Two Moduli Spaces

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Outline: Two Examples.

The object of this talk will be to describe two examples of smooth group actions on smooth manifolds.

Easier Example (Divisors on $\mathbb{P}^1$):

The group $G(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$ of Möbius automorphisms of the Riemann sphere $\mathbb{P}^1$ acts on the space $\mathfrak{D}_n$ of effective divisors of degree n on $\mathbb{P}^1$, with quotient space $\mathfrak{D}_n/G(\mathbb{P}^1)$.

Much Harder Example (Curves in $\mathbb{P}^2$):

The group $G(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{C})$ of projective automorphisms of the complex projective plane $\mathbb{P}^2$, acts on the projective compactification $\mathfrak{C}_n$ of the space of algebraic curves of degree n in $\mathbb{P}^2$, with quotient space $\mathfrak{C}_n/G(\mathbb{P}^2)$.

In both cases, some parts of the quotient space are beautiful objects to study, but other parts are rather nasty.

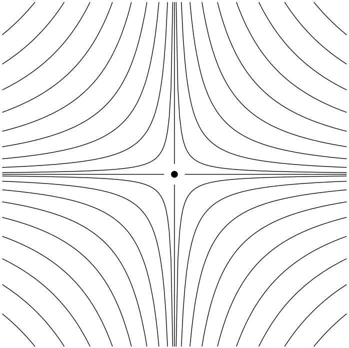
Basic Problem: Which parts are which?

A Toy Example

The additive group G of real numbers acts on $\mathbb{R}^2$ by

$$\mathbf{g}_t(x, y) = (e^t x,\ e^{-t} y).$$

Most orbits are smooth curves; but the origin is a single point orbit.



If we remove the origin, then the quotient space

$$\left(\mathbb{R}^2 \setminus \{(0,0)\}\right)/G$$

is locally a smooth manifold.

But it is only locally Hausdorff.

Part 1. The Space $\mathfrak{D}_n$ of Degree n Divisors on $\mathbb{P}^1$.

Definition: An ***effective divisor*** $\mathcal{D}$ of degree n on the Riemann sphere $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ is a formal sum

$$\mathcal{D} \;=\; m_1\langle \mathbf{p}_1\rangle + \cdots + m_k\langle \mathbf{p}_k\rangle \;,$$

where the $m_j > 0$ are integers with $\sum_j m_j = n$, and the $\mathbf{p}_j$ are distinct points of $\mathbb{P}^1$.

Each such $\mathcal{D}$ can be identified with the set of zeros, counted with multiplicity, for some non-zero homogeneous polynomial

$$\Phi(x,y) \;=\; c_0x^n + c_1x^{n-1}y + \cdots + c_ny^n \;.$$

It follows that the space $\mathfrak{D}_n$ of all such divisors is isomorphic to the projective space $\mathbb{P}^n(\mathbb{C})$.

The group $G = G(\mathbb{P}^1)$ of Möbius automorphisms of $\mathbb{P}^1$ acts on $\mathfrak{D}_n$.

Two integer invariants under the action of G:

- *The number of points* $k = \#|\mathcal{D}|$ *in the* ***support***

$$|\mathcal{D}| = \{\mathbf{p}_1, \ldots, \mathbf{p}_k\} \subset \mathbb{P}^1 \;.$$

- *The maximum* $m_{\mathbf{max}} = \mathbf{max}\{m_1, \ldots, m_k\}$ *of the multiplicities of the various points of* $|\mathcal{D}|$.

Definition. The ***stabilizer*** $G_{\mathcal{D}}$ of a divisor $\mathcal{D}$ is the subgroup of G consisting of all $\mathbf{g} \in G$ with $\mathbf{g}(\mathcal{D}) = \mathcal{D}$.

Lemma. *The stabilizer $G_{\mathcal{D}}$ is finite if and only if the support $|\mathcal{D}| \subset \mathbb{P}^1$ contains at least three elements.*

Proof. For any $\mathcal{D}$ there is a natural homomorphism $G_{\mathcal{D}} \to \mathscr{S}_{|\mathcal{D}|}$, where $\mathscr{S}_{|\mathcal{D}|}$ is the symmetric group consisting of all permutations of the finite set $|\mathcal{D}|$.
If $\#|\mathcal{D}| \geq 3$, since any Möbius transformation which fixes three distinct points must be the identity, it follows that $G_{\mathcal{D}}$ maps isomorphically onto a subgroup of $\mathscr{S}_{|\mathcal{D}|}$.

Now suppose that $\#|\mathcal{D}| \leq 2$. After a Möbius transformation, we may assume that $|\mathcal{D}| \subset \{0, \infty\}$. (Here I am identifying the Riemann sphere with $\mathbb{C} \cup \{\infty\}$.) The group $G_{\mathcal{D}}$ then contains infinitely many transformations of the form

$$\mathbf{g}_\kappa(z) = \kappa z \quad \text{with} \quad \kappa \neq 0 . \quad \square$$

The Moduli Space for Divisors.

Let $\mathfrak{D}_n^{\text{fstab}}$ be the open subset of $\mathfrak{D}_n$ consisting of all divisors with finite stabilizer ($\iff$ all divisors with $\#|\mathcal{D}| \geq 3$).

Definition. The quotient $\mathfrak{M}_n = \mathfrak{D}_n^{\text{fstab}}/G$ will be called the ***moduli space*** for divisors, under the action of G.

> **Proposition 1.** *This quotient space $\mathfrak{M}_n$ is a T_1 -**space**, that is:*
>
> *Every point of $\mathfrak{M}_n$ is a closed subset,*
>
> $\iff$ *Every G-orbit* $(\!(\mathcal{D})\!) = \{\mathbf{g}(\mathcal{D}) \; ; \; \mathbf{g} \in G\}$
>
> *in $\mathfrak{D}_n^{\text{fstab}}$ is closed as a subset of $\mathfrak{D}_n^{\text{fstab}}$.*

In other words, every $\mathcal{D}' \in \mathfrak{D}_n$ which belongs to the topological boundary $\overline{(\!(\mathcal{D})\!)} \smallsetminus (\!(\mathcal{D})\!)$ must have infinite stabilizer.

To prove Proposition 1, we must study elements of G which are "close to infinity" in G.

Distortion Lemma for Möbius Transformations.

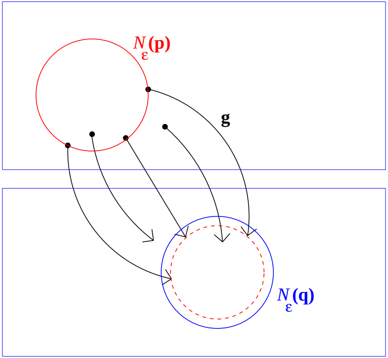
Using the spherical metric on $\mathbb{P}^1$, let $N_\varepsilon(\mathbf{p})$ be the open ε-neighborhood of $\mathbf{p}$.

Lemma. *For any $\varepsilon > 0$ there is a large compact set*

$$K = K_\varepsilon \subset G$$

with the following property:
For any $\mathbf{g} \notin K$, there are (not necessarily distinct) points $\mathbf{p}$ and $\mathbf{q}$ such that

$$\mathbf{g}(N_\varepsilon(\mathbf{p})) \cup N_\varepsilon(\mathbf{q}) = \mathbb{P}^1 .$$



Thus points outside of $N\varepsilon(\mathbf{p})$ map inside $N_\varepsilon(\mathbf{q})$.

(**Proof Outline.** The proof for the group of diagonal transformations $\mathbf{d}(x:y) = (\kappa\, x : y)$ is easy. But any $\mathbf{g} \in G$ can be written as a product $\mathbf{g} = \mathbf{r} \circ \mathbf{d} \circ \mathbf{r}'$ where $\mathbf{r}$ and $\mathbf{r}'$ are rotations of the Riemann sphere and $\mathbf{d}$ is diagonal. . . .)

Proof of Proposition 1: Points of $\mathfrak{M}_n$ are closed.

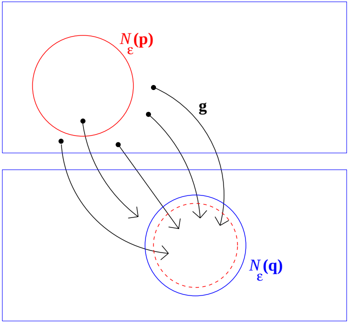
To prove: *Every G-orbit $((\mathcal{D})) \subset \mathfrak{D}_n^{\text{fstab}}$ is closed as a subset of $\mathfrak{D}_n^{\text{fstab}}$.*

Choose ε small enough so that any two points of $|\mathcal{D}|$ have distance $> 2\varepsilon$ from each other.

$\implies$ No ε-ball contains more than one point of $|\mathcal{D}|$.

Given any $\mathbf{g} \notin K_\varepsilon$, choose $\mathbf{p}$ and $\mathbf{q}$ as in the Distortion Lemma. It follows that:

all but possibly one of the points of $\mathbf{g}(|\mathcal{D}|)$ lie in $N_\varepsilon(\mathbf{q})$.



Now suppose that we are given a sequence of points $\mathbf{g}_j(\mathcal{D}) \in ((\mathcal{D}))$ converging to $\mathcal{D}' \in \mathfrak{D}_n$.

Case 1. If all $\mathbf{g}_j \in K \subset G$, then

$$\mathcal{D}' \in ((\mathcal{D})).$$

Case 2. If $\mathbf{g}_j \in K_{\varepsilon_j}$ with $\varepsilon_j \to 0$, then $|\mathcal{D}'|$ has at most two points, so $\mathcal{D}' \notin \mathfrak{D}_n^{\text{fstab}}$. □

The Cases $n \leq 4$ are very special.

$\mathfrak{M}_3$ is a single point.

$\mathfrak{M}_4 \cong \mathbb{P}^1$ is a 2-sphere.

Proof Outline: Four distinct points in $\mathbb{P}^1$ determine a 2-fold branched covering which is an elliptic curve; characterized by the classical invariant $j(\mathcal{C}) \in \mathbb{C}$. Thus the open subset corresponding to divisors with four distinct points is canonically isomorphic to $\mathbb{C}$.

But there is one other G-orbit

$$((2\langle \mathbf{p} \rangle + \langle \mathbf{q} \rangle + \langle \mathbf{r} \rangle)) \subset \mathfrak{D}_4^{\text{fstab}}$$

consisting of divisors with only three distinct points.

It follows easily that $\mathfrak{M}_4$ is homeomorphic to the one point compactification $\mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$.

Theorem. *For* $n \geq 5$, $\mathfrak{M}_n$ *has a unique maximal open subset* $\mathfrak{M}_n^{\text{Haus}}$ *which is Hausdorff.*

Here $\mathfrak{M}_n^{\text{Haus}}$ *is the set of all images* $\pi(\mathcal{D}) \in \mathfrak{M}_n$ *where* $\mathcal{D}$ *is a divisor with maximum multiplicity* $m_{\textbf{max}} < n/2$

(where $\pi : \mathfrak{D}_n^{\text{fstab}} \to \mathfrak{M}_n$ *denotes the projection map).*

$\mathfrak{M}_n^{\text{Haus}}$ *is compact if* n *is odd;*
but non-compact if n *is even.*

$\mathfrak{M}_n^{\text{Haus}}$ *is an orbifold of complex dimension* $n-3$.

Points of $\mathfrak{M}_n$ *outside of* $\mathfrak{M}_n^{\text{Haus}}$
are not even locally Hausdorff.

Partial Proof Outline.

Definition. The action of a Lie group G on a space X is ***proper*** if, for every $x, y \in X$, there are neighborhoods U and V so that the set of group elements with $\mathbf{g}(U) \cap V \neq \emptyset$ has compact closure within G.

Standard Theorem. The quotient X/G of a Hausdorff space under a proper action is a Hausdorff space.

Using the Distortion Theorem, one can show that the action of $G(\mathbb{P}^1)$ on the space of divisors with $m_{\mathbf{max}} < n/2$ is proper.

Non Locally Hausdorff Points for $n > 4$

To fix ideas, let $n = 5$. Consider two divisors of the form

$$\mathcal{D} = \mathcal{D}_2 + 3\langle\infty\rangle \qquad \text{and} \qquad \mathcal{D}' = \mathcal{D}_3 + 2\langle\infty\rangle$$

in $\mathfrak{D}_5$, where

$$\mathcal{D}_2 = \langle\mathbf{p}\rangle + \langle\mathbf{q}\rangle \qquad \text{and} \qquad \mathcal{D}_3 = \langle\mathbf{p}'\rangle + \langle\mathbf{q}'\rangle + \langle\mathbf{r}'\rangle\ .$$

Let $\mathbf{g}_\kappa(z) = \kappa^2/z$, with $\kappa \gg 1$;

so that $|z| < \kappa \iff |\mathbf{g}_\kappa(z)| > \kappa$.

Then the two divisors $\mathcal{D}_2 + \mathbf{g}_\kappa(\mathcal{D}_3)$ and $\mathcal{D}_3 + \mathbf{g}_\kappa(\mathcal{D}_2)$ belong to the same G-orbit.

As $\kappa \to \infty$, the first converges to $\mathcal{D}$
and the second converges to $\mathcal{D}'$.

Thus every neighborhood of $\pi(\mathcal{D}) \in \mathfrak{M}_5$
intersects every neighborhood of $\pi(\mathcal{D}')$.

Since $\mathcal{D}'$ can be arbitrarily close to $\mathcal{D}$, this proves that $\mathfrak{M}_5$ is not locally Hausdorff at the point $\pi(\mathcal{D})$.

Part 2. Curves in the Projective Plane.

Definition. An ***effective 1-cycle*** of degree $n \geq 1$ on the complex projective plane $\mathbb{P}^2$ is a formal sum

$$\mathcal{C} = m_1 \cdot \mathcal{C}_1 + \cdots + m_k \cdot \mathcal{C}_k \, ,$$

where each $\mathcal{C}_j$ is an irreducible complex curve, where the $m_j \geq 1$ are integers, and where $n = \sum_j m_j \deg(\mathcal{C}_j)$.

The space $\mathfrak{C}_n$ of all effective 1-cycles can be given the structure of a complex projective space of dimension $n(n+3)/2$. (In fact each non-zero homogeneous polynomial $\Phi(x, y, z)$ of degree n has a zero locus consisting of irreducible curves $\mathcal{C}_j$, each counted with some multiplicity $m_j \geq 1$; yielding a 1-cycle.)

The group $G = G(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{C})$ of all automorphisms of $\mathbb{P}^2$ acts on $\mathbb{P}^2$ and hence on the space $\mathfrak{C}_n$.

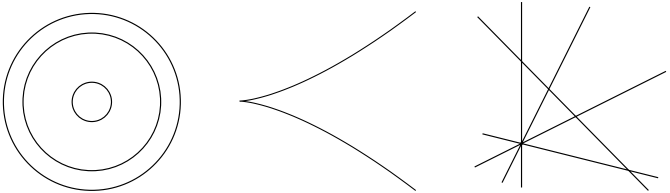
The stabilizer $G_{\mathcal{C}}$ of $\mathcal{C} \in \mathfrak{C}_n$ is just the group consisting of all projective automorphisms which map $\mathcal{C}$ to itself.

This stabilizer $G_{\mathcal{C}}$ may be either finite or infinite.

W-curves (and cycles).

Curves with infinite stabilizer were first studied by Felix Klein and Sophus Lie, who called them ***W-curves***.

Some examples:



Let $\mathfrak{W}_n \subset \mathfrak{C}_n$ be the algebraic set consisting of all cycles with infinite stabilizer. ($\mathfrak{W}_n$ is a union of finitely many maximal irreducible subvarieties of $\mathfrak{C}_n$, of varying dimension.)

Note: $\mathcal{C}$ has finite stabilizer if and only if the G-orbit $((\mathcal{C})) \subset \mathfrak{C}_n$ has dimension 8.

In fact $\dim((\mathcal{C})) + \dim(G_\mathcal{C}) = \dim(G) = 8$,
where $\dim(G_\mathcal{C}) = 0 \iff G_\mathcal{C}$ is finite.

The Moduli Space $\mathbb{M}_n$.

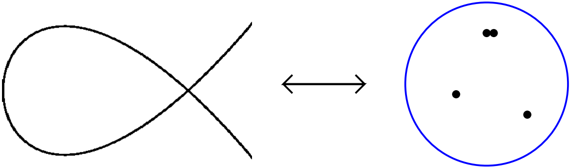
The complement $\mathfrak{C}_n^{\text{fstab}} = \mathfrak{C}_n \smallsetminus \mathfrak{W}_n$ is the open set consisting of all cycles with ***finite stabilizer.***

Definition. The quotient space $\mathbb{M}_n = \mathfrak{C}_n^{\text{fstab}}/G$, will be called the ***moduli space*** for plane cycles of degree n.

Examples. $\mathbb{M}_1 = \mathbb{M}_2 = \emptyset$.

The moduli space $\mathbb{M}_3$ for cubic curves in $\mathbb{P}^2$ is canonically isomorphic to the moduli space $\mathfrak{M}_4$ for divisors in $\mathbb{P}^1$.

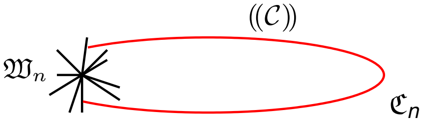
Each has two “ramified points” corresponding to points with extra symmetry (= larger stabilizer). Each also has one “improper point” where the group action is not proper.



Thus $\mathbb{M}_3 \cong \mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$.

$\mathbb{M}_n$ is a T_1-space.

Cartoon of $\mathfrak{C}_n$, showing a typical G-orbit in red:



Theorem. The topological boundary of any G-orbit in $\mathfrak{C}_n$ is contained in the closed subset $\mathfrak{W}_n$.

[Ghizzetti 1936; Aluffi and Faber 2010.]

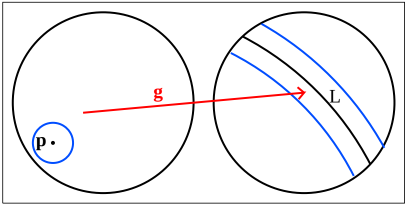
$\Longrightarrow$ Every G-orbit of cycles with finite stabilizer is closed as a subset of $\mathfrak{C}_n^{\text{fstab}}$.

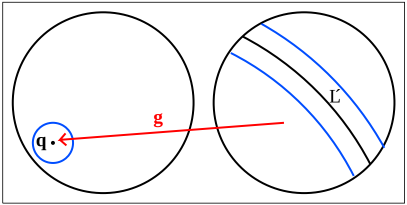
$\Longrightarrow$ Every point in $\mathbb{M}_n$ is a closed set.

The Distortion Lemma for $\mathbb{P}^2$.

Lemma. *Given* $\varepsilon > 0$ *there exists a compact set* $K_\varepsilon \subset G(\mathbb{P}^2)$ *with the following property.*

For any $\mathbf{g} \notin K_\varepsilon$ *there exists either:*





(1) a point $\mathbf{p} \in \mathbb{P}^2$ and a line $L \subset \mathbb{P}^2$ such that

$$\mathbf{g}\big(N_\varepsilon(\mathbf{p})\big) \cup N_\varepsilon(L) = \mathbb{P}^2$$

(so that $\mathbf{g}$ maps every point outside of $N_\varepsilon(\mathbf{p})$ into $N_\varepsilon(L)$),

or **(2)** a line $L' \subset \mathbb{P}^2$ and a point $\mathbf{q} \in \mathbb{P}^2$ such that

$$\mathbf{g}\big(N_\varepsilon(L)\big) \cup N_\varepsilon(\mathbf{q}) = \mathbb{P}^2$$

(so that $\mathbf{g}$ maps every point outside of $N_\varepsilon(L')$ into $N_\varepsilon(\mathbf{q})$).

The Genus Invariant of a Singularity.

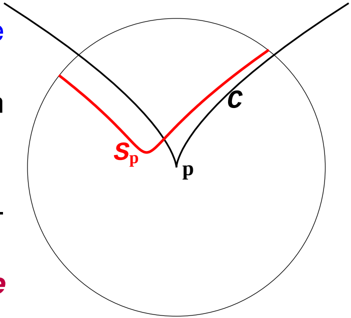
Let **p** be a singular point of a complex curve $\mathcal{C} \subset \mathbb{P}^2$. Let N_ε be a small round ball centered at **p**.

If $\mathcal{C}'$ is a smooth curve which closely approximates $\mathcal{C}$, then

$$\mathcal{S}_p = \mathcal{C}' \cap \overline{N}_\varepsilon$$

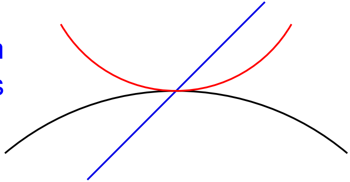
is a compact connected Riemann-surface-with-boundary.

Its genus $\mathfrak{g}(\mathcal{S}_p)$ will be called ***the genus of the singularity*** $\mathbf{p} \in \mathcal{C}$.



Examples: For a cusp singularity $x^p = y^q$ the genus is $(p-1)(q-1)/2$.

If $\mathcal{C}$ is locally the union of k smooth branches $\mathcal{B}_j$, then the genus is $-1 + \sum_{i<j} \mathcal{B}_i \cdot \mathcal{B}_j$.



Two Properties of the Genus.

Monotonicity. Suppose that

$$\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_k \ \subset \ \mathcal{S}' ,$$

where the $\mathcal{S}_j$ are disjoint compact Riemann-surfaces-with-boundary, and $\mathcal{S}'$ is another compact Riemann surface, possibly with boundary. Then

$$\mathfrak{g}(\mathcal{S}) \ := \ \sum \mathfrak{g}(\mathcal{S}_j) \ \leq \ \mathfrak{g}(\mathcal{S}) .$$

Scissors and Paste. Suppose that k disjoint embedded curves cut the closed Riemann surface $\mathcal{S}$ into ℓ subspaces with boundary $\mathcal{S}_j$. Then

$$\mathfrak{g}(\mathcal{S}) \ = \ k + 1 - \ell \ + \ \sum \mathfrak{g}(\mathcal{S}_j) .$$

(This follows from the Euler characteristic identity

$$\chi(\mathcal{S}) = \sum \chi(\mathcal{S}_j) \ .)$$

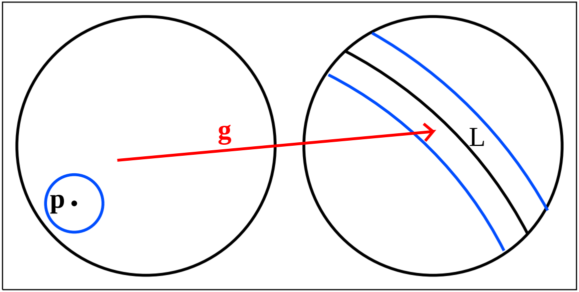
A Hypothesis which implies Proper Action.

For any line $L \subset \mathbb{P}^2$ and any specified curve $\mathcal{C}$ we can form the intersection $\mathcal{S}_L = \mathcal{C}' \cap \overline{N}_\varepsilon(L)$, where ε is small and $\mathcal{C}'$ is a very close generic approximation to $\mathcal{C}$.

Lemma. *If*

$$\mathfrak{g}\left(\overline{\mathcal{C}' \setminus \mathcal{S}_{\mathbf{p}}}\right) > \mathfrak{g}(\mathcal{S}_L)$$

for **every** $\mathbf{p} \in |\mathcal{C}|$ *and* **every** $L \subset \mathbb{P}^2$*,*
then the action of G is locally proper at $\mathcal{C}$.



Let $\mathcal{S}^*_{\mathbf{p}} = \overline{\mathcal{C}' \setminus \mathcal{S}_p}$. Then

$$\mathcal{S}_{\mathbf{p}} \cup \mathcal{S}^*_p = \mathcal{C}' , \qquad \mathcal{S}_p \cap \mathcal{S}^*_p = \big(\text{union of } k \text{ circles}\big) .$$

Therefore

$$\mathbf{g}(\mathcal{S}_{\mathbf{p}}) + \mathbf{g}(\mathcal{S}^*_p) + k - \ell + 1 = \mathfrak{g}(\mathcal{C}') = \binom{n-1}{2} .$$

Here $\ell \geq 2$ is the number of components of $\mathcal{S}_{\mathbf{p}}$ plus the number of components of $\mathcal{S}^*_{\mathbf{p}}$.

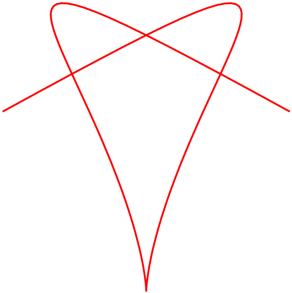
Define the ***augmented genus*** of $\mathcal{S}_p$ to be

$$\mathfrak{g}^+(\mathcal{S}_p) = \mathfrak{g}(\mathcal{S}_{\mathbf{p}}) + k - 1 .$$

Together the the Lemma, this formula yields:

Theorem. *If* $\mathfrak{g}^+(\mathcal{S}_{\mathbf{p}}) + \mathfrak{g}(\mathcal{S}_L) < \mathfrak{g}(\mathcal{C}')$ *for every* $\mathbf{p} \in \mathcal{C}$ *and every* $L \subset \mathbb{P}^2$, *then the action of* G *is locally proper at* $\mathcal{C}$.

Sample Corollary.



Let $\mathfrak{U}_n \subset \mathfrak{C}_n$ be the open set consisting of curves with no singularities other than simple double points and cubic cusps.

Corollary. *If $n \geq 4$ then the action of $G(\mathbb{P}^2)$ is locally proper throughout $\mathfrak{U}_n$.*

In fact the action is proper throughout $\mathfrak{U}_N$, so the quotient space $\mathfrak{U}_n / G(\mathbb{P}^2) \subset \mathbb{M}_n$ is a Hausdorff space.

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