THE SET OF (-1)-CURVES ON A CERTAIN CLASS OF ELLIPTIC SURFACES

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Let X be the elliptic surface obtained from a general pencil of cubic curves in $\mathbb{P}^2$. In other words, X is the blow-up of $\mathbb{P}^2$ along the nine points that form the base locus of the pencil. A (-1)-curve on X is by definition a smooth rational curve $C \subseteq X$ with self-intersection $C \cdot C = -1$. In this short note, we shall determine all such curves. We shall be using the following two maps:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{P}^2 \\ \mathbb{P}^1 & \end{tikzcd}$$

For a point $u \in \mathbb{P}^1$, we shall let $F_u = p^{-1}(u)$ be the corresponding fiber of the map p. All but twelve of the fibers are smooth cubic curves in $\mathbb{P}^2$; the other twelve are cubic curves with a single node.

Classes of curves. Let P_i, for $i = 1, 2, \ldots, 9$, be the nine points in the base locus; the corresponding exceptional divisors in X will be denoted by E_i. The Neron-Severi group of the surface X has rank ten; it is generated by $\lambda = f^*[L]$, and the nine classes $\varepsilon_i = [E_i]$. Thus the class of any irreducible curve C in the surface can be uniquely written in the form

$$[C] = b \cdot \lambda - \sum_{i=1}^{9} a_i \cdot \varepsilon_i,$$

with integer coefficients $b, a_1, \ldots, a_9$. Note that b is the degree of the image $f(C)$, hence nonnegative; also, we have $a_i \geq 0$, unless $C = E_i$.

Conditions on (-1)-curves. Since X is a blow-up of $\mathbb{P}^2$, its canonical class is given by the formula

$$K_X = f^* K_{\mathbb{P}^2} + \sum_{i=1}^{9} [E_i] = -3\lambda + \sum_{i=1}^{9} \varepsilon_i. \tag{1}$$

Now let $C \subseteq X$ be a (-1)-curve; we then have $C \cdot C = -1$. Moreover, C is a smooth rational curve, hence isomorphic to $\mathbb{P}^1$, and so

$$-2 = \deg K_C = (K_X + C) \cdot C = K_X \cdot C + C \cdot C = K_X \cdot C - 1$$

by adjunction. Thus $K_X \cdot C = -1$, too. Consequently, an irreducible and nonsingular curve $C \subseteq X$ is a (-1)-curve precisely when $K_X \cdot C = -1$ and $C \cdot C = -1$.

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This translates into the following two numerical conditions:

$$\sum_{i=1}^{9} a_i = 3b - 1 \qquad \text{and} \qquad \sum_{i=1}^{9} a_i^2 = b^2 + 1 \tag{2}$$

Since every (-1)-curve can be contracted smoothly (by Castelnuovo's criterion), it has to be the only curve in its class; thus C is uniquely determined by the ten parameters $b, a_1, \dots, a_9$. (But, of course, not all possible values of these actually correspond to curves.)

Another useful fact is that every (-1)-curve on X is a section of $p\colon X \to \mathbb{P}^1$. Indeed, the class of any fiber of p is precisely equal to $-K_X$ by (1), and if C is a (-1)-curve, then $K_X \cdot C = -1$, and so C has intersection number 1 with the fibers of p. Since it can obviously not be contained in any fiber, it has to meet every fiber in exactly one point, and thus has to be the image of a section. (Note that each of the twelve nodal fibers, while being rational, has self-intersection 0.)

A transformation. The general fiber of the pencil is a smooth cubic curve in $\mathbb{P}^2$, and can be made into an abelian group by choosing a point (to represent the unit element in the group). Even without such a choice, it is possible to assign to any two points P and Q on the curve a third, by seeing where the line through the first two meets the curve a third time. (If $P = Q$, the tangent line to the curve has to be used.) We shall call this third point the composition of P and Q, and denote it by $P * Q$. The operation $*$ is not associative, but when applied twice, it can be expressed in terms of the group law on the cubic curve. Indeed, take any three points P, Q, and R on a fiber, and let $A = P * Q$, and $B = A * R$. Then P, Q, and A lie on a line, as do A, R, and B, and so we have $P + Q + A = A + R + B$ in the group law on the curve. Thus

$$B = (P * Q) * R = P + (Q - R),$$

independently of the choice of unit element.

This operation also allows one to compose (-1)-curves on X. Indeed, suppose C_1, C_2, and C_3 are three such curves; say C_i is the image of a section $s_i\colon \mathbb{P}^1 \to X$. Then we can form a new section $s = s_1 * s_2$ by the fiberwise rule

$$s(u) = s_1(u) * s_2(u).$$

Of course, this works just on smooth fibers, but since $\mathbb{P}^1$ is a smooth curve, the resulting rational section naturally extends over the twelve points where F_u is singular. The image of s is then another (-1)-curve, and we shall denote it by the symbol $C_1 * C_2$. By applying this operation twice, we obtain a new rational curve

$$\bigl(C_1 * C_2\bigr) * C_3 = C_1 + (C_2 - C_3)$$

from any three given ones.

Formulas. Now let C be a (-1)-curve, with coefficients as in (2), and fix any number j between 1 and 9. We shall work out the class of the curve $C' = C * E_j$, which we write as

$$[C'] = b' \cdot \lambda - \sum_{i=1}^{9} a_i' \cdot \varepsilon_i.$$

Say C is the image of a section $s\colon \mathbb{P}^1 \to X$; also let us assume, to make things simpler, that C is not equal to any of the exceptional divisors E_i. Then the coefficient

$a_i = C' \cdot E_i$ is the number of points in the intersection of the two curves C' and E_i. We get such a point in one of the cubic curves F_u precisely when the line through $s(u) = C \cap F_u$ and $P_j = C \cap E_j$ meets F_u a third time in the point P_i. In other words, we have

$$a_i' = \#\bigl\{\, u \in \mathbb{P}^1 \bigm| P_i,\, P_j, \text{ and } f(C \cap F_u) \text{ are collinear in } \mathbb{P}^2 \,\bigr\},$$

at least when $i \neq j$. This implies that

$$a_i' = b - a_i - a_j.$$

Indeed, the image curve $f(C)$ in $\mathbb{P}^2$ is a curve of degree b, and thus intersects the line L through P_i and P_j in exactly b points. Since $C \cdot E_i = a_i$, and $C \cdot E_j = a_j$, the curve $f(C)$ already has $a_i + a_j$ points of intersection with the line L at P_i and P_j; thus a_i', being the number of other points of intersection, has to equal $b - a_i - a_j$.

On the other hand, the coefficients $b', a_1', \dots, a_9'$ of the curve C' also have to satisfy the two conditions in (2). This allows us to determine a_j' and b' as well. A somewhat tedious computation gives their values as $a_j' = 1 + b - 2a_j$, and $b' = 1 + 2b - 3a_j$. (There is a second solution to the two equations; but as it has fractional coefficients, it is not relevant here.) We thus have

$$a_i' = b - a_i - a_j + \bigl[i = j\bigr] \qquad \text{and} \qquad b' = 2b - 3a_j + 1, \tag{3}$$

where we define $\bigl[i = j\bigr]$ to equal 1 if $i = j$, and to equal 0 otherwise.

Reducing the degree. We are now ready to determine all (-1)-curves on the surface X. To begin with, there are the nine exceptional divisors $E_1, \dots, E_9$. In addition, for each $j \neq k$, we can take the strict transform of the line connecting P_j and P_k, and get a (-1)-curve. (Altogether, there are 36 of these.) In fact, this curve is nothing but $E_j * E_k$, as can easily be seen; numerically, this also follows from the equations in (3) (which, however, were derived under the assumption that C is not one of the exceptional divisors).

The question we want to investigate, is whether all (-1)-curves can be obtained from these by the operations introduced above. As we shall see in a minute, this is almost the case. The most transparent method is to work backwards—we start with an arbitrary (-1)-curve C, and try to express it as a sum of various E_j.

For $j \neq k$, consider the new curve $C'' = C + (E_j - E_k)$. Letting

$$[C''] = b'' \cdot \lambda - \sum_{i=1}^{9} a_i'' \cdot \varepsilon_i,$$

and using (3) twice, we find that

$$a_i'' = a_i + \bigl(a_j - a_k\bigr) + 1 + \bigl[i = j\bigr] - \bigl[i = k\bigr],$$

while the new degree is given by

$$b'' = b + 3\bigl(a_j - a_k + 1\bigr).$$

Now we observe that the value of b'' can be reduced by this operation, unless $|a_j - a_k| \leq 1$ for all j and k. We can thus do induction on the degree b''; all that remains to do is classify those curves C for which all $|a_j - a_k| \leq 1$.

But this is easily done. Say m of the coefficients a_i are equal to $a + 1$, and the remaining $9 - m$ are equal to a. Then (2) gives us the two conditions

$$m(a+1) + (9-m)a = 3b - 1 \qquad \text{and} \qquad m(a+1)^2 + (9-m)a^2 = b^2 + 1.$$

The first condition shows that $m \equiv -1$ modulo 3. It is then easy to solve both equations for a and b; there are exactly three possibilities: (1) Either $m = 2$, in which case $a = 0$ and $b = 1$; this means that C is the strict transform of a line through two of the points. (2) Or $m = 5$, and then $a = 0$ and $b = 2$; such a C is the strict transform of a conic through five of the points. (3) Or, again, $m = 8$, and then $a = -1$ and $b = 0$, and C is one of the exceptional divisors. With a little bit of extra work, one sees that any two curves in the same category are related to each other by a sequence of transformations as above (for suitable choices of j and k), but that it is not possible to pass from one of the three categories to the other. (This follows by looking at the formula for b'' modulo 3.)

In summary, we have proved the following result.

Proposition. *Let C be an arbitrary (-1)-curve on the elliptic surface X. Then C can be written in the form*

$$C = C_0 + \bigl(E_{j_1} - E_{k_1}\bigr) + \cdots + \bigl(E_{j_N} - E_{k_N}\bigr),$$

where C_0 is one of three possible curves: (1) The strict transform of a line through two of the nine points; (2) the strict transform of a conic through five of the nine points; (3) one of the exceptional divisors.