Spheres

John Milnor

Institute for Mathematical Sciences

Stony Brook University (www.math.sunysb.edu)

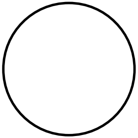
ABEL LECTURE

Oslo, May 25, 2011

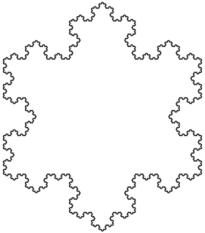
Examples of Spheres:

The **standard sphere** $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the locus

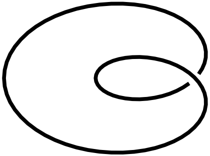
$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1 .$$



The standard 1-sphere $\mathbb{S}^1$.

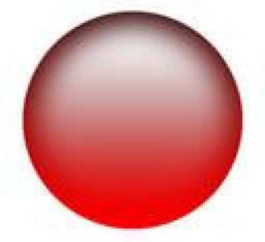


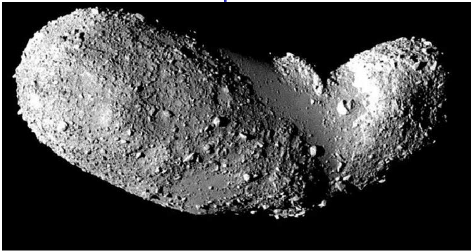
A **topological** 1-sphere.



A **smooth** 1-sphere.

Standard, Topological, and Smooth 2-spheres

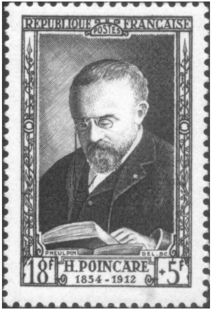




Asteroid Itokawa,
Japan Aerospace Agency

Dancing Bear by Anita Issaluk,
Chesterfield Inlet, Nunavut

Topological Characterization of Spheres



Poincaré’s Question in 1904

(Oevre VI, p.498):

“Est-il possible que le groupe fondemental de V se réduise à la substitution identique, et que pourtant V ne soit pas simplement connexe?”

It took 100 years to find the answer:

Theorem GPH. *A closed n-dimensional manifold M^n is homeomorphic to $\mathbb{S}^n$ $\iff$ it has the same homotopy type as $\mathbb{S}^n$*

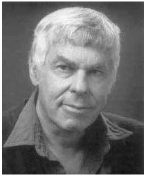
$\iff$ any proper subset can be shrunk to a point within M^n.

This is a compilation of work by many different people over 150 years!

For dimensions $n \leq 2$ it is classical.

(Compare: Francis and Weeks, 1999.)

High Dimensional Cases.



Steve Smale made the first breakthrough in 1961, giving a proof for **smooth** n-manifolds with $n > 4$.

John Stallings and E. C. Zeeman, using a different method, proved this for **Piecewise Linear** manifolds with $n > 4$.

Max Newman and E. H. Connell modified the Stallings argument to cover all **topological** manifolds of dimension $n > 4$.

The case $n = 4$ is much harder.

Mike Freedman proved the 4-dimensional theorem in 1982, using wildly non-differentiable methods.

In fact, he classified all possible **closed, oriented, simply-connected topological 4-manifolds**, using just two invariants:

- the quadratic form $x \mapsto x \cup x$, where

$$x \in H^2(M^4) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}, \qquad x \cup x \in H^4(M^4) \cong \mathbb{Z},$$

- and an invariant in $\mathbb{Z}/2$ which is zero when M^4 is smooth.

(**Note:** *I will always use homology or cohomology with* **integer coefficients**.)

The hardest case: $n = 3$

Bill Thurston's **Geometrization Conjecture** suggested an effective description of all possible closed 3-manifolds.

Richard Hamilton introduced the **Ricci flow** method in an attempt to prove the Geometrization Conjecture.

Grisha Perelman managed to overcome all of the many difficulties with this method !

QED for Theorem GPH.

Smooth Spheres

Suppose we translate Poincaré's question somewhat differently:

Consider a **smooth** manifold M^n, and ask whether it is **diffeomorphic** to the standard sphere $\mathbb{S}^n$.

We might try to use the following:

Lemma. *Any homeomorphism* $f : M^n \to \mathbb{S}^n$ *can be uniformly approximated by a smooth map* $M^n \to \mathbb{S}^n$.

Question: Can a homeomorphism between smooth manifolds always be approximated by a diffeomorphism?

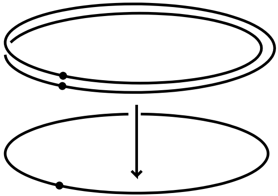
The answer is **No** !

Sphere Bundles over Spheres

In the middle 1950s, I was completely stunned by an apparent contradiction in mathematics.

Consider 3-sphere bundles over the 4-sphere:

$$\begin{array}{ccc} \mathbb{S}^3 & \subset & M^7 \\ & & \downarrow \\ & & \mathbb{S}^4 . \end{array}$$



I found examples where M^7 was a sphere by a topological argument; but couldn't be by a differentiable argument.

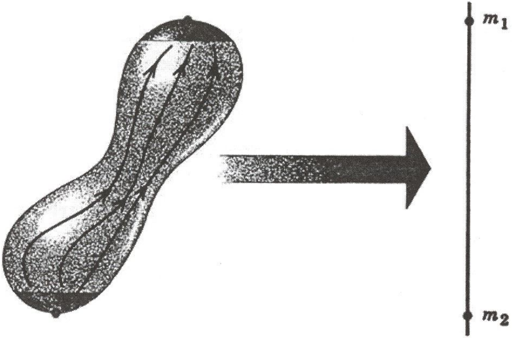
The only way out of this apparent contradiction was to assume that M^7 was homeomorphic to $\mathbb{S}^7$, but not diffeomorphic to $\mathbb{S}^7$.

To understand such examples, we need methods for

proving homeomorphism;

and for **disproving diffeomorphism**.

Proving Homeomorphism: George Reeb's Criterion



Theorem: *Let M^n be a smooth closed manifold. If there is a Morse function $M^n \to \mathbb{R}$ with only two critical points, then M is a topological n-sphere.*

We want to prove that certain $\mathbb{S}^3$-bundles over $\mathbb{S}^4$ are not diffeomorphic to $\mathbb{S}^7$.
The proof will be based on a linear equation

$$45\ \sigma(M^8) = 7\, p_2\langle M^8\rangle - p_1^2\langle M^8\rangle\,.$$

relating three different integer invariants for a **smooth closed oriented 8-manifold**.

I Must Answer Three Questions:

- What are these three invariants?
- How does one prove such a relation between them?
- What does this have to do with 7-dimensional manifolds?

The Signature $\sigma(M^{4k})$.

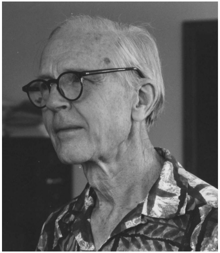
For any closed oriented $4k$-dimensional manifold we can form the **signature** $\sigma(M^{4k})$ of the quadratic form

$$x \mapsto x^2 = x \cup x \quad \text{from} \quad H^{2k}(M^{4k})/(\text{torsion}) \quad \text{to} \quad H^{4k}(M^{4k}) \xrightarrow{\cong} \mathbb{Z}\,.$$

Simply diagonalize this form over the real numbers, and count the number of positive diagonal entries minus the number of negative ones.

This is an integer valued topological invariant.

The definition of the **Pontrjagin numbers** $p_2\langle M^8\rangle$ and $p_1^2\langle M^8\rangle$ is more complicated, and requires several steps.



Hassler Whitney showed that any smooth M^n has an essentially unique embedding $M^n \xrightarrow{\subset} \mathbb{R}^L$ provided that the dimension L is large enough ($L > 2n+1$).

Hermann Grassmann studied the manifold $G_n(\mathbb{R}^L)$ consisting of all n-dimensional planes through the origin in $\mathbb{R}^L$.

Let $\mathbf{G}_n$ be the limit as $L \to \infty$,

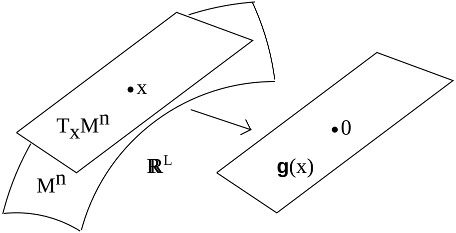
$$G_n(\mathbb{R}^{n+1}) \subset G_n(\mathbb{R}^{n+2}) \subset \cdots \subset \mathbf{G}_n .$$

The (Generalized) Gauss Map

For a smooth manifold $M^n \subset \mathbb{R}^L$, the **"Gauss map"**

$$\mathbf{g} = \mathbf{g}_{M^n} : M^n \to G_n(\mathbb{R}^L) \subset \mathbf{G}_n$$

sends each $x \in M^n$ to the tangent n-plane $T_x M^n$, translated to the origin.



The Characteristic Homology Class

Every closed oriented M^n has a **fundamental homology class**

$$\mu \in H_n(M^n).$$

For any smooth $M^n \subset \mathbb{R}^{n+L}$, the Gauss map $\mathbf{g} : M^n \to \mathbf{G}_n$ induces a homomorphism

$$\mathbf{g}_* : H_n(M^n) \to H_n(\mathbf{G}_n).$$

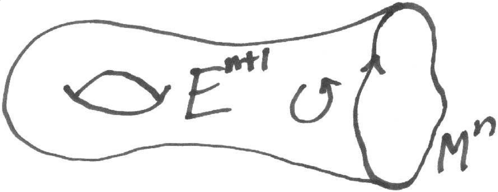
If M^n is closed and oriented, then the fundamental homology class $\mu \in H_n(M^n)$ is defined,

and maps to a "**characteristic homology class**"

$$\langle M^n \rangle = \mathbf{g}_*(\mu) \in H_n(\mathbf{G}_n).$$

When is a manifold a boundary?

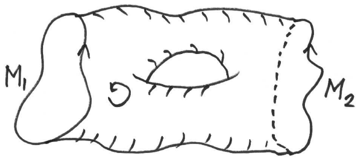
René Thom's Question: Given a smooth oriented closed manifold M^n, when does there exist a smooth oriented compact manifold-with-boundary E^{n+1} such that $\partial E^{n+1} = M^n$?



Theorem. M^n is a boundary if and only if its characteristic homology class $\langle M^n \rangle \in H_n(\mathbf{G}_n)$ is zero.

(Proved by Thom up to elements of finite order. C. T. C. Wall took care of 2-primary elements; Sergei Novikov and I took care of elements of odd order.)

The Cobordism Group Ω_n



Two closed oriented n-manifolds are **oriented cobordant** if their disjoint union, suitably oriented, is the boundary of a compact oriented $(n+1)$-manifold.

The set of all cobordism classes of smooth oriented closed n-manifolds forms an **abelian group** Ω_n,

with the disjoint union as sum operation.

Corollary. The correspondence

$$\text{(cobordism class of } M^n) \quad \mapsto \quad \langle M^n \rangle \in H_n(\mathbf{G}_n)$$

embeds Ω_n as a subgroup $\Omega_n \xrightarrow{\subset} H_n(\mathbf{G}_n)$ of finite index.

Pontrjagin Numbers

Lev Pontrjagin had introduced what we would now describe as cohomology classes

$$p_i \in H^{4i}(\mathbf{G}_n)\,.$$

Modulo elements of finite order, these generate the cohomology ring $H^*(\mathbf{G}_n)$.

In particular, the cohomology group $H^8(\mathbf{G}_8)$ is generated by the two elements p_2 and $p_1^2 = p_1 \cup p_1$ (together with elements of finite order).

For any smooth oriented closed manifold M^8, we can evaluate these two cohomology classes on the characteristic homology class $\langle M^8\rangle \in H_8(\mathbf{G}_8)$.
This yields the two **Pontrjagin numbers**

$$p_2\langle M^8\rangle\,,\ \ p_1^2\langle M^8\rangle \ \in\ \mathbb{Z}\,.$$

The Signature Formula

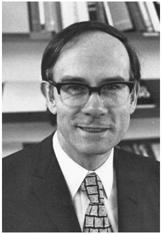
Lemma (Thom). If M^{4k} is a closed, smooth, oriented $4k$-manifold, then the signature $\sigma(M^{4k})$ is a cobordism invariant; yielding a homomorphism

$$\sigma : \Omega_{4k} \to \mathbb{Z}\,.$$

Corollary. The signature of M^{4k} can be expressed as a linear combination of Pontrjagin numbers, with **rational coefficients**,

$$\sigma(M^{4k}) = \sum a(i_1, \ldots, i_h)\, p_{i_1} \cdots p_{i_h} \langle M^{4k} \rangle\,,$$

to be summed over all $0 < i_1 \leq i_2 \leq \cdots \leq i_h$ with sum k.



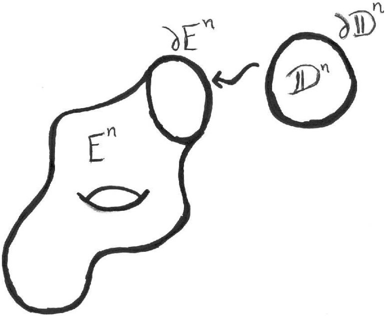
Hirzebruch computed these rational coefficients in terms of

Bernoulli numbers

Let E^n be a smooth compact n-manifold, bounded by a smooth manifold homeomorphic to $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$.

Choosing a homeomorphism $f : \mathbb{S}^{n-1} \to \partial E^n$, we can paste $\mathbb{D}^n$ onto E^n to obtain a closed **topological manifold**

$M^n = E^n \cup_f \mathbb{D}^n$.



If f is a **diffeomorphism**, then

$M^n = E^n \cup_f \mathbb{D}^n$ can be made into a **smooth manifold.**

The Obstruction to Smoothness.

Now consider the case $n = 8$.

The signature of $M^8 = E^8 \cup_f \mathbb{D}^8$ can be computed from the cohomology of the pair $(E^8, \partial E^8)$.

Similarly, the Pontrjagin number $p_1^2\langle M^8\rangle$ can be computed from knowledge of E^8 as a smooth manifold.

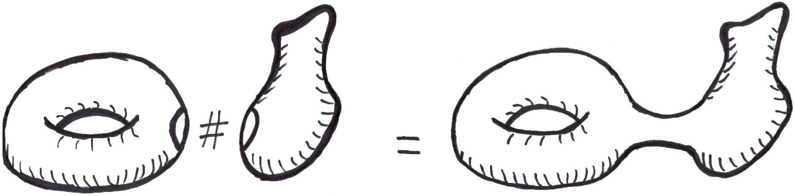
We can then solve for

$$p_2\langle M^8\rangle = \frac{45\,\sigma(M^8) + p_1^2\langle M^8\rangle}{7}.$$

Whenever this quotient is not an integer, we have proved that ∂E^8 cannot be diffeomorphic to $\mathbb{S}^7$.

Higher Dimensions: The Connected Sum

If M_1 and M_2 are smooth, oriented, connected n-manifolds, then the **connected sum** $M_1 \# M_2$ is a new smooth, oriented, connected n-manifold.



This operation is well defined up to orientation preserving diffeomorphism. Thus we obtain a commutative, associative semigroup $\mathscr{M}_n$ of oriented diffeomorphism classes; with the class of $\mathbb{S}^n$ as identity element, $M^n \# \mathbb{S}^n \cong M^n$.

Invertibility: Is $M^n \# N^n \cong \mathbb{S}^n$ for some N^n?

Barry Mazur:

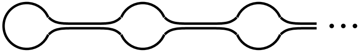
(1) M^n is invertible

$\Leftrightarrow$ **(2)** $M^n \setminus \{\text{point}\} \cong \mathbb{R}^n$

$\Rightarrow$ (**3**) M^n is a topological sphere.

Proof that **(1)** $\Longrightarrow$ **(2)**,
using "**infinite connected sums**".

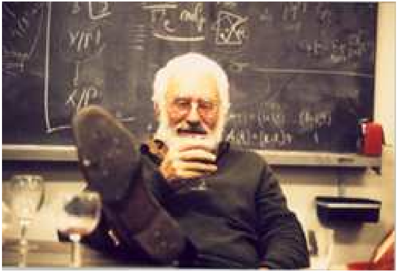
First consider the sum $\mathbb{S}^n \# \mathbb{S}^n \# \mathbb{S}^n \# \cdots \cong \mathbb{R}^n$



$$\begin{aligned} (M\#N)\#(M\#N)\#\cdots &\cong \mathbb{S}^n\#\mathbb{S}^n\#\cdots \cong \mathbb{R}^n \\ \cong M\#(N\#M)\#(N\#M)\#\cdots &\cong M\#\mathbb{R}^n \cong M\setminus\{\text{point}\}\,. \end{aligned}$$

Thus **(1)** $\Longrightarrow$ **(2)**. The proof that (**2**) $\Longrightarrow$ (**1**) is not difficult.
Since (**2**) $\Longrightarrow$ (**3**), this proves the Lemma.

Work with Michel Kervaire: the semigroup $\mathscr{S}_n$



The oriented diffeomorphism classes of smooth manifolds **homeomorphic** to $\mathbb{S}^n$ form a sub-semigroup $\mathscr{S}_n \subset \mathscr{M}_n$.

For example

$$\mathscr{S}_1 = \mathscr{S}_2 = \mathscr{S}_3 = 0\,.$$

Theorem. This semigroup $\mathscr{S}_n$ is a **finite abelian group** for $n > 4$, with

$$\mathscr{S}_5 = \mathscr{S}_6 = 0\,,$$

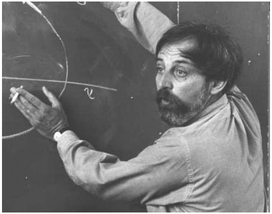
but:

$\mathscr{S}_7$	$\mathscr{S}_8$	$\mathscr{S}_9$	$\mathscr{S}_{10}$	$\mathscr{S}_{11}$	$\mathscr{S}_{12}$	$\mathscr{S}_{13}$	$\cdots$
$\mathbb{Z}/28$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/6$	$\mathbb{Z}/992$	0	$\mathbb{Z}/3$	$\cdots$

Three Necessary Ingredients for our Work

Witold Hurewicz introduced higher homotopy groups.

Jean-Pierre Serre developed the algebraic machinery needed to compute these groups

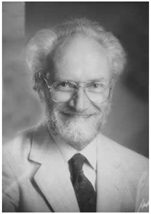


Raoul Bott computed the homotopy groups of the infinite rotation group **SO**.

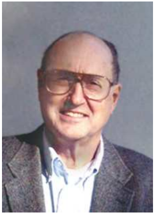
Further Developments by Many People.

Frank Adams

Greg Brumfiel

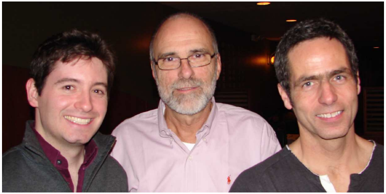


Bill Browder



Mark Mahowald

and for the latest developments:



Mike Hill, Doug Ravenel and Mike Hopkins

The group $\mathscr{S}_n$ is now completely known for $n \leq 64$,

EXCEPT FOR THE CASE $n = 4$!

Simon Donaldson: If M^4 is smooth, simply-connected, with positive definite quadratic form, then its quadratic form

$$\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \quad \mapsto \quad \mathbb{Z}$$

can be diagonalized $\quad \Longrightarrow$
M^4 is homeomorphic to a connected sum

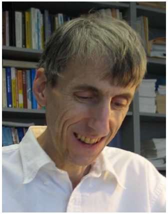
$$\mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2 .$$

But there are **many** unimodular quadratic forms which cannot be diagonalized; **hence there are many topological 4-manifolds which cannot be given any differentiable structure.**

The combination of Donaldson's methods and Freedman's methods had amazing consequences:

The Four Dimensional Jungle.

Robert Friedman and John Morgan: A closed topological 4-manifold can have **infinitely many** essentially distinct differentiable structures.



Cliff Taubes: The topological space $\mathbb{R}^4$ has **uncountably many** essentially distinct differentiable structures.

All other dimensions are better behaved:

For $n \neq 4$, a compact topological n-manifold can have only finitely many essentially distinct differentiable structures;

and the topological space $\mathbb{R}^n$ has a unique differentiable structure up to diffeomorphism.

(*Proved by putting together results from Moise, Stallings, Cerf, Kirby and Siebenmann, Munkres, Hirsch, Smale, Kervaire and Milnor.*)

Recall that $\mathscr{S}_n$ is a finite abelian group **for n $\neq$ 4.**

What is known about $\mathscr{S}_4$?

It is a commutative, associative semigroup with unit,

and has at most countably many elements.

What isn't known? **Everything else:**

Is $\mathscr{S}_4$ trivial?

Is it finite?

Finitely generated?

Is it a group?

? ? ? ? ?

For further information:

- J. Milnor, “Differential Topology, 46 years later”,
  Notices AMS, June-July 2011.

- J. Milnor, “Introduction to Part 1, Exotic Spheres”,
  in Collected Papers III, AMS 2007.

Both available in:
http://www.math.sunysb.edu/$\sim$jack/PREPRINTS