Practice Midterm

Spring 2019 MAT 342: Applied Complex Analysis

Instructions: Answer all questions below. You may not use books, notes, calculators, or cell phones. Write your name and student ID in each page that you hand in.

Problem 1.

(i) What does it mean for a function $f\colon S \to \mathbb{C}$ to be differentiable at a point $z_0 \in S$, where S contains a neighborhood of z_0?

(ii) Give the definition of an analytic function.

(iii) What is the largest domain in which the function $f(z) = \frac{e^z}{z^2+2i}$ is analytic and why is it analytic there?

Problem 2.

(i) State the Cauchy-Riemann equations for a function $f(z) = u(z) + iv(z)$.

(ii) Using the Cauchy-Riemann equations, examine whether the function $f(z) = \overline{z}^2$ is differentiable at any point z, in which case, compute $f'(z)$. Is f analytic at any point of $\mathbb{C}$?

Justify carefully all your claims.

Problem 3.

(i) State the coincidence principle.

(ii) Show that the only entire function $g\colon \mathbb{C} \to \mathbb{C}$ satisfying the equation

$$\sin^2 z + g(z) = 1$$

for all $z \in \mathbb{C}$ is the function $g(z) = \cos^2 z$.

Problem 4. Consider the branch of the logarithm defined by $\log z = \ln r + i\theta$, where $z = re^{i\theta}$, $r > 0$, and $7\pi/3 < \theta < 13\pi/3$. Using that branch, write the following numbers in (x, y)-coordinates:

(i) $(-2)^i$ (ii) i^i.

Is it true in general that $z^i \cdot w^i = (z \cdot w)^i$?

Problem 5.

(i) Give the domain and the formula (in polar coordinates) of the principal branch of z^{-1-2i}.

(ii) Let $f(z)$ be the function in part (i) and C be the contour $z = e^{i\theta}$, $-\pi \leq \theta \leq \pi$. Compute

$$\int_C f(z)dz.$$

Problem 6. Consider the set $S = \{z : \mathrm{Re}(z^2) < 0\}$.

(i) Is the set S open, closed, or neither?

(ii) Is S connected? Justify your answer.

(iii) Find the image of the set S under the principal branch of the logarithm.

Problem 7.

(i) Show that the function $\mathrm{Re}(e^{z^2+1})$ is harmonic on $\mathbb{C}$.

(ii) If $f(z)$ is analytic in a domain D, is it true that $\overline{f(z)}$ is also analytic in that domain? If yes, then provide a proof. If no, then give an example that justifies your claim.