Short Course on
Floer homology and applications

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This short course consists of five sections. In §1 we discuss the Arnold conjecture and its history. General references are [HZ94, MS95, L04]; for some of the exercises it is useful to also consult [GP74, Lefschetz theory] and [MZ05, Corollary to the Poincaré-Birkhoff theorem asserting existence of infinitely many periodic points].

In §2 we discuss the finite dimensional toy model for Floer homology – the Morse-Witten complex associated to a Morse function f on a closed Riemannian manifold M. It nicely illustrates the main steps in the construction of Floer homology and provides many useful pictures to keep in mind. As a corollary we obtain the Morse inequalities: The number of critical points of f is bounded below by the sum of the Betti numbers of M. We will present the dynamical systems approach [W06b]; see [Sch93] for a global analysis point of view which more closely parallels the analytic setup of Floer homology. For basic intersection theory of submanifolds (transversality and orientations) see e.g. [GP74].

In §3 we define Floer homology for a closed symplectic manifold (M, ω) and show that it is naturally isomorphic to singular homology of M. Here, for simplicity, we assume that (M, ω) is symplectically aspherical, i.e. $[\omega]$ vanishes on $\pi_2(M)$. Now the Arnold conjecture is a simple corollary. Our main references are [HZ94, S99]. Furthermore, we refer to [SZ92, RS93, W96, Conley-Zehnder index].

In §4 we introduce Floer homology for a class of noncompact, but exact (hence aspherical), symplectic manifolds, namely cotangent bundles T^*M. It is naturally isomorphic to singular homology of the free loop space of

M. We indicate three methods of this calculation, see [W05], and present some details of the heat flow method. The main analytical difficulty is to deform Floer's elliptic PDE on T^*M into the parabolic heat equation on the closed Riemannian manifold M keeping track of the solutions. Here we meet the Morse-Witten complex of lecture 1 again, but this time the manifold is the infinite dimensional free loop space ΛM and the Morse function is the classical action functional whose critical points are (perturbed) closed geodesics in M. Flow lines are provided by the solutions to the heat equation. At this point we have identified Floer homology with this version of Morse homology [W10]. To actually calculate the latter we profit from the fact that many tools from finite dimensional dynamical systems are still available for parabolic PDE's [H81]. On the other hand, having only a *semi*flow on ΛM requires new methods to construct a suitable Morse filtration – the key object to obtain a natural isomorphism to singular homology. Here Conley index theory enters.

In §5 we shall explain the main ideas behind symplectic homology. We discuss the application in [W06a], namely existence of noncontractible periodic orbits for any compactly supported Hamiltonian on $(T^*M, -d\lambda)$ which is sufficiently large over the zero section (assuming that M is not simply connected).

The references below will be used throughout the lectures. Some of them are surveys providing references to the original research papers. Some of our references are available on www.math.sunysb.edu/∼joa.

References

[GP74] V. Guillemin, A. Pollack, *Differential Topology*, Prentice-Hall, Englewood Cliffs 1974.

[H81] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, Berlin, 1981, third printing 1993.

[HZ94] H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Verlag, Basel 1994.

[L04] F. Laudenbach, Symplectic geometry and Floer homology, *Ensaios Mat.* **7**, Soc. Brasil. Mat., Rio de Janeiro, 2004, 1–50.

[MS95] D. McDuff and D.A. Salamon, *Introduction to symplectic topology*, Oxford University Press, Oxford 1995.

[Mi65] J. Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, 1965.

[MZ05] J. Moser and E. Zehnder, *Notes on Dynamical Systems*, Courant Lecture Notes **12**, AMS 2005.

[P01] L. Polterovich, *The geometry of the group of symplectic diffeomorphisms*, Lectures in mathematics, Birkhäuser Verlag, Basel 2001.

[RS93] J.W. Robbin and D.A. Salamon, The Maslov index for paths, *Topology* **32** (1993), 827–44.

[RS95] J.W. Robbin and D.A. Salamon, The spectral flow and the Maslov index, *Bulletin of the LMS* **27** (1995), 1–33.

[S99] D.A. Salamon, Lectures on Floer Homology, In *Symplectic Geometry and Topology*, edited by Y. Eliashberg and L. Traynor, IAS/Park City Mathematics Series, Vol **7**, 1999, pp 143–230.

[SZ92] D.A. Salamon and E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, *Comm. Pure Appl. Math.* **45** (1992), 1303–60.

[Sch93] M. Schwarz, *Morse homology*, PM **111**, Birkhäuser Verlag, Basel 1993.

[W96] J. Weber, Topology of Sp($2\mathbb{R}$) and the Conley-Zehnder index, Preprint, 1996. www.math.sunysb.edu/~joa/UP/sp2r.ps.gz

[W05] J. Weber, Three approaches towards Floer homology of cotangent bundles, *J. Symplectic Geom.* **3** no.4 (2005), 671–701.

[W06a] J. Weber, Noncontractible periodic orbits in cotangent bundles and Floer homology, *Duke Math. J.* **133** no.3 (2006), 527–568.

[W06b] J. Weber, The Morse-Witten complex via dynamical systems, *Expo. Math.* **24** (2006), 127–159.

[W10] J. Weber, The heat flow and the homology of the loop space, Habilitation thesis, HU Berlin, 2010.