The Geometry of Cubic Maps

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CELEBRATING LINDA KEEN

Parameter Space

PROBLEM: To study cubic polynomial maps F with a marked critical point which is periodic under F.

Normal form:

If F is monic and centered, then it is uniquely determined by the marked critical point a and the associated critical value $v = F(a)$:

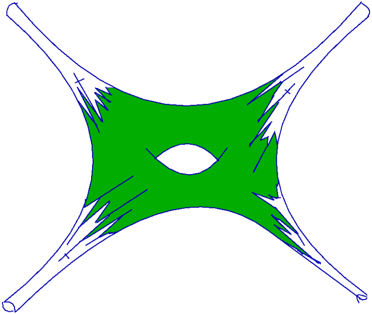
$$F(z) = (z - a)^2(z + 2a) + v\,.$$

Thus the **parameter space** for the family of all such maps $F = F_{a,\,v}$ is the set of pairs $(a, v) \in \mathbb{C}^2$,

where $F'(a) = 0$ and $F(a) = v$.

The Curve $\mathcal{S}_p$

The **period p curve** $\mathcal{S}_p$ consists of those $(a, v) \in \mathbb{C}^2$ such that the marked critical point a has period *exactly* p.



A caricature of $\mathcal{S}_p = \mathcal{C}(\mathcal{S}_p) \cup E_1 \cup \cdots \cup E_{N_p}$

The ***connectedness locus*** $\mathcal{C}(\mathcal{S}_p)$ is compact.

Each ***escape region*** E_h is open, with $E_h \cong \mathbb{C} \smallsetminus \overline{\mathbb{D}}$.

More precisely, E_h is naturally a μ_h-fold covering of $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$,

where $\mu_h \geq 1$ is called the ***multiplicity*** of E_h.

Computation of the degree

The simplest invariant of $\mathcal{S}_p$ is the ***degree*** $\deg(\mathcal{S}_p)$.

Note that the disjoint union $\bigcup_{n|p} \mathcal{S}_n$ is the zero set of a polynomial function

$$(a, v) \mapsto F_{a,v}^{\circ p}(a) - a$$

of degree 3^{p-1}. Hence $3^{p-1} = \sum_{n|p} \deg(\mathcal{S}_n)$.

It follows that $\deg(\mathcal{S}_p)$ grows exponentially:

$$\deg(\mathcal{S}_p) \sim 3^{p-1} \qquad \text{as} \quad p \to \infty .$$

The degree can also be described as the number of escape regions **counted with multiplicity**:

$$\deg(\mathcal{S}_p) = \sum_{h=1}^{N_p} \mu_h .$$

The smooth compactification $\overline{\mathcal{S}}_p$

THEOREM: $\mathcal{S}_p$ is a smooth affine curve

(conjecturally always connected).

Adding an ideal "point at infinity" ∞_h to each E_h, we obtain a **smooth** compact complex 1-manifold

$$\overline{\mathcal{S}}_p = \mathcal{S}_p \cup \infty_1 \cup \cdots \cup \infty_{N_p} .$$

We would like to compute the "genus" of this curve. But will settle for computing the Euler characteristic, since we don't know that it is connected.

Computation of the Euler characteristic of $\overline{\mathcal{S}}_p$

The computation will proceed in six steps:

Step 1. $\mathcal{S}_p$ is a translation surface. In other words, we can construct a nowhere vanishing holomorphic 1-form dt on $\mathcal{S}_p$.

Step 2. Some classical geometry. Considered as a 1-form on $\overline{\mathcal{S}}_p$, dt is meromorphic, with zeros or poles only at the ideal points. The Euler characteristic can then be expressed as a sum, with one integer contribution from each ideal point.

Step 3. The Branner-Hubbard Puzzle for maps in E_h. In order to understand asymptotic behavior near ∞_h, we must first study the dynamics of maps $F \in E_h$.

Step 4. The Kiwi Puzzle: Non-Archimedean Dynamics. This will convert the Branner-Hubbard dynamic information into asymptotic information about the differences $a - F^{\circ j}(a)$.

Step 5. Local Computation: The contribution of each ∞_h to $\chi(\overline{\mathcal{S}}_p)$.

Step 6. A Global Identity. This will help piece the complicated local information together into a relatively simple formula.

Step 1. $\mathcal{S}_p$ is a translation surface

Define the Hamiltonian function $H_p : \mathbb{C}^2 \to \mathbb{C}$ by

$$H_p(a, v) \;=\; F^{\circ p}(a) - a\,, \qquad \text{where} \quad F = F_{a,v}\,.$$

This vanishes everywhere on $\mathcal{S}_p$, with $dH_p \neq 0$ on $\mathcal{S}_p$.

Let $t \mapsto (a,\; v)$ be any solution to the Hamiltonian differential equation

$$\frac{da}{dt} \;=\; \frac{\partial H_p}{\partial v}\,, \qquad \frac{dv}{dt} \;=\; -\,\frac{\partial H_p}{\partial a}\,.$$

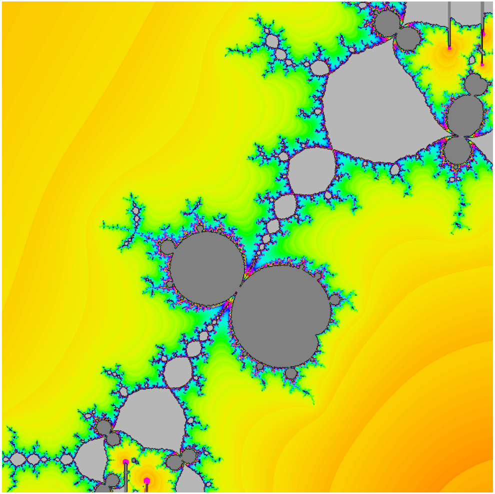
The local solutions $\quad t \;\mapsto\; (a, v) \;=\; \big(a(t),\; v(t)\big)$ are holomorphic, with $\frac{dH_p}{dt} \;=\; \frac{\partial H_p}{\partial a}\frac{da}{dt} \;+\; \frac{\partial H_p}{\partial v}\frac{dv}{dt} \;\equiv\; 0$.

Hence they lie in curves $H_p = \text{constant}\,.$

Those solutions which lie in $\mathcal{S}_p$ provide a local holomorphic parametrization, unique up to a translation, $t \mapsto t + \text{constant}$.

Equivalently, the holomorphic 1-form dt ***on*** $\mathcal{S}_p$ ***is well defined and non-zero everywhere.***

A typical parameter space picture



This is a region in the t-plane for the period 4 curve $\mathcal{S}_4$.

Step 2. Some Classical Geometry

Consider dt as a **meromorphic 1-form** on the closed manifold $\overline{\mathcal{S}}_p$. It has zeros or poles only at the ideal points ∞_h. We can compute

$$\chi(\overline{\mathcal{S}}_p) = \#(\text{poles}) - \#(\text{zeros}).$$

More precisely, if

$$\mathrm{ord}(dt, \infty_h) = \begin{cases} \text{the multiplicity of the pole at } \infty_h, \\ \text{or minus the multiplicity of the zero}, \end{cases}$$

then

$$\chi(\overline{\mathcal{S}}_p) = \sum_{j=1}^{N_p} \mathrm{ord}(dt, \infty_h).$$

The winding number

LEMMA. There exists a local parameter η for $\overline{\mathcal{S}}_p$ around ∞_h, and a local integral t of dt, so that

$$t = \eta^{w_h} . \quad \text{(In particular } \oint_{\infty_h} dt = 0 .\text{)}$$

Here w_h is a non-zero integer to be called the ***winding number*** of E_h. Then

$$dt = w_h \, \eta^{w_h - 1} \, d\eta .$$

Therefore $\quad \operatorname{ord}(dt, \infty_h) = 1 - w_h .$

Hence

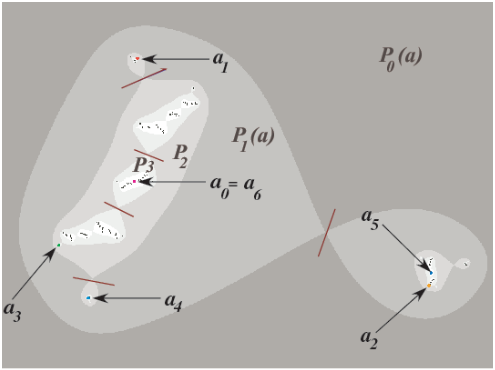
$$\chi(\overline{\mathcal{S}}_p) = \sum_{j=1}^{N_p} (1 - w_h) = N_p - \sum w_h .$$

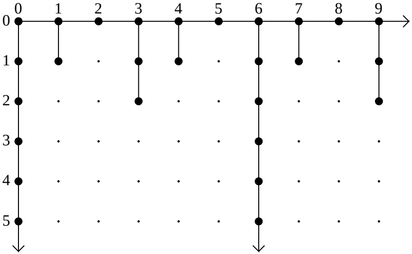
THEOREM: $\sum_h w_h = (p-2) \deg(\mathcal{S}_p)$.

Euler characteristic of the period p curve $\overline{\mathcal{S}}_p$

- Period 1: $\chi = +2$
- Period 2: $\chi = +2$
- Period 3: $\chi = 0$
- Period 4: $\chi = -28$
- Period 5: $\chi = -184$ (by Laura De Marco)
- Period 6: $\chi = -784$
- Period 7: $\chi = -3236$
- Period 8: $\chi = -11848$
- Period 9: $\chi = -42744$
- Period 10: $\chi = -147948$
- Period 11: $\chi = -505876$
- Period 12: $\chi = -1694848$
- Period 13: $\chi = -5630092$
- Period 14: $\chi = -18491088$
- Period 15: $\chi = -60318292$
- Period 16: $\chi = -195372312$
- Period 17: $\chi = -629500300$
- Period 18: $\chi = -2018178780$

Step 3. The Branner-Hubbard Puzzle. Let $a_j = F^{\circ j}(a)$





Step 4. The Kiwi Puzzle: Non-Archimedian Dynamics

Instead of working over the field of complex numbers, work with a different algebraically closed field which is complete with a well behaved norm:

Let $\mathbb{S}$ be the field of all formal infinite series

$$\mathbf{z} \;=\; c_0\xi^{r_0} + c_1\xi^{r_1} + \cdots .$$

Here each c_j is to be an algebraic number, $c_j \in \overline{\mathbb{Q}}$,
while $r_0 < r_1 < \cdots$ are rational numbers tending to $+\infty$,
and ξ is a formal indeterminate.

If $c_0 \neq 0$, then the ***norm*** is defined by $\|\mathbf{z}\| = \exp(-r_0) > 0$ (or $\log\|\mathbf{z}\| = -r_0$); while $\|\mathbf{0}\| = 0$.

This norm is multiplicative $\|\mathbf{z}_1\mathbf{z}_2\| = \|\mathbf{z}_1\|\,\|\mathbf{z}_2\}$,

and ***non-archimedian***:

$$\|\mathbf{z}_1 + \mathbf{z}_2\| \;\leq\; \max\left\{\|\mathbf{z}_1\|,\; \|\mathbf{z}_2\|\right\| .$$

Application to an escape region E_h.

Replace the marked critical point $a \in \mathbb{C}$ by a constant element $\mathbf{a} = \xi^{-1} \in \mathbb{S}$, with $\log \|\mathbf{a}\| = +1$.

Define $\mathbf{F_v} : \mathbb{S} \to \mathbb{S}$ by

$$\mathbf{F}(\mathbf{z}) \;=\; (\mathbf{z} - \mathbf{a})^2\,(\mathbf{z} + 2\mathbf{a}) \;+\; \mathbf{v}\,.$$

Here $\mathbf{v} \in \mathbb{S}$ must be chosen so that the orbit

$$\mathbf{F} : \mathbf{a} = \mathbf{a}_0 \mapsto \mathbf{a}_1 \mapsto \mathbf{a}_2 \mapsto \cdots$$

has period p. (For each such choice the series are locally convergent, and determine a parametrization of a neighborhood of some $\infty_j \in \overline{\mathcal{S}}_p$.)

Define the Green's function $G : \mathbb{S} \to [0, \infty)$ by

$$G(\mathbf{z}) \;=\; \lim_{n\to\infty} \frac{1}{3^n} \log^+ \left\|\mathbf{F}^{\circ n}(\mathbf{z})\right\|\,.$$

Then, as usual, $\quad G\big(\mathbf{F}(\mathbf{z})\big) \;=\; 3\,G(\mathbf{z}).$

Construction of the puzzle

For the “escaping” critical point $-\mathbf{a}$, it is easy to check that $G(-\mathbf{a}) = +1$.)

The locus $\{\mathbf{z} \in \mathbb{S} \;;\; G(\mathbf{z}) < 1/3^{n-1}\}$ is a union of finitely many round balls, which are called ***puzzle pieces*** of level n.

The region $G(\mathbf{z}) > 1/3^n$ in each such puzzle piece is a round ***annulus*** $\mathbb{A}$ of the form

$$\mathbb{A} = \{\mathbf{z} \,;\, R_1 < \|\mathbf{z} - \widehat{\mathbf{z}}\| < R_2\}, \text{ with } \textbf{modulus } \log(R_2/R_1)\,.$$

Such an annulus of level n surrounding the point $\widehat{\mathbf{z}} = \mathbf{a}_j$ will be denoted by $\mathbb{A}_{nj}$.

There is an associated marked grid, where the grid at level n for a_j is ***marked*** if and only if $\mathbb{A}_{nj} = \mathbb{A}_{n0}$.

Kiwi's Theorem

Let $\mathbf{F} : \mathbb{S} \to \mathbb{S}$ be the formal map corresponding to an escape region $E_h \subset \mathcal{S}_p$.

THEOREM: The Kiwi marked grid for $\mathbf{F}$ is identical to the Branner-Hubbard marked grid for any classical map $F \in E_h$.

Furthermore, this marked grid determines and is determined by the sequence of norms $\|\mathbf{a} - \mathbf{a}_j\|$.

In particular,

$$\log \|\mathbf{a} - \mathbf{a}_j\| \;=\; 3 \;-\; \sum_{\mathbb{A}_{n\,j} = \mathbb{A}_{n\,0}} \mathrm{mod}(\mathbb{A}_{n\,0}) \;\leq\; 1\,.$$

Here $\log \|\mathbf{a} - \mathbf{a}_j\| \;=\; r$ if and only if, for $F = F_{a,\,v} \in E_h$,

$$a \,-\, a_j \;\sim\; c\, a^r \qquad \text{as} \quad |a| \to \infty\,,$$

where $c \neq 0$ is some constant in $\overline{\mathbb{Q}} \subset \mathbb{C}$.

Step 5. Local Computation

We must compute the winding number w_h in terms of the asymptotic behavior of the critical orbit $\{a_j\}$.

The key step in the proof is to estimate the derivative $\frac{da}{dt}$ which relates the 1-form dt to the critical orbit.

Passing to formal power series, a non-trivial computation shows that

$$\left\|\frac{d\mathbf{a}}{d\mathbf{t}}\right\| = \prod_{j=1}^{p-1} \|\mathbf{a}(\mathbf{a}-\mathbf{a}_j)\| .$$

The rest of the argument is just elementary calculus, using the fact that

$$\|\mathbf{t}\| = \|\text{local uniformizing parameter}\|^{w_j} = \|\mathbf{a}\|^{-w_j/\mu_j} .$$

The local formula:

LEMMA. For each escape region E_h, with multiplicity μ_h, the winding number can be expressed as a sum

$$w_h = (p-2)\mu_h + \sum_{j=1}^{p-1} \mu_h \log \|a - a_j\| .$$

Hence

$$\sum_{h=1}^{N_p} w_h = (p-2)\mathrm{deg}(\mathcal{S}_p) + \sum_h \sum_j \left(\mu_h \log \|a - a_j\|\right) .$$

LEMMA: $\sum_h \left(\mu_h \log \|a - a_j\|\right) = 0$ for each j.

This will complete the proof that

$$\sum w_h = (p-2)\mathrm{deg}(\mathcal{S}_p) .$$

Step 6. The Global Identity

For any complex number a_0, consider the intersection of the line $\{(a,\ v)\ ;\ \ a = a_0\}$ with the affine curve $\mathcal{S}_p$.
Generically, the number of intersection points is equal to $\deg(\mathcal{S}_p)$. Number them as $(a_0,\ v_k)$ with $1 \le k \le \deg(\mathcal{S}_p)$.
Each such pair determines a map $F_{a_0,\,v_k} : \mathbb{C} \to \mathbb{C}$.
Let $a_0 \mapsto a_{1\,k} \mapsto a_{2\,k} \mapsto \cdots$ be the period p critical orbit under $F_{a_0,\,v_k}$.

LEMMA. For each fixed $0 < j < p$, the product

$$\prod_k (a_0 - a_{j\,k})$$

is a non-zero complex constant, independent of a_0.

Proof. This product is holomorphic as a function of a_0, with no zeros or poles. □

(Perhaps this product always equals $+1$??)

Now assume that $|a_0|$ is large.

Then the line $\{(a,\ v)\ ;\ \ a=a_0\}$ intersects each escape region E_h exactly μ_h times.
Number the intersection points as $(a_0,\ v_{h,\ell})$ with $1\le\ell\le\mu_h$, and number the corresponding orbit points as $a_{j\,h\,\ell}$. Then

$$\prod_{h=1}^{N_p}\prod_{\ell=1}^{\mu_h}\big(a-a_{j\,h\,\ell}\big)\ =\ \text{constant}\neq 0\,.$$

Now pass to formal power series.The norm $\|\mathbf{a}-\mathbf{a}_{j\,h\,\ell}\|$ is independent of ℓ, so we can write simply

$$\prod_h\|\mathbf{a}-\mathbf{a}_{j\,h}\|^{\mu_h}\ =\ 1\,.$$

Hence

$$\sum_h\mu_h\log\|\mathbf{a}-\mathbf{a}_{j\,h}\|\ =\ 0\qquad\text{for each }\ j\,.$$

This completes the proof! □

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  Part I, in "Complex Dynamics Families and Friends", ed. D. Schleicher, A. K. Peters 2009, pp. 333-411.

  Part II: Escape Regions (with Bonifant and Kiwi), Conformal Geometry and Dynamics **14** (2010) 68-112 and 190-193.

  Part III: External rays (with Bonifant), in preparation.