ON COMPUTING PICARD-FUCHS EQUATIONS

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This is a short guide to computing the Picard-Fuchs equation for the example of the elliptic surface

$$F(x,y,z) = t(x^3 + y^3 + z^3) - 3xyz = 0. \tag{1}$$

I will describe two different methods.

As for notation, $E \subseteq \mathbb{P}^2 \times \mathbb{P}^1$ is the elliptic surface defined by this equation, E_t the fiber at t. With $B = \mathbb{P}^1 \setminus \{0, 1, \mu, \mu^2\}$, the part of the surface that lies over B is a fiber bundle whose fibers are smooth elliptic curves.

We know that $H^{1,0}$ of each smooth fiber is one-dimensional. Let $\omega = \omega(t)$ be a holomorphic 1-form (with a dependance on t) that is defined in all smooth fibers of E. A formula for ω will be given below. As we know, the homology of nearby fibers in the family can be identified; thus a 1-cycle γ in a single fiber can also be considered as an 1-cycle in all nearby fibers, and we can study how the *periods*

$$\pi(t) = \int_\gamma \omega(t),$$

depend on the parameter t. They satisfy a second-order differential equation, the *Picard-Fuchs equation.*

I will show how to obtain this equation, how to solve it, and what the behavior of its solutions is near $t = 0$. On the way, I will give an introduction to Griffiths' theory of residues. The section titled 'The brute-force approach' on page 2 gives a computation assuming no background knowledge. A better method, using residues, is presented on page 8. In addition, Theorem 2 on page 7 gives a useful formula for computing Hodge numbers of smooth hypersurfaces in $\mathbb{P}^n$.

General remarks about periods. In the example of the elliptic surface (1) above, what information is contained in the periods? As was said before, the family of elliptic curves is locally trivial topologically—every smooth fiber is just a torus. On the other hand, the complex structure changes from torus to torus. It is determined by knowing the period lattice on each torus (that is, τ), and so this change can be measured by looking at the periods.

To say this again in a different way: Each fiber E_t has a two-dimensional first cohomology group $H^1(E_t, \mathbb{C}) \simeq \mathbb{C}^2$; locally on B, these are constant (by Ehresmann's fibration theorem, see p. 23 in Lamotke's notes) and can be combined into a rank two vector bundle on B. But inside $H^1(E_t, \mathbb{C})$, we have the one-dimensional subspace $H^{1,0}(E_t)$ determined by the complex structure. We can therefore ask how this one-dimensional subspace moves around in the fixed two-dimensional space as t changes (see Fig. 1).

Let α and β be the two basic cycles on the torus. The position of a point in $H^1(E_t, \mathbb{C})$, i.e. of a cohomology class, is measured exactly by what its values on these two basic cycles are. Or, representing cohomology by differential forms, by

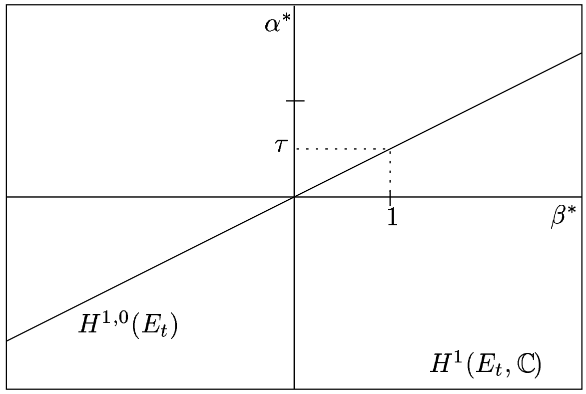

FIGURE 1. The subspace $H^{0,1}(E_t) \subseteq H^1(E_t, \mathbb{C})$

its integrals (or periods) over these cycles. Now $H^{1,0}(E_t)$ is a complex line in the direction of $\omega(t)$, and slope of this line is precisely

$$\tau = \int_\alpha \omega(t) \Bigg/ \int_\beta \omega(t),$$

Thus knowing the periods tells us how the complex structure changes between fibers.

We see that the Picard-Fuchs equation, by describing the behavior of the periods, also gives us information on the complex structures in fibers.

THE BRUTE-FORCE APPROACH

Without using much theory, we can find the Picard-Fuchs equation by a brute-force computation. This section describes how.

Normal form. The main idea is to put the equation in normal form; this depends on the fact that each fiber is a cubic. One gets (using *Mathematica*, say)

$$y^2 = tx^3 + \frac{1}{4}(1-4t^3)x^2 + \frac{1}{3}t^2(t^3-1)x - \frac{1}{27}t(t^3-1)^2. \tag{2}$$

Since we are interested in the behavior near $t = 0$, coordinate changes that involve division by t are excluded. For this reason, we cannot quite get the equation in Weierstrass form.

The holomorphic 1-form ω is then

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{tx^3 + \frac{1}{4}(1-4t^3)x^2 + \frac{1}{3}t^2(t^3-1)x - \frac{1}{27}t(t^3-1)^2}};$$

we study its periods

$$\pi = \int_\gamma \omega.$$

As said above, π satisfies a second-order differential equation. Thus, we are looking for an equation of the form

$$\frac{d^2\pi}{dt^2} + B\frac{d\pi}{dt} + C\pi = 0,$$

where B and C are rational functions of t. We determine B and C so that the 1-form

$$\eta = \frac{d^2\omega}{dt^2} + B\frac{d\omega}{dt} + C\omega$$

is closed, say equal to $d\phi$ for some function ϕ. This is sufficient, because Stokes' theorem gives us

$$\frac{d^2\pi}{dt^2} + B\frac{d\pi}{dt} + C\pi = \int_\gamma \eta = \int_\gamma d\phi = \int_{\partial\gamma} \phi = 0.$$

Because the equation is in normal form, we have a fixed coordinate x. We can therefore compute the derivatives of π by differentiating under the integral sign. For example,

$$\frac{d\pi}{dt} = \int_\gamma \frac{d\omega}{dt}.$$

Computations. Now the question is how to find suitable B and C. I will not detail the computations—they are long and best done with *Mathematica*. But the idea is simple:

(A) Compute the necessary derivatives of ω and write out η explicitly. It has the form

$$\eta = \frac{P(x)}{y^5}dx,$$

with a certain polynomial $P(x)$ of degree six in x (and coefficients in $\mathbb{C}[t]$).

(B) Subtract suitable multiples of exact forms

$$d\Big(\frac{x^k}{y^3}\Big)$$

for $k = 4, 3, \ldots, 0$, until the polynomial in the numerator has degree one.

(C) Determine B and C by setting the remaining two coefficients in the numerator equal to zero and solving.

This rather laborious process gives

$$B = \frac{4t^3 - 1}{t(t^3 - 1)} \qquad \text{and} \qquad C = \frac{2t}{t^3 - 1};$$

the Picard-Fuchs equation is therefore

$$\frac{d^2\pi}{dt^2} + \frac{4t^3 - 1}{t(t^3 - 1)}\frac{d\pi}{dt} + \frac{2t}{t^3 - 1}\pi = 0.$$

The Cohomology of a Hypersurface

The aim of this section is to explain how to compute the cohomology of a smooth projective hypersurface. This will provide us with another method for getting the Picard-Fuchs equation.

Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface, defined by an equation $F = 0$, say. From the Lefschetz theory, we know all of its cohomology groups except for the middle one, $H^{n-1}(X, \mathbb{C})$. In his papers on periods of rational integrals, Griffiths showed that the primitive part of this group, $\mathrm{Prim}^{n-1}(X)$, can be computed using residues of rational differential forms with poles along X.

Main result. Let us write $A^p_k(X)$ for the rational p-forms on $\mathbb{P}^n$ with poles only along X and of order at most k. Since $dA^p_k(X) \subseteq A^{p+1}_{k+1}(X)$, we can define the *rational de Rham groups*,

$$\mathcal{K}_k(X) = A^n_k(X)/dA^{n-1}_{k-1}(X).$$

Note that $\mathcal{K}_k(X) \subseteq \mathcal{K}_{k+1}(X)$.

We also need to remember the Hodge filtration

$$H^{n-1}(X,\mathbb{C}) = F^0H^{n-1}(X,\mathbb{C}) \supseteq F^1H^{n-1}(X,\mathbb{C}) \supseteq \cdots \supseteq F^{n-1}H^{n-1}(X,\mathbb{C}) \supseteq 0$$

on the complex cohomology of X. One has

$$F^pH^{n-1}(X,\mathbb{C}) \simeq H^{n-1,0}(X) + H^{n-2,1}(X) + \cdots + H^{p,n-1-p}(X);$$

the groups $H^{p,q}(X,\mathbb{C})$ are defined using forms of type (p,q) and satisfy

$$H^{q,p}(X,\mathbb{C}) = \overline{H^{p,q}(X,\mathbb{C})}.$$

We write $F^p\mathrm{Prim}^{n-1}(X)$ for the primitive part of $F^pH^{n-1}(X,\mathbb{C})$, i.e. for the kernel of the Lefschetz map.

The main result is then the following.

Theorem 1. *Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface. There are natural residue maps* $\mathrm{Res}\colon \mathcal{K}_k(X) \to F^{n-k}H^{n-1}(X,\mathbb{C})$. *These give isomorphisms between $\mathcal{K}_k(X)$ and the primitive cohomology $F^{n-k}\mathrm{Prim}^{n-1}(X)$, and the following diagram commutes.*

$$\begin{array}{ccccccc}
\mathcal{K}_1(X) & \longrightarrow & \mathcal{K}_2(X) & \longrightarrow & \cdots & \longrightarrow & \mathcal{K}_n(X) \\
\downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} & & & & \downarrow{\scriptstyle \mathrm{Res}} \\
F^{n-1}\mathrm{Prim}^{n-1}(X) & \longrightarrow & F^{n-2}\mathrm{Prim}^{n-1}(X) & \longrightarrow & \cdots & \longrightarrow & F^0\mathrm{Prim}^{n-1}(X)
\end{array}$$

The next few sections explain this result.

Rational Forms. In his paper, Griffiths gives a general formula for rational forms on $\mathbb{P}^n$. For our purposes, we only need to know what n-forms and $(n-1)$-forms look like.

A rational n-form on $\mathbb{A}^n$ has the shape

$$\frac{A(z_1,\ldots,z_n)}{B(z_1,\ldots,z_n)}\, dz_1 \wedge \cdots \wedge dz_n.$$

for certain polynomials A and B. If we homogenize this expression, replacing z_i by z_i/z_0, we obtain something like

$$\frac{Q(z_0,\ldots,z_n)}{P(z_0,\ldots,z_n)}\,\Omega$$

where

$$\Omega = \sum_{i=0}^{n} (-1)^i z_i\, dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n$$

and P, Q are homogeneous polynomials with $\deg P = \deg Q + \deg \Omega = \deg Q + (n+1)$. (To make the entire expression homogeneous of degree 0.)

If we want an n-form with a single pole of order k along X, we need to write

$$\frac{Q}{F^k}\,\Omega$$

with $\deg Q = k \deg F - (n+1)$.

Similarly, a rational $(n-1)$-form on $\mathbb{P}^n$ with a pole along X has the shape

$$\alpha = \sum_{i<j} (-1)^{i+j} \frac{z_j A_i - z_i A_j}{F^k}\, dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$$

for $A_0, \ldots, A_n$ homogeneous and such that $\deg A_i = k \deg F - n$.

For later use, we also want $d\alpha$. A short computation gives

$$d\alpha = -k \frac{\sum_i A_i \frac{\partial F}{\partial z_i}}{F^{k+1}} \Omega + \frac{\sum_i \frac{\partial A_i}{\partial z_i}}{F^k} \Omega. \tag{3}$$

Residues. We need to define the residue maps used in Theorem 1. To motivate the definition, let us consider residues as they appear in one-variable complex analysis. Let $f(z)$ be a meromorphic function on $\mathbb{C}$, with a pole of some order at a point P. The residue $\mathrm{Res}_P\, f(z)$ of f at P is a certain number; the residue theorem states that if T is a small loop around P, then

$$\frac{1}{2\pi i} \int_T f(z) dz = \mathrm{Res}_P\, f(z).$$

Concentrating on the 1-form $f(z)dz$ instead of on the function $f(z)$, we could say that Res takes a meromorphic 1-form with a pole at P, and sends it to a 0-form on P, which is just a number. If we view P as a hypersurface in $\mathbb{C}$ or in $\mathbb{P}^1$, we are in a situation similar to the one above.

Now we generalize to higher dimensions. X is again a smooth hypersurface in $\mathbb{P}^n$. Let ω be some rational n-form with a pole along X. We define its residue, which is a class in $H^{n-1}(X, \mathbb{C})$, as follows: For any $(n-1)$-cycle γ in X, let $T(\gamma)$ be a small tube around γ—X has codimension 2, and the tube is locally just $\gamma \times \mathbb{S}^1$. Choose the radius of the tube small enough, so that it lies entirely in the complement of X. We get a map

$$\gamma \mapsto \frac{1}{2\pi i} \int_{T(\gamma)} \omega,$$

and this defines an element $\mathrm{Res}\,\omega \in \mathrm{Hom}(H_{n-1}(X, \mathbb{Z}), \mathbb{C}) \simeq H^{n-1}(X, \mathbb{C})$. By its very definition, it satisfies

$$\frac{1}{2\pi i} \int_{T(\gamma)} \omega = \int_\gamma \mathrm{Res}\,\omega.$$

How can residues be found in practice? A simple computation, using the one-variable residue theorem, shows that if X is defined locally by $w = 0$, and if ω is of the form $\alpha \wedge dw/w + \beta$, with α and β holomorphic, then $\mathrm{Res}\,\omega = \alpha|_X$.

In general, we have to reduce the order of the pole first, and we have to glue the local descriptions together using partitions of unity. To wit:

(A) Let $w = 0$ be a local equation for X. Write ω as $dw/w^k \wedge \rho$ for some number $k \geq 1$.

(B) Use the formula

$$\frac{dw}{w^k} \wedge \rho = \frac{1}{k-1}\left(\frac{d\rho}{w^{k-1}} - d\Big(\frac{\rho}{w^{k-1}}\Big)\right) \equiv \frac{d\rho}{(k-1)w^{k-1}}$$

to lower the order of the pole, modulo an exact form, at least if $k \geq 2$.

(C) Glue these local pieces, $d\rho/(k-1)w^{k-1}$, using a partition of unity, to get a globally defined form with a pole of smaller order. Note that this form may not be rational anymore, because the partitions of unity are not. It can still be written as in (A), though.

(D) When a pole of order one is reached, use the formula given above, namely

$$\operatorname{Res}\left(\alpha \wedge \frac{dw}{w} + \beta\right) = \alpha|_X.$$

An example to illustrate how this works. Let $X \subseteq \mathbb{P}^2$ be the surface from (1) (see p. 1). We compute the residue of

$$\omega = \frac{\Omega}{F} = \frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{t(x^3+y^3+z^3) - 3xyz},$$

which should give a holomorphic 1-form on X (more correctly, a cohomology class in $H^{1,0}(X,\mathbb{C})$). To find the residue, we need an expression involving dF/F. At each point of X, at least one partial derivative of F is non-zero; let us assume that we are in a neighborhood of a point with $\partial F/\partial x \neq 0$ and find the residue there first. So we compute

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

and use it to eliminate dx from ω. A short computation gives

$$\omega = \frac{ydz - zdy}{\partial F/\partial x} \wedge \frac{dF}{F} + \frac{3dy \wedge dz}{\partial F/\partial x}.$$

Since the second term is holomorphic near our point, we get

$$\operatorname{Res}\omega = \frac{ydz - zdy}{\partial F/\partial x} = \frac{ydz - zdy}{3(tx^2 - yz)}. \tag{4}$$

Technically, this gives the residue only on the open set where $\partial F/\partial x \neq 0$. On the other two open sets, one has similar formulas, with x, y, z permuted cyclically, and to get the global residue, we are supposed to combine all these using a partition of unity. But as it happens, all three already represent the same (Kähler) differential, because one has the relations $dF = 0$ and $F = 0$ on X. So the partitions of unity have no real effect here—the residue is globally given by (4), with the understanding that we should choose a different representative near points where the denominator is zero.

Combinatorial description. We return to the computation of cohomology. The isomorphisms in Theorem 1 above actually give a nice combinatorial description of the middle cohomology of X.

First of all, we have

$$\operatorname{Prim}^{p,n-1-p}(X) \simeq \frac{F^p \operatorname{Prim}^{n-1}(X)}{F^{p+1}\operatorname{Prim}^{n-1}(X)}$$

by definition of the Hodge filtration. Because of Theorem 1, this is in turn isomorphic to

$$\frac{\mathcal{K}_{n-p}(X)}{\mathcal{K}_{n-p-1}(X)} \simeq \frac{A^n_{n-p}(X)}{dA^{n-1}_{n-p-1}(X) + A^n_{n-p-1}(X)}.$$

Now let us try to express this in terms of polynomials. Let $d = \deg F$ be the homogeneous degree of the polynomial F defining X. We have seen above that elements of $A^n_{n-p}(X)$ are of the form $Q\Omega/F^{n-p}$, hence can be represented by homogeneous polynomials Q of degree $d(n-p)-(n+1)$. This means that we have an isomorphism

$$\mathbb{C}[z_0,\dots,z_n]_{d(n-p)-(n+1)} \to A^n_{n-p}(X), \qquad Q \mapsto \frac{Q}{F^{n-p}}\Omega.$$

(The subscript refers to the grading on $\mathbb{C}[z_0,\dots,z_n]$.) We can compose this with a projection and the residue map to get

$$\nu\colon \mathbb{C}[z_0,\dots,z_n]_{d(n-p)-(n+1)} \to \frac{A^n_{n-p}(X)}{dA^{n-1}_{n-p-1}(X)+A^n_{n-p-1}(X)} \simeq \operatorname{Prim}^{p,n-1-p}(X).$$

I claim that the kernel of this map consists exactly of polynomials from the *Jacobian ideal*

$$J(F) = \left(\frac{\partial F}{\partial z_0},\dots,\frac{\partial F}{\partial z_n}\right).$$

To see why, look back at (3) on page 5: with degrees adjusted to our situation, we find that elements of $dA^{n-1}_{n-p-1}(X)$ are of the form

$$-(n-p-1)\frac{\sum_i A_i\frac{\partial F}{\partial z_i}}{F^{n-p}}\Omega + \frac{\sum_i \frac{\partial A_i}{\partial z_i}}{F^{n-p-1}}\Omega,$$

while elements of $A^n_{n-p-1}(X)$ are of the form

$$\frac{PF}{F^{n-p}}\Omega.$$

Together, this says that the kernel of ν consists exactly of all polynomials Q that can be written as

$$Q = -(n-p-1)\sum_i A_i\frac{\partial F}{\partial z_i} + F\cdot\sum_i\frac{\partial A_i}{\partial z_i} + PF.$$

Because of Euler's identity,

$$d\cdot F = \deg F\cdot F = \sum_{i=0}^{n}\frac{\partial F}{\partial z_i}z_i,$$

any such Q lies in the Jacobian ideal. But we can in fact get any element (of the correct degree) from that ideal in this way, by a good choice of A_i and P. So the kernel is as claimed.

We can thus restate Theorem 1 as follows.

Theorem 2. *Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface, defined by an equation $F=0$ of homogeneous degree d. Let $R(F)$ denote the graded ring $\mathbb{C}[z_0,\dots,z_n]/J(F)$. Then*

$$\operatorname{Prim}^{p,n-1-p}(X) \simeq R(F)_{(n-p)d-(n+1)}.$$

The isomorphism associates to a homogeneous polynomial Q the cohomology class $\operatorname{Res} Q\Omega/F^{n-p}$.

An Example. As an example, we compute the cohomology of the quintic threefold $X \subseteq \mathbb{P}^4$ defined by

$$F(z_0, \ldots, z_4) = z_0^5 + \cdots + z_4^5.$$

The only interesting piece of the cohomology is $H^3(X, \mathbb{C})$; since X has odd dimension, all of this group is primitive. We use Theorem 2, with $d = 5$, $n = 4$ and $p = 3$ to get

$$H^{3,0}(X, \mathbb{C}) \simeq R(F)_0 \simeq \mathbb{C}.$$

To compute $H^{2,1}(X, \mathbb{C})$, we apply the result again:

$$H^{2,1}(X, \mathbb{C}) \simeq R(F)_5.$$

The dimension is found by counting—there are $\binom{5+4}{4} = 126$ homogenous polynomials of degree 5 in $z_0, \ldots, z_4$; the Jacobian ideal accounts for $5 \cdot 5 = 25$ of them, and therefore $H^{2,1}(X, \mathbb{C}) \simeq \mathbb{C}^{101}$.

The residue approach

Our second method for computing the Picard-Fuchs equation is more general. Remember that E is defined by (1) on page 1; we are going to derive the Picard-Fuchs equation again, this time using residues.

Relevant facts. In our case,

$$\Omega = x dy \wedge dz - y dx \wedge dz + z dx \wedge dy.$$

As we have seen on page 6, the residue of $\Omega_0(t) = \Omega/F$ is a holomorphic 1-form $\omega(t)$; it was computed above as

$$\omega(t) = \operatorname{Res} \Omega_0(t) = \frac{y dz - z dy}{3(tx^2 - yz)}$$

Let us introduce the additional 3-forms

$$\Omega_k(t) = \left(\frac{d}{dt}\right)^k \left(\frac{\Omega}{F}\right) = (-1)^k k! \frac{\left(x^3 + y^3 + z^3\right)^k}{F^{k+1}} \Omega.$$

We let

$$\pi(t) = \int_\gamma \omega(t),$$

be a period of $\omega(t)$. The following facts, partly extracted from the general discussion above, are relevant to our computation.

(A) We have the following identity:

$$\frac{d^k \pi(t)}{dt^k} = \int_{T(\gamma)} \Omega_k(t)$$

(B) Modulo exact forms,

$$\frac{A \frac{\partial F}{\partial x} + B \frac{\partial F}{\partial y} + C \frac{\partial F}{\partial z}}{F^k} \Omega \equiv \frac{\Omega}{(k-1) F^{k-1}} \cdot \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right).$$

(C) By Theorem 2,

$$H^{p,1-p}(X, \mathbb{C}) \simeq R(F)_{3(1-p)},$$

where $R(F) = \mathbb{C}[x, y, z]/J(F)$. A homogeneous polynomial Q corresponds to the cohomology class $\operatorname{Res} Q\Omega/F^{2-p}$.

We are always going to use (C) in the following way: If we know that $\operatorname{Res} Q\Omega/F^{2-p}$ is zero (for example because it lands in a cohomology group that is zero), then $Q \in J(F)$.

As for proving these statements, (C) is an immediate consequence of Theorem 2. (B) is obtained by a simple, if somewhat tedious, computation. (A) follows from the definition of residues: We have, by definition,

$$\pi(t) = \int_\gamma \omega(t) = \int_\gamma \operatorname{Res} \omega(t) = \frac{1}{2\pi i} \int T(\gamma)\Omega_0(t).$$

Since $T(\gamma)$ can, locally, be chosen independent of t, we can now differentiate under the integral sign to obtain (A).

Computations. To get the Picard-Fuchs equation, it is therefore sufficient to find a relation of the form

$$\Omega_2(t) + B(t)\Omega_1(t) + C(t)\Omega_0(t) \equiv 0,$$

modulo exact forms. Then (A) shows that $\pi(t)$ satisfies the differential equation

$$\frac{d^2\pi}{dt^2} + B\frac{d\pi}{dt} + C\pi = 0.$$

We begin with

$$\Omega_2(t) = \frac{2\left(x^3+y^3+z^3\right)^2}{F^3}\Omega.$$

By (D), with $p=-1$, its residue would land in $H^{-1,2}(X,\mathbb{C})$, which is zero. Therefore the numerator has to lie in the Jacobian ideal. We know that it can be written as a combination of partial derivatives of F; a Gröbner basis calculation (for example) shows that

$$\left(x^3+y^3+z^3\right)^2 = \left(\frac{x^3+y^3+z^3}{t} + \frac{3xyz}{t^2}\right)F + \frac{3x^2y}{t^3-1}\left(\frac{z}{t^2}\frac{\partial F}{\partial x} + \frac{x}{t}\frac{\partial F}{\partial y} + y\frac{\partial F}{\partial z}\right).$$

If we substitute into the expression for $\Omega_2(t)$ and use (B) above, we get

$$\Omega_2(t) \equiv \left(\frac{2}{t}\left(x^3+y^3+z^3\right) + \frac{6txyz}{t^3-1} + \frac{3x^3}{t(t^3-1)}\right)\frac{\Omega}{F^2}.$$

This form has a pole of order two, and so does

$$\Omega_1(t) = -\frac{x^3+y^3+z^3}{F^2}\Omega.$$

As $H^{0,1}(X,\mathbb{C})$ is one-dimensional, (D) shows that there has to exist some $B=B(t)$ such that if $B\cdot\Omega_1(t)$ is added to the above, we get something of the form $Q\Omega/F^2$ with $Q\in J(F)$. We try to determine B accordingly.

Substituting, one finds that for arbitrary B,

$$Q = \left(\frac{2}{t} - B\right)\left(x^3+y^3+z^3\right) + \frac{3}{t(t^3-1)}x^3 + \frac{6t}{t^3-1}xyz.$$

After some replacements, one arrives at

$$Q = \frac{2-Bt}{t^2}F + \frac{x}{t^2(t^3-1)}\frac{\partial F}{\partial x} + \frac{3}{t}\left(\frac{4t^3-1}{t(t^3-1)} - B\right)xyz;$$

this lies in the Jacobian ideal precisely when

$$B = \frac{4t^3 - 1}{t(t^3 - 1)},$$

and for this choice of B, we have

$$\Omega_2(t) + B\Omega_1(t) \equiv -\frac{2t^3 + 1}{t^2(t^3 - 1)} \frac{\Omega}{F} + \frac{x}{t^2(t^3 - 1)} \frac{\partial F}{\partial x} \frac{\Omega}{F^2}$$

Using (B) again, then replacing Ω/F by $\Omega_0(t)$, we finally arrive at

$$\Omega_2(t) + B\Omega_1(t) \equiv -\frac{2t}{t^3 - 1}\Omega_0(t),$$

that is, we determined the values

$$B = \frac{4t^3 - 1}{t(t^3 - 1)} \qquad \text{and} \qquad C = \frac{2t}{t^3 - 1}.$$

Of course, the result is the same as before!

Solving the Picard-Fuchs equation

Now that we have found the equation

$$\frac{d^2\pi}{dt^2} + \frac{4t^3 - 1}{t(t^3 - 1)} \frac{d\pi}{dt} + \frac{2t}{t^3 - 1}\pi = 0,$$

let us solve it. We note first that it is of second-order and of the form

$$\frac{d^2\pi}{dt^2} + \frac{P(t)}{t} \frac{d\pi}{dt} + \frac{Q(t)}{t^2}$$

for functions

$$P = \frac{4t^3 - 1}{t^3 - 1} \qquad \text{and} \qquad Q = \frac{2t^3}{t^3 - 1}$$

that are holomorphic near $t = 0$. This means that the equation has a *regular singular point* at $t = 0$. We can compute the indicial equation, which is

$$r(r - 1) + P(0)r + Q(0) = r^2;$$

the fact that it has a double root at $r = 0$ means the following, according to the general theory of such equations: The two-dimensional space of solutions is spanned by $\pi_1(t)$, which is holomorphic near $t = 0$, and $\pi_2(t)$, which is of the form $\pi_1(t) \cdot \log t + \rho(t)$ for some holomorphic function $\rho(t)$. Note that $\pi_2(t)$ is multi-valued because of the logarithmic term.

The holomorphic solution can be found as a power series; if we try

$$\pi_1(t) = \sum_{n=0}^{\infty} a_n t^n,$$

then a short computation gives the recursive relation

$$a_{n+3}(n + 3)^2 = a_n(n + 2)(n + 1).$$

From this, one finds that

$$\pi_1(t) = \sum_{n=0}^{\infty} \frac{(3n)!}{3^{3n}(n!)^3} t^{3n}$$

is the holomorphic solution with value 1 at $t = 0$.

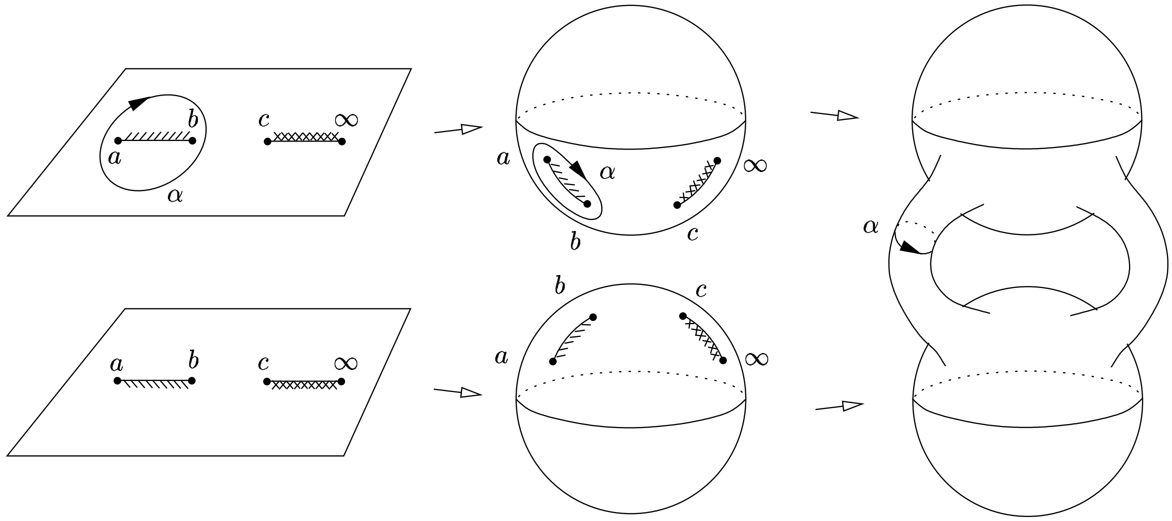

FIGURE 2. The position of α on the usual two-sheeted cover of E_t

Geometric Interpretation. Now, using our knowledge about the monodromy near $t = 0$, let us try to interpret this result. The two cycles that generate $H_1(E_t, \mathbb{Z})$ in a smooth fiber behave differently as $t \to 0$. One of them, say α, is a vanishing cycle and invariant under the monodromy action. The other one, β, does not vanish and is mapped to $\beta + 3\alpha$ when t moves around 0 once.

It follows that the period of $\omega(t)$ corresponding to α is single-valued around $t = 0$, because there is no monodromy. It therefore has to be a certain multiple of $\pi_1(t)$. On the other hand, the period for β has monodromy and must contain a logarithmic term.

To find out what multiple of $\pi_1(t)$ gives the period for α, we need to do a small residue computation. We take a look at the normal form of F derived in (2) on page 2, especially at its right-hand side. As $t \to 0$, it converges to $x^2/4$, and so one of its zeroes moves off to infinity whereas the other two approach the origin. We will call these zeroes $a(t)$, $b(t)$ and $c(t)$. Note that, together with ∞, they give the four points of order two on each curve.

If we let α be the (negatively oriented) unit circle in one of the two sheets of E_t, then for sufficiently small t, only $a(t)$ and $b(t)$ lie inside that circle. Fig. 2 shows that α (which circles $a(t)$ and $b(t)$) is in fact the vanishing cycle. Computing the integral of $\omega(t)$ along α will thus give us the required period

$$\int_\alpha \omega(t) = \int_{-\mathbb{S}^1} \frac{dx}{\sqrt{tx^3 + \frac{1}{4}(1-4t^3)x^2 + \frac{1}{3}t^2(t^3-1)x - \frac{1}{27}t(t^3-1)^2}}.$$

In this expression, we now let $t \to 0$ to see what value the period should have at $t = 0$. We obtain

$$\int_{-\mathbb{S}^1} \frac{2dx}{x} = -2 \cdot 2\pi i = -4\pi i$$

by the residue theorem. It follows that the period of the vanishing cycle is given by

$$-4\pi i \pi_1(t) = -4\pi i \sum_{n=0}^{\infty} \frac{(3n)!}{3^{3n}(n!)^3} t^{3n}$$

near $t = 0$.