||d\xi|_{\hat{g}} - |d\xi|_{\hat{\delta}}|}{|d\xi|_{\hat{g}}|d\xi|_{\hat{\delta}}} \lesssim |x|^{-(n-3)}/|d\xi|_{\hat{g}}.$$

Now we estimate

$$\begin{aligned} |(A_{\hat{g}} - A_{\hat{\delta}})(X,Y)| \leq C|x|^{-(n-3)}|\frac{\partial_{i,j}\xi X^i Y^j}{|d\xi|_{\hat{g}}}| + C|\Gamma||X|_{\hat{g}}|Y|_{\hat{g}} \\ &\lesssim |x|^{-(n-3)}|A_{\hat{g}}(X,Y) + \Gamma^k_{i,j}\frac{\partial_k \xi}{|d\xi|_{\hat{g}}} X^i Y^j| + |x|^{-(n-2)}|X|_{\hat{g}}|Y|_{\hat{g}}, \\ &\lesssim |x|^{-(n-3)}|A_{\hat{g}}(X,Y)| + (|x|^{-(n-2)} + |x|^{-(n-2)(n-3)})|X|_{\hat{g}}|Y|_{\hat{g}}, \\ &\lesssim |x|^{-(n-2)}(|x||A_{\hat{g}}(X,Y)| + |X|_{\hat{g}}|Y|_{\hat{g}}), \\ &\lesssim |x|^{-(n-2)}|X|_{\hat{g}}|Y|_{\hat{g}} \qquad \text{use Equation(9.2)} \end{aligned}$$

As a consequence, we have that

$$H_{\hat{\delta}} = O(|x|^{n-2}).$$

Step 3 Show that choosing R larger, we can write $\hat{\Sigma}_\infty$ as the graph of a function $u \in C_0^{1,\frac{1}{n}}(\mathbb{R}^{n-1} \setminus B_R(0))$

We apply Allard's regularity theorem in the form [[?], Theorem 23.1] with $p = n$ and $T = \{\hat{\theta} = 0\} \cong \mathbb{R}^{n-1}$. The L^n norm estimate is estimated by

$$(\int_{B_\rho(x)\cap\hat{\Sigma}_\infty} |H_{\hat{\delta}}|^n d\mathcal{H}^{n-1})^{\frac{1}{n}} \rho^{\frac{1}{n}} \lesssim \rho^{-(n-3)} \leq \epsilon_n,$$

where ϵ_n is the constant appearing in Allard's theorem. Here we use $|H_{\hat{\delta}}| = O(|x|^{-(n-2)})$, which is crucial.

Step 4 The strategy is to write $H_{\hat{g}} = 0$ as a prescribed curvature equation in Euclidean space and to Bootstrap this decay rate.

We define $\xi(x,\hat{\theta}) = u(x) - \hat{\theta}$, so that $\hat{\Sigma}_\infty = \{\xi = 0\}$ and use Equation (9.4) to get a elliptic PDE abut u whose all coefficient involve ∇u and u. Adjusting the decay rate, we use the same argument in [Proposition 4.6 and Proposition 4.13-Theorem 4.17, [24]] to complete the proof of Theorem 9.2.

We now use Theorem 9.2 to complete the proof for Theorem 9.1.

Proof. Let γ be the induced metric on Σ_∞ and $\hat{\gamma}$ the lifting metric of $\gamma|_{\mathcal{E}\cap\Sigma_\infty}$ on $\hat{\Sigma}_\infty$. We use the standard graphical coordinate y_i associated $x \to (x, u(x))$. The coordinate vector field are given by

$$\frac{\partial}{\partial y_i} = \frac{\partial}{\partial x_i} + \frac{\partial u}{\partial x_i}\frac{\partial}{\partial \hat{\theta}}$$

From Theorem 9.2, we have that

$$\begin{aligned}
\hat{\gamma}(\partial_{y_i}, \partial_{y_j}) &= \hat{g}(\partial_{y_i}, \partial_{y_j}) \\
&= \hat{g}(\partial_{x_i}, \partial_{x_j}) + \frac{\partial u}{\partial x_i}\hat{g}(\partial_{x_j}, \partial_{\hat{\theta}}) + \frac{\partial u}{\partial x_j}\hat{g}(\partial_{x_i}, \partial_{\hat{\theta}}) + \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\hat{g}(\partial_{\hat{\theta}}, \partial_{\hat{\theta}}) \\
&= (1 + \frac{m}{|x|^{n-3}})_{i,j} + C^2_{n-2} + C^2_{n-3}C^2_{n-2} + C^2_{n-3}C^2_{n-2} + C^2_{n-3}C^2_{n-3} \\
&= (1 + \frac{m}{|x|^{n-3}})\delta_{i,j} + C^2_{n-2}
\end{aligned}$$

We can conclude that $\hat{\gamma}$ is an Schwarzschild end and its AMD mass is equal to m. From Corollary 8.8, $\hat{\Sigma}_\infty \to \Sigma_\infty \cap \mathcal{E}$ is isometric. Then, γ is Schwarzschild and its ADM mass is m. $\square$

Corollary 9.3. *Let Σ_∞ be a stable minimal hypersurface. Then, we have that $|A_{\Sigma_\infty}| = O(|x|^{-(n-2)})$.*

9.2. **Stability of Σ_∞.** Let (M^n, g) be AF manifold assumed in Theorem 1.2 and Σ_∞ a complete stable minimal hypersurface constructed in Proposition 8.3. From Theorem 9.1, Σ^{n-1}_∞ is an asymptotically Euclidean $(n-1)$-manifold with a unique end. Use Remark 8.4 and Inequality (8.5) to get a stronger stability inequality as follows

Theorem 9.4. *For any constant $a \in \mathbb{R}$ and any function $u - a \in W^{1,2}_{\frac{n-3}{2}}$, one has*

$$\int_{\Sigma_\infty} \frac{1}{2}(R_M + |A|^2)\phi^2 dx \le \int_{\Sigma_\infty} |\nabla\phi|^2 + \frac{1}{2}R_{\Sigma_\infty}\phi^2 dx. \tag{9.5}$$

The proof follows from the inequality (8.4) and the same argument of stronger stability in [13] and [24].

Proposition 9.5. *Let Σ^{n-1}_∞ be a complete stable minimal hypersurface with a unique Schwarzschild end as in Theorem 9.1. If $R_M \ge 0$, then the ADM mass of Σ_∞ is nonnegative.*

The volume estimate $\mathrm{Vol}(\Sigma_\infty \cap (M \setminus \mathcal{E})) \le C$ implies that Σ_∞ has a unique end. From Theorem 9.1, Σ_∞ is a complete manifold with a unique Schwarzschild end

Proof. We seek a conformal factor $w > 0$, $w \to 1$ at ∞, so that $w^{\frac{4}{n-3}}g_{\Sigma_\infty}$ is scalar flat. This is equivalent to solving $L(v) = -R_{\Sigma_\infty}$, where $v = w - 1$, with $w \to 0$ at ∞.

The scalar curvature of Σ_∞ is $O(|x|^{-n})$ and integrable. It lies in $L^p_{n-1-\epsilon}$ for any $\epsilon \in (0, 1)$ and $p \ge 1$, which we can fix $p \ge n$. We now claim that

$$L = -\frac{2(n-2)}{n-3}\Delta + \frac{1}{2}R_{\Sigma_\infty} : W^{2,p}_{n-3-\epsilon} \to L^p_{n-1-\epsilon} \text{ is invertible.}$$

Suppose $\phi \in W^{2,p}_{n-3-\epsilon}$ satisfies $L(\phi) = 0$. By the elliptic regularity and Sobolev embedding, $\phi \in C^1_{n-3-\epsilon} \cap C^\infty_{loc} \subset W^{2,2}_{\frac{n-3}{2}}$. The inequality (9.5) shows that

$$\int_{\Sigma_\infty} \frac{1}{2}(R_M + |A|^2)\phi^2 dx + \int_{\Sigma_\infty} \frac{(n-1)}{n-3}|\nabla\phi|^2 \le \int_{\Sigma_\infty} \frac{2(n-2)}{n-3}|\nabla\phi|^2 + \frac{1}{2}R_{\Sigma_\infty}\phi^2 dx.$$

The right hand side is equal to $\int L(\phi)\phi dx = 0$, which follows from $\phi \in W^{2,p}_{n-2-\epsilon}$. That is to say,

$$\int_{\Sigma_\infty} \frac{1}{2}(R_M + |A|^2)\phi^2 + \frac{(n-1)}{n-3}|\nabla\phi|^2 dx = 0$$

The non-negativity of R_M implies that $\nabla\phi = 0$. Namely, it vanishes identically, since $\phi \in C^1_{n-3-\epsilon}$. By the standard Fredholm-type theorem in asymptotic analysis [[23], Theorem A.40, A4.2], L is invertible.

Let v be the solution so obtained and set $w = v + 1$. Using the fact that the conformal laplacian is strictly positive for any smooth with compact support, it is an easy matter to verify that $w > 0$. By the asymptotic analysis (see [23]), $w - (1 + \frac{c_\infty}{|x|^{n-3}}) \in C^2_{n-3+\epsilon}$. The metric $w^{\frac{4}{n-3}} g_{\Sigma_\infty}$ is asymptotical Euclidean with scalar flat and its mass is

$$\begin{aligned} m_{ADM}(w^{\frac{4}{n-3}} g) &= m(g_{\Sigma_\infty}) + c_n c_\infty \\ &= m(g_{\Sigma_\infty}) - c_n(\int_{\Sigma_\infty} \frac{2(n-2)}{n-3}|\nabla w|^2 + \frac{1}{2}R_{\Sigma_\infty} w^2 dx) \end{aligned} \tag{9.6}$$

The positive mass theorem [13] shows that $m(w^{\frac{4}{n-3}} g_{\Sigma_\infty}) \geq 0$ and the stronger stability (see Inequality (9.5)) show that the integral part is also non-negative. We conclude that the ADM mass of Σ_∞ is non-negative. $\square$

9.3. **Proof of Theorem 1.2.** We first recall the density of ADM mass on AF manifolds,

Proposition 9.6. [7] *Let $(M^n, g, \mathcal{E})$ be a complete AF n-manifold with non-negative scalar curvature. For any $\epsilon > 0$, there is a complete metric g' on M satisfying that*

- *the scalar curvature $R_{g'}$ is non-negative;*
- *the space $(\mathcal{E}, g'|_{\mathcal{E}})$ is a Schwarzchild-like AF end;*
- $|m_{ADM}(M, g, \mathcal{E}) - m_{ADM}(M, g', \mathcal{E})| \leq \epsilon$

We now complete the proof of Theorem 1.2

Proof. Suppose that the ADM mass of (M, g^n) is negative. From Proposition 9.6, we may assume that the AF end $\mathcal{E}$ of (M, g) is Schwarzchild-like and $m(M, g, \mathcal{E}) < 0$.

Let Σ_∞ be a complete stable minimal hypersurface constructed in Section 2. Theorem 9.1 implies that Σ_∞ is asymptotically Euclidean with a unique Schwarzchild-like end and its ADM mass $m(\Sigma_\infty, g_{\Sigma_\infty})$ is equal to the ADM mass $m(M, g, \mathcal{E})$. Namely, $m(\Sigma_\infty, g_{\Sigma_\infty}) < 0$.

Since $R_M \geq 0$, Proposition 9.5 tells that $m(\Sigma_\infty, g_{\Sigma_\infty}) \geq 0$, which leads to a contradiction with the last paragraph.

The rigidity result follows from the original proof of Schoen and Yau [29]. $\square$

10. Example

Appendix I: Uniform L^∞-estimate for w_i

Let $(M^4, g, \mathcal{E})$ and f be assumed as in Theorem 4.5. The function w_i satisfies that

$$\begin{cases} \Delta_g w_i - f w_i = f \text{ in } U_i \\ w_i = 0, \text{ on } \partial U_i \end{cases}$$

From (4.4), it enjoys a uniform L^6-estimate

$$(\int_{\mathcal{E}} |w_i|^6)^{\frac{1}{6}} \leq C_1(||f||_{L^{\frac{6}{5}}}, C)$$

We will use Nash-Moser iteration to get a uniform L^∞-estimate.

Multiplying w_i^{p-1} on the two sides of differential equation, we do the integration by part, where $p \geq 2$.

$$\begin{aligned}
\frac{4(p-1)}{p^2}\int_{\mathcal{E}} |\nabla w_i^{\frac{p}{2}}|^2 &= -\int_{\mathcal{E}} f w_i^p - \int_{\mathcal{E}} f w_i^{p-1} \\
&\leq ||f||_{L^\infty}\int_{\mathcal{E}} w_i^p + ||f||_{L^\infty}^{1-\frac{2}{p}}\int_{\mathcal{E}} |f|^{\frac{2}{p}} w_i^{p-1} \\
&\leq ||f||_{L^\infty}\int_{\mathcal{E}} w_i^p + ||f||_{L^\infty}^{1-\frac{2}{p}} ||f||_{L^2}^{\frac{2}{p}} (\int_{\mathcal{E}} w_i^p)^{\frac{p-1}{p}} \\
&\leq (||f||_{L^\infty} + ||f||_{L^\infty}^{1-\frac{2}{p}} ||f||_{L^2}^{\frac{2}{p}}) \max\{1, ||w_i||_{L^p}^p\} \\
&\leq (\max\{||f||_{L^2}, 1\} + 1)\max\{||f||_{L^\infty}, 1\}\max\{1, ||w_i||_{L^p}\} \\
&\leq C_0(\max\{1, ||w_i||_{L^p}^p\})^p
\end{aligned}$$

where $C_0 := (\max\{||f||_{L^2}, 1\} + 1)\max\{||f||_{L^\infty}, 1\} + 2/C_0$, and C_0 is defined in Lemma 4.4. Remark that $||f||_{L^2}$ can be bounded by $||f||_{L^\infty}$ and $||f||_{L^{\frac{6}{5}}}$ From Lemma 4.4, it follows that

$$\frac{4(p-1)}{p^2 C_0}(\int_{\mathcal{E}} w_i^{3p})^{\frac{1}{3}} \leq C(\max\{1, ||w_i||_p\})^p$$

The choice of C_0 gives that

$$(\max\{1, ||w_i||_{L^{3p}}\})^p \leq \frac{CC_0p^2}{4(p-1)}(\max\{1, ||w_i||_p\})^p \leq \frac{C_0Cp}{2}(\max\{1, ||w_i||_p\})^p$$

Let $a_n = \max\{1, ||w_i||_{L^{6\cdot 3^n}}\}$. Then, we have that

$$\begin{cases} a_{n+1} \leq (\frac{C_0C}{2})^{\frac{1}{6\cdot 3^n}}(\frac{1}{6\cdot 3^n})^{\frac{1}{6\cdot 3^n}} a_n \\ a_0 \leq C_1 \end{cases}$$

Since $C_3 := \Pi_{n=0}^{\infty}(\frac{C_0C}{2})^{\frac{1}{6\cdot 3^n}}(\frac{1}{6\cdot 3^n})^{\frac{1}{6\cdot 3^n}}$ is finite, then we have that

$$\limsup_{n\to\infty} a_n \leq C_3C_1$$

Namely, $||w_i||_{L^\infty(\mathcal{E})}$ is uniformly bounded by $||f||_{L^\infty}$, $||f||_{L^{\frac{6}{5}}}$

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