Practice Midterm

Spring 2019 MAT 342: Applied Complex Analysis

Instructions: Answer all questions below. You may not use books, notes, calculators, or cell phones. Write your name and student ID in each page that you hand in.

Problem 1.

(i) What does it mean for a function $f\colon S \to \mathbb{C}$ to be differentiable at a point $z_0 \in S$, where S contains a neighborhood of z_0?

(ii) Give the definition of an analytic function.

(iii) What is the largest domain in which the function $f(z) = \frac{e^z}{z^2+2i}$ is analytic and why is it analytic there?

Solution: (i) A function f defined in a set $S \subset \mathbb{C}$ is differentiable at a point $z_0 \in S$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

(ii) A function f is analytic in an open set S if it is differentiable at every point of S.

(iii) The function e^z is analytic in $\mathbb{C}$ with $(e^z)' = e^z$ and the function $z^2 + 2i$ is analytic in $\mathbb{C}$, since it is a polynomial. By the quotient rule, the function $f(z) = \frac{e^z}{z^2+2i}$ is analytic everywhere except at the points where the denominator is zero. We have $z^2 + 2i = 0$ if $z^2 = -2i = 2e^{i3\pi/2}$, so $z = \sqrt{2}e^{i3\pi/4}$ or $z = \sqrt{2}e^{i(3\pi/4+\pi)} = -\sqrt{2}e^{i3\pi/4}$. Therefore, f is analytic everywhere in $\mathbb{C}$, except at the points $\sqrt{2}e^{i3\pi/4}$ and $-\sqrt{2}e^{i3\pi/4}$.

Problem 2.

(i) State the Cauchy-Riemann equations for a function $f(z) = u(z)+iv(z)$.

(ii) Using the Cauchy-Riemann equations, examine whether the function $f(z) = \overline{z}^2$ is differentiable at any point z, in which case, compute $f'(z)$. Is f analytic at any point of $\mathbb{C}$?

Justify carefully all your claims.

Solution: (i) If $z = (x, y)$, then the Cauchy-Riemann equations for f are:

$$u_x(x, y) = v_y(x, y)$$
$$u_y(x, y) = -v_x(x, y).$$

(ii) For $z = x + iy$ we have $f(z) = \overline{z}^2 = (x - iy)^2 = x^2 - y^2 - 2ixy$. Therefore $u(x, y) = x^2 - y^2$ and $v(x, y) = -2xy$. We will find the points at which the Cauchy-Riemann equations hold. We have $u_x(x, y) = 2x$, $u_y(x, y) = -2y$, $v_x(x, y) = -2y$, and $v_y(x, y) = -2x$. If the Cauchy-Riemann equations hold, then we have $2x = -2x$ and $-2y = 2y$, so $x = y = 0$. Hence, the point $z = 0$ is the only point at which the Cauchy-Riemann equations hold. Since the partial derivatives of u and v exist and are continuous in a neighborhood of 0, we conclude that $f'(0)$ exists, and $f'(0) = u_x(0, 0) + iv_x(0, 0) = 0$.

The function f is not analytic at 0 since it is only differentiable at 0 but it is not differentiable in a neighborhood of 0. Moreover, f is not analytic at any other point of $\mathbb{C}$, since it is not differentiable at these points.

Problem 3.

(i) State the coincidence principle.

(ii) Show that the only entire function $g\colon \mathbb{C} \to \mathbb{C}$ satisfying the equation

$$\sin^2 z + g(z) = 1$$

for all $z \in \mathbb{C}$ is the function $g(z) = \cos^2 z$.

Solution: (i) Let f, g be functions that are analytic in a domain D such that $f(z) = g(z)$ for all points z lying in a line segment I contained in D. Then $f(z) = g(z)$ for all $z \in D$.

(ii) Note that $\sin^2 x + \cos^2 x = 1$ for all $x \in \mathbb{R}$. Therefore, the analytic function $g(z) = 1 - \sin^2 z$ is equal to the analytic function $\cos^2 z$ when $z = x$ is a real number. In particular, the two functions are equal to each other on the line segment $I = \{x + iy : y = 0, 0 \le x \le 1\}$. By the coincidence principle we conclude that $g(z) = \cos^2 z$ for all $z \in \mathbb{C}$.

Problem 4. Consider the branch of the logarithm defined by $\log z = \ln r + i\theta$, where $z = re^{i\theta}$, $r > 0$, and $7\pi/3 < \theta < 13\pi/3$. Using that branch, write the following numbers in (x, y)-coordinates:

(i) $(-2)^i$ (ii) i^i.

Is it true in general that $z^i \cdot w^i = (z \cdot w)^i$?

Solution: (i) Note that $-2 = 2e^{i\pi} = 2e^{i3\pi}$. Since $7\pi/3 < 3\pi < 13\pi/3$, we have $\log(-2) = \ln 2 + 3\pi i$, so

$$(-2)^i = e^{i\log(-2)} = e^{i(\ln 2 + 3\pi i)} = e^{i\ln 2}e^{-3\pi} = e^{-3\pi}\cos(\ln 2) + ie^{-3\pi}\sin(\ln 2).$$

(ii) We have $i = e^{i\pi/2} = e^{i5\pi/2}$. Since $7\pi/3 < 5\pi/2 < 13\pi/3$, we have $\log(i) = i5\pi/2$. Hence,

$$i^i = e^{i\log i} = e^{i\cdot i5\pi/2} = e^{-5\pi/2}.$$

Note, however, that $(-2i)^i \neq (-2)^i \cdot i^i$. Indeed, we have $-2i = 2e^{i3\pi/2} = 2e^{i7\pi/2}$, so $\log(-2i) = \ln 2 + i7\pi/2$. We have

$$(-2i)^i = e^{i\log(-2i)} = e^{i\ln 2}e^{-7\pi/2} \text{ and}$$
$$(-2)^i \cdot i^i = e^{i\ln 2}e^{-11\pi/2},$$

which are not equal to each other.

Problem 5.

(i) Give the domain and the formula (in polar coordinates) of the principal branch of z^{-1-2i}.

(ii) Let $f(z)$ be the function in part (i) and C be the contour $z = e^{i\theta}$, $-\pi \leq \theta \leq \pi$. Compute
$$\int_C f(z)dz.$$

Solution: (i) $z^{-1-2i} = e^{(-1-2i)\operatorname{Log} z}$, where $\operatorname{Log} z = \ln r + i\theta$ and $z = re^{i\theta}$, $r > 0$, $-\pi < \theta < \pi$.

(ii) We have $z(t) = e^{it}$ and $z'(t) = ie^{it}$, $-\pi \leq t \leq \pi$. For $-\pi < t < \pi$ we have $f(e^{it}) = e^{(-1-2i)\cdot it}$, so

$$\int_C f(z)dz = \int_{-\pi}^{\pi} f(z(t))z'(t)dt = \int_{-\pi}^{\pi} f(e^{it})ie^{it}dt = \int_{-\pi}^{\pi} e^{(-1-2i)\cdot it} \cdot ie^{it}dt$$
$$= i\int_{-\pi}^{\pi} e^{2t}dt = i\frac{e^{2t}}{2}\bigg|_{-\pi}^{\pi} = i\frac{e^{2\pi} - e^{-2\pi}}{2}.$$

Problem 6. Consider the set $S = \{z : \operatorname{Re}(z^2) < 0\}$.

(i) Is the set S open, closed, or neither?

(ii) Is S connected? Justify your answer.

(iii) Find the image of the set S under the principal branch of the logarithm.

Solution: (i) Note that S is the set of points (x, y) such that $-y < x < y$. The boundary of the set S is the union of the lines $y = x$ and $y = -x$. No point of these lines is contained in S. Since S does not contain any of its boundary points, it is open and not closed.

(ii) The set S is not connected. The reason is that any polygonal path that connects a point $z_1 \in S$ with $\operatorname{Im}(z_1) > 0$ to a point $z_2 \in S$ with $\operatorname{Im}(z_2) < 0$ has to intersect the boundary of S and cannot be entirely contained in S.

(iii) Note that S consists of rays of the form $re^{i\theta}$, $r > 0$, $\pi/4 < \theta < 3\pi/4$, or $-3\pi/4 < \theta < -\pi/4$. The principal branch of the logarithm is defined by $\operatorname{Log} z = \ln r + i\theta$, where $r > 0$ and $-\pi < \theta < \pi$. For fixed θ, each ray $re^{i\theta}$, $r > 0$, is mapped to $\ln r + i\theta$, which represents a horizontal line passing through $i\theta = (0, \theta)$. Taking into account all admissible angles θ, we see that the image of S is the union of two infinite horizontal strips: $\mathbb{R} \times (-3\pi/4, -\pi/4)$ and $\mathbb{R} \times (\pi/4, 3\pi/4)$.

Problem 7.

(i) Show that the function $\text{Re}(e^{z^2+1})$ is harmonic on $\mathbb{C}$.

(ii) If $f(z)$ is analytic in a domain D, is it true that $\overline{f(z)}$ is also analytic in that domain? If yes, then provide a proof. If no, then give an example that justifies your claim.

Solution: (i) The function e^{z^2+1} is analytic in $\mathbb{C}$, since it is the composition of two analytic functions. The function $\text{Re}(e^{z^2+1})$ is the real part of an analytic function, so it is harmonic.

(ii) The statement is false. The function $f(z) = z^2$ is analytic in $\mathbb{C}$. However, the function $f(z) = \overline{z}^2$ is not analytic anywhere; see problem 2 for the justification.