HYPERSURFACES CONTAINING CERTAIN SUBVARIETIES

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Let X be a nonsingular projective variety, defined over the complex numbers, and let L be an ample line bundle on X. If $Z \subseteq X$ is a closed subvariety, some sufficiently big power of L will have sections that vanish on Z. A natural question is how nice the zero scheme of such sections can be. We answer this question in certain special cases.

1. The case of a nonsingular subvariety

In this section, we assume that $Z \subseteq X$ is a nonsingular subvariety of codimension $p \geq 1$. If $i \colon Z \to X$ denotes the inclusion, then the co-normal bundle

$$N^\vee = N^\vee_{Z\subseteq X} = i^* I_Z$$

is locally free on Z, of rank p. The sections of L that vanish along Z make up the space $H^0(X, L \otimes I_Z)$, and because of the exactness of

$$0 \to L \otimes I_Z^2 \to L \otimes I_Z \to i^* L \otimes N^\vee \to 0,$$

this vector space sits in a three-term exact sequence

$$H^0(X, L \otimes I_Z) \xrightarrow{\rho} H^0(Z, i^* L \otimes N^\vee) \to H^1(X, L \otimes I_Z^2). \tag{1}$$

The following lemma shows what singularities the zero scheme of a section will have along Z.

Lemma 1. *Let $s \in H^0(X, L \otimes I_Z)$ be a nontrivial section. The zero scheme $V = V(s)$ of s is singular at a point $x \in Z$ if, and only if, the section $\rho(s)$ of the vector bundle $i^* L \otimes N^\vee$ vanishes at x. More precisely, the zero scheme of $\rho(s)$ is the intersection of Z with the singular locus of V.*

Proof. Let U be a sufficiently small open neighborhood, in X, of an arbitrary point of Z, and let $z_1, \dots, z_n$ be holomorphic coordinates on U, such that the subvariety $Z \cap U$ is defined by the equations $z_1 = \dots = z_p = 0$. We may assume that L is trivial on U, with non-vanishing section e.

The section $s = f \cdot e$ then gives a holomorphic function $f = f(z_1, \dots, z_n)$ on U that vanishes along $Z \cap U$, and thus satisfies

$$f(0, \dots, 0, z_{p+1}, \dots, z_n) = 0 \qquad \text{for all } z_{p+1}, \dots, z_n.$$

The partial derivatives $\partial f / \partial z_i$, for $i = p+1, \dots, n$, therefore vanish identically along $Z \cap U$, and so the singularities of V along Z are determined by the vanishing of the first p partial derivatives of f alone.

Now the vector bundle $i^* L \otimes N^\vee$ is also trivial on U, and a local frame is given by the p vectors $e \otimes dz_i$, for $i = 1, \dots, p$. In this frame, the section $\rho(s)$ is expressed as

$$\rho(s) = \rho(f \cdot e) = \sum_{i=1}^{p} \frac{\partial f}{\partial z_i}(0, \dots, 0, z_{p+1}, \dots z_n) \cdot e \otimes dz_i;$$

consequently, the vanishing of df at any point of $Z \cap U$ is equivalent to that of $\rho(s)$, and the lemma is proved. □

To make use of the Lemma, we will take L sufficiently ample, to guarantee that

- the space $H^1(L \otimes I_Z^2) = 0$;
- the bundle $i^* L \otimes N^\vee$ is globally generated.

The first condition, together with the exact sequence in (1), implies that a general point of $H^0(X, L \otimes I_Z)$ maps to a general point in $H^0(Z, i^* L \otimes N^\vee)$; by Bertini's theorem, the zero scheme of such a section is nonsingular outside of Z.

The second condition implies that a general point in $H^0(Z, i^* L \otimes N^\vee)$ is a regular section; in particular, the zero scheme of such a section represents the top Chern class

$$c_p\bigl(i^* L \otimes N^\vee\bigr)$$

of the vector bundle.

In conclusion, we get the following result.

Proposition 2. *Assume that the line bundle L is sufficiently ample. Then a general section of $L \otimes I_Z$ has its only singularities along a codimension p subscheme of Z, whose class is the top Chern class of $i^* L \otimes N_{Z \subseteq X}^\vee$.*

If $Z \subseteq X$ is a nonsingular curve in a threefold, for instance, a general section will have no singularities at all; thus a nonsingular curve on a threefold is always contained in a smooth hypersurface section. On the other hand, if Z is a nonsingular surface in a fourfold, we have to allow a certain finite number of singular points in a hypersurface containing it; but if the hypersurface is general enough, these singular points can be taken as nodes.