MAT 342 Practice Final

Applied Complex Analysis

May 2019

Name: Solutions

Student ID: ______________________

Instructions:

There are 10 problems on 12 pages (plus this cover sheet) in this exam. Make sure that you have them all.

Answer all questions below. You may use the backs of pages, but in this case indicate where your work for each problem is located. You must show all of your work and provide **complete justifications** for all of your claims. Insufficient justifications will not receive full credit. Use **non-erasable pen**; do not use pencil.

Leave all answers in exact form (that is, do not approximate π, square roots, and so on.)

You may **not** use books, notes, calculators, cell phones or other electronic devices.

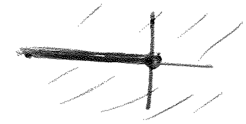
Problem	1	2	3	4	5	6	7	8	9	10	Total
Points	10	10	10	10	10	10	10	10	10	10	100
Score											

Name: ______________________ Student ID: ______________________

[10 pts] **Problem 1.**

(i) Show that the function $\ln|z|$ is harmonic everywhere except at 0.

$\ln|z|$ is the real part of $\mathrm{Log}\, z$, so it is harmonic in $\mathbb{C} \setminus \{z = x+iy : x \le 0, y = 0\}$



$\ln|z|$ is also the real part of the branch of $\log z$ defined by $\log z = \ln|z| + i\vartheta$, $z = re^{i\vartheta}$, $r > 0$, $0 < \vartheta < 2\pi$. Therefore, $\ln|z|$ is also harmonic in the domain $r > 0$, $0 < \vartheta < 2\pi$.

It follows that $\ln|z|$ is harmonic for all $z \neq 0$.

(ii) Consider a function $f(z) = u(z) + iv(z)$ defined in a domain D. If u and v are harmonic in D, is it true that f is analytic in D?

The answer is no.

The functions $u(x,y) = x$, $v(x,y) = x$ are harmonic in $\mathbb{C}$, because they satisfy Laplace's equation:

$u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.

However $f(z) = u(z) + iv(z) = x + ix$ is not analytic in $\mathbb{C}$, since u, v do not satisfy the Cauchy–Riemann equations:

$$\left.\begin{aligned} u_x &= 1 \\ v_y &= 0 \end{aligned}\right\} \; u_x \neq v_y.$$

Name: ______________________ Student ID: ______________________

[10 pts] **Problem 2.**

(i) When is an isolated singular point of an analytic function called removable?

An isolated singular point z_0 of a function f is called removable if the Laurent expansion of f in an annulus $0<|z-z_0|<R$ has the form

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

(ii) Show that the function

$$f(z) = \begin{cases} \frac{\tan z}{z}; & z \neq 0,\ z \neq \pi/2 + k\pi,\ k \in \mathbb{Z} \\ 1, & z = 0 \end{cases}$$

is analytic in its domain.

The function $\tan z = \frac{\sin z}{\cos z}$ is analytic everywhere except at the points z, where $\cos z = 0$. These are precisely the points $z = \pi/2 + k\pi$, $k \in \mathbb{Z}$.
Therefore $f(z)$ is analytic when $z \neq \pi/2 + k\pi$, $k \in \mathbb{Z}$ and when $z \neq 0$. We only have to show that f is analytic at $z=0$.
The Maclaurin expansion of $\tan z$ is

$$\tan z = \tan 0 + (\tan' 0)z + \cdots = z + \cdots, \quad |z| < r.$$

For $z \neq 0$, $0<|z|<r$, we have:

$$f(z) = \frac{\tan z}{z} = \frac{z + \cdots}{z} = 1 + \cdots$$

$$= \sum_{n=0}^{\infty} a_n z^n$$

for some a_n, $n = 0, 1, \ldots$

Consider the function $S(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < r$

This power serier is convergent when $0 < |z| < r$, since it is equal to the function $f(z)$.

When $z = 0$ we have $S(0) = 1$ (by substituting $z = 0$).

Therefore, the series $S(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for all $|z| < r$. It follows that the function $S(z)$ is analytic in $|z| < r$.

Note that $S(z) = \begin{cases} f(z), & z \neq 0 \\ 1, & z = 0 \end{cases} = f(z)$.

Therefore $S(z) = f(z)$ and f is analytic in $|z| < r$, so in particular f is analytic at $z = 0$.

Name: ______________________ **Student ID:** ______________________

[10 pts] **Problem 3.** (*No credit will be given for just stating the Laurent expansion without showing all required work. You can use known expansions such as the expansion of* $\sin z$.)

(i) What type of singularity does the function

$$f(z) = \frac{1}{z^4 \sin z}$$

have at the point 0? Find the first 3 terms in the Laurent expansion of f that is valid in some annulus $0 < |z| < R$. What is the residue of f at 0?

We have $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right)$ for $z \in \mathbb{C}$.

Hence, $f(z) = \frac{1}{z^4 \sin z} = \frac{1}{z^5} \cdot \frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots}$

for $0 < |z| < R$ (for some $R > 0$).

We perform the long division:

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots \;\Big)\; 1 \qquad \text{quotient: } 1 + \frac{z^2}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)z^4$$

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots$$

$$+\frac{z^2}{3!} - \frac{z^4}{5!} + \dots$$

$$\frac{z^2}{3!} - \frac{z^4}{(3!)^2}$$

$$\left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)z^4 + \dots$$

Therefore, $f(z) = \frac{1}{z^5}\left(1 + \frac{z^2}{6} + \frac{7z^4}{360} + \dots\right) = \frac{1}{z^5} + \frac{1}{6z^3} + \frac{7}{360z} + \dots$

$\operatorname{Res}_{z=0} f(z) = \frac{7}{360}$.

Name: ______________________ Student ID: ______________________

(ii) Suppose that a function $g(z)$ has the Laurent expansion

$$g(z) = \frac{-1}{z^2} + \frac{2i}{z} + 1 + i + z + 3iz^2 + \dots$$

in an annulus $0 < |z| < R$. Find the value of the following limit, if it exists:

$$\lim_{z \to 0} g(z).$$

In case it does not exist, explain the reason.

Note that g has a pole of order 2 at $z=0$. It follows that g can be written as $g(z) = \frac{\varphi(z)}{z^2}$, where φ is analytic at 0 and $\varphi(0) \neq 0$.

Therefore $\lim_{z\to 0} g(z) = \lim_{z \to 0} \frac{\varphi(z)}{z^2} =$ "$\frac{\varphi(0)}{0}$" $= \infty$.

<u>Alternatively</u>: Since g has a pole (of order 2) at 0, we have $\lim_{z\to 0} g(z) = \infty$.

Name: ______________________ Student ID: ______________________

[10 pts] **Problem 4.** (*No credit will be given for just stating the expansions without showing all required work. You can use known expansions such as e^z etc.*)
Consider the function

$$g(z) = \frac{1}{1+z^2}.$$

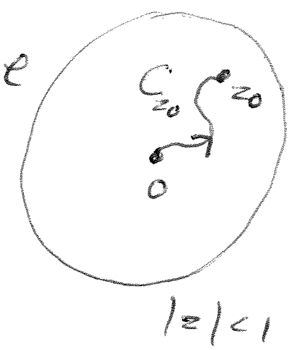
Let $G(z)$ be the antiderivative of $g(z)$ defined in the the unit disk $|z| < 1$ such that $G(0) = 0$. Find the Maclaurin expansion of $G(z)$.

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-1\cdot z^2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot z^{2n}$$

for $|z| < 1$.

Let C_{z_0} be a path in the disk $|z| < 1$ starting at 0 and ending at a point $|z_0| < 1$.

Then, since G is an antiderivative of g we have



$$G(z_0) = G(z_0) - G(0) = \int_{C_{z_0}} g(z)\, dz$$

$$= \int_{C_{z_0}} \sum_{n=0}^{\infty} (-1)^n z^{2n}\, dz = \sum_{n=0}^{\infty} (-1)^n \int_{C_{z_0}} z^{2n}\, dz$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{2n+1}$$

Name: ____________________ Student ID: ____________________

[10 pts] **Problem 5.**

(i) Let f be an analytic function in a domain D. State the Cauchy Integral Formula for f.

If C is a simple closed contour in D that surrounds a point z_0 and f is analytic in all points inside C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0}.$$

(ii) Let C be the positively oriented circle $|z| = 3$ and consider the function

$$g(z) = \int_C \frac{2w^2 - e^w}{w - z}\,dw.$$

Show that the function $g(z)$ is analytic when $|z| < 3$ and $|z| > 3$.

Consider the function $h(w) = 2w^2 - e^w$, which is entire. By the Cauchy integral formula, if $|z| < 3$, then

$$g(z) = \int_C \frac{h(w)}{w-z}\,dw = 2\pi i \cdot h(z) = 2\pi i\left(2z^2 - e^z\right).$$

Therefore g is analytic when $|z| < 3$.

If $|z| > 3$ then the function $f(w) = \dfrac{h(w)}{w-z}$ is analytic at all points in the circle of radius 3 since $w - z \neq 0$. By the Cauchy-Goursat theorem we have $g(z) = \int_C f(w)\,dw = 0$

So g is also analytic when $|z|>3$.

Name: ______________________ Student ID: ______________________

[10 pts] **Problem 6.**

(i) State Liouville's theorem.

Let f be an entire function. If $|f|$ is bounded then f is constant.

(ii) Suppose that f is an entire function and v is the imaginary part of f. If v is bounded, then show that the function f is constant.

We write $f = u + iv$.

By assumption, we have $|v| \leq M$ for some constant $M > 0$.

Consider the entire function $g(z) = e^{if(z)} =$

$= e^{i(u(z)+iv(z))} = e^{iu(z) - v(z)} = e^{iu(z)} \cdot e^{-v(z)}$

Then $|g(z)| = |e^{iu(z)}| \cdot e^{-v(z)} = e^{-v(z)} \leq e^{M}$.

Since $|g|$ is bounded, Liouville's theorem implies that g is constant. Therefore, $|g| = e^{-v}$ is constant, so v is constant.

Equivalenty, $v_x = 0$ and $v_y = 0$.

By the Cauchy-Riemann equations we have $u_x = v_y = 0$ and $u_y = -v_x = 0$.

Therefore, u is also constant.

It follows that $f = u + iv$ is constant.

Name: ______________________ Student ID: ______________________

[10 pts] **Problem 7.**

(i) Determine the number of zeros, counting multiplicities, of the polynomial

$$z^6 - 5z^4 + z^3 - 2z = 0$$

inside the circle $|z| = 1$.

Let $f(z) = -5z^4$ and note that $|f(z)| = 5$ for $|z|=1$.
Let $g(z) = z^6 + z^3 - 2z$ and note that
$|g(z)| \leq |z|^6 + |z|^3 + 2|z| = 4$ for $|z| = 1$.
Since f and g are entire and $|f(z)| > |g(z)|$ when $|z| = 1$, Rouché's Theorem implies that f and $f+g$ have the same number of zeros in the circle $|z|=1$.
Therefore the original equation has 4 solutions inside the circle $|z|=1$.

(ii) Consider the function

$$f(z) = \frac{(2z-1)^7}{z^3}$$

and denote by C the unit circle with the counter-clockwise orientation. How many times does the image of C under f wind around the origin and in what orientation?

The function f has a pole of order 3 at 0 and a zero of order 7 at $1/2$. Moreover, f is analytic and non-zero in C.
Since C surrounds $Z=7$ zeros of f and $P=3$ poles of f, by the Argument Principle we have $\frac{1}{2\pi} \Delta_C \arg f(z) = Z - P = 7 - 3 = 4$.
That is, the image of C under f winds around the origin 4 times in the counter-clockwise orientation

Name: ______________________ **Student ID:** ______________________

[10 pts] **Problem 8.** Using the residue at infinity evaluate the integral

$$\int_C \frac{5z - z^2}{(z-5)(z-1)^2} dz,$$

where C is the positively oriented circle $|z| = 6$.

$$f(z) = \frac{5z - z^2}{(z-5)(z-1)^2}$$

$$\frac{1}{z^2} f(1/z) = \frac{1}{z^2} \frac{\frac{5}{z} - \frac{1}{z^2}}{(\frac{1}{z} - 5)(\frac{1}{z} - 1)^2}$$

$$= \frac{1}{z^2} \frac{5z - 1}{(\frac{1}{z} - 5)(1-z)^2} = \frac{1}{z} \cdot \frac{5z - 1}{(1-5z)(1-z)^2}$$

$$= \frac{\varphi(z)}{z},$$ where φ is analytic at 0 and $\varphi(0) = \frac{5 \cdot 0 - 1}{(1 - 5 \cdot 0)(1-0)^2} = -1$

Hence $\frac{1}{z^2} f(1/z)$ has a simple pole at 0 with residue $\varphi(0) = -1$.

Since $f(z)$ is analytic outside the circle $|z| = 6$, we can use the residue at infinity and we have:

$$\int_C f(z)\, dz = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f(1/z)\right) = 2\pi i \cdot (-1) = -2\pi i$$

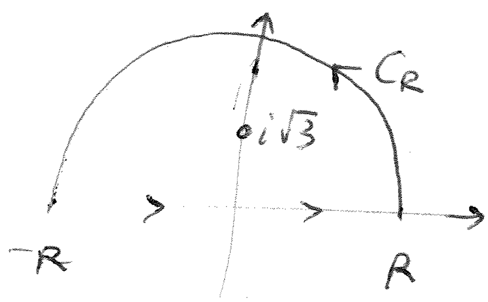
Name: ______________________ Student ID: ______________________

[10 pts] **Problem 9.** Calculate the improper integral

$$\int_0^\infty \frac{x \sin 2x}{x^2+3}\,dx.$$

Justify carefully all steps.

Define $f(z) = \dfrac{z e^{2iz}}{z^2+3}$ and consider the contour C:



The function f has two singular points: $\pm i\sqrt{3}$

Only the point $i\sqrt{3}$ is surrounded by C. We have

$C_R: |z| = R$
$R > \sqrt{3}$

$$f(z) = \frac{z e^{2iz}}{(z-i\sqrt{3})(z+i\sqrt{3})} = \frac{\varphi(z)}{z-i\sqrt{3}}, \text{ where } \varphi(z) = \frac{z e^{2iz}}{z+i\sqrt{3}}.$$

Since φ is analytic at $i\sqrt{3}$ and $\varphi(i\sqrt{3}) = \dots = \dfrac{e^{-2\sqrt{3}}}{2} \neq 0$, we conclude that f has a simple pole at $i\sqrt{3}$ with

$$\operatorname*{Res}_{z=i\sqrt{3}} f(z) = \varphi(i\sqrt{3}) = \frac{e^{-2\sqrt{3}}}{2}.$$

By the <u>Residue Theorem</u>: $\displaystyle\int_C f(z)\,dz = 2\pi i \cdot \frac{e^{-2\sqrt{3}}}{2}$

Equivalently:

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = \pi i e^{-2\sqrt{3}}$$

We take imaginary parts (since $\operatorname{Im}(e^{2ix}) = \sin 2x$):

$$\operatorname{Im} \int_{-R}^{R} f(x)\,dx + \operatorname{Im} \int_{C_R} f(z)\,dz = \pi e^{-2\sqrt{3}}$$

$$\Rightarrow \int_{-R}^{R} \frac{x \sin 2x}{x^2+3}\,dx + \operatorname{Im} \int_{C_R} f(z)\,dz = \pi e^{-2\sqrt{3}}.$$

Since $\frac{x \sin 2x}{x^2+3}$ is an even function, we have:

$$2\int_{0}^{R} \frac{x \sin 2x}{x^2+3}\,dx = \pi e^{-2\sqrt{3}} - \operatorname{Im} \int_{C_R} f(z)\,dz.$$

If we show that $\int_{C_R} f(z)\,dz \xrightarrow{R\to\infty} 0$, then

$$\boxed{\int_{0}^{\infty} \frac{x \sin 2x}{x^2+3}\,dx = \frac{\pi e^{-2\sqrt{3}}}{2}}$$

<u>Proof that</u> $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$:

$$\int_{C_R} f(z)\,dz = \int_{C_R} \frac{z}{z^2+3} \cdot e^{2iz}\,dz.$$

For $z \in C_R$ we have $\left|\frac{z}{z^2+3}\right| = \frac{R}{|z^2+3|} \leq \frac{R}{R^2-3} =: M_R$

Since $\lim_{R\to\infty} M_R = 0$, by Jordan's lemma we have $\lim_{R\to\infty} \int_{C_R} \frac{z}{z^2+3} e^{2iz}\,dz = 0$, as claimed.

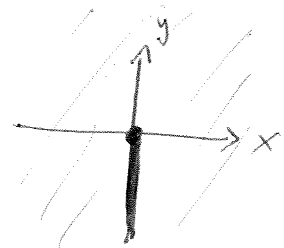
[15 pts] **Problem 10.**

[5 pts] (i) Give the formula in polar coordinates for the branch of $z^{-1/2}$ that is defined in the complement of the negative imaginary axis including the origin, so that $(-1)^{-1/2} = -i$. Using that branch, describe the largest domain in which the function

$$f(z) = \frac{z^{-1/2}}{z^2+1}$$

is analytic.

$z^{-1/2} = e^{-\frac{1}{2}\log z}$, where $\log z = \ln r + i\theta$,



$z = re^{i\theta}$, $r > 0$, $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

Then $(-1)^{-1/2} = e^{-\frac{1}{2}\log(-1)} = e^{-\frac{1}{2}(\ln 1 + i\pi)} = e^{-i\pi/2} = -i$.

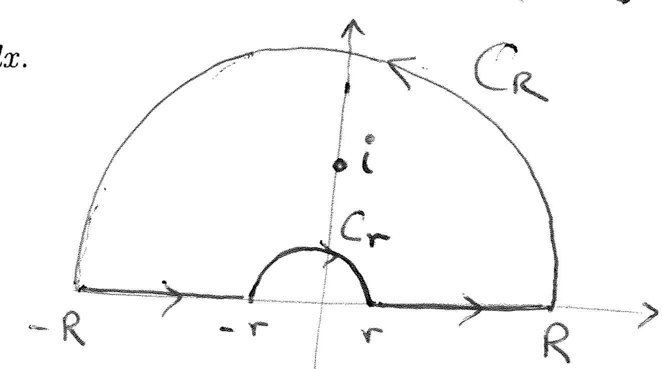
f is analytic in the domain of $z^{-1/2}$, except at the points where $z^2+1=0$. Therefore f is analytic in $\mathbb{C} \setminus \{z = x+iy : z = i$ or $x = 0$ and $y \le 0\}$

[10 pts] (ii) Calculate the improper integral

$$\int_0^\infty \frac{x^{-1/2}}{x^2+1}dx.$$

Justify carefully all steps.

Consider the contour C:



The function f has only one singular point $z = i$ that is surrounded by the contour C. We have:

$f(z) = \dfrac{z^{-1/2}}{(z-i)(z+i)} = \dfrac{\varphi(z)}{z-i}$, where $\varphi(z) = \dfrac{z^{-1/2}}{z+i}$

is analytic at i and $\varphi(i) = \dfrac{i^{-1/2}}{2i} =$

Extra page for **Problem 10** (ii).

$$= \frac{e^{-\frac{1}{2}\log i}}{2i} = \frac{e^{-\frac{1}{2}(\ln 1 + i\pi/2)}}{2i} = \frac{e^{-i\pi/4}}{2i} \neq 0$$

Therefore, f has a simple pole at $z=i$ with

$$\operatorname*{Res}_{z=i} f(z) = \varphi(i) = \frac{e^{-i\pi/4}}{2i}.$$

By the <u>Residue Theorem</u> we have:

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = 2\pi i \frac{e^{-i\pi/4}}{2i} = \pi e^{-i\pi/4}$$

$$\rightarrow \int_{-R}^{-r} f(x)\,dx + \int_r^R f(x)\,dx + \int_{C_r} f(z)\,dz + \int_{C_R} f(z)\,dz = \pi e^{-i\pi/4}.$$

- $$\int_r^R f(x)\,dx = \int_r^R \frac{x^{-1/2}}{x^2+1}\,dx$$

- $$\int_{-R}^{-r} f(x)\,dx = \int_{-R}^{-r} \frac{e^{-\frac{1}{2}\log(x)}}{x^2+1}\,dx \overset{\substack{x<0\\ \vartheta=\pi}}{=\!=} \int_{-R}^{-r} \frac{e^{-\frac{1}{2}(\ln|x| + i\pi)}}{x^2+1}\,dx =$$

$$= \int_{-R}^{-r} \frac{|x|^{-1/2} \cdot e^{-i\pi/2}}{x^2+1}\,dx \overset{u=-x}{=\!=} e^{-\frac{i\pi}{2}} \int_r^R \frac{u^{-1/2}\,du}{u^2+1}$$

If we show that $\displaystyle\int_{C_r} f(z)\,dz \xrightarrow{r\to 0} 0$ and $\displaystyle\int_{C_R} f(z)\,dz \xrightarrow{R\to\infty} 0$

then :

$$e^{-i\pi/2}\int_0^\infty \frac{x^{-1/2}\,dx}{x^2+1} + \int_0^\infty \frac{x^{-1/2}\,dx}{x^2+1} = \pi\, e^{-i\pi/4}$$

$$\Rightarrow \int_0^\infty \frac{x^{-1/2}\,dx}{x^2+1} = \frac{\pi\, e^{-i\pi/4}}{1+e^{-i\pi/2}} = \frac{\pi}{e^{i\pi/4}+e^{-i\pi/4}}$$

$$= \frac{\pi}{2\cos \pi/4} = \frac{\pi}{\sqrt{2}}$$

<u>Proof that</u> $\displaystyle\int_{C_R} f(z)\,dz \xrightarrow{R\to\infty} 0$:

$$|f(z)| = \frac{|z^{-1/2}|}{|z^2+1|} \overset{|z|=R}{=} \frac{e^{-\frac{1}{2}\ln R}\,\cancel{|e^{-\frac{1}{2}i\theta}|}^{\,1}}{|z^2+1|} \leq \frac{R^{-1/2}}{R^2-1} = M_R$$

$$\left|\int_{C_R} f(z)\,dz\right| \leq M_R \cdot \text{length}(C_R) = \frac{R^{-1/2}}{R^2-1}\cdot \pi R$$

$$= \frac{\pi R^{1/2}}{R^2-1}$$

Since $\displaystyle\lim_{R\to\infty} \frac{\pi R^{1/2}}{R^2-1} = 0$, we have $\displaystyle\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$

Proof that $\displaystyle\int_{C_r} f(z)\,dz \xrightarrow{r\to 0} 0$:

$$|f(z)| = \frac{|z^{-1/2}|}{|z^2+1|} \overset{|z|=r}{=\!=} \frac{e^{-\frac{1}{2}\ln r}\cancelto{1}{|e^{-\frac{1}{2}i\theta}|}}{|z^2+1|}$$

$$\leq \frac{r^{-1/2}}{1-r^2} = M_r$$

$$\left|\int_{C_r} f(z)\,dz\right| \leq M_r \cdot \text{length}(C_r)$$

$$= \frac{r^{-1/2}}{1-r^2} \cdot \pi r = \frac{\pi r^{1/2}}{1-r^2}$$

Since $\displaystyle\lim_{r\to 0} \frac{\pi r^{1/2}}{1-r^2} = 0$, we have

$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = 0$$