Points on the Circle: from Pappus to Thurston

John Milnor

Work with Araceli Bonifant

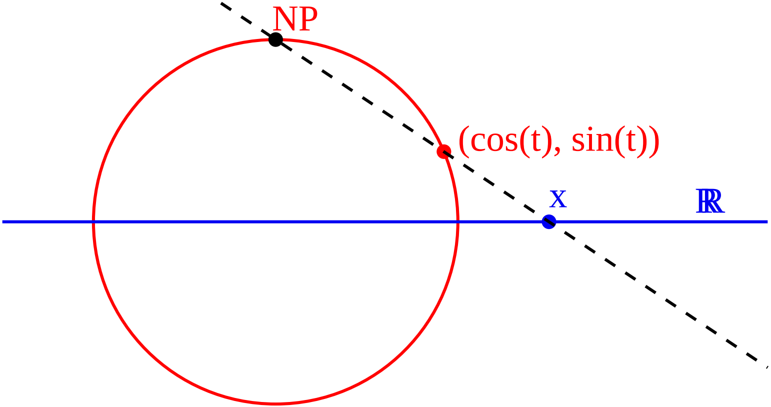
Simons Center, Stony Brook University

MAY 9, 2018

With help from

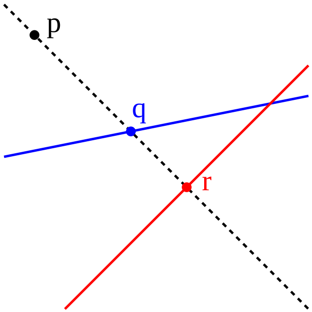
Sam Grushevsky, Dusa McDuff, Dennis Sullivan

The Circle: $\widehat{\mathbb{R}} \cong \mathbb{P}^1(\mathbb{R})$



Our “circle” will be the real projective line, $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Projective (= fractional linear) transformations



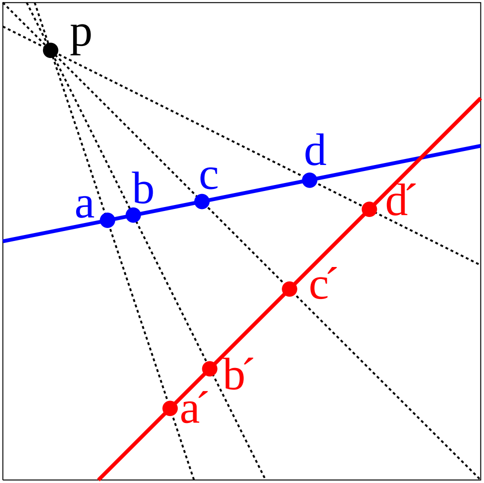
Geometrically: *Projecting from p, the point q on the blue line maps to the point r on the red line.*

Algebraically: A map from $\widehat{\mathbb{R}}$ to $\widehat{\mathbb{R}}$ is ***fractional linear*** if it has the form

$$x \mapsto \frac{ax+b}{cx+d} \qquad \text{with} \quad ad-bc \neq 0\,.$$

Essential Property: The action of the group of fractional linear transformations on $\widehat{\mathbb{R}}$ is ***three point simply transitive***.

Pappus of Alexandria (4th century).



Pappus defined a numerical invariant, computed from the distances between four points on a line;
and proved that it is invariant under projective transformations.

Cross-Ratio: Four Points on the Projective Line.

Definition (non-standard): For $a, b, c, d \in \mathbb{R}$, let

$$\mathbf{cr}(a, b, c, d) = \mathbf{cr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{(a-b)(c-d)}{(a-c)(b-d)} \in \widehat{\mathbb{R}}\,.$$

$$= \frac{\text{product of row differences}}{\text{product of column differences}}$$

Restriction: At least 3 of the 4 variables must be distinct.

There is a unique continuous extension to the case $a, b, c, d \in \widehat{\mathbb{R}}$. Then:

$$\mathbf{cr}(a,b,c,d) = 0 \iff a = b \text{ or } c = d,$$

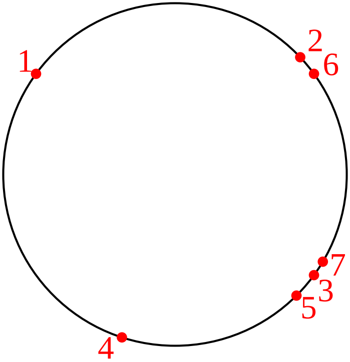
$$\mathbf{cr}(a,b,c,d) = \infty \iff a = c \text{ or } b = d,$$

$$\mathbf{cr}(a,b,c,d) = 1 \iff a = d \text{ or } b = c,$$

$$\mathbf{cr}(a,b,c,d) \in \widehat{\mathbb{R}} \setminus \{0,1,\infty\} \iff a,b,c,d \text{ all distinct.}$$

$$\mathbf{cr}(1, \infty, 0, x) = x \quad \text{for all } x\,.$$

Many Points on $\widehat{\mathbb{R}}$.



Definition. The ***moduli space*** $\mathcal{M}_{0,n}(\mathbb{R}) = \mathcal{M}_n$ is the space of equivalence classes of ordered n-tuples $(p_1, \cdots, p_n)$ of distinct points of $\widehat{\mathbb{R}}$ modulo the action of the group of fractional linear transformations.

Thus $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$ represent the same point of $\mathcal{M}_n$ if and only if there is a fractional linear transformation $\mathbf{g}$ such that

$$\mathbf{g}(p_j) = q_j \quad \text{for every } j .$$

Embedding $\mathcal{M}_n$ into a product of many circles.

Easy Lemma. *The n-tuples $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$ represent the same point of $\mathcal{M}_n$ if and only if:*

$$\mathbf{cr}(p_h, p_i, p_j, p_k) = \mathbf{cr}(q_h, q_i, q_j, q_k)$$

for every $1 \le h < i < j < k \le n$.

Thus we can embed $\mathcal{M}_n$ into the $\binom{n}{4}$-fold product of circles

$$\widehat{\mathbb{R}}^{\binom{n}{4}} = \prod_{0 \le h < i < j < k \le n} \widehat{\mathbb{R}},$$

sending the equivalence class of $(p_1, \cdots, p_n)$ into the $\binom{n}{4}$-tuple of cross-ratios $\mathbf{cr}(p_h, p_i, p_j, p_k)$, where

$$1 \le h < i < j < k \le n.$$

A Non-Standard Definition of $\overline{\mathcal{M}}_n$

Theorem (McDuff and Salamon). *The closure* $\overline{\mathcal{M}}_n$ *of* $\mathcal{M}_n$ *within the torus*

$$\widehat{\mathbb{R}}^{\binom{n}{4}} = \prod_{0\le h<i<j<k\le n} \widehat{\mathbb{R}}$$

is a smooth, compact, real-algebraic manifold of dimension $n-3$*.*

Intuitive Proof that $\overline{\mathcal{M}}_n$ is a real-algebraic set.
Since the p_j are all distinct, we can put p_1, p_2, p_3 at $1, \infty, 0$, so that

$$\mathbf{cr}(p_1, p_2, p_3, p_k) = p_k \qquad \text{for all} \quad k\ .$$

Thus $p_4, p_5, \ldots, p_n$ are $n-3$ independent variables, and determine all of the $\binom{n}{4}$ coordinate cross-ratios. Clearing denominators, we get a set of $\binom{n}{4}-3$ defining polynomial equations.

The Simplest Cases $n = 3, \;\; 4$.

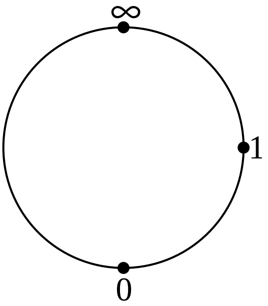
By definition, $\mathcal{M}_3 = \overline{\mathcal{M}}_3$ is a single point.

The subset $\mathcal{M}_4 \subset \mathbb{R}^{\binom{4}{4}} = \mathbb{R}$ is clearly just

$$\widehat{\mathbb{R}} \setminus \{0,\, 1,\, \infty\} \quad = \quad \mathbb{R} \setminus \{0,\, 1\}\;;$$

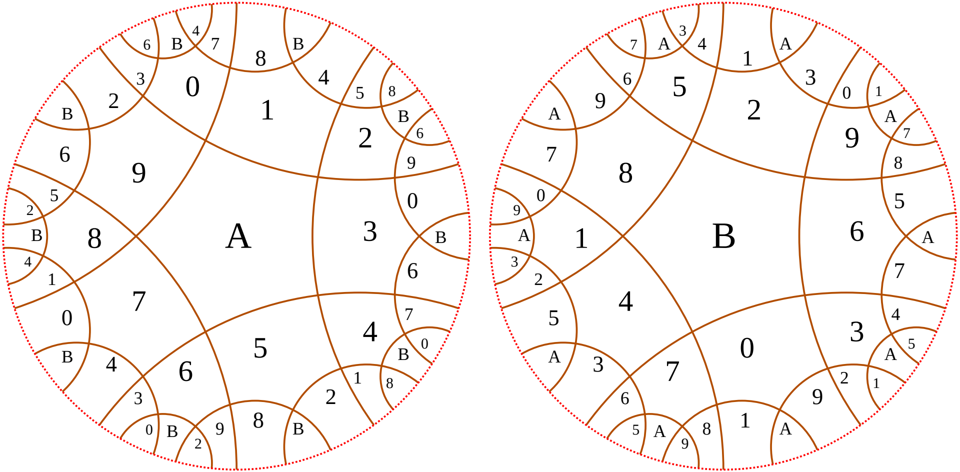
and its closure within $\widehat{\mathbb{R}}$ is the entire circle: $\overline{\mathcal{M}}_4 \cong \widehat{\mathbb{R}}$.

We should think of $\overline{\mathcal{M}}_4$ as a cell complex with three vertices and three edges:



$\overline{\mathcal{M}}_5$ is a “hyperbolic dodecahedron”,

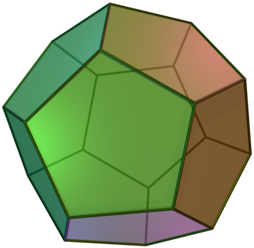
covered by twelve ***right angled hyperbolic pentagons.***

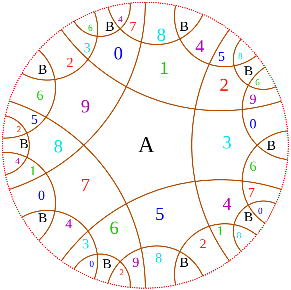


The interiors of the twelve pentagons

are the twelve connected components of $\mathcal{M}_5$.

Euclidean versus Hyperbolic Dodecahedra.





Both have isometry group of order 120:

$\mathfrak{A}_5 \oplus (\mathbb{Z}/2)\,;$ $\qquad\qquad \mathfrak{S}_5$

In both cases, each face has an “opposite” face:

In the hyperbolic case, $\quad A \leftrightarrow B\,, \quad j \longleftrightarrow j+5 \pmod{10}\;.$

Euler Characteristic:

$$\chi = 12 - 30 + 20 = 2\,, \qquad \chi = 12 - 30 + 15 = -3\;.$$

Hyperbolic case $\quad\Longrightarrow\quad \overline{\mathcal{M}}_5$ is non-orientable;
with no fixed point free involution.

Why Twelve Pentagons in $\overline{\mathcal{M}}_5$?

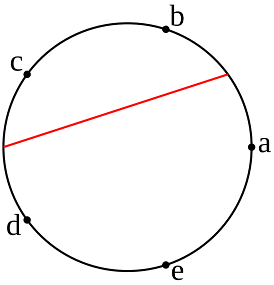
Each top dimensional cell in $\mathcal{M}_n$ corresponds to one of the

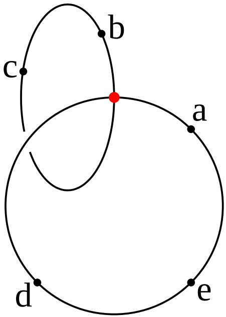
$$\frac{(n-1)!}{2}$$

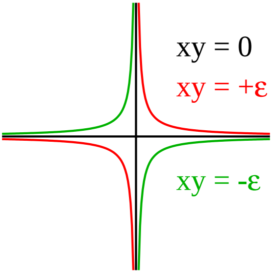
different ways of arranging the labels $1,2,3,\ \ldots,n$ in cyclic order (up to orientation) around the circle.

Thus $\overline{\mathcal{M}}_5$ has $4!/2 = 12$ two-cells.

Within $\overline{\mathcal{M}}_5$ there are five different ways that two neighboring points can cross over each other to pass to a different face; hence five edges to each 2-cell.





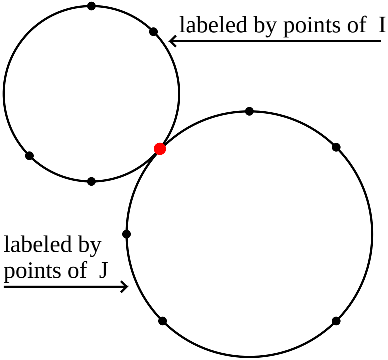


The Embedding $\varphi_{\mathbf{I},\mathbf{J}} : \overline{\mathcal{M}}_{r+1} \times \overline{\mathcal{M}}_{s+1} \hookrightarrow \overline{\mathcal{M}}_n$.

Let

$$\{1,\, 2,\, \ldots,\, n\} = \mathbf{I} \cup \mathbf{J}$$

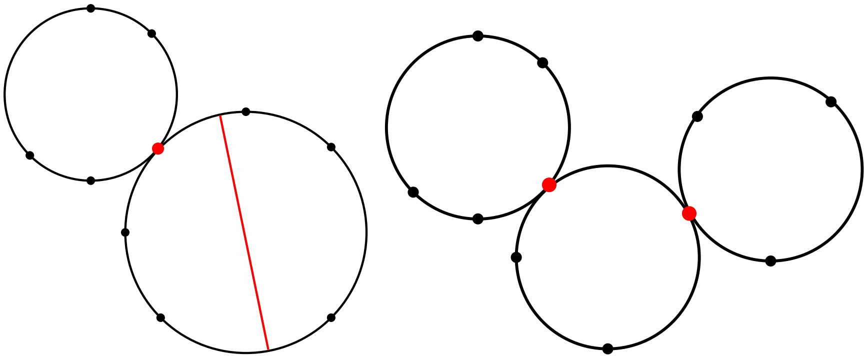
be a partition into a set $\mathbf{I}$ with $r \geq 2$ elements,
and a disjoint set $\mathbf{J}$ with $s \geq 2$ elements, where $r + s = n$



The image of $\varphi_{\mathbf{I},\mathbf{J}}$ is a union of codimension one faces;

and every codimension one face of $\overline{\mathcal{M}}_n$ is included in the image of $\varphi_{\mathbf{I},\mathbf{J}}$ for just one partition $\{\mathbf{I},\, \mathbf{J}\}$ of $\{1, 2, \ldots, n\}$.

Iterating this construction.



Mumford Stability Condition:

Each circle must have at least three distinguished points.

Cross-ratios and the Image of $\varphi_{\mathbf{I},\mathbf{J}}$.

For each $\mathbf{x} \in \overline{\mathcal{M}}_n$ and each list of distinct numbers $h,\ i,\ j,\ k$ in $\{1, 2, \ldots, n\}$, define the ***limiting cross-ratio***

$$\mathbf{cr}_{h,\,i,\,j,\,k}(x) \ \in \ \widehat{\mathbb{R}}$$

to be the limit, for any sequence of points $\mathbf{x}^\eta \in \mathcal{M}_n$ converging to $\mathbf{x}$, of the cross-ratios $\mathbf{cr}(p^\eta_h,\ p^\eta_i,\ p^\eta_j,\ p^\eta_k)$, where each $(p^\eta_1,\ \ldots,\ p^\eta_n) \in \widehat{\mathbb{R}}^n$ is a representative for the class $\mathbf{x}^\eta \in \mathcal{M}_n$.

Assertion. *The point $\mathbf{x} \in \overline{\mathcal{M}}_n$ belongs to the image,*

$$\mathbf{x} \ \in \ \varphi_{\mathbf{I},\mathbf{J}}(\overline{\mathcal{M}}_{r+1} \times \overline{\mathcal{M}}_{s+1}) \ \ \subset \ \ \overline{\mathcal{M}}_n\ ,$$

if and only if

$$\mathbf{cr}_{i,\,i',\,j,\,j'}(\mathbf{x}) \ = \ 0$$

for every $i,\ i' \in \mathbf{I}$ and every $j,\ j' \in \mathbf{J}$.

Example: $\overline{\mathcal{M}}_5$.

The are $\binom{5}{2} = 10$ partitions of $\{1, 2, 3, 4, 5\}$ into subsets of order two and three. Hence there are ten embeddings

$$\overline{\mathcal{M}}_3 \times \overline{\mathcal{M}}_4 \;\cong\; \widehat{\mathbb{R}} \;\hookrightarrow\; \overline{\mathcal{M}}_5 \;.$$

These correspond to ten closed geodesics, each made up of three edges.

Thus there are $10 \times 3 = 30$ edges in $\overline{\mathcal{M}}_5$.

Each of these geodesics also contains three vertices, Here each vertex is counted twice since it belongs to two different geodesics, so there are $10 \times 3/2 = 15$ vertices.

Thus verifying that $\chi \;=\; 12 - 30 + 15 \;=\; -3$.

Example: $\overline{\mathcal{M}}_6$

The are $\binom{6}{2} = 15$ partitions of $\{1, 2, 3, 4, 5, 6\}$ into subsets **I**, **J** of order two and four. Hence there are fifteen embeddings

$$\overline{\mathcal{M}}_3 \times \overline{\mathcal{M}}_5 \;\cong\; \overline{\mathcal{M}}_5 \quad\hookrightarrow\quad \overline{\mathcal{M}}_6\ ;$$

where each copy of $\overline{\mathcal{M}}_5$ is made up of twelve pentagons.

Similarly there are $\binom{6}{3}/2 = 10$ partitions into two subsets of order three, yielding ten embeddings of the torus

$$\overline{\mathcal{M}}_4 \times \overline{\mathcal{M}}_4 \quad\hookrightarrow\quad \overline{\mathcal{M}}_6\ .$$

Each copy of the torus is made up of $3 \times 3 = 9$ squares.

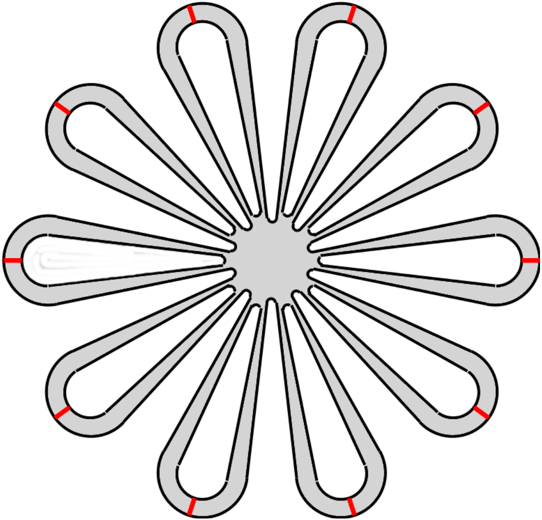
(Thus the 2-skeleton of $\overline{\mathcal{M}}_6$ consists of
$15 \times 12 = 180$ pentagons, plus $10 \times 9 = 90$ squares.)

==============================

According to Thurston, every smooth closed 3-manifold can be cut along embedded 2-spheres, tori, and/or Klein bottles into pieces, each of which has a locally homogeneous geometry.

Jaco-Shalen-Johannson Decomposition of $\overline{\mathcal{M}}_6$.

Theorem. If we cut $\overline{\mathcal{M}}_6$ open along its ten embedded tori, then the remainder can be given the structure of a complete hyperbolic manifold of finite volume with twenty infinite cusps.

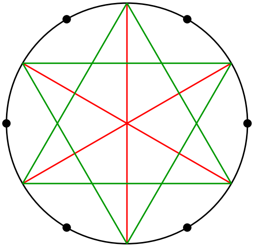


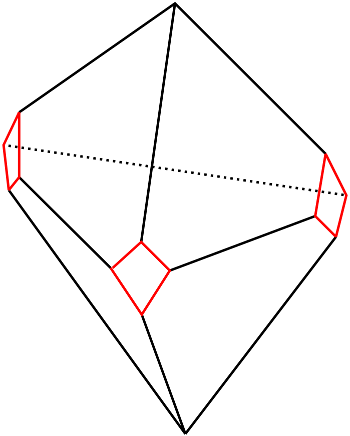
Corollary: The fundamental group $\pi_1(\overline{\mathcal{M}}_6)$ maps onto a free group on ten generators.

But $\pi_1(\overline{\mathcal{M}}_6)$ also contains free abelian groups $\mathbb{Z} \oplus \mathbb{Z}$.

Proof Outline.

Each of the 60 3-cells in $\overline{\mathcal{M}}_6$ is bounded by 6 pentagons & 3 squares.



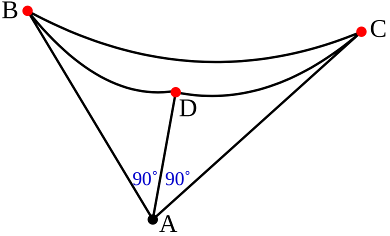


(Take the union of two tetrahedra with a face in common, and chop off three of the corners.)

We want 60 copies of this 3-cell to fit together to form a smooth manifold.

Thus we need all dihedral angles to be 90° !

Constructing a Model 3-Cell.



In Hyperbolic 3-Space, choose three orthogonal lines of length ℓ starting at the point A. Then their convex closure is a tetrahedron with dihedral angle 90° along three of the edges.

We need the dihedral angles along edges between B, C and D to be 45°. For ℓ finite, these angles are always $> 45^\circ$.

But as $\ell \to \infty$ these dihedral angles will tend to 45°.

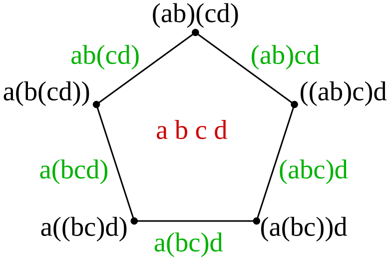
Two copies yield a model 3-cell; but only by collapsing the three squares to points, and pushing them out to the sphere at infinity.

Thus $\overline{\mathcal{M}}_6$ with the 10 tori (or 90 squares) removed is a hyperbolic manifold tiled by 120 ideal tetrahedra.

Concluding Remark: The Associahedron.

55 years ago Stasheff, while studying associativity for spaces with a continuous product operation, invented a sequence of objects A_n which we call ***associahedra***.

A_3

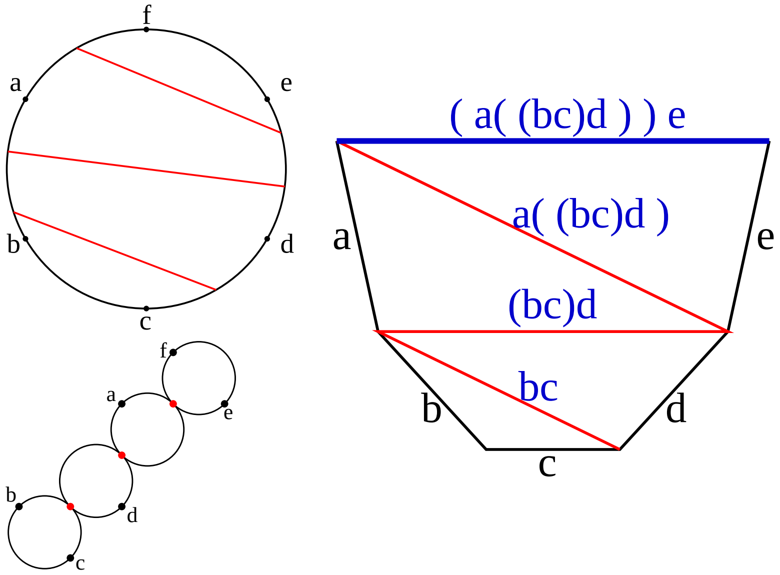


A_4

The vertices of A_n correspond to the many ways of making sense of an n-fold non-associative product.

Theorem. *Each top-dimensional cell of* $\overline{\mathcal{M}}_n$ *is isomorphic as a cell complex to* A_{n-1} *.*

The top cells of $\overline{\mathcal{M}}_n$ are Associahedra: Proof Idea.



Some References

The associahedron:

J. Stasheff, *Homotopy associativity* $\cdots$, 1963.

Thurston geometrization conjecture:

W. Thurston, *3-dimensional manifolds, Kleinian groups* $\cdot\cdot$, 1982.

Construction and smoothness of $\overline{\mathcal{M}}_n(\mathbb{C})$:

F. Knudsen, *Projectivity of the moduli space* $\cdots$, 1983.

Topology of $\overline{\mathcal{M}}_n(\mathbb{C})$:

S. Keel, *Intersection theory of Moduli space* $\cdots$, 1992.

Proof of Thurston's conjecture:

B. Kleiner and J. Lott, *Notes on Perelman* $\cdots$, 2006.

Topology of $\overline{\mathcal{M}}_n(\mathbb{R})$:

P. Etingof, A. Henriques, J. Kamnitzer, E.Rains,
The cohomology ring of the real locus $\cdots$, 2010.

Construction of $\overline{\mathcal{M}}_n(\mathbb{C})$ using cross-ratios:

D. McDuff and D. Salamon, "J-holomorphic curves $\cdots$
(Appendix D)," 2012.

From phylogenetic trees to origami folding and more:

S. Devadoss and J. Morava, *Navigation in tree spaces,* 2015.