Algebraic non-hyperbolicity of hyperkähler manifolds with Picard rank greater than one

Ljudmila Kamenova, Misha Verbitsky

Abstract. A projective manifold is algebraically hyperbolic if the degree of any curve is bounded from above by its genus times a constant, which is independent from the curve. This is a property which follows from Kobayashi hyperbolicity. We prove that hyperkähler manifolds are non algebraically hyperbolic when the Picard rank is at least 3, or if the Picard rank is 2 and the SYZ conjecture on existence of Lagrangian fibrations is true. We also prove that if the automorphism group of a hyperkähler manifold is infinite then it is algebraically non-hyperbolic.

1 Introduction

In [V] M. Verbitsky proved that all hyperkähler manifolds are Kobayashi non-hyperbolic. It is interesting to inquire if projective hyperkähler manifolds are also algebraically non-hyperbolic (Definition 2.7). For a given projective manifold algebraic non-hyperbolicity implies Kobayashi non-hyperbolicity. We prove algebraic non-hyperbolicity for projective hyperkähler manifolds with infinite group of automorphisms.

Theorem 1.1: Let M be a projective hyperkähler manifold with infinite automorphism group. Then M is algebraically non-hyperbolic.

If a projective hyperkähler manifold has Picard rank at least three, we show that it is algebraically non-hyperbolic. For the case when the Picard rank equals to two we need an extra assumption in order to prove algebraic non-hyperbolicity. The SYZ conjecture states that a nef parabolic line bundle on a hyperkähler manifold gives rise to a Lagrangian fibration (Conjecture 2.4).

Theorem 1.2: Let M be a projective hyperkähler manifold with Picard rank ρ. Assume that either $\rho > 2$, or $\rho = 2$ and the SYZ conjecture holds. Then M is algebraically non-hyperbolic.

2 Basic notions

Definition 2.1: A *hyperkähler manifold of maximal holonomy* (or *irreducible holomorphic symplectic*) manifold M is a compact complex Kähler manifold with $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$, where σ is everywhere non-degenerate. From now on we would tacitly assume that our hyperkähler manifolds are of maximal holonomy.

Due to results of Matsushita, holomorphic maps from hyperkähler manifolds are quite restricted.

Theorem 2.2: (Matsushita, [Mat]) Let M be a hyperkähler manifold and $f\colon M \to B$ a proper surjective morphism with a smooth base B. Assume that f has connected fibers and $0 < \dim B < \dim M$. Then f is Lagrangian and $\dim_{\mathbb{C}} B = n$, where $\dim_{\mathbb{C}} M = 2n$.

Following Theorem 2.2, we call the surjective morphism $f\colon M \to B$ a *Lagrangian fibration* on the hyperkähler manifold M. A dominant map $f\colon M \dashrightarrow B$ is a *rational Lagrangian fibration* if there exists a birational map $\varphi\colon M \dashrightarrow M'$ between hyperkähler manifolds such that the composition $f \circ \varphi^{-1}\colon M' \to B$ is a Lagrangian fibration. J.-M. Hwang proved that if the base B of a hyperkähler Lagrangian fibration is smooth, then $B \cong \mathbb{P}^n$ (see [Hw]).

Definition 2.3: Given a hyperkähler manifold M, there is a non-degenerate primitive form q on $H^2(M, \mathbb{Z})$, called the *Beauville-Bogomolov-Fujiki form* (or the *"BBF form"* for short), of signature $(3, b_2 - 3)$, and satisfying the *Fujiki relation*

$$\int_M \alpha^{2n} = c \cdot q(\alpha)^n \qquad \text{for } \alpha \in H^2(M, \mathbb{Z}),$$

with $c > 0$ a constant depending on the topological type of M. This form generalizes the intersection pairing on K3 surfaces. A detailed description of the form can be found in [Be], [Bog] and [F].

Notice that given a Lagrangian fibration $f\colon M \to \mathbb{P}^n$, if h is the hyperplane class on $\mathbb{P}^n$, and $\alpha = f^*h$, then α belongs to the birational Kähler cone of M and $q(\alpha) = 0$. The SYZ conjecture states that the converse is also true.

Conjecture 2.4: [SYZ] If L is a line bundle on a hyperkähler manifold M with $q(L) = 0$, and such that $c_1(L)$ belongs to the birational Kähler cone of M, then L defines a rational Lagrangian fibration.

This conjecture is known for deformations of Hilbert schemes of points on K3 surfaces (Bayer–Macrì [BM]; Markman [Mar]), and for deformations of the generalized Kummer varieties $K_n(A)$ (Yoshioka [Y]).

Definition 2.5: A negative class $\alpha \in H^{1,1}(M, \mathbb{Z})$ (i.e., $q(\alpha) < 0$) is called an *MBM class* if for some isometry $\gamma \in SO(H^2(M, \mathbb{Z}))$ in the monodromy group, $\gamma(\alpha)^\perp \subset H^{1,1}(M, \mathbb{Z})$ contains a face of the Kähler cone of a birational model M' of M.

Geometrically, the MBM classes are negative integral $(1, 1)$-classes that are represented by minimal rational curves on deformations of M after identifying $H_2(M, \mathbb{Q})$ with $H^2(M, \mathbb{Q})$ via the BBF form (Amerik-Verbitsky, [AV1]).

Definition 2.6: The *Kobayashi pseudometric* on M is the maximal pseudometric d_M such that all holomorphic maps $f\colon (D, \rho) \to (M, d_M)$ are distance decreasing, where (D, ρ) is the unit disk with the Poincaré metric.

A manifold M is *Kobayashi hyperbolic* if d_M is a metric, otherwise it is called *Kobayashi non-hyperbolic.* In [V] M. Verbitsky proved that all hyperkähler manifolds are Kobayashi non-hyperbolic. In [KLV] together with S. Lu we proved that the Kobayashi pseudometric vanishes identically for K3 surfaces and for hyperkähler manifolds deformation equivalent to Lagrangian fibrations under some mild assumptions. In [De] Demailly introduced the following notion.

Definition 2.7: A projective manifold M is *algebraically hyperbolic* if for any Hermitian metric h on M there exists a constant A such that for any holomorphic map $\varphi\colon C \to M$ from a curve of genus g to M we have that $2g - 2 \geqslant A \int_C \varphi^* \omega_h$, where ω_h is the Kähler form of h.

In this paper all varieties we consider are smooth and projective. For projective varieties, Kobayashi hyperbolicity implies algebraic hyperbolicity ([De]). Here we explore non-hyperbolic properties of projective hyperkähler manifolds. Algebraic non-hyperbolicity implies Kobayashi non-hyperbolicity.

3 Main Results

Proposition 3.1: Let M be a hyperkähler manifold admitting a (rational) Lagrangian fibration. Then M is algebraically non-hyperbolic.

Proof: We use the fact that the fibers of a Lagrangian fibrations are abelian varieties ([Mat]). The isogeny self-maps on an abelian variety provide curves of fixed genus and arbirary large degrees, and therefore they are algebraically non-hyperbolic.

An alternative way of proving this proposition is by using the following result whose proof was suggested by Prof. K. Oguiso.

Lemma 3.2: If a hyperkähler manifold M admits a Lagrangian fibration, then there exists a rational curve on M.

Indeed, in [HO] J.-M. Hwang and K. Oguiso give a Kodaira-type classification of the general singular fibers of a holomorphic Lagrangian fibration. All of the general singular fibers are covered by rational curves. The locus of singular fibers is non-empty (e.g., Proposition 4.1 in [Hw]), and therefore there is a rational curve on M.

According to Lemma 3.2, M contains a rational curve, and therefore, M is algebraically non-hyperbolic. This finishes the proof of Proposition 3.1. ■

Lemma 3.3: Let M be a projective hyperkähler manifold with infinite automorphism group Γ. Consider the natural map $f : \Gamma \longrightarrow \mathrm{Aut}(H^{1,1}(M))$. Then the elements of the Kähler cone have infinite orbits with respect to $f(\Gamma)$.

Proof: See the discussion in section 2 of [O2]. ■

Lemma 3.4: Let M be a projective hyperkähler manifold, and Γ its automorphism group. Consider the natural map $g : \Gamma \longrightarrow \mathrm{Aut}(H^2_{tr}(M)) \times \mathrm{Aut}(H^{1,1}(M))$. Then $g(\Gamma)$ is finite in the first component $\mathrm{Aut}(H^2_{tr}(M))$.

Proof: This has been proven by Oguiso, see [O1]. The idea is that the BBF form restricted to the transcendental part $H^2_{tr}(M)$ is of K3-type. Then we can apply Zarhin's theorem (Theorem 1.1.1 in [Z]) to deduce that $g(\Gamma) \subset \mathrm{Aut}(H^2_{tr}(M))$ is finite. ■

Theorem 3.5: Let M be a projective hyperkähler manifold with infinite automorphism group. Then M is algebraically non-hyperbolic.

Proof: For any Kähler class w on M, its orbit is infinite by Lemma 3.3. Fix a polarization w on M with normalization $q(w) = 1$. For a given constant $C > 0$ consider the set

$$\mathcal{D}_C = \{x \in H^{1,1}(M, \mathbb{Z}) \mid q(x) \geqslant 0, \quad q(x, w) \leqslant C\}.$$

Notice that $\mathcal{D}_C$ is compact. Indeed, $y = x - q(x, w)w$ is orthogonal to w with respect to the BBF form q. The quadratic form q is of type $(1, b_2 - 1)$ on $H^{1,1}(M, \mathbb{Z})$ and since $q(w) > 0$, the restriction $q|_{w^\perp}$ is negative-definite. A direct computation shows that $q(y) = q(x) - 2q(x, w)^2 + q(x, w)^2 q(w) = q(x) - q(x, w)^2 \geqslant -C^2$. The set $\mathcal{D}_C$ is equivalent to the set of elements $\{y \in w^\perp | q(y) \geqslant -C^2\}$, which is compact because $q|_{w^\perp}$ is negative-definite. Since the set $\mathcal{D}_C$ is compact, $\sup_{x \in \Gamma \cdot \eta} \deg x = \infty$, which means there is a class of a curve η with $q(\eta) > 0$. However, all curves in the orbit $\Gamma \cdot \eta$ have constant genus. Since their degrees could be arbitrarily high, then M is algebraically non-hyperbolic. ■

Lemma 3.6: Let M be a hyperkähler manifold such that the positive cone does not coincide with the Kähler cone. Then M contains a rational curve.

Proof: There exists an MBM class as in Definition 2.5. This implies that M admits a rational curve (see Corollary 2.11 in [AV2]). ■

Theorem 3.7: Let M be a hyperkähler manifold with Picard rank ρ. Assume that either $\rho > 2$ or $\rho = 2$ and the SYZ conjecture holds. Then M is algebraically non-hyperbolic.

Proof: Notice that the Hodge lattice $H^{1,1}(M, \mathbb{Z})$ of a hyperkähler manifold has signature $(1, k)$. Therefore, for $\rho \geqslant 2$, the Hodge lattice contains a vector with positive square, and M is projective ([Hu]) First, consider the case when $\rho > 2$. If the Kähler cone coincides with the positive cone, then the automorphism group $\mathrm{Aut}(M)$ is commensurable with the group of isometries $SO(H^2(M, \mathbb{Z}))$ (Theorem 2.17 in [AV3]) preserving the Hodge type. By Lemma 3.4, this group is commensurable with the group of isometries of the Hodge lattice $H^{1,1}(M, \mathbb{Z})$. By Borel and Harish-Chandra's theorem ([BHC]), if $\rho > 2$, any arithmetic subgroup of $SO(1, \rho - 1)$ is a lattice. However, Borel density theorem implies that any lattice in a non-compact simple Lie group is Zariski dense ([Bor]). Therefore, for $\rho > 2$, $SO(H^{1,1}(M, \mathbb{Z}))$ is infinite. In this case $\mathrm{Aut}(M)$ is also infinite and we can apply Theorem 3.5. On the other hand, if the Kähler cone does not coincide with the positive cone, then by Lemma 3.6 there is a rational curve on M. Therefore, M is algebraically non-hyperbolic.

Now let $\rho = 2$. Assume the positive cone and the Kähler cone coincide. If there is no $\eta \in H^{1,1}(M, \mathbb{Z})$ with $q(\eta) = 0$, then by Theorem 87 in [Di], $SO(H^{1,1}(M, \mathbb{Z}))$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore, both $SO(H^{1,1}(M, \mathbb{Z}))$ and $\mathrm{Aut}(M)$ are infinite and we can apply Theorem 3.5. If there is $\eta \in H^{1,1}(M, \mathbb{Z})$ with $q(\eta) = 0$, then the SYZ conjecture implies that η defines a rational fibration on M and we could apply Proposition 3.1. If $\rho = 2$ and the positive and the Kähler cones are different (i.e., the positive cone is divided into Kähler chambers), then there is a nef class $\eta \in H^{1,1}(M, \mathbb{Z})$ with $q(\eta) = 0$. Since we assumed that the SYZ conjecture holds, the class η defines a Lagrangian fibration on M. Applying Proposition 3.1 we conclude that M is algebraically non-hyperbolic. ■

Remark 3.8: We conjecture that all projective hyperkähler manifolds are algebraically non-hyperbolic. However, our proof fails for manifolds with Picard rank 1.

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Ljudmila Kamenova
Department of Mathematics, 3-115
Stony Brook University
Stony Brook, NY 11794-3651, USA,
kamenova@math.sunysb.edu

Misha Verbitsky
Laboratory of Algebraic Geometry,
National Research University HSE,
Faculty of Mathematics, 7 Vavilova Str. Moscow, Russia,
verbit@mccme.ru, also:
Université libre de Bruxelles.