Lecture 1

Introduction. Our topic this semester is the "generic vanishing theorem" and its applications. In the late 1980s, Green and Lazarsfeld studied the cohomology of topologically trivial line bundles on smooth complex projective varieties (and compact Kähler manifolds). The generic vanishing theorem is one of their main results: it says that, under certain conditions on the variety, the cohomology of a generically chosen line bundle is trivial in all degrees less than the dimension of the variety. This theorem, and related results by Green and Lazarsfeld, are a very useful tool in the study of irregular varieties and abelian varieties. Here are some examples of its applications:

(1) Singularities of theta divisors on principally polarized abelian varieties (Ein-Lazarsfeld).
(2) Numerical characterization of abelian varieties up to birational equivalence (Chen-Hacon)
(3) Birational geometry of varieties of Kodaira dimension zero (Ein-Lazarsfeld, Chen-Hacon)
(4) Inequalities among Hodge numbers of irregular varieties (Lazarsfeld-Popa)
(5) M-regularity on abelian varieties (Pareschi-Popa)
(6) Holomorphic one-forms on varieties of general type (Popa-Schnell, Villadsen)

There are now two completely different proofs for the generic vanishing theorem. The original one by Green and Lazarsfeld used deformation theory and classical Hodge theory; there is also a more recent one by Hacon, based on derived categories and Mukai's "Fourier transform" for abelian varieties. Here is a list of the sources:

[GL87] M. Green and R. Lazarsfeld, *Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville*, Invent. Math. **90** (1987), no. 2, 389–407.

[GL91] ———, *Higher obstructions to deforming cohomology groups of line bundles*, J. Amer. Math. Soc. **4** (1991), no. 1, 87–103.

[Sim93] C. Simpson, *Subspaces of moduli spaces of rank one local systems*, Ann. Sci. École Norm. Sup. (4) **26** (1993), 361–401.

[Hac04] C. D. Hacon, *A derived category approach to generic vanishing*, J. Reine Angew. Math. **575** (2004), 173–187.

[PS13] M. Popa and C. Schnell, *Generic vanishing theory via mixed Hodge modules*, Forum Math. Sigma **1** (2013), e1, 60.

The following articles contain various examples and applications of the theory. We will be discussing most of them over the course of the semester.

[Bea92] A. Beauville, *Annulation du H^1 pour les fibrés en droites plats*, Complex algebraic varieties (Bayreuth, 1990), Lecture Notes in Math., vol. 1507, Springer, Berlin, 1992, pp. 1–15.

[EL97] L. Ein and R. Lazarsfeld, *Singularities of theta divisors and the birational geometry of irregular varieties*, J. Amer. Math. Soc. **10** (1997), no. 1, 243–258.

[CH01] J. A. Chen and C. D. Hacon, *Characterization of abelian varieties*, Invent. Math. **143** (2001), no. 2, 435–447.

[Par12] G. Pareschi, *Basic results on irregular varieties via Fourier-Mukai methods*, Current developments in algebraic geometry, Math. Sci. Res. Inst. Publ., vol. 59, Cambridge Univ. Press, Cambridge, 2012, pp. 379–403.

[PS14] M. Popa and C. Schnell, *Kodaira dimension and zeros of holomorphic one-forms*, Ann. of Math. (2) **179** (2014), no. 3, 1109–1120.

[Sch22] C. Schnell, *The Fourier-Mukai transform made easy*, Pure Appl. Math. Q. **18** (2022), no. 4, 1749–1770.

[CJ18] J. A. Chen and Z. Jiang, *Positivity in varieties of maximal Albanese dimension*, J. Reine Angew. Math. **736** (2018), 225–253.

Vanishing theorems and their applications. Before I introduce the work of Green and Lazarsfeld, let me say a few words about the role of vanishing theorems in modern algebraic geometry. Vanishing theorems are very important because many questions can be phrased in terms of coherent sheaves, their global sections, and their cohomology. Here are three typical cases:

(1) *Lifting sections.* Suppose we have a short exact sequence of coherent sheaves
$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$
on an algebraic variety X, and need to know whether or not every global section of $\mathscr{F}''$ can be lifted to a global section of $\mathscr{F}$. The long exact sequence
$$\cdots \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}'') \to H^1(X, \mathscr{F}') \to \cdots$$
in cohomology shows that the vanishing of $H^1(X, \mathscr{F}')$ is sufficient.

(2) *Existence of sections.* Suppose we have a coherent sheaf $\mathscr{F}$ on an algebraic variety X, and need to know whether or not $\mathscr{F}$ has nontrivial global sections. If $H^i(X, \mathscr{F})$ happens to vanish for every $i > 0$, then
$$\dim H^0(X, \mathscr{F}) = \chi(X, \mathscr{F})$$
is equal to the Euler characteristic of $\mathscr{F}$, which can typically be computed with the help of the Riemann-Roch theorem.

(3) *Vanishing of obstructions.* Suppose we need to do some global construction on an algebraic variety X, but only know that it works locally. In this situation, the obstruction to the global problem is often an element of some sheaf cohomology group; the vanishing of this cohomology group is therefore sufficient to get a solution.

The most famous vanishing theorem is the *Kodaira vanishing theorem* for ample line bundles on complex projective varieties. (In this course, we will only consider algebraic varieties that are defined over the complex numbers; I will therefore not explicitly mention that assumption from now on.)

Kodaira Vanishing Theorem. *Let L be an ample line bundle on a smooth projective variety X. Then $H^i(X, \omega_X \otimes L) = 0$ for every $i > 0$.*

The *Nakano vanishing theorem* extends this to the other sheaves of differential forms Ω_X^p, but the result is not as strong as in the case of the canonical bundle.

Nakano Vanishing Theorem. *Under the same assumptions on X and L, one has $H^q(X, \Omega_X^p \otimes L) = 0$ for every $p, q \in \mathbb{N}$ with $p + q > \dim X$.*

Here are a few elementary applications, along the lines of what I said above.

Example 1.1. If L is an ample line bundle on an abelian variety A, then $H^0(A, L) \neq 0$. Indeed, the canonical bundle ω_A is trivial, and so all higher cohomology groups of L vanish by Kodaira's theorem. Together with the Riemann-Roch theorem,
$$\dim H^0(A, L) = \chi(A, L) = \frac{L^g}{g!} \neq 0,$$
where $g = \dim A$.

Example 1.2. Infinitesimal deformations of a smooth projective variety are parametrized by $H^1(X, \mathscr{T}_X)$, where $\mathscr{T}_X$ is the tangent sheaf; the obstructions to extending an infinitesimal deformation to an actual deformation lie in $H^2(X, \mathscr{T}_X)$. Suppose that X is a Fano manifold of dimension n, which means that ω_X^{-1} is an ample line bundle. Because $\omega_X \otimes \mathscr{T}_X \simeq \Omega_X^{n-1}$, we obtain

$$H^2(X, \mathscr{T}_X) \simeq H^2(X, \Omega_X^{n-1} \otimes \omega_X^{-1}) = 0$$

from the Nakano vanishing theorem. Fano manifolds are therefore unobstructed.

It turns out that one can relax the assumptions in the Kodaira vanishing theorem and allow L to be only nef and big (= a birational version of being ample). Recall that L is *nef* if, for every curve $C \subseteq X$, the intersection number $L \cdot C \geq 0$. It is called *big* if the function $m \mapsto \dim H^0(X, L^{\otimes m})$ grows like $m^{\dim X}$; when L is nef, this is equivalent to having $L^{\dim X} > 0$ (by the Riemann-Roch theorem).

Kawamata-Viehweg Vanishing Theorem. *Let X be a smooth projective variety. If L is nef and big, then $H^i(X, \omega_X \otimes L) = 0$ for every $i > 0$.*

This result has had a great influence on birational geometry; in fact, there is even a more general version of the Kawamata-Viehweg vanishing theorem that involves $\mathbb{R}$-divisors and multiplier ideals.

The generic vanishing theorem. All of the vanishing theorems above depend on the fact that the first Chern class $c_1(L)$ is (in some sense) positive. The question that motivated the generic vanishing theorem is what happens when we consider instead line bundles with $c_1(L) = 0$. We may call such line bundles "topologically trivial", because the underlying smooth line bundle is trivial. They are parametrized by the points of $\mathrm{Pic}^0(X)$, which is an abelian variety of dimension $\dim H^1(X, \mathscr{O}_X)$ when X is smooth and projective (and a compact complex torus when X is a compact Kähler manifold).

To get a feeling for what can happen, let us consider a few simple examples.

Example 1.3. We clearly have $H^0(X, L) = 0$ unless $L \simeq \mathscr{O}_X$; in fact, any nontrivial global section must be everywhere nonzero because $c_1(L) = 0$.

Example 1.4. Let $n = \dim X$. Serre duality shows that

$$\dim H^n(X, L) = \dim H^0(X, \omega_X \otimes L^{-1}),$$

and in many situations (for example, on curves of genus at least two), the cohomology group on the right is nonzero for every $L \in \mathrm{Pic}^0(X)$.

Example 1.5. Let X be a smooth projective variety, and suppose that we have a morphism $f \colon X \to C$ with connected fibers to a curve of genus at least two. Then $f_*\mathscr{O}_X \simeq \mathscr{O}_C$, and if we look at the first cohomology of f^*L, we get (from the Leray spectral sequence) an exact sequence

$$0 \to H^1\bigl(C, L\bigr) \to H^1\bigl(X, f^*L\bigr) \to H^0\bigl(C, R^1 f_*\mathscr{O}_X \otimes L\bigr) \to 0.$$

Since $H^1(C, L)$ is nonzero for every $L \in \mathrm{Pic}^0(C)$, this means that $\mathrm{Pic}^0(X)$ contains a whole subvariety isomorphic to $\mathrm{Pic}^0(C)$ where the first cohomology is nontrivial. This example was studied very carefully by Beauville; it will appear again later in the course.

The lesson to draw from these three examples is that we cannot expect to have a good vanishing theorem that works for every $L \in \mathrm{Pic}^0(X)$, because there may be special line bundles whose cohomology does not vanish for geometric reasons. Moreover, even if we exclude those special line bundles, the group $H^{\dim X}(X, L)$ will typically be nonzero. The following example shows another interesting phenomenon.

Example 1.6. Suppose that $X = C \times Y$, where C is a curve of genus at least two, and $H^1(Y, \mathscr{O}_Y) = 0$. In that case, any topologically trivial line bundle on X is of the form $p_1^* L$, for some $L \in \operatorname{Pic}^0(C)$, and consequently

$$H^1\bigl(X, p_1^* L\bigr) \simeq H^1\bigl(C, L \otimes p_{1*}\mathscr{O}_X\bigr) \simeq H^1(C, L) \neq 0.$$

Of course, X is "one-dimensional", at least from the point of view of $\operatorname{Pic}^0(X)$.

We can make this last observation more precise. Recall that the *Albanese variety* $\operatorname{Alb}(X)$ is the abelian variety dual to $\operatorname{Pic}^0(X)$; after choosing a base point on X, one has a canonical morphism

$$\operatorname{alb}\colon X \to \operatorname{Alb}(X),$$

with the property that $\operatorname{alb}^*\colon \operatorname{Pic}^0\bigl(\operatorname{Alb}(X)\bigr) \to \operatorname{Pic}^0(X)$ is an isomorphism. As far as topologically trivial line bundles are concerned, what matters is therefore not the dimension of X, but the so-called *Albanese dimension* $\dim \operatorname{alb}(X)$. (In the example above, $\operatorname{Alb}(X)$ is the Jacobian of C, and $\operatorname{alb}(X)$ is isomorphic to the curve C.) With that in mind, here is the result by Green and Lazarsfeld.

Generic Vanishing Theorem. *Let X be a compact Kähler manifold. Then one has $H^i(X, L) = 0$ for $0 \leq i < \dim \operatorname{alb}(X)$ and general $L \in \operatorname{Pic}^0(X)$.*

In the course of the semester, we are going to see three different proofs for this theorem, including the original one by Green and Lazarsfeld, which is based on Hodge theory. The generic vanishing theorem suggests to look at the exceptional sets

$$S_m^i(X) = \bigl\{\, L \in \operatorname{Pic}^0(X) \bigm| \dim H^i(X, L) \geq m \,\bigr\};$$

note that they are closed analytic subsets of $\operatorname{Pic}^0(X)$ by the semicontinuity theorem (and algebraic when X is projective). Green and Lazarsfeld proved several other remarkable results about the structure of those sets; their results, in turn, led to many spectacular applications in algebraic geometry, especially for abelian varieties and irregular varieties (= varieties that admit nontrivial morphisms to abelian varieties).

Outline of the course. Let me now give a brief outline of what we are going to do this semester. First, we will review in some detail the basic results of Hodge theory on compact Kähler manifolds, including the case of coefficients in a topologically trivial line bundle. Next, we will prove the generic vanishing theorem and several other results about the sets $S_m^i(X)$, following the two papers by Green and Lazarsfeld. We will then discuss geometric applications of the theory, for example, the famous paper by Ein and Lazarsfeld in which they study singularities of theta divisors on principally polarized abelian varieties. Up to this point, we will mostly be working with complex manifolds.

In the second part of the course, we will switch over to algebraic varieties. After a brief review of some technical results about derived categories, we will discuss another proof of the generic vanishing theorem, due to Hacon, that uses Mukai's "Fourier transform" on the derived category of an abelian variety. The proof singles out a certain class of coherent sheaves on abelian varieties, called *GV-sheaves*, and after studying their properties, we shall go over some additional geometric applications, including the birational characterization of abelian varieties by Chen and Hacon.

Lecture 2

The idea behind Hodge theory. We shall begin with a review of basic Hodge theory, in the setting of a compact Kähler manifold X. In the process, we will also review the analytic description of topologically trivial line bundles on X, and of $\operatorname{Pic}^0(X)$ and $\operatorname{Alb}(X)$. A good knowledge of Hodge theory is useful not only for this course, but for every algebraic geometer who works with complex algebraic varieties.

Here is a brief outline of what we will need:

(1) Hodge theory on compact oriented Riemannian manifolds. The main theorem is that classes in de Rham cohomology are uniquely represented by harmonic forms; this fundamental fact is proved using results about partial differential equations from analysis.
(2) Hodge theory on compact Kähler manifolds. The Kähler condition guarantees that the harmonic theory is compatible with the complex structure; this is proved with the help of the Kähler identities, which are identities between certain operators on differential forms.
(3) Hodge theory with coefficients in a line bundle whose first Chern class is zero. This case is very similar to the previous one, because the Kähler identities can easily be generalized to this setting.

Let us begin by discussing the case of a compact smooth manifold. Hodge theory tries to solve the problem of finding good representatives for classes in de Rham cohomology. Recall that if M is a smooth manifold, we have the space of smooth real-valued k-forms $A^k(M,\mathbb{R})$, and the exterior derivative d maps $A^k(M,\mathbb{R})$ to $A^{k+1}(M,\mathbb{R})$. The de Rham cohomology groups of M are

$$H^k_{dR}(M,\mathbb{R}) = \frac{\ker\bigl(d\colon A^k(M,\mathbb{R}) \to A^{k+1}(M,\mathbb{R})\bigr)}{\operatorname{im}\bigl(d\colon A^{k-1}(M,\mathbb{R}) \to A^k(M,\mathbb{R})\bigr)}.$$

A class in $H^k_{dR}(M,\mathbb{R})$ is represented by a closed k-form ω, but ω is far from unique, since $\omega+d\psi$ represents the same class for every $\psi \in A^{k-1}(M,\mathbb{R})$. The only exception is the group $H^0_{dR}(M,\mathbb{R})$, whose elements are the locally constant functions.

From now on, we shall assume that M is compact and oriented, of dimension $n = \dim M$. Then $H^n_{dR}(M,\mathbb{R}) \simeq \mathbb{R}$, and once we choose a Riemannian metric g on M, we have the volume form $vol(g) \in A^n(M,\mathbb{R})$; because

$$\int_M vol(g) = \operatorname{vol}(M) \neq 0,$$

its class in $H^n_{dR}(M,\mathbb{R})$ is nonzero. Every class in $H^n_{dR}(M,\mathbb{R})$ therefore does have a distinguished representative, namely a multiple of $vol(g)$. It turns out that, once we have chosen a Riemannian metric, the same is actually true for every cohomology class. Let me explain why.

Recall that g defines an inner product on every tangent space T_pM. It induces inner products on the spaces $\bigwedge^k T_p^*M$, and by integrating over M, we obtain an inner product on the space of forms $A^k(M,\mathbb{R})$. Given a cohomology class in $H^k_{dR}(M,\mathbb{R})$, we can then look for a representative of minimal norm. It is not clear that such a representative exists, but suppose for a moment that we have $\omega \in A^k(M,\mathbb{R})$ with $d\omega = 0$, and such that $\|\omega\| \leq \|\omega + d\psi\|$ for every $\psi \in A^{k-1}(M,\mathbb{R})$. From the inequality

$$\|\omega\|^2 \leq \|\omega + td\psi\|^2 = (\omega + td\psi, \omega + td\psi) = \|\omega\|^2 + 2t(\omega, d\psi) + t^2\|d\psi\|^2,$$

valid for every $t \in \mathbb{R}$, we deduce that $(\omega, d\psi) = 0$. Consequently, ω has minimal size iff it is perpendicular to the space $dA^{k-1}(M,\mathbb{R})$ of d-exact forms. This shows that

ω is unique in its cohomology class, because an exact form that is perpendicular to the space of exact forms is necessarily zero.

An equivalent (but more useful) formulation is the following: Define the *adjoint* operator $d^*\colon A^k(M,\mathbb{R}) \to A^{k-1}(M,\mathbb{R})$ by the condition that

$$(d^*\alpha, \beta) = (\alpha, d\beta)$$

for all $\alpha \in A^k(M,\mathbb{R})$ and all $\beta \in A^{k-1}(M,\mathbb{R})$. Then ω is perpendicular to the space of d-exact forms iff $d^*\omega = 0$. Since also $d\omega = 0$, we can combine both conditions into one by defining the *Laplacian* $\Delta = d \circ d^* + d^* \circ d$; from

$$(\Delta\omega, \omega) = (dd^*\omega + d^*d\omega, \omega) = \|d\omega\|^2 + \|d^*\omega\|^2,$$

we see that ω is d-closed and of minimal norm iff ω is *harmonic*, in the sense that $\Delta\omega = 0$. To summarize:

Proposition 2.1. *Let (M,g) be a compact connected Riemannian manifold, and let $\omega \in A^k(M,\mathbb{R})$ be smooth k-form. The following conditions are equivalent:*

(1) $d\omega = 0$ and ω is of minimal norm in its cohomology class.
(2) $d\omega = 0$ and ω is perpendicular to the space of d-exact forms.
(3) $d\omega = 0$ and $d^*\omega = 0$, or equivalently, $\Delta\omega = 0$.

If ω satisfies any of these conditions, it is unique in its cohomology class, and is called a harmonic form *with respect to the given metric.*

On $\mathbb{R}^n$ with the usual Euclidean metric, $\Delta f = -\sum_i \partial^2 f / \partial x_i^2$ for $f \in A^0(M,\mathbb{R})$, which explains the terminology. In general, the Laplacian $\Delta\colon A^k(M,\mathbb{R}) \to A^k(M,\mathbb{R})$ is an example of an elliptic differential operator, and this fact plays a key role in the theory.

Some linear algebra. Let us pause for a moment and examine the definition of the inner product on the space of forms. This is basically a problem in linear algebra. Let V be a real vector space of dimension n, with inner product $g\colon V \times V \to \mathbb{R}$. (The example we have in mind is $V = T_pM$, with the inner product g_p coming from the Riemannian metric.) The inner product yields an isomorphism

$$\varepsilon\colon V \to V^*, \quad v \mapsto g(v, -),$$

between V and its dual space $V^* = \operatorname{Hom}(V,\mathbb{R})$. If $e_1, \dots, e_n$ is an orthonormal basis for V, then $\varepsilon(e_1), \dots, \varepsilon(e_n)$ is the dual basis in V^*. It becomes an orthonormal basis if we endow V^* with the inner product induced by the isomorphism ε.

All the spaces $\bigwedge^k V$ also acquire inner products, by setting

$$g(u_1 \wedge \dots \wedge u_k, v_1 \wedge \dots \wedge v_k) = \det\bigl(g(u_i, v_j)\bigr)_{i,j=1}^k$$

and extending bilinearly. These inner products have the property that, for any orthonormal basis $e_1, \dots, e_n \in V$, the vectors

$$e_{i_1} \wedge \dots \wedge e_{i_k}$$

with $i_1 < i_2 < \dots < i_k$ form an orthonormal basis for $\bigwedge^k V$.

Now suppose that V is in addition oriented, which means that we choose a generator of the one-dimensional real vector space $\bigwedge^n V$. Recall that the fundamental element $\phi \in \bigwedge^n V$ is the unique positive vector of length 1; we have $\phi = e_1 \wedge \dots \wedge e_n$ for any positively-oriented orthonormal basis.

Definition 2.2. The *$*$-operator* is the unique linear operator $*\colon \bigwedge^k V \to \bigwedge^{n-k} V$ with the property that $\alpha \wedge *\beta = g(\alpha, \beta) \cdot \phi$ for any $\alpha, \beta \in \bigwedge^k V$.

Note that $\alpha \wedge *\beta$ belongs to $\bigwedge^n V$, and is therefore a multiple of the fundamental element ϕ. The $*$-operator is most conveniently defined using an orthormal basis $e_1, \dots, e_n$ for V: for any permutation σ of the set $\{1, \dots, n\}$, we have

$$e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)} = \operatorname{sgn}(\sigma) \cdot e_1 \wedge \cdots \wedge e_n = \operatorname{sgn}(\sigma) \cdot \phi,$$

and consequently

$$*\big(e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)}\big) = \operatorname{sgn}(\sigma) \cdot e_{\sigma(k+1)} \wedge \cdots \wedge e_{\sigma(n)}.$$

This relation shows that $*$ takes an orthonormal basis to an orthonormal basis, and is therefore an isometry: $g(*\alpha, *\beta) = g(\alpha, \beta)$.

Lemma 2.3. *We have* $** \alpha = (-1)^{k(n-k)}\alpha$ *for any* $\alpha \in \bigwedge^k V$.

Proof. Let $\alpha, \beta \in \bigwedge^k V$. By definition of the $*$-operator, we have

$$\begin{aligned}(** \alpha) \wedge (*\beta) &= (-1)^{k(n-k)}(*\beta) \wedge (** \alpha) = (-1)^{k(n-k)} g(*\beta, *\alpha) \cdot \phi \\ &= (-1)^{k(n-k)} g(\alpha, \beta) \cdot \phi = (-1)^{k(n-k)} \alpha \wedge *\beta.\end{aligned}$$

This being true for all β, we conclude that $** \alpha = (-1)^{k(n-k)}\alpha$. ☐

It follows that $*\colon \bigwedge^k V \to \bigwedge^{n-k} V$ is an isomorphism; this may be viewed as an abstract form of Poincaré duality (which says that on a compact oriented manifold, $H^k_{dR}(M, \mathbb{R})$ and $H^{n-k}_{dR}(M, \mathbb{R})$ are dual vector spaces for every $0 \leq k \leq n$).

Harmonic forms and the Hodge theorem. Let (M, g) be a Riemannian manifold that is compact, oriented, and of dimension n. At every point $p \in M$, we have an inner product g_p on the real tangent space $T_p M$, and therefore also on the cotangent space $T^*_p M$ and on each $\bigwedge^k T^*_p M$. In other words, each vector bundle $\bigwedge^k T^*M$ carries a natural Euclidean metric g. This allows us to define an inner product on the space of smooth k-forms $A^k(M, \mathbb{R})$ by the formula

$$(\alpha, \beta)_M = \int_M g\big(\alpha, \beta\big) vol(g).$$

The individual operators $*\colon \bigwedge^k T^*_p M \to \bigwedge^{n-k} T^*_p M$ at each point $p \in M$ determine a a linear mapping

$$*\colon A^k(M, \mathbb{R}) \to A^{n-k}(M, \mathbb{R}).$$

By definition, we have $\alpha \wedge *\beta = g(\alpha, \beta) vol(g)$ as elements of $A^n(M, \mathbb{R})$, and so the inner product can also be expressed by the simpler formula

$$(\alpha, \beta)_M = \int_M \alpha \wedge *\beta.$$

It has the advantage of hiding the terms coming from the metric.

We already know that the exterior derivative d is a first-order linear differential operator. Since the bundles in question carry Euclidean metrics, there is a unique adjoint; the $*$-operator allows us to write down a simple formula for it.

Proposition 2.4. *The adjoint* $d^*\colon A^k(M, \mathbb{R}) \to A^{k-1}(M, \mathbb{R})$ *is given by*

$$d^* = -(-1)^{n(k+1)} * d * .$$

Proof. Fix $\alpha \in A^{k-1}(M, \mathbb{R})$ and $\beta \in A^k(M, \mathbb{R})$. By Stokes' theorem, the integral of $d\big(\alpha \wedge *\beta\big) = d\alpha \wedge *\beta + (-1)^{k-1}\alpha \wedge d(*\beta)$ over M is zero, and therefore

$$(d\alpha, \beta)_M = \int_M d\alpha \wedge *\beta = (-1)^k \int_M \alpha \wedge d * \beta = (-1)^k \int_M \alpha \wedge *\big(*^{-1} d * \beta\big)$$

This shows that the adjoint is given by the formula $d^*\beta = (-1)^k *^{-1} d * \beta$. Since $d * \beta \in A^{n-k+1}(M, \mathbb{R})$, we can use the identity from Lemma 2.3 to compute that

$$d^*\beta = (-1)^k (-1)^{(n-k+1)(k-1)} * d * \beta,$$

from which the assertion follows because $k^2 + k$ is an even number. □

The same method can be used to find adjoints for other operators.

Definition 2.5. For each $0 \leq k \leq n$, we define the *Laplace operator*

$$\Delta \colon A^k(M, \mathbb{R}) \to A^k(M, \mathbb{R})$$

by the formula $\Delta = d \circ d^* + d^* \circ d$. A k-form $\omega \in A^k(M, \mathbb{R})$ is called *harmonic* if $\Delta\omega = 0$, and we let $\mathcal{H}^k(M, \mathbb{R}) = \ker \Delta$ be the space of all harmonic k-forms.

More precisely, Δ is a second-order linear differential operator from the vector bundle $\bigwedge^k T^*M$ to itself. It is easy to see that Δ is formally self-adjoint; indeed, the adjointness of d and d^* shows that

$$(\Delta\alpha, \beta)_M = (d\alpha, d\beta)_M + (d^*\alpha, d^*\beta)_M = (\alpha, \Delta\beta)_M.$$

By computing a formula for Δ in local coordinates, one shows that Δ is an elliptic operator. Because M is compact, one can then apply several deep theorems about elliptic operators from analysis and obtain the following result.

Theorem 2.6. *The space of harmonic k-forms $\mathcal{H}^k(M, \mathbb{R})$ is finite-dimensional. Moreover, one has a direct-sum decomposition*

$$A^k(M, \mathbb{R}) = \mathcal{H}^k(M, \mathbb{R}) \oplus \operatorname{im}\bigl(\Delta \colon A^k(M, \mathbb{R}) \to A^k(M, \mathbb{R})\bigr), \tag{2.7}$$

orthogonal with respect to the inner product on $A^k(M, \mathbb{R})$.

This is the only point where hard analysis is needed; all the other results in Hodge theory follow from this one by more-or-less algebraic methods. For example, we can now state and prove the Hodge theorem.

Theorem 2.8. *Let (M, g) be a compact and oriented Riemannian manifold. Then the natural map $\mathcal{H}^k(M, \mathbb{R}) \to H^k_{dR}(M, \mathbb{R})$ is an isomorphism; in other words, every de Rham cohomology class contains a unique harmonic form.*

Proof. Recall that a form ω is harmonic if and only if $d\omega = 0$ and $d^*\omega = 0$; this follows from the identity $(\Delta\omega, \omega)_M = \|d\omega\|_M^2 + \|d^*\omega\|_M^2$. In particular, harmonic forms are automatically closed, and therefore define classes in de Rham cohomology. We have to show that the resulting map $\mathcal{H}^k(M, \mathbb{R}) \to H^k_{dR}(M, \mathbb{R})$ is bijective.

To prove the injectivity, suppose that $\omega \in \mathcal{H}^k(M, \mathbb{R})$ is harmonic and d-exact, say $\omega = d\psi$ for some $\psi \in A^{k-1}(M, \mathbb{R})$. Then

$$\|\omega\|_M^2 = (\omega, d\psi)_M = (d^*\omega, \psi)_M = 0,$$

and therefore $\omega = 0$. Note that this part of the proof is elementary.

To prove the surjectivity, take an arbitrary cohomology class and represent it by some $\alpha \in A^k(M, \mathbb{R})$ with $d\alpha = 0$. The decomposition in (2.7) shows that we have

$$\alpha = \omega + \Delta\beta = \omega + dd^*\beta + d^*d\beta.$$

with $\omega \in \mathcal{H}^k(M, \mathbb{R})$ harmonic and $\beta \in A^k(M, \mathbb{R})$. Since $d\omega = 0$, we get $0 = d\alpha = dd^*d\beta$, and therefore

$$\|d^*d\beta\|_M^2 = (d^*d\beta, d^*d\beta)_M = (d\beta, dd^*d\beta)_M = 0,$$

proving that $d^*d\beta = 0$. This shows that $\alpha = \omega + dd^*\beta$, and so the harmonic form ω represents the original cohomology class. □

Note. The space of harmonic forms $\mathcal{H}^k(M,\mathbb{R})$ depends on the Riemannian metric, because the definition of the operators d^* and Δ involves the metric. The result above shows that, nevertheless, the dimension of $\mathcal{H}^k(M,\mathbb{R})$ is independent of the choice of metric.

Hodge theory on complex manifolds. Let X be a complex manifold of dimension n. We denote by $A^k(X)$ the space of smooth complex-valued differential k-forms on X. Because of the complex structure, we get a decomposition

$$A^k(X) = \bigoplus_{k=p+q} A^{p,q}(X),$$

where $A^{p,q}(X)$ is the space of differential forms of type (p,q); recall that a differential form has type (p,q) if, in local holomorphic coordinates $z_1,\dots,z_n$, it can be expressed as

$$\alpha = \sum_{i_1,\dots,i_p,j_1,\dots,j_q} \alpha_{i_1,\dots,i_p,j_1,\dots,j_q} dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$$

with smooth functions $\alpha_{i_1,\dots,i_p,j_1,\dots,j_q}$. This does not depend on the particular choice of coordinates, because we are only considering holomorphic coordinate changes. The exterior derivative

$$d\colon A^k(X) \to A^{k+1}(X)$$

decomposes by type into $d = \partial + \bar{\partial}$, with

$$\partial\colon A^{p,q}(X) \to A^{p+1,q}(X) \quad \text{and} \quad \bar{\partial}\colon A^{p,q}(X) \to A^{p,q+1}(X).$$

From $d^2 = 0$ one gets $\partial^2 = 0$, $\bar{\partial}^2 = 0$, and $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

Using those differential operators, one can define two kinds of cohomology groups on a complex manifold X. The first is *de Rham cohomology*, defined as

$$H^k_{dR}(X,\mathbb{C}) = \frac{\ker\bigl(d\colon A^k(X) \to A^{k+1}(X)\bigr)}{\operatorname{im}\bigl(d\colon A^{k-1}(X) \to A^k(X)\bigr)}.$$

By the Poincaré lemma, the complex of sheaves of smooth forms is a soft resolution of the constant sheaf $\mathbb{C}$; this makes $H^k_{dR}(X,\mathbb{C})$ canonically isomorphic to $H^k(X,\mathbb{C})$. The second is *Dolbeault cohomology*, defined as

$$H^{p,q}(X) = \frac{\ker\bigl(\bar{\partial}\colon A^{p,q}(X) \to A^{p,q+1}(X)\bigr)}{\operatorname{im}\bigl(\bar{\partial}\colon A^{p,q-1}(X) \to A^{p,q}(X)\bigr)}.$$

By the holomorphic version of the Poincaré lemma, one has

$$H^{p,q}(X) \simeq H^q(X, \Omega_X^p).$$

We are now going to extend Hodge theory to this setting.

To begin with, we need a Hermitian metric h on the holomorphic tangent bundle of X; in other words, a smoothly varying family of Hermitian inner products on the holomorphic tangent spaces. It induces a Riemannian metric g on the tangent bundle of the smooth manifold X. In local holomorphic coordinates $z_1,\dots,z_n$, the metric h is given by an $n\times n$-matrix H of complex numbers

$$h_{j,k} = h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right),$$

Hermitian symmetric and positive definite at every point. If we let $z_j = x_j + iy_j$, and agree that $x_1,\dots,x_n,y_1,\dots,y_n$ is a positively oriented coordinate system on the underlying smooth manifold, then the corresponding Riemannian metric is given by the matrix

$$\begin{pmatrix} \operatorname{Re} H & \operatorname{Im} H \\ -\operatorname{Im} H & \operatorname{Re} H \end{pmatrix}$$

with respect to the frame $\partial/\partial x_1, \dots, \partial/\partial x_n, \partial/\partial y_1, \dots, \partial/\partial y_n$. In particular, X is also a compact oriented Riemannian manifold.

Now back to Hodge theory and the problem of choosing good representative for cohomology classes. Using the Riemannian metric induced by h, we define the Laplace operator $\Delta\colon A^k(X) \to A^k(X)$ and the subspace of harmonic forms $\mathcal{H}^k(X) \subseteq A^k(X)$ in the same way as last time, and Theorem 2.8 (with coefficients in $\mathbb{C}$) shows that

$$H^k(X, \mathbb{C}) \simeq H^k_{dR}(X, \mathbb{C}) \simeq \mathcal{H}^k(X).$$

Now we would like our theory of harmonic forms to interact nicely with the complex structure on X: for instance, if $\alpha \in \mathcal{H}^k(X)$ is harmonic, we would like each summand in the decomposition $\alpha = \sum \alpha^{p,q}$ to be harmonic, too. But there is no reason why this should be the case if h is an arbitrary Hermitian metric. What we should do instead is to consider only Hermitian metrics that are compatible with the complex structure; they are called *Kähler metrics*. Next time, we will see how the Kähler condition leads to many nice results about harmonic forms, including the Hodge decomposition.

Exercises.

Exercise 2.1. Let $x_1, \dots, x_n$ be a local coordinate system on a Riemannian manifold (M, g); the Riemannian metric can then be described by the $n \times n$-matrix with entries

$$g_{i,j} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Using the local trivialization of $\bigwedge^k T^*M$ by the set of k-forms $dx_{i_1} \wedge \dots \wedge dx_{i_k}$ with $i_1 < \dots < i_k$, find formulas for d^* and Δ. Conclude that Δ is an elliptic differential operator of second order.

Exercise 2.2. Show that the two operators $*$ and Δ commute with each other. Use this to prove the Poincaré duality theorem: on a compact oriented Riemannian manifold of dimension n, the pairing

$$H^k_{dR}(M, \mathbb{R}) \otimes H^{n-k}_{dR}(M, \mathbb{R}) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta,$$

is nondegenerate.

Lecture 3

Hodge theory on complex manifolds. Let X be a compact complex manifold of dimension n. Our task today is to extend Hodge theory to this setting. We choose a Hermitian metric h on the holomorphic tangent bundle of X. Considered as a real vector space of dimension $2n$, the holomorphic tangent space is canonically isomorphic to the tangent space of X, viewed as a smooth manifold of dimension $2n$. Under this identification, our Hermitian metric induces a Riemannian metric $g = \operatorname{Re} h$ on the tangent bundle of the smooth manifold X, as well as a differential form $\omega = -\operatorname{Im} h \in A^2(X, \mathbb{R}) \cap A^{1,1}(X)$. In local holomorphic coordinates $z_1, \dots, z_n$, the metric h is given by an $n \times n$-matrix H of complex numbers

$$h_{j,k} = h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right),$$

Hermitian symmetric and positive definite at every point. Then

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\bar{z}_k,$$

and g is given by the matrix

$$\begin{pmatrix} \operatorname{Re} H & \operatorname{Im} H \\ -\operatorname{Im} H & \operatorname{Re} H \end{pmatrix}$$

with respect to the basis $\partial/\partial x_1, \dots, \partial/\partial x_n, \partial/\partial y_1, \dots, \partial/\partial y_n$, where $z_j = x_j + iy_j$.

We showed last time that $H^k_{dR}(X, \mathbb{C})$ is isomorphic to the space of harmonic forms $\mathcal{H}^k(X)$ (with respect to the Riemannian metric g). In order for the theory of harmonic forms to interact well with the complex structure on X, it is necessary to assume that the Hermitian metric h is Kähler. We shall see below how this condition leads to the following results about harmonic forms:

(1) Every harmonic form $\alpha \in \mathcal{H}^k(X)$ is both ∂-closed and $\bar{\partial}$-closed.
(2) If we expand $\alpha \in \mathcal{H}^k(X)$ by type as
$$\alpha = \sum_{p+q=k} \alpha^{p,q},$$
then each $\alpha^{p,q}$ is again harmonic.
(3) Every class in $H^{p,q}(X)$ contains a unique harmonic (p,q)-form; in particular, all holomorphic $(p,0)$-forms are harmonic.

Kähler manifolds. In this section, we recall the definition of a Kähler metric and discuss some of its consequences.

Definition 3.1. A *Kähler metric* on a complex manifold is a Hermitian metric whose associated $(1,1)$-form is closed. A complex manifold that admits at least one Kähler metric is called a *Kähler manifold.*

This condition is easy to write down—it only takes four symbols—but hard to understand. An equivalent condition is that the complex structure, viewed as an endomorphism of the tangent bundle, should be parallel with respect to the Levi-Cività connection of g; this means that, as we move from point to point, the complex structure changes in a way that is compatible with the Riemannian metric g.

Example 3.2. The standard Euclidean metric on $\mathbb{C}^n$ has associated $(1,1)$-form

$$\frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j,$$

and is therefore Kähler.

Example 3.3. A typical example is the Fubini-Study metric on complex projective space $\mathbb{P}^n$. The associated $(1,1)$-form is, after pulling back via the quotient map $q\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, given by the formula

$$q^* \omega_{\mathbb{P}^n} = \frac{i}{2\pi} \partial\bar{\partial} \log\bigl(|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2\bigr).$$

The formula shows that $d\omega_{\mathbb{P}^n} = 0$, which means that $\omega_{\mathbb{P}^n}$ is a closed form.

Example 3.4. It is easy to see that a complex submanifold of a Kähler manifold is again Kähler; in fact, the restriction of a Kähler metric remains a Kähler metric. It follows that every projective complex manifold is a compact Kähler manifold.

We shall now look at the Kähler condition in local holomorphic coordinates $z_1, \dots, z_n$ on X. With $h_{j,k} = h(\partial/\partial z_j, \partial/\partial z_k)$, the associated $(1,1)$-form is given by the formula

$$\omega = \frac{i}{2} \sum_{j,k=1}^n h_{j,k} dz_j \wedge d\bar{z}_k.$$

Note that the matrix with entries $h_{j,k}$ is necessarily Hermitian-symmetric, and therefore, $h_{k,j} = \overline{h_{j,k}}$. Now we compute that

$$d\omega = \frac{i}{2} \sum_{j,k,l} \frac{\partial h_{j,k}}{\partial z_l} dz_l \wedge dz_j \wedge d\bar{z}_k + \frac{i}{2} \sum_{j,k,l} \frac{\partial h_{j,k}}{\partial \bar{z}_l} dz_j \wedge d\bar{z}_k \wedge d\bar{z}_l,$$

and so $d\omega = 0$ iff $\partial h_{j,k}/\partial z_l = \partial h_{l,k}/\partial z_j$ and $\partial h_{j,k}/\partial \bar{z}_l = \partial h_{j,l}/\partial \bar{z}_k$. The second condition is actually equivalent to the first (which can be seen by conjugating), and this proves that the metric h is Kähler iff

$$\frac{\partial h_{j,k}}{\partial z_l} = \frac{\partial h_{l,k}}{\partial z_j} \tag{3.5}$$

for every $j, k, l \in \{1, \dots, n\}$. We can use this condition to show that, in suitable local coordinates, any Kähler metric looks to first order like the Euclidean metric on $\mathbb{C}^n$.

Lemma 3.6. *A Hermitian metric h is Kähler iff, at every point $x \in X$, there is a holomorphic coordinate system $z_1, \dots, z_n$ centered at x such that*

$$\omega = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j + O(|z|^2).$$

Proof. One direction is very easy: If we can find such a coordinate system centered at a point x, then $d\omega$ vanishes at x; this being true for every $x \in X$, it follows that $d\omega = 0$, and so h is Kähler.

Conversely, assume that $d\omega = 0$, and fix a point $x \in X$. Let $z_1, \dots, z_n$ be arbitrary holomorphic coordinates centered at x, and set $h_{j,k} = h(\partial/\partial z_j, \partial/\partial z_k)$; since we can always make a linear change of coordinates, we may clearly assume that $h_{j,k}(0) = \mathrm{id}_{j,k}$ is the identity matrix. Using that $h_{j,k} = \overline{h_{k,j}}$, we then have

$$h_{j,k} = \mathrm{id}_{j,k} + E_{j,k} + \overline{E_{k,j}} + O(|z|^2),$$

where each $E_{j,k}$ is a linear form in $z_1, \dots, z_n$. Since h is Kähler, (3.5) shows that $\partial E_{j,k}/\partial z_l = \partial E_{l,k}/\partial z_j$; this condition means that there exist quadratic functions $q_j(z)$ such that $E_{k,j} = \partial q_j/\partial z_k$ and $q_j(0) = 0$. Now let

$$w_j = z_j + q_j(z);$$

because the Jacobian $\partial(w_1, \dots, w_n)/\partial(z_1, \dots, z_n)$ is the identity matrix at $z = 0$, the functions $w_1, \dots, w_n$ give holomorphic coordinates in a small enough neighborhood of the point x. By construction,

$$dw_j = dz_j + \sum_{k=1}^n \frac{\partial q_j}{\partial z_k} dz_k = dz_j + \sum_{k=1}^n E_{k,j} dz_k.$$

and so we have, up to second-order terms,

$$\begin{aligned}
\frac{i}{2}\sum_{j=1}^n dw_j \wedge d\bar{w}_j &\equiv \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j + \frac{i}{2}\sum_{j,k=1}^n dz_j \wedge \overline{E_{k,j}} d\bar{z}_k + \frac{i}{2}\sum_{j,k=1}^n E_{k,j} dz_k \wedge d\bar{z}_j \\
&= \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j + \frac{i}{2}\sum_{j,k=1}^n \bigl(E_{j,k} + \overline{E_{k,j}}\bigr) dz_j \wedge d\bar{z}_k \\
&\equiv \frac{i}{2}\sum_{j,k=1}^n h_{j,k} dz_j \wedge d\bar{z}_k.
\end{aligned}$$

which shows that $\omega = \frac{i}{2}\sum_{j,k} dw_j \wedge d\bar{w}_k + O(|w|^2)$ in the new coordinate system. $\square$

This lemma is extremely useful for proving results about arbitrary Kähler metrics. One consequence is that any statement about a Kähler manifold that involves the metric and its first derivatives (but no derivatives of higher order) is true in general once it is true for the Euclidean metric on $\mathbb{C}^n$. We shall use this method below to prove the so-called Kähler identities.

The Kähler identities. Let (X, h) be a Kähler manifold, and denote by $g = \operatorname{Re} h$ the associated Riemannian metric; we sometimes refer to the associated $(1,1)$-form $\omega \in A^{1,1}(X)$ as the *Kähler form*. Since ω is closed, it defines a class in $H^2_{dR}(X, \mathbb{C})$; if X is compact, this class is necessarily nonzero. Indeed, a local calculation shows that $\omega^{\wedge n} = n!vol(g)$ is a multiple of the volume form; if we integrate this identity over X, we get $\int_X \omega^{\wedge n} = n! \operatorname{vol}(X) \neq 0$, and so ω cannot be exact.

In fact, we have a whole collection of operators on X; our next goal is to establish several relations among them, collectively known as the Kähler identities. To begin with, we have the operators

$$d\colon A^k(X) \to A^{k+1}(X), \quad \partial\colon A^{p,q}(X) \to A^{p+1,q}(X), \quad \bar{\partial}\colon A^{p,q}(X) \to A^{p,q+1}(X).$$

There are also the adjoints

$$d^*\colon A^k(X) \to A^{k-1}(X), \quad \partial^*\colon A^{p,q}(X) \to A^{p-1,q}(X), \quad \bar{\partial}^*\colon A^{p,q}(X) \to A^{p,q-1}(X)$$

with respect to the inner product $g(\alpha, \beta) = \int_X \alpha \wedge *\beta$. We also get two additional operators from the Kähler form ω: taking the wedge product with ω defines the so-called *Lefschetz operator*

$$L_\omega\colon A^{p,q}(X) \to A^{p+1,q+1}(X), \quad \alpha \mapsto \omega \wedge \alpha;$$

note that ω has type $(1,1)$. We also define its adjoint

$$\Lambda_\omega\colon A^k(X) \to A^{k-2}(X)$$

by the condition that $g(L_\omega \alpha, \beta) = g(\alpha, \Lambda_\omega \beta)$; we can get a formula for Λ_ω involving the $*$-operator by noting that

$$g(L_\omega \alpha, \beta) \cdot vol(g) = \omega \wedge \alpha \wedge *\beta = \alpha \wedge (\omega \wedge *\beta) = \alpha \wedge (L_\omega * \beta);$$

consequently, $\Lambda_\omega \beta = *^{-1} L_\omega * \beta = (-1)^k * L_\omega * \beta$ because $*^2 = (-1)^k \operatorname{id}$ by Lemma 2.3. Now it turns out that when the metric is Kähler, the adjoints ∂^* and $\bar{\partial}^*$ can be expressed in terms of ∂, $\bar{\partial}$, and Λ_ω.

Theorem 3.7. *On a Kähler manifold (X, h), the following identities are true:*

$$[\Lambda_\omega, \bar\partial] = -i\partial^* \quad \text{and} \quad [\Lambda_\omega, \partial] = i\bar\partial^*.$$

Since the two identities are conjugates of each other, it suffices to prove the second one. Moreover, both involve only the metric h and its first derivatives, and so they hold on a general Kähler manifold as soon as they are known on $\mathbb{C}^n$ with the Euclidean metric h. In this metric, dz_j is orthogonal to every $d\bar z_k$, and to every dz_k with $k \neq j$, while

$$h(dz_j, dz_j) = h(dx_j + idy_j, dx_j + idy_j) = g(dx_j, dx_j) + g(dy_j, dy_j) = 2.$$

More generally, we have $h(dz_J \wedge d\bar z_K, dz_J \wedge d\bar z_K) = 2^{|J|+|K|}$ for any pair of multi-indices $J, K \subseteq \{1, \dots, n\}$.

To facilitate the computation, we introduce a few additional but more basic operators on the spaces $A^{p,q} = A^{p,q}(\mathbb{C}^n)$. First, define

$$e_j \colon A^{p,q} \to A^{p+1,q}, \quad \alpha \mapsto dz_j \wedge \alpha$$

as well as its conjugate

$$\bar e_j \colon A^{p,q} \to A^{p,q+1}, \quad \alpha \mapsto d\bar z_j \wedge \alpha.$$

We then have

$$L_\omega \alpha = \omega \wedge \alpha = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar z_j \wedge \alpha = \frac{i}{2} \sum_{j=1}^n e_j \bar e_j \alpha.$$

Using the induced Hermitian inner product on forms, we then define the adjoint

$$e_j^* \colon A^{p,q} \to A^{p-1,q}$$

by the condition that $h(e_j \alpha, \beta) = h(\alpha, e_j^* \beta)$.

Lemma 3.8. *The adjoint e_j^* has the following properties:*

(1) If $j \notin J$, then $e_j^*(dz_J \wedge d\bar z_K) = 0$, while $e_j^*(dz_j \wedge dz_J \wedge d\bar z_K) = 2dz_J \wedge d\bar z_K$.
(2) $e_k e_j^* + e_j^* e_k = 2\,\mathrm{id}$ in case $j = k$, and 0 otherwise.

Proof. By definition, we have

$$h(e_j^* dz_J \wedge d\bar z_K, dz_L \wedge d\bar z_M) = h(dz_J \wedge d\bar z_K, dz_j \wedge dz_L \wedge d\bar z_M),$$

and since dz_j occurs only in the second term, the inner product is always zero, proving that $e_j^* dz_J \wedge d\bar z_K = 0$. On the other hand,

$$\begin{aligned} h(e_j^* dz_j \wedge dz_J \wedge d\bar z_K, dz_L \wedge d\bar z_M) &= h(dz_j \wedge dz_J \wedge d\bar z_K, dz_j \wedge dz_L \wedge d\bar z_M) \\ &= 2h(dz_J \wedge d\bar z_K, dz_L \wedge d\bar z_M), \end{aligned}$$

which is nonzero exactly when $J = L$ and $K = M$. From this identity, it follows that $e_j^* dz_j \wedge dz_J \wedge d\bar z_K = 2dz_J \wedge d\bar z_K$, establishing (1).

To prove (2) for $j = k$, observe that since $dz_j \wedge dz_j = 0$, we have

$$e_j^* e_j \big(dz_J \wedge d\bar z_K\big) = \begin{cases} 0 & \text{if } j \in J, \\ 2dz_J \wedge d\bar z_K & \text{if } j \notin J, \end{cases}$$

while

$$e_j e_j^* \big(dz_J \wedge d\bar z_K\big) = \begin{cases} 2dz_J \wedge d\bar z_K & \text{if } j \in J, \\ 0 & \text{if } j \notin J. \end{cases}$$

Taken together, this shows that $e_j e_j^* + e_j^* e_j = 2\,\mathrm{id}$. Finally, let us prove that $e_k e_j^* + e_j^* e_k = 0$ when $j \neq k$. By (1), this is clearly true on $dz_J \wedge d\bar z_K$ in case $j \notin J$. On the other hand,

$$e_k e_j^* \big(dz_j \wedge dz_J \wedge d\bar z_K\big) = 2e_k \big(dz_J \wedge d\bar z_K\big) = 2dz_k \wedge dz_J \wedge d\bar z_K$$

and

$$e_j^* e_k \big(dz_j \wedge dz_J \wedge d\bar{z}_K \big) = e_j^* \big(dz_k \wedge dz_j \wedge dz_J \wedge d\bar{z}_K \big) = -2 dz_k \wedge dz_J \wedge d\bar{z}_K,$$

and the combination of the two proves the asserted identity. □

We also define the differential operator

$$\partial_j \colon A^{p,q} \to A^{p,q}, \quad \sum_{J,K} \varphi_{J,K} dz_J \wedge d\bar{z}_K \mapsto \sum_{J,K} \frac{\partial \varphi_{J,K}}{\partial z_j} dz_J \wedge d\bar{z}_K$$

and its conjugate

$$\bar{\partial}_j \colon A^{p,q} \to A^{p,q}, \quad \sum_{J,K} \varphi_{J,K} dz_J \wedge d\bar{z}_K \mapsto \sum_{J,K} \frac{\partial \varphi_{J,K}}{\partial \bar{z}_j} dz_J \wedge d\bar{z}_K.$$

Clearly, both commute with the operators e_j and e_j^*, as well as with each other. As before, let ∂_j^* be the adjoint of ∂_j, and $\bar{\partial}_j^*$ that of $\bar{\partial}_j$; integration by parts (against compactly supported forms) proves the following lemma.

Lemma 3.9. *We have $\partial_j^* = -\bar{\partial}_j$ and $\bar{\partial}_j^* = -\partial_j$.*

We now turn to the proof of the crucial identity $[\Lambda_\omega, \partial] = i\bar{\partial}^*$.

Proof. All the operators in the identity can be expressed in terms of the basic ones, as follows. Firstly, $L_\omega = \frac{i}{2} \sum e_j \bar{e}_j$, and so the adjoint is given by the formula $\Lambda_\omega = -\frac{i}{2} \sum \bar{e}_j^* e_j^*$. Quite evidently, we have $\partial = \sum \partial_j e_j$ and $\bar{\partial} = \sum \bar{\partial}_j \bar{e}_j$, and after taking adjoints, we find that $\partial^* = -\sum \bar{\partial}_j e_j^*$ and that $\bar{\partial}^* = -\sum \partial_j \bar{e}_j^*$. Using these expressions, we compute that

$$\Lambda_\omega \partial - \partial \Lambda_\omega = -\frac{i}{2} \sum_{j,k} \Big(\bar{e}_j^* e_j^* \partial_k e_k - \partial_k e_k \bar{e}_j^* e_j^* \Big) = -\frac{i}{2} \sum_{j,k} \partial_k \Big(\bar{e}_j^* e_j^* e_k - e_k \bar{e}_j^* e_j^* \Big).$$

Now $\bar{e}_j^* e_j^* e_k - e_k \bar{e}_j^* e_j^* = \bar{e}_j^* (e_j^* e_k + e_k e_j^*)$, which equals $2\bar{e}_j^*$ in case $j = k$, and is zero otherwise. We conclude that

$$\Lambda_\omega \partial - \partial \Lambda_\omega = -i \sum_j \partial_j \bar{e}_j^* = i\bar{\partial}^*,$$

which is the Kähler identity we were after. □

These two basic identities lead to many wonderful relations between various operators on a Kähler manifold; we shall discuss here only the most important one.

Theorem 3.10. *On a Kähler manifold, the Laplace operator satisfies*

$$\frac{1}{2} \Delta = \partial \partial^* + \partial^* \partial = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}.$$

Proof. To simplify the notation, we set $\Box = \partial \partial^* + \partial^* \partial$. By definition,

$$\Delta = dd^* + d^* d = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}).$$

In order for this to equal $2\Box$, a lot of terms will have to cancel, obviously. According to the second identity in Theorem 3.7, we have $\bar{\partial}^* = i\partial\Lambda_\omega - i\Lambda_\omega \partial$, and therefore

$$\begin{aligned} \Delta &= (\partial + \bar{\partial})(\partial^* - i\Lambda_\omega \partial + i\partial\Lambda_\omega) + (\partial^* - i\Lambda_\omega \partial + i\partial\Lambda_\omega)(\partial + \bar{\partial}) \\ &= \partial\partial^* + \bar{\partial}\partial^* - i\bar{\partial}\Lambda_\omega\partial + i\bar{\partial}\partial\Lambda_\omega + \partial^*\partial + \partial^*\bar{\partial} - i\Lambda_\omega\partial\bar{\partial} + i\partial\Lambda_\omega\bar{\partial}. \end{aligned}$$

Now $\partial^* \bar{\partial} = i[\Lambda_\omega, \bar{\partial}]\bar{\partial} = -i\bar{\partial}\Lambda_\omega\bar{\partial} = -\partial^*\bar{\partial}$ by the other Kähler identity. The above formula consequently simplifies to

$$\begin{aligned} \Delta &= \Box - i\bar{\partial}\Lambda_\omega\partial + i\bar{\partial}\partial\Lambda_\omega - i\Lambda_\omega\partial\bar{\partial} + i\partial\Lambda_\omega\bar{\partial} = \Box - i\bar{\partial}\Lambda_\omega\partial - i\partial\bar{\partial}\Lambda_\omega + i\Lambda_\omega\bar{\partial}\partial + i\partial\Lambda_\omega\bar{\partial} \\ &= \Box + i\partial(\Lambda_\omega\bar{\partial} - \bar{\partial}\Lambda_\omega) + i(\Lambda_\omega\bar{\partial} - \bar{\partial}\Lambda_\omega)\partial = \Box + \partial\partial^* + \partial^*\partial = 2\Box. \end{aligned}$$

The second formula for Δ follows from this by conjugation. □

It is really this formula for the Laplace operator that we were after; the identities in Theorem 3.7 are just an intermediate result. As a matter of fact, we could have also proved the identity $\Delta = 2(\partial\partial^* + \partial^*\partial)$ by a calculation on $\mathbb{C}^n$: although the formula does involve second derivatives of the metric, those terms can be shown to be the same on both sides. But it is considerably easier to prove the more basic identity $[\Lambda_\omega, \partial] = i\bar{\partial}^*$ and then use algebra to get the result.

Corollary 3.11. *On a Kähler manifold, one has $\Delta A^{p,q}(X) \subseteq A^{p,q}(X)$. Moreover, any harmonic form is both ∂-closed and $\bar{\partial}$-closed.*

Proof. This follows from the identity in Theorem 3.10, because $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ clearly preserves the type of a form. If $\Delta\alpha = 0$, then we have $(\partial\partial^* + \partial^*\partial)\alpha = 0$; but then

$$0 = g(\alpha, \partial\partial^*\alpha) + g(\alpha, \partial^*\partial\alpha) = g(\partial^*\alpha, \partial^*\alpha) + g(\partial\alpha, \partial\alpha),$$

and therefore $\partial\alpha = 0$. A similar argument proves that $\bar{\partial}\alpha = 0$. $\square$

The Hodge decomposition. We have seen that the Laplace operator Δ preserves the type of a form. It follows that if a form $\alpha \in A^k(X)$ is harmonic, then its components $\alpha^{p,q} \in A^{p,q}(X)$ are also harmonic. Indeed, we have

$$0 = \Delta\alpha = \sum_{p+q=k} \Delta\alpha^{p,q},$$

and since each $\Delta\alpha^{p,q}$ belongs again to $A^{p,q}(X)$, we see that $\Delta\alpha^{p,q} = 0$.

Corollary 3.12. *On a compact Kähler manifold X, the space of harmonic forms decomposes by type as*

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X),$$

where $\mathcal{H}^{p,q}(X)$ is the space of (p,q)-forms that are harmonic.

Since we know from Theorem 2.8 that every cohomology class contains a unique harmonic representative, we now obtain the famous *Hodge decomposition* of the de Rham cohomology of a compact Kähler manifold. We state it in a way that is independent of the choice of Kähler metric.

Theorem 3.13. *Let X be a compact Kähler manifold. Then the cohomology of X admits a direct sum decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}, \tag{3.14}$$

with $H^{p,q}$ equal to the set of those cohomology classes that are represented by a d-closed form of type (p,q). We have $H^{q,p} = \overline{H^{p,q}}$, where complex conjugation is with respect to the real structure $H^k(X, \mathbb{R})$; moreover, $H^{p,q}$ is isomorphic to the Dolbeault cohomology group $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$.

Proof. Since X is a Kähler manifold, it admits a Kähler metric h, and we can consider forms that are harmonic for this metric. Such forms are always closed, and therefore define cohomology classes in $H^k_{dR}(X, \mathbb{C}) \simeq H^k(X, \mathbb{C})$. Let us prove first that the subspace $H^{p,q}$ is precisely the image of $\mathcal{H}^{p,q}(X)$. It is clear from the definition that $\mathcal{H}^{p,q}(X)$ maps into $H^{p,q}$. Now consider a class in $H^{p,q}$, represented by closed form $\alpha \in A^{p,q}(X)$. By Theorem 2.6, we have a unique decomposition

$$\alpha = \alpha_0 + \Delta\beta$$

with α_0 harmonic. Taking (p,q)-components, and using the fact that Δ preserves the type of a form, we conclude that $\alpha_0 \in \mathcal{H}^{p,q}(X)$ and $\beta \in A^{p,q}(X)$. Now the class of α_0 is equal to the class of α, and so $\mathcal{H}^{p,q}(X) \simeq H^{p,q}$, as claimed.

By Theorem 2.8, every class in $H^k_{dR}(X, \mathbb{C})$ contains a unique harmonic form. We therefore obtain the asserted decomposition of $H^k(X, \mathbb{C})$ from Corollary 3.12. Now Δ is a real operator, and so the conjugate of a harmonic (p, q)-form is a harmonic (q, p)-form; this clearly implies that $\overline{H^{p,q}} = H^{q,p}$. Finally, every harmonic form is automatically $\bar\partial$-closed, and so we have $H^{p,q} \simeq \mathcal{H}^{p,q}(X) \simeq H^{p,q}(X)$. □

Recall the definition of the sheaf Ω_X^p holomorphic p-forms: its sections are smooth $(p, 0)$-forms that can be expressed in local coordinates as

$$\alpha = \sum_{j_1 < \cdots < j_p} f_{j_1,\dots,j_p} dz_{j_1} \wedge \cdots \wedge dz_{j_p},$$

with locally defined holomorphic functions $f_{j_1,\dots,j_p}$. This expression shows that $\bar\partial\alpha = 0$. A useful (and not obvious) fact is that on a compact Kähler manifold, every global holomorphic form is harmonic, and hence closed.

Corollary 3.15. *On a compact Kähler manifold X, every holomorphic form is harmonic, and so we get an embedding $H^0(X, \Omega_X^p) \hookrightarrow H^p(X, \mathbb{C})$ whose image is precisely the subspace $H^{p,0}$.*

Proof. If $\alpha \in A^{p,0}(X)$ is holomorphic, it satisfies $\bar\partial\alpha = 0$; on the other hand, we trivially have $\bar\partial^*\alpha = 0$ because there are no forms of type $(p, -1)$. Thus $\Delta\alpha = 2(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\alpha = 0$, and so α is indeed harmonic. □

Example 3.16. For $H^1(X, \mathbb{C})$, the Hodge decomposition is

$$H^1(X, \mathbb{C}) \simeq H^{1,0}(X) \oplus H^{0,1}(X) \simeq H^0(X, \Omega_X^1) \oplus H^1(X, \mathscr{O}_X).$$

Consequently, any cohomology class can be uniquely written in the form $\omega_1 + \overline{\omega_2}$, where ω_1 and ω_2 are holomorphic one-forms. It also follows that

$$\dim H^1(X, \mathbb{C}) = 2 \dim H^0(X, \Omega_X^1).$$

The principle of two types. Another property of compact Kähler manifolds that is used very often in complex geometry is the following "principle of two types" (sometimes also called the $\partial\bar\partial$-lemma).

Proposition 3.17. *Let X be a compact Kähler manifold, and let α be a smooth form that is both ∂-closed and $\bar\partial$-closed. If α is also either ∂-exact or $\bar\partial$-exact, then it can be written as $\alpha = \partial\bar\partial\beta$.*

Proof. We shall suppose that $\alpha = \bar\partial\phi$. Let $\phi = \phi_0 + \Delta\psi$ be the decomposition given by (2.7), with ϕ_0 harmonic. We then have $\bar\partial\phi_0 = 0$ on account of Corollary 3.11. Using the previously mentioned identity $\bar\partial\partial^* = -\partial^*\bar\partial$, we compute that

$$\alpha = \bar\partial\phi = \bar\partial\Delta\psi = 2\bar\partial(\partial\partial^* + \partial^*\partial)\psi = -2\partial\bar\partial(\partial^*\psi) - 2\partial^*\bar\partial\partial\psi.$$

Now $\partial\alpha = 0$, and so the form $\partial^*\bar\partial\partial\psi$ belongs to $\ker\partial \cap \operatorname{im}\partial^* = \{0\}$. Consequently, we have $\alpha = \partial\bar\partial\beta$ with $\beta = -2\partial^*\psi$. □

Exercises.

Exercise 3.1. Let (V, h) be an n-dimensional complex vector space with a Hermitian inner product. Show that $\operatorname{Re} h$ defines an inner product on the underlying real vector space $V_{\mathbb{R}}$, and that $-\operatorname{Im} h$ is naturally an element of $\bigwedge^2 V_{\mathbb{R}}^*$. Now take any basis $v_1, \dots, v_n \in V$, and denote by $v_1^*, \dots, v_n^* \in V^*$ be the dual basis. Show that

$$-\operatorname{Im} h = \frac{i}{2} \sum_{j,k=1}^{n} h_{j,k} v_j^* \overline{v_k^*},$$

where $h_{j,k} = h(v_j, v_k)$. Conclude that $-\operatorname{Im} h$ is a vector of type $(1, 1)$.

Exercise 3.2. Prove that the Laplace operator commutes with the operators L_ω and Λ_ω. Deduce that the Kähler form ω is a harmonic form. Since harmonic forms are closed, this shows that the metric is Kähler if and only if ω is harmonic.

Exercise 3.3. Show that the operator $*$ maps $A^{p,q}(X)$ into $A^{n-q,n-p}(X)$. Prove that Δ commutes with $*$, and deduce that

$$H^q(X, \Omega_X^p) \simeq H^{n-q}(X, \Omega_X^{n-p}),$$

which is a version of Serre duality.

Exercise 3.4. Consider the double complex $A^{\bullet,\bullet}(X)$, with differentials ∂ and $\bar\partial$; note that $\partial\bar\partial = -\bar\partial\partial$. Since the associated simple complex is $\bigl(A^\bullet(X), d\bigr)$, we get a spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Longrightarrow H^{p+q}(X, \mathbb{C}),$$

called the Hodge-de Rham spectral sequence. Prove that this spectral sequence degenerates at E_1 when X is a compact Kähler manifold.

Exercise 3.5. Let X be a compact Kähler manifold. The point of this exercise is to consider the cohomology of the complex $\bigl(A^\bullet(X), u\partial + v\bar\partial\bigr)$, for different values of $[u,v] \in \mathbb{P}^1$; if you listened to Sabbah's lecture last week, you might find this question interesting. In the chart $\mathbb{C}_u$, we consider the complex

$$\bigl(A^\bullet(X) \otimes_{\mathbb{C}} \mathbb{C}[u], u\partial + \bar\partial\bigr),$$

and in the chart $\mathbb{C}_v$, the complex

$$\bigl(A^\bullet(X) \otimes_{\mathbb{C}} \mathbb{C}[v], \partial + v\bar\partial\bigr).$$

Show that $\alpha \otimes f(u) \mapsto \alpha \otimes v^{\deg \alpha} f(v^{-1})$ is compatible with the two differentials. Then prove that the k-th cohomology of both complexes determines a coherent sheaf on $\mathbb{P}^1$ that is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(k)^{\oplus \dim H^k(X,\mathbb{C})}$.

Lecture 4

Application to vanishing theorems. Today, we will take a break from Hodge theory, and discuss vanishing theorems again. The original proof of the Kodaira vanishing theorem was differential-geometric, but there is also a very beautiful proof using the results in Hodge theory from last time. My goal is to present that proof, as well as a useful generalization by Kollár. Here is the statement again.

Kodaira Vanishing Theorem. *Let L be an ample line bundle on a smooth projective variety X. Then $H^i(X, \omega_X \otimes L) = 0$ for every $i > 0$.*

If we replace L by L^N for sufficiently large N, the cohomology groups vanish by Serre's theorem (which is proved by elementary arguments). The idea of the proof is to reduce the problem to that special case with the help of a geometric construction and Hodge theory.

Covering construction. Let me describe the geometric construction first; for later use, we will work in a more general setting. We fix a smooth projective variety X, a line bundle L, and an integer $N \geq 2$, and assume that there is a nontrivial section $s \in H^0(X, L^N)$ whose associated divisor $D = Z(s)$ is nonsingular. In this situation, one can construct a branched covering

$$\pi \colon Y \to X$$

of degree N, by "extracting N-th roots of s". Abstractly, Y can be obtained by taking the spectrum of the sheaf of $\mathscr{O}_X$-algebras

$$\bigoplus_{i=0}^{N-1} L^{-i};$$

the multiplication is defined by using the morphism $L^{-N} \to \mathscr{O}_X$ given by s. For a more geometric definition, recall that the actual line bundle is given by $\mathbb{V}(L) \to X$, where $\mathbb{V}(L)$ is the spectrum of $\mathrm{Sym}(L^{-1})$. Taking N-th powers gives a morphism

$$\begin{tikzcd} \mathbb{V}(L) \arrow[r] \arrow[dr] & \mathbb{V}(L^N) \arrow[d] \\ & X \arrow[u, bend right, "s"'] \end{tikzcd} \tag{4.1}$$

and Y can be defined as the preimage of $s(X)$ inside $\mathbb{V}(L)$. I will not bother to show that the two definitions are equivalent, because we are going to adopt a more useful definition in local coordinates. (Along the way, we will discover that the local definition is equivalent to the other two.)

We can cover X by affine open sets U on which L is trivial; let

$$\phi_U \colon L\big|_U \to \mathscr{O}_U$$

be the local trivialization, and set $\ell_U = \phi_U^{-1}(1)$. Now $s\big|_U = f_U \ell_U^N$ is represented by a regular function $f_U \in H^0(U, \mathscr{O}_X)$; note that f_U is a local equation for D. To extract the N-th root of the section, we take a new variable t_U, and define

$$\tilde{U} = \operatorname{Spec} \frac{H^0(U, \mathscr{O}_X)[t_U]}{(t_U^N - f_U)} \to U.$$

Note that this is a finite morphism of degree N.

Now we glue these various affine varieties together to obtain Y. Given another affine open set $V \subseteq X$ as above, we have a transition function

$$g_{U,V} = (\phi_U \circ \phi_V^{-1})(1) = \phi_U(\ell_V) \in H^0(U \cap V, \mathscr{O}_X^\times),$$

satisfying $g_{U,V}\ell_U = \ell_V$ on $U \cap V$. An easy calculation shows that $f_U = g_{U,V}^N f_V$; we can therefore glue $\tilde{U}$ and $\tilde{V}$ according to the rule

$$t_U = g_{U,V} t_V.$$

To check that this produces a well-defined variety Y, we have to use the cocycle condition $g_{U,V} g_{V,W} = g_{U,W}$ on $U \cap V \cap W$. By construction, Y comes with a finite morphism $\pi \colon Y \to X$ that ramifies exactly over the divisor D.

Note. The formula $t_U = g_{U,V} t_V$ shows that the functions $t_U \in H^0(\tilde{U}, \mathscr{O}_Y)$ determine a global section of the line bundle $\pi^* L$. The divisor of this section is mapped isomorphically to D under π. This can be used to show that Y embeds into $\mathbb{V}(L)$, relating our local construction with the geometric one from above.

Let us now analyze the construction a bit more carefully. On the one hand, we would like to show that Y is again a smooth projective variety; on the other hand, we need to describe the cohomology of the sheaves $\mathscr{O}_Y$ and Ω_Y^p in terms of X.

Proposition 4.2. *The variety Y is smooth and projective, and we have*

$$\pi_* \mathscr{O}_Y \simeq \bigoplus_{i=0}^{N-1} L^{-i}.$$

Moreover, for every $p \geq 1$, we have

$$\pi_* \Omega_Y^p \simeq \Omega_X^p \oplus \bigoplus_{i=1}^{N-1} \Omega_X^p(\log D) \otimes L^{-i},$$

where $\Omega_X^p(\log D)$ is the sheaf of logarithmic differential forms.

Proof. By construction, the coherent sheaf $\pi_* \mathscr{O}_Y$ corresponds, over the affine open set U, to the $H^0(U, \mathscr{O}_X)$-algebra

$$H^0(\tilde{U}, \mathscr{O}_Y) = \frac{H^0(U, \mathscr{O}_X)[t_U]}{(t_U^N - f_U)} \simeq \bigoplus_{i=0}^{N-1} H^0(U, \mathscr{O}_X)\, t_U^i.$$

Note that we can describe the i-th summand in a coordinate-free way as follows: the group of N-th roots of unity acts on Y, by sending t_U to ζt_U for a primitive N-th root of unity ζ; the i-th summand is precisely the ζ^i-eigenspace of this action. We therefore obtain a well-defined locally free $\mathscr{O}_X$-module of finite rank, or in other words, a line bundle on X. To see that this line bundle is L^{-i}, note that the transition functions are given by $g_{U,V}^{-i}$, because $t_U^i = g_{U,V}^i t_V^i$ by our gluing rule. This proves the formula for $\pi_* \mathscr{O}_Y$.

To prove the remaining assertions, we have to use the fact that D is nonsingular. We can choose each affine open set $U \subseteq X$ in such a way that $f_U = x_1$ is part of a coordinate system $x_1, \dots, x_n$ on U; here $n = \dim X$. Now $\tilde{U} \subseteq Y$ has coordinates $y_1, \dots, y_n$, where $t_U = y_1$, and the morphism $\tilde{U} \to U$ is given by the formula

$$(y_1, \dots, y_n) \mapsto (y_1^N, y_2, \dots, y_n).$$

This shows that Y is again smooth; being finite over a projective variety, it is automatically projective as well. For simplicity, we prove the second formula only for $p = 1$. Then $\Omega_X^1(\log D)$ is generated on U by the differential forms

$$\frac{dx_1}{x_1}, dx_2, \dots, dx_n.$$

By construction, $\pi_* \Omega_Y^1$ corresponds to the $H^0(U, \mathscr{O}_X)$-module

$$H^0(\tilde{U}, \Omega_Y^1) = \bigoplus_{k=1}^{n} H^0(\tilde{U}, \mathscr{O}_Y)\, dy_k = \bigoplus_{k=1}^{n} \bigoplus_{i=0}^{N-1} H^0(U, \mathscr{O}_X)\, y_1^i dy_k.$$

We have $y_1^i dy_k = y_1^i dx_k$ for $k \geq 2$, and

$$y_1^i dy_1 = y_1^{i+1}\frac{dy_1}{y_1} = \frac{1}{N} y_1^{i+1}\frac{dx_1}{x_1}.$$

By grouping the summands according to their degree in y_1, which corresponds to looking at the eigenspaces for the action by the N-th roots of unity, we then obtain the desired formula for $\pi_* \Omega_Y^1$. □

Now we can draw some conclusions about X from the covering construction. The isomorphisms in Proposition 4.2 give us direct sum decompositions

$$\begin{aligned} H^j(Y, \mathscr{O}_Y) &\simeq H^j(X, \mathscr{O}_X) \oplus \bigoplus_{i=1}^{N} H^j(X, L^{-i}) \\ H^j(Y, \Omega_Y^1) &\simeq H^j(X, \Omega_X^1) \oplus \bigoplus_{i=1}^{N-1} H^j\big(X, \Omega_X^1(\log D) \otimes L^{-i}\big). \end{aligned}$$

Consider the exterior derivative $d\colon \mathscr{O}_Y \to \Omega_Y^1$, which is a $\mathbb{C}$-linear morphism of sheaves. Now comes the crucial observation from Hodge theory: because Y is a smooth projective variety, the induced mapping on cohomology

$$d\colon H^j(Y, \mathscr{O}_Y) \to H^j(Y, \Omega_Y^1)$$

is zero! The reason is that elements of $H^j(Y, \mathscr{O}_Y)$ can be represented by harmonic $(0, j)$-forms, and every harmonic form is automatically d-closed. It is easy to see that d is compatible with the decomposition above; it follows that

$$H^j(X, L^{-1}) \to H^j\big(X, \Omega_X^1(\log D) \otimes L^{-1}\big)$$

is also zero. If we compose with the residue mapping $\operatorname{Res}_D\colon \Omega_X^1(\log D) \to \mathscr{O}_D$, we find that the induced mapping

$$H^j(X, L^{-1}) \to H^j(D, \mathscr{O}_D \otimes L^{-1}) \tag{4.3}$$

is zero, too.

Lemma 4.4. *Up to a factor of N, the mapping in (4.3) is the restriction mapping.*

Proof. We work in local coordinates. The summand L^{-1} in the decomposition corresponds to elements of $H^0(\tilde{U}, \mathscr{O}_Y)$ of the form $f y_1$, for $f \in H^0(U, \mathscr{O}_X)$. Now

$$d(f y_1) = df y_1 + f dy_1 = y_1 \left(df + \frac{f}{N}\frac{dx_1}{x_1} \right)$$

is clearly a section of the summand $\Omega_X^1(\log D) \otimes L^{-1}$. Moreover,

$$\operatorname{Res}_{x_1=0} d(f y_1) = y_1 \frac{f}{N}\bigg|_{x_1=0},$$

and this agrees with the restriction of $y_1 f$ after multiplying by N. This argument proves that the composition

$$\operatorname{Res}_D \circ \pi_*(d)\colon L^{-1} \to \Omega_X^1(\log D) \otimes L^{-1} \to \mathscr{O}_D \otimes L^{-1}$$

is equal to the restriction mapping up to a factor of N; we now get the desired result by passing to cohomology. □

To see what (4.3) means, recall that we have a short exact sequence

$$0 \to \mathscr{O}_X(-D) \otimes L^{-1} \to L^{-1} \to \mathscr{O}_D \otimes L^{-1} \to 0.$$

The long exact sequence in cohomology and (4.3) show that

$$H^j\big(X, \mathscr{O}_X(-D) \otimes L^{-1}\big) \to H^j(X, L^{-1})$$

is surjective; this mapping is locally given by multiplying by the defining equation of D. After applying Serre duality, and remembering that $\mathscr{O}_X(D) \simeq L^N$, we obtain the following result.

Theorem 4.5. *Let X be a smooth projective variety, L a line bundle on X, and $s \in H^0(X, L^N)$ a nontrivial section whose divisor is smooth. Then the mapping*

$$H^j(X, \omega_X \otimes L) \to H^j\big(X, \omega_X \otimes L^{N+1}\big)$$

induced by multiplying by s is injective.

This result is a special case of Kollár's injectivity theorem.

Proof of Kodaira's vanishing theorem. We can now prove the Kodaira vanishing theorem very easily. Suppose that L is an ample line bundle on a smooth projective variety X. Since L^N is very ample for large N, it certainly has global sections whose divisors are smooth. Theorem 4.5 therefore gives us an injection

$$H^j(X, \omega_X \otimes L) \hookrightarrow H^j\big(X, \omega_X \otimes L^{N+1}\big).$$

But for sufficiently large values of N, the group on the right-hand side vanishes for $j > 0$ by Serre's theorem. Consequently, $H^j(X, \omega_X \otimes L) = 0$ for $j > 0$, as desired. (I do not know who first discovered this wonderful proof, but I learned it from the book *Lectures on vanishing theorems* by Esnault and Viehweg.)

Kollár's vanishing theorem. In the proof above, we only used one special case of Theorem 4.5. We will now see how it can be applied to prove a powerful generalization of the Kodaira vanishing, due to Kollár. Instead of the canonical bundle, Kollár takes a morphism $f \colon X \to Y$ from a smooth projective variety X to an arbitrary projective variety Y, and considers the higher direct image sheaves $R^i f_* \omega_X$. Very surprisingly, they still satisfy the same vanishing theorem.

Theorem 4.6. *Let $f \colon X \to Y$ be a morphism from a smooth projective variety X to a projective variety Y, and let L be an ample line bundle on Y. Then one has*

$$H^j(Y, R^i f_* \omega_X \otimes L) = 0$$

for every $i \in \mathbb{N}$ and every $j > 0$.

In fact, Kollár proved several other results about the sheaves $R^i f_* \omega_X$, for example that they are torsion-free sheaves on Y (when f is surjective). We will come back to this point later.

For now, let us prove Theorem 4.6 by adapting the proof of the Kodaira vanishing theorem. Fix a sufficiently large integer N, with the property that L^N is very ample. For a generic section $s \in H^0(Y, L^N)$, the preimage of $H = Z(s)$ under f is a smooth divisor in X; this is a consequence of the Bertini theorem. Now we apply Theorem 4.5 to the line bundle $f^* L$ and the divisor $D = f^* H$; the result is that

$$H^j(X, \omega_X \otimes f^* L) \to H^j\big(X, \omega_X \otimes f^* L^{N+1}\big) \tag{4.7}$$

is injective. With the help of the Leray spectral sequence, we can turn this into a result about the higher direct image sheaves $R^i f_* \omega_X$.

Denote by $g \colon D \to H$ the restriction of $f \colon X \to Y$, as in the following diagram:

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "g"] & X \arrow[d, "f"] \\ H \arrow[r, hook] & Y \end{tikzcd}$$

Because D is also smooth, we can assume (by induction on the dimension) that the vanishing of $H^j(H, R^i g_* \omega_D \otimes L)$ for $i \geq 0$ and $j > 0$ is already known. By adjunction, the canonical bundle of D is given by $\omega_D \simeq \omega_X \otimes f^*L^N\big|_D$. We therefore have a short exact sequence

$$0 \to \omega_X \otimes f^*L \to \omega_X \otimes f^*L^{N+1} \to \omega_D \otimes g^*L \to 0,$$

and after pushing forward to Y and using the projection formula, we obtain a long exact sequence

$$\cdots \to R^i f_*\omega_X \otimes L \to R^i f_*\omega_X \otimes L^{N+1} \to R^i g_*\omega_D \otimes L \to \cdots$$

Now observe that the morphism $R^f_*\omega_X \otimes L \to R^i f_*\omega_X \otimes L^{N+1}$ is injective when the section s is chosen sufficiently general. Indeed, the morphism is obtained by tensoring the short exact sequence

$$0 \to L \to L^{N+1} \to \mathscr{O}_H \otimes L^{N+1} \to 0$$

with the sheaf $R^i f_*\omega_X$; but if H is sufficiently transverse to the support of $R^i f_*\omega_X$, then multiplication by s remains injective. The conclusion is that

$$0 \to R^i f_*\omega_X \otimes L \to R^i f_*\omega_X \otimes L^{N+1} \to R^i g_*\omega_D \otimes L \to 0$$

is exact (and also that $R^i g_*\omega_D \otimes L \simeq R^i f_*\omega_X \otimes \mathscr{O}_H \otimes L^{N+1}$).

Now we can complete the proof of Theorem 4.6. By Serre's theorem and induction, we know that the higher cohomology groups of the sheaves $R^i f_*\omega_X \otimes L^{N+1}$ and $R^i g_*\omega_D \otimes L$ vanish. This already shows that

$$H^j(Y, R^i f_*\omega_X \otimes L) = 0$$

for $j \geq 2$. To deal with the remaining case $j = 1$, we use the Leray spectral sequence

$$E_2^{j,i} = H^j(Y, R^i f_*\omega_X \otimes L) \Longrightarrow H^{j+i}(X, \omega_X \otimes f^*L).$$

We know that $E_2^{j,i} = 0$ for $j \geq 2$; this implies the E_2-degeneration of the spectral sequence, and shows in particular that

$$E_2^{1,i} = H^1(Y, R^i f_*\omega_X \otimes L)$$

injects into $H^{i+1}(X, \omega_X \otimes f^*L)$. Now consider the commutative diagram

$$\begin{tikzcd}
H^1(Y, R^i f_*\omega_X \otimes L) \arrow[r, hook] \arrow[d] & H^{i+1}(X, \omega_X \otimes f^*L) \arrow[d, hook] \\
H^1(Y, R^i f_*\omega_X \otimes L^{N+1}) \arrow[r] & H^{i+1}(X, \omega_X \otimes f^*L^{N+1})
\end{tikzcd}$$

where the vertical arrows are induced by multiplication by s and $f^*(s)$, respectively. The horizontal arrow is injective by the spectral sequence argument from above; the vertical arrow by (4.7). Since the group on the bottom-left is zero for $N \gg 0$, we conclude that $H^1(Y, R^i f_*\omega_X \otimes L) = 0$, too.

Exercises.

Exercise 4.1. Show that our construction of the branched covering $\pi\colon Y \to X$ is equivalent to the geometric one in (4.1).

Exercise 4.2. Let L be a very ample line bundle on a projective variety X, and let $\mathscr{F}$ be a coherent sheaf on X. Show that for a sufficiently general section $s \in H^0(X, L)$, the induced morphism $\mathscr{F} \to \mathscr{F} \otimes L$ is injective.

Exercise 4.3. Under the same assumptions as in Theorem 4.5, show that the natural mapping $H^q(X, \Omega_X^p \otimes L^{-1}) \to H^q(D, \Omega_D^p \otimes L^{-1})$ is zero for $p, q \geq 0$. Use this to prove the Nakano vanishing theorem: if L is an ample line bundle on a smooth projective variety X, then $H^q(X, \Omega_X^p \otimes L) = 0$ for $p + q > \dim X$.

LECTURE 5

Albanese variety. Before we can start talking about the generic vanishing theorem, we also have to review the complex-analytic description of holomorphic line bundles, of the Picard variety, and of the Albanese variety. This is because we want to prove the theorem on arbitrary compact Kähler manifolds, where we cannot use the definitions from algebraic geometry.

Let me first say a few words about the Albanese variety; you have probably seen the definition before. Let X be a compact Kähler manifold, and choose a base point $x_0 \in X$. Analytically, the *Albanese variety* of X is defined as the quotient

$$\operatorname{Alb}(X) = \frac{\operatorname{Hom}\bigl(H^0(X, \Omega_X^1), \mathbb{C}\bigr)}{\operatorname{im}\Bigl(\pi_1(X, x_0) \to \operatorname{Hom}\bigl(H^0(X, \Omega_X^1), \mathbb{C}\bigr)\Bigr)}$$

This definition requires some explanation. Every homotopy class in $\pi_1(X, x_0)$ can be represented by a smooth mapping $c\colon [0,1] \to X$ with $c(0) = c(1) = x_0$; by integration, it defines a linear functional

$$H^0(X, \Omega_X^1) \to \mathbb{C}, \quad \omega \mapsto \int_0^1 c^*\omega$$

on the space of holomorphic one-forms. By Stokes' theorem, the functional only depends on the homotopy class of c (because holomorphic forms are closed).

Lemma 5.1. $\operatorname{Alb}(X)$ *is a compact complex torus of dimension* $g = \dim H^0(X, \Omega_X^1)$.

Proof. Because $\operatorname{Hom}\bigl(H^0(X, \Omega_X^1), \mathbb{C}\bigr)$ is an abelian group, the homomorphism

$$\pi_1(X, x_0) \to \operatorname{Hom}\bigl(H^0(X, \Omega_X^1), \mathbb{C}\bigr)$$

factors through $H_1(X, \mathbb{Z})$. We need to show that the image of $\pi_1(X, x_0)$ is a lattice; this is equivalent to the induced homomorphism

$$H_1(X, \mathbb{R}) \simeq H_1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \to \operatorname{Hom}\bigl(H^0(X, \Omega_X^1), \mathbb{C}\bigr)$$

being an isomorphism of real vector spaces. By Hodge theory, both sides have real dimension $2g$, and so it suffices to prove injectivity. But this is clear, because every class in $H^1(X, \mathbb{R})$ is uniquely represented by $\omega + \overline{\omega}$ for some choice of $\omega \in H^0(X, \Omega_X^1)$, and because the pairing between $H_1(X, \mathbb{R})$ and $H^1(X, \mathbb{R})$ is nondegenerate. $\square$

When X is a smooth projective variety, one can show that $\operatorname{Alb}(X)$ is also projective, and therefore an abelian variety. We will come back to this point later.

The reason for introducing the Albanese variety is that there is always a holomorphic mapping from X to $\operatorname{Alb}(X)$, the so-called *Albanese mapping*. For $x \in X$, we can choose a path from x_0 to x, and define a linear functional

$$H^0(X, \Omega_X^1) \to \mathbb{C}, \quad \omega \mapsto \int_{x_0}^x \omega. \tag{5.2}$$

Its image in $\operatorname{Alb}(X)$ is independent of the choice of path; in this way, we obtain a well-defined mapping

$$\operatorname{alb}\colon X \to \operatorname{Alb}(X),$$

called the *Albanese mapping* of X. It is not hard to show that alb is holomorphic: it suffices to know that the integral in (5.2) depends holomorphically on x, which is obvious. By construction, the Albanese mapping takes the base point x_0 to the unit element of the complex torus $\operatorname{Alb}(X)$.

Lemma 5.3. *The differential of* alb *at a point $x \in X$ is the mapping*

$$T_x X \to \operatorname{Hom}\bigl(H^0(X, \Omega_X^1), \mathbb{C}\bigr), \quad v \mapsto \bigl(\omega \mapsto \omega(v)\bigr).$$

Consequently, $\operatorname{alb}^* \Omega^1_{\operatorname{Alb}(X)} \to \Omega^1_X$ *is simply the evaluation morphism*

$$H^0(X, \Omega_X^1) \otimes \mathscr{O}_X \to \Omega_X^1.$$

Proof. The tangent space to $\operatorname{Alb}(X)$ at any point is canonically isomorphic to $\operatorname{Hom}\bigl(H^0(X, \Omega_X^1), \mathbb{C}\bigr)$; we get the first assertion by differentiating the integral in (5.2). Dually, the codifferential of alb is the mapping

$$H^0(X, \Omega_X^1) \to T_x^* X, \quad \omega \mapsto \omega\big|_{T_x X}.$$

Because the holomorphic cotangent bundle of $\operatorname{Alb}(X)$ is canonically isomorphic to $H^0(X, \Omega_X^1) \otimes \mathscr{O}_{\operatorname{Alb}(X)}$, we then get the second assertion. □

A useful consequence is that the pullback mapping

$$\operatorname{alb}^* \colon H^0\bigl(\operatorname{Alb}(X), \Omega^1_{\operatorname{Alb}(X)}\bigr) \to H^0(X, \Omega_X^1)$$

is an isomorphism. This means that every holomorphic one-form on X is the pullback of a holomorphic one-form from $\operatorname{Alb}(X)$. Since we know from Hodge theory that every class in $H^1(X, \mathbb{C})$ can be uniquely written as the sum of a holomorphic one-form and the conjugate of a holomorphic one-form,

$$\operatorname{alb}^* \colon H^1\bigl(\operatorname{Alb}(X), \mathbb{C}\bigr) \to H^1(X, \mathbb{C})$$

is also an isomorphism. Thus $\operatorname{Alb}(X)$ is a "geometric realization" of $H^1(X, \mathbb{C})$.

Holomorphic line bundles. Our next goal is to describe analytically all holomorphic line bundles whose first Chern class is trivial. We will see that there is a nice global way of doing this that avoids the use of open coverings and transition functions. We denote by $\operatorname{Pic}^0(X)$ the set of all holomorphic line bundles with trivial first Chern class on X. You have probably seen the formula

$$\operatorname{Pic}^0(X) = \frac{H^1(X, \mathscr{O}_X)}{H^1\bigl(X, \mathbb{Z}(1)\bigr)}$$

coming from the exponential sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \longrightarrow 0.$$

Here $\mathbb{Z}(1) = 2\pi i \cdot \mathbb{Z}$ is the kernel of the exponential mapping. In those terms, what we are looking for is a simple way to take a point in the quotient and write down the corresponding line bundle.

Let L be a holomorphic line bundle on a complex manifold X. The complex structure on L is uniquely determined by the differential operator

$$\bar\partial_L \colon A^0(X, L) \to A^{0,1}(X, L)$$

from smooth sections of L to smooth $(0,1)$-forms with coefficients in L; one has $\bar\partial_L^2 = 0$, and the holomorphic sections of L are precisely the smooth sections in the kernel of $\bar\partial_L$. To see why, cover X by open sets U_j over which L is trivial, and let $g_{j,k} \in H^0(U_j \cap U_k, \mathscr{O}_X^\times)$ be the holomorphic transition functions. A smooth section $s \in A^0(X, L)$ is given by a collection of smooth functions $s_j \colon U_j \to \mathbb{C}$ with

$$s_j = g_{j,k} \cdot s_k,$$

and because $g_{j,k}$ is holomorphic, we have

$$\bar\partial s_j = g_{j,k} \cdot \bar\partial s_k.$$

This means that the $(0,1)$-forms $\bar\partial s_j$ patch together into a well-defined element $\bar\partial_L s \in A^{0,1}(X, L)$. Since $\bar\partial^2 = 0$, it is clear that $\bar\partial_L^2 = 0$.

Now let us suppose that the first Chern class of L is trivial.

Lemma 5.4. *Let L be a holomorphic line bundle on a complex manifold X. If $c_1(L) \in H^2\big(X, \mathbb{Z}(1)\big)$ is zero, then L is trivial as a smooth line bundle.*

Proof. Let $\mathcal{A}_X$ denote the sheaf of smooth functions, and consider the following two exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}(1) \arrow[r] \arrow[d, equal] & \mathscr{O}_X \arrow[r, "\exp"] \arrow[d, hook] & \mathscr{O}_X^\times \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathbb{Z}(1) \arrow[r] & \mathcal{A}_X \arrow[r, "\exp"] & \mathcal{A}_X^\times \arrow[r] & 0 \end{tikzcd}$$

Now $\mathcal{A}_X$ is a soft sheaf (= admits partitions of unity), and so its higher cohomology is trivial; therefore $H^1(X, \mathcal{A}_X^\times) \simeq H^2\big(X, \mathbb{Z}(1)\big)$. Thus $c_1(L) = 0$ if and only if L has a nowhere vanishing smooth global section. $\square$

Assuming that $c_1(L) = 0$, we can therefore find a nowhere vanishing global section $s \in A^0(X, L)$. We have $\bar{\partial}_L s = \tau \otimes s$ for some $\tau \in A^{0,1}(X)$; note that $\bar{\partial}_L^2 = 0$ implies that τ must be $\bar{\partial}$-closed. Concretely, the section s is given by a collection of nowhere vanishing smooth functions $s_j \in A^0(U_j)$ with $s_j = g_{j,k} s_k$, and we have

$$\tau\big|_{U_j} = \frac{\bar{\partial} s_j}{s_j} \in A^{0,1}(U_j).$$

Every smooth section of L is of the form ϕs for $\phi \in A^0(X)$, and by the Leibniz rule

$$\bar{\partial}_L(\phi s) = (\bar{\partial}\phi + \tau\phi) \otimes s,$$

which means that we can describe L by the operator $\bar{\partial} + \tau$. In fact, only the class of τ in the Dolbeault cohomology group $H^{0,1}(X) \simeq H^1(X, \mathscr{O}_X)$ matters: if we change our choice of smooth section to $e^f s$, for a smooth function f, then τ is replaced by $\tau + \bar{\partial} f$.

From now on, we assume that X is a compact Kähler manifold. We can then choose the trivialization in such a way that $\tau \in A^{0,1}(X)$ is harmonic, and therefore d-closed. To summarize, holomorphic line bundles on X with trivial first Chern class can be obtained by endowing the trivial bundle $X \times \mathbb{C}$ with the complex structure coming from the operator $\bar{\partial} + \tau$, where $\tau \in \mathcal{H}^{0,1}(X)$ is a harmonic $(0,1)$-form. This information is enough to determine the holomorphic sections of L, and therefore L itself: by definition,

$$H^0(U, L) = \big\{\, s \in A^0(U) \;\big|\; \bar{\partial} s + s\tau = 0 \,\big\}$$

gives the space of holomorphic sections on an open set $U \subseteq X$. Conversely, one can show that this rule always defines a holomorphic line bundle (as long as $\bar{\partial}\tau = 0$).

Our description of holomorphic line bundles with trivial first Chern class is also very convenient for computing cohomology.

Lemma 5.5. *Let L be a holomorphic line bundle with trivial first Chern class, and suppose that $\bar{\partial} + \tau$ for $\tau \in \mathcal{H}^{0,1}(X)$. Then the complex*

$$A^{p,0}(X) \to A^{p,1}(X) \to \cdots \to A^{p,n}(X)$$

with differential $\bar{\partial} + \tau$, computes the cohomology of $\Omega_X^p \otimes L$.

Proof. By the holomorphic Poincaré lemma (applied locally), the complex

$$\mathcal{A}_X^{p,0} \to \mathcal{A}_X^{p,1} \to \cdots \to \mathcal{A}_X^{p,n},$$

with differential $\bar{\partial} + \tau$, is a resolution of $\Omega_X^p \otimes L$. Since the sheaves $\mathcal{A}_X^{p,q}$ admit partitions of unity, they are acyclic for the global sections functor. The assertion follows by taking global sections. $\square$

Since we are interested in describing line bundles up to isomorphism, we also have to decide when a line bundle of the form $\bar\partial + \tau$ is trivial. The condition is that there exists a nowhere vanishing smooth function $f \in A^0(X)$ such that $\bar\partial f + f\tau = 0$. It is not immediately clear what this condition says about τ, so we will postpone this problem until we have developed some more theory.

Hodge theory for line bundles. The results about the cohomology of a compact Kähler manifold can be extended without much difficulty to cohomology groups of the form

$$H^q\big(X, \Omega_X^p \otimes L\big),$$

where $L \in \operatorname{Pic}^0(X)$ is a line bundle with trivial first Chern class. As explained above, we can realize L as the trivial smooth bundle $X \times \mathbb{C}$, endowed with the complex structure given by the operator $\bar\partial + \tau$, where $\tau \in \mathcal{H}^{0,1}(X)$ is harmonic.

Now observe that the trivialization also induces a Hermitian metric on L, given by the simple formula

$$h(s_1, s_2) = s_1 \overline{s_2}$$

for $s_1, s_2 \in A^0(X)$. A basic fact in differential geometry is that any holomorphic vector bundle with a Hermitian metric has a unique connection.

Definition 5.6. Let $(\mathscr{E}, h)$ be a holomorphic vector bundle with a Hermitian metric. The *Chern connection* is the unique connection $\nabla \colon A^0(\mathscr{E}) \to A^1(\mathscr{E})$ with the following two properties:

(a) The $(0,1)$-part of ∇ is equal to the operator $\bar\partial_{\mathscr{E}}$.
(b) The connection ∇ satisfies the Leibniz rule

$$dh(s_1, s_2) = h\big(\nabla s_1, s_2\big) + h\big(s_1, \nabla s_2\big).$$

The idea is that $\nabla = \nabla^{1,0} + \nabla^{0,1}$; the first condition says that $\nabla^{0,1} = \bar\partial_{\mathscr{E}}$, and then the second condition uniquely determines $\nabla^{1,0}$.

Lemma 5.7. *With respect to the trivialization from above, the Chern connection on (L, h) is given by the formula*

$$\nabla \colon A^0(X) \to A^1(X), \quad \nabla s = \big(d + \tau - \bar\tau\big)s.$$

In particular, we have $\nabla \circ \nabla = 0$, and so ∇ is integrable.

Proof. Since $\nabla^{0,1} = \bar\partial + \tau$, it suffices to show that the above connection preserves the metric. This is an easy computation:

$$\begin{aligned} h(\nabla s_1, s_2) + h(s_1, \nabla s_2) &= (ds_1 + \tau s_1 - \bar\tau s_1)\overline{s_2} + s_1 \overline{(ds_2 + \tau s_2 - \bar\tau s_2)} \\ &= ds_1 \overline{s_2} + (\tau - \bar\tau) \cdot s_1 \overline{s_2} + (\bar\tau - \tau) \cdot s_1 \overline{s_2} \\ &= dh(s_1, s_2). \end{aligned}$$

Since both τ and $\bar\tau$ are closed, it is obvious that $\nabla \circ \nabla = 0$. $\square$

In fact, Hodge theory works more generally for any holomorphic vector bundle $\mathscr{E}$ with Hermitian metric h, whose associated Chern connection ∇ is integrable. The condition $\nabla \circ \nabla = 0$ implies that there are local trivializations for $\mathscr{E}$ that are flat with respect to ∇. If that is the case, analytic continuation of solutions to the equation $\nabla s = 0$ defines a representation of the fundamental group of X, and the existence of the metric h is equivalent to this representation being unitary.

Example 5.8. Let us compute the monodromy representation for the connection $\nabla = d + \tau - \bar\tau$ on the trivial line bundle $X \times \mathbb{C}$. Let $p \colon Y \to X$ be the universal covering space, and fix base points $x_0 \in X$ and $y_0 \in p^{-1}(x_0)$. We need to solve the equation

$$df + p^*(\tau - \bar\tau)f = 0;$$

since Y is simply connected, the solution is given by the integral

$$f = \exp \int_{y_0}^{y} p^*(\bar{\tau} - \tau).$$

Now let γ be a (smooth) closed loop based at the point x_0, and let $\tilde{\gamma}$ be the unique lifting to a path in Y starting at the point y_0. Then

$$\exp \int_{\tilde{\gamma}(0)}^{\tilde{\gamma}(1)} p^*(\bar{\tau} - \tau) = \exp \int_{\gamma} (\bar{\tau} - \tau),$$

and so the representation is

$$\rho\colon \pi_1(X, x_0) \to \mathbb{C}^\times, \quad \rho\big([\gamma]\big) = \exp \int_{\gamma} (\bar{\tau} - \tau).$$

The integral is a purely imaginary complex number, and so ρ takes values in the circle $U(1)$. This calculation suggests that the line bundle L is trivial if and only if the harmonic one-form $\bar{\tau} - \tau$ has periods in $\mathbb{Z}(1) = 2\pi i \cdot \mathbb{Z}$.

Exercises.

Exercise 5.1. Prove the following universal property of the Albanese mapping: For every holomorphic mapping $f\colon X \to T$ to a compact complex torus T that takes the base point $x_0 \in X$ to the unit element of T, there is a unique factorization

$$\begin{tikzcd} X \arrow[r, "\mathrm{alb}"] \arrow[rr, bend right, "f"'] & \mathrm{Alb}(X) \arrow[r, "g"] & T \end{tikzcd}$$

with $g\colon \mathrm{Alb}(X) \to T$ a holomorphic group homomorphism.

Exercise 5.2. Let X be a complex manifold, and let $\tau \in A^{0,1}(X)$ satisfy $\bar{\partial}\tau = 0$.

(1) Show that the operator $\bar{\partial} + \tau$ defines a holomorphic line bundle L; in other words, show that the rule
$$L(U) = \left\{\, s \in A^0(U) \;\middle|\; \bar{\partial}s + s\tau = 0 \,\right\}$$
produces a locally free sheaf of $\mathscr{O}_X$-modules of rank one.

(2) Similarly, show that the complex of sheaves
$$\mathcal{A}_X^{p,0} \xrightarrow{\bar{\partial}+\tau} \mathcal{A}_X^{p,1} \xrightarrow{\bar{\partial}+\tau} \cdots \xrightarrow{\bar{\partial}+\tau} \mathcal{A}_X^{p,n}$$
is a resolution of $\Omega_X^p \otimes L$.

Exercise 5.3. Show that any holomorphic line bundle on X with trivial first Chern class is the pullback of a holomorphic line bundle from $\mathrm{Alb}(X)$.

Lecture 6

Hodge theory for line bundles. Let X be a compact Kähler manifold. Recall from last time that we can describe holomorphic line bundles with trivial first Chern class by operators of the form $\bar\partial + \tau$, where $\tau \in A^{0,1}(X)$ is harmonic. I forgot to say this last time, but τ being harmonic is in fact equivalent to $d\tau = 0$: in one direction, every harmonic form is closed; in the other direction, we can decompose by type to get $\bar\partial\bar\tau = \overline{\partial\tau} = 0$, which means that τ is the conjugate of a holomorphic one-form. Recall also that the line bundle has a natural connection $\nabla = d + \tau - \bar\tau$, which is integrable because $\nabla \circ \nabla = 0$. We still have to decide when exactly $\bar\partial + \tau$ defines the trivial line bundle; Hodge theory will solve that problem for us.

In order to extend the results of Hodge theory to cohomology groups with coefficients in L, the key observation is that the Kähler identities in Theorem 3.7 continue to hold for the operators $\bar\partial + \tau$ and $\partial - \bar\tau$.

Lemma 6.1. *Let X be a Kähler manifold, and let $\tau \in \mathcal{H}^{0,1}(X)$ be a harmonic $(0,1)$-form. Then the following identities are true:*

$$[\Lambda_\omega, \bar\partial + \tau] = -i(\partial - \bar\tau)^* \quad \textit{and} \quad [\Lambda_\omega, \partial - \bar\tau] = i(\bar\partial + \tau)^*.$$

Proof. Since we already know that $[\Lambda_\omega, \partial] = i\bar\partial^*$, it suffices to show that

$$[\Lambda_\omega, \bar\tau] = -i\tau^*.$$

This identity involves no derivatives, and so it is again enough to prove it in the case of the Euclidean metric on $\mathbb{C}^n$. Here we have

$$\tau = \sum_{k=1}^n f_k \bar e_k$$

for smooth functions $f_1, \dots, f_n$, and therefore

$$\bar\tau = \sum_{k=1}^n \overline{f_k} e_k \quad \text{and} \quad \tau^* = \sum_{k=1}^n \overline{f_k} \bar e_k^*.$$

Recall from our earlier considerations that $\Lambda_\omega = -\frac{i}{2}\sum \bar e_j^* e_j^*$. We compute that

$$\begin{aligned}
[\Lambda_\omega, \bar\tau] = \Lambda_\omega\bar\tau - \bar\tau\Lambda_\omega &= -\frac{i}{2}\sum_{j,k=1}^n \bar e_j^* e_j^* \overline{f_k} e_k + \frac{i}{2}\sum_{j,k=1}^n \overline{f_k} e_k \bar e_j^* e_j^* \\
&= -\frac{i}{2}\sum_{j,k=1}^n \bar e_j^* \overline{f_k}\bigl(e_j^* e_k + e_k e_j^*\bigr) = -i\sum_{k=1}^n \bar e_k^* \overline{f_k} = -i\tau^*.
\end{aligned}$$

Here we used the relations $e_k \bar e_j^* + \bar e_j^* e_k = 0$ and $e_j^* e_k + e_k e_j^* = 2\,\mathrm{id}_{j,k}$ from Lemma 3.8. This proves the second identity; as before, the first follows by conjugation. □

With these identities in hand, everything goes through as in the case of the operators ∂ and $\bar\partial$. We define the Laplace operator

$$\nabla\nabla^* + \nabla^*\nabla = (d + \tau - \bar\tau)(d + \tau - \bar\tau)^* + (d + \tau - \bar\tau)^*(d + \tau - \bar\tau),$$

and observe that it is a second-order elliptic operator: in fact, its second-order terms are exactly the same as those of $\Delta = dd^* + d^*d$. By the same calculation as before, the Kähler identities imply that

$$\begin{aligned}
\frac{1}{2}\bigl(\nabla\nabla^* + \nabla^*\nabla\bigr) &= (\bar\partial + \tau)(\bar\partial + \tau)^* + (\bar\partial + \tau)^*(\bar\partial + \tau) \\
&= (\partial - \bar\tau)(\partial - \bar\tau)^* + (\partial - \bar\tau)^*(\partial - \bar\tau),
\end{aligned}$$

and so $\nabla\nabla^* + \nabla^*\nabla$ preserves the type of forms. By a version of the analysis result in Theorem 2.6, the space $A^{p,q}(X)$ therefore decomposes into the direct sum of the kernel and the image of this operator; we call forms in the kernel ∇-*harmonic*. We can use this to prove the following version of the Hodge theorem.

Theorem 6.2. *Every class in $H^q\big(X, \Omega_X^p \otimes L\big)$ is uniquely represented by a smooth form $\alpha \in A^{p,q}(X)$ that satisfies*

$$(\bar\partial + \tau)\alpha = (\bar\partial + \tau)^*\alpha = (\partial - \bar\tau)\alpha = (\partial - \bar\tau)^*\alpha = 0.$$

All global holomorphic sections of $\Omega_X^p \otimes L$ lie in the kernel of $\nabla = d + \tau - \bar\tau$.

Proof. Recall that the cohomology groups of $\Omega_X^p \otimes L$ are computed by the complex

$$A^{p,0}(X) \to A^{p,1}(X) \to \cdots \to A^{p,n}(X)$$

with differential $\bar\partial + \tau$. Using the decomposition of $A^{p,q}(X)$ into the kernel and the image of $(\bar\partial + \tau)(\bar\partial + \tau)^* + (\bar\partial + \tau)^*(\bar\partial + \tau)$, we see that every cohomology class in $H^q(X, \Omega_X^p \otimes L)$ is uniquely represented by a form which is both $(\bar\partial + \tau)$-closed and $(\bar\partial - \tau)^*$-closed. The above identity implies that it is also $(\partial + \bar\tau)$-closed and $(\partial - \bar\tau)^*$-closed. If $\alpha \in A^{p,0}(X)$ satisfies $(\bar\partial + \tau)\alpha = 0$, then the condition $(\bar\partial + \tau)^*\alpha = 0$ holds by default, and so $(\partial - \bar\tau)\alpha = 0$ as well. $\square$

This theory has one application which is very surprising at first glance. Namely, suppose that $f \in A^0(X)$ is a smooth function such that $(\bar\partial + \tau)f = 0$. By the above, we automatically have $(\partial - \bar\tau)f = 0$ as well. This means that the two differential equations are somehow coupled to each other, due to the fact that X is a compact Kähler manifold. Trying to prove this directly is an interesting problem.

Corollary 6.3. *We have an isomorphism of complex vector spaces*

$$\overline{H^q\big(X, \Omega_X^p \otimes L\big)} \simeq H^p\big(X, \Omega_X^q \otimes L^{-1}\big).$$

Proof. Let $\alpha \in A^{p,q}(X)$ be the unique ∇-harmonic representative of a class in $H^q\big(X, \Omega_X^p \otimes L\big)$. Then $(\bar\partial + \tau)\alpha = 0$ and $(\partial - \bar\tau)\alpha = 0$. After complex conjugation, we obtain $(\bar\partial + \tau)\bar\alpha = 0$ and $(\partial - \tau)\bar\alpha = 0$. Since the operator $\bar\partial + \tau$ corresponds to the holomorphic line bundle L^{-1}, we conclude that $\bar\alpha \in A^{q,p}(X)$ is the harmonic representative of a class in $H^p\big(X, \Omega_X^q \otimes L^{-1}\big)$. $\square$

More about holomorphic line bundles. Now let us return to the question when a holomorphic line bundle of the form $\bar\partial - \tau$ is trivial. This happens exactly when there is a nowhere vanishing smooth function $f \in A^0(X)$ such that $\bar\partial f + f\tau = 0$. Because X is compact Kähler Hodge theory shows that automatically also

$$\partial f - f\bar\tau = 0.$$

If we combine the two relations, we obtain

$$\bar\tau - \tau = \frac{df}{f}.$$

By Stokes' theorem, this happens exactly when all periods of the closed one-form $\bar\tau - \tau$ belong to $\mathbb{Z}(1) = 2\pi i \cdot \mathbb{Z} \subseteq \mathbb{C}$: indeed, f is necessarily given by the integral

$$f = \exp\left(\int_{x_0}^{x} \frac{df}{f}\right) = \exp\left(\int_{x_0}^{x} (\bar\tau - \tau)\right),$$

which is well-defined exactly when the integral of $\bar\tau - \tau$ over any closed loop takes values in $\mathbb{Z}(1)$. In terms of cohomology, this means that the cohomology class of τ should lie in the image of

$$H^1\big(X, \mathbb{Z}(1)\big) \to H^1(X, \mathbb{C}) \to H^1(X, \mathscr{O}_X).$$

To summarize what we have said so far:

Theorem 6.4. *Let X be a compact Kähler manifold. Every holomorphic line bundle with trivial first Chern class can be represented by an operator of the form $\bar\partial + \tau$, where $\tau \in A^{0,1}(X)$ satisfies $d\tau = 0$ (equivalently, is harmonic). The line bundle is trivial if and only if all periods of the one-form $\bar\tau - \tau$ lie in $\mathbb{Z}(1)$.*

Picard variety and Poincaré bundle. According to the discussion above, the set of isomorphism classes of holomorphic line bundles with trivial first Chern class on a compact Kähler manifold X is isomorphic to

$$\operatorname{Pic}^0(X) = \frac{\mathcal{H}^{0,1}(X)}{\left\{ \tau \in \mathcal{H}^{0,1}(X) \;\middle|\; \bar\tau - \tau \text{ has periods in } \mathbb{Z}(1) \right\}}.$$

For $\tau \in \mathcal{H}^{0,1}(X)$, the class $[\tau]$ in the quotient corresponds to the smooth line bundle $X \times \mathbb{C}$, with complex structure given by the operator $\bar\partial + \tau$.

$\operatorname{Pic}^0(X)$ is called the *Picard variety* of X; as in the case of the Albanese variety, one can show that it is a compact complex torus of dimension g. Note that

$$\operatorname{Pic}^0(X) \simeq \frac{H^1(X, \mathscr{O}_X)}{H^1\big(X, \mathbb{Z}(1)\big)},$$

which agrees with the description coming from the exponential sequence

$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \longrightarrow 0$$

Our next goal is the construction of the universal line bundle on $X \times \operatorname{Pic}^0(X)$, sometimes called the *Poincaré bundle* of X. This will justify considering $\operatorname{Pic}^0(X)$ as the "moduli space" of holomorphic line bundles with trivial first Chern class; it will also be useful when we study how the cohomology of such line bundles depends on the line bundle.

Proposition 6.5. *Fix a base point $x_0 \in X$. Then there exists a holomorphic line bundle P on $X \times \operatorname{Pic}^0(X)$ with the following properties:*

(a) For any $y \in \operatorname{Pic}^0(X)$, the restriction of P to $X \times \{y\}$ is the holomorphic line bundle on X corresponding to y.
(b) The restriction of P to $\{x_0\} \times \operatorname{Pic}^0(X)$ is the trivial line bundle.

Moreover, P is unique up to isomorphism.

Proof. Since $c_1(P) \neq 0$, we cannot directly use the analytic formalism from above. Instead, we shall first construct a suitable holomorphic line bundle on $X \times \mathcal{H}^{0,1}(X)$, and then descend it to the quotient. To find the correct formulas, I reverse-engineered the description of the Poincaré bundle in terms of Appell-Humbert data from Mumford's book.

Let $V = \mathcal{H}^{0,1}(X)$, and let $\Gamma \subseteq V$ denote the set of those $\tau \in V$ for which the periods of $\bar\tau - \tau$ belong to $\mathbb{Z}(1)$. Then $\operatorname{Pic}^0(X) = V/\Gamma$. On $X \times V$, we have the smooth line bundle $X \times V \times \mathbb{C}$, with complex structure given by the operator

$$\bar\partial_{\tilde P} = \bar\partial + \sum_{j=1}^{g} t_j \cdot p_1^* \varepsilon_j;$$

here $\varepsilon_1, \dots, \varepsilon_g \in V$ is any basis, and $t_1, \dots, t_g$ denotes the corresponding system of holomorphic coordinates on V. You can check that the $(0,1)$-form in this formula is independent of the choice of basis; we might call it the "universal $(0,1)$-form" on the product $X \times V$, because its restriction to $X \times \{\tau\}$ is equal to τ. From this construction, it is clear that the restriction of this line bundle to $X \times \{\tau\}$ is equal to the holomorphic line bundle with complex structure defined by $\bar\partial + \tau$, which is what we want.

It remains to descend the line bundle to $X \times V/\Gamma$. For every $\gamma \in \Gamma$, the periods of $\overline{\gamma} - \gamma$ belong to $\mathbb{Z}(1)$, and so

$$f_\gamma(x) = \exp \int_{x_0}^{x} (\overline{\gamma} - \gamma);$$

is a well-defined smooth function on X; by construction, it satisfies $\bar{\partial} f_\gamma = -f_\gamma \cdot \gamma$.

The action (by translation) of Γ on V extends to an action on the trivial smooth line bundle $X \times V \times \mathbb{C}$, defined as follows:

$$\gamma \cdot (x, \tau, z) = (x, \tau + \gamma, f_\gamma(x) \cdot z) .$$

The quotient by this action is therefore a smooth line bundle on $X \times V/\Gamma$. To construct P, it now suffices to show that $\bar{\partial}_{\tilde{P}}$ is compatible with this action, and therefore descends to an operator $\bar{\partial}_P$ on our smooth line bundle on $X \times V/\Gamma$.

You can check easily that Γ acts on the space of sections $A^0(X \times V)$ according to the formula

$$(\gamma^* s)(x, \tau) = f_\gamma(x) \cdot s(x, \tau - \gamma).$$

The operator $\bar{\partial}_P$ is compatible with this action, because

$$\begin{aligned} \bar{\partial}_{\tilde{P}}(\gamma^* s)(x,\tau) &= \bar{\partial} f_\gamma(x) \cdot s(x, \tau - \gamma) + f_\gamma(x) \cdot \bar{\partial}_{\tilde{P}} s(x, \tau - \gamma) \\ &= f_\gamma(x) \cdot (\bar{\partial}_{\tilde{P}} - \gamma) s(x, \tau - \gamma), \end{aligned}$$

whereas

$$\begin{aligned} (\gamma^* \bar{\partial}_{\tilde{P}} s)(x,\tau) &= f_\gamma(x) \cdot (\bar{\partial}_{\tilde{P}} s)(x, \tau - \gamma) \\ &= f_\gamma(x) \cdot (\bar{\partial}_{\tilde{P}} - \gamma) s(x, \tau - \gamma). \end{aligned}$$

It is obvious from the construction that (a) is satisfied; as for (b), note that

$$\gamma \cdot (x_0, \tau, z) = (x_0, \tau + \gamma, z),$$

which means that the restriction of P to $\{x_0\} \times V/\Gamma$ is trivial. □

The generic vanishing theorem. Having reviewed the analytic description of $\mathrm{Alb}(X)$ and $\mathrm{Pic}^0(X)$, as well as the basic results in Hodge theory, we are now ready to come back to the generic vanishing theorem. Let X be a compact Kähler manifold. The examples we looked at in the first lecture suggested that there is no vanishing theorem that works for all line bundles in $\mathrm{Pic}^0(X)$; and that the possible range for a vanishing theorem is not determined by $\dim X$, but rather by the Albanese dimension $\dim \mathrm{alb}(X)$, where $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ is the Albanese mapping of X (for some choice of base point).

To state the result, let us define the following subsets of $\mathrm{Pic}^0(X)$:

$$\begin{aligned} S_m^i(X) &= \left\{ L \in \mathrm{Pic}^0(X) \;\middle|\; \dim H^i(X, L) \geq m \right\} \\ S^i(X) &= \left\{ L \in \mathrm{Pic}^0(X) \;\middle|\; H^i(X, L) \neq 0 \right\} \end{aligned}$$

What Green and Lazarsfeld actually proved is the following more precise version of the generic vanishing theorem.

Theorem 6.6. *Let X be a compact Kähler manifold. Then*

$$\operatorname{codim}_{\mathrm{Pic}^0(X)} S^i(X) \geq \dim \mathrm{alb}(X) - i,$$

where $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ *is the Albanese mapping of X.*

For $i < \dim \mathrm{alb}(X)$, we therefore have $H^i(X, L) = 0$ for general L, where "general" means on the complement of the proper analytic subset $S^i(X)$.

The idea of the proof is to study the deformation theory of cohomology groups of the form $H^i(X, L)$. More precisely, we need to understand how $H^i(X, L)$ changes when we move the line bundle L; this question is related to the infinitesimal properties of the loci $S^i_m(X)$. We will first develop this deformation theory abstractly, and then use Hodge theory in order to apply it to our problem.

Proof of the generic vanishing theorem, Part 1. To get started, we need a good model for computing the cohomology groups $H^i(X,L)$ when $L \in \operatorname{Pic}^0(X)$ is allowed to vary. Fix a base point $x_0 \in X$, and let P be the Poincaré bundle on $X \times \operatorname{Pic}^0(X)$, constructed in Proposition 6.5. Let

$$p_1 \colon X \times \operatorname{Pic}^0(X) \to X \quad \text{and} \quad p_2 \colon X \times \operatorname{Pic}^0(X) \to \operatorname{Pic}^0(X)$$

be the projections to the two factors. For $y \in \operatorname{Pic}^0(X)$, let P_y denote the restriction of P to $X \times \{y\}$; by construction, P_y is the holomorphic line bundle corresponding to the point y. The cohomology groups $H^i(X, P_y)$ are clearly related to the higher direct image sheaves $R^i p_{2*} P$. Locally on $\operatorname{Pic}^0(X)$, we can always find a bounded complex of vector bundles that computes the higher direct image sheaves, with the help of the following general result from complex analysis.

Theorem 6.7. *Let $f \colon X \to Y$ be a proper morphism of complex manifolds, and let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Then locally on Y, there exists a bounded complex $E^\bullet$ of holomorphic vector bundles, with the following property: for every coherent sheaf $\mathscr{G}$ on Y, one has*

$$R^i f_*\big(\mathscr{F} \otimes f^*\mathscr{G}\big) \simeq \mathcal{H}^i\big(E^\bullet \otimes \mathscr{G}\big),$$

and the isomorphism is functorial in $\mathscr{G}$.

You will remember from Hartshorne's book (Section III.12) that a similar result is true in the algebraic setting (where it can be proved by using Čech cohomology). In the analytic case, the proof is a lot more difficult, and so I cannot present it here. In any case, we will later calculate the higher direct image sheaves $R^i p_{2*} P$ explicitly in terms of harmonic forms.

If we apply Theorem 6.7 to our situation, we obtain locally on $Y = \operatorname{Pic}^0(X)$ a bounded complex of vector bundles $E^\bullet$. For any point $y \in Y$, we denote by

$$E^\bullet(y) = E^\bullet / \mathfrak{m}_y E^\bullet \simeq E^\bullet \otimes_{\mathscr{O}_Y} \mathbb{C}(y)$$

the complex of vector spaces obtained by restricting $E^\bullet$ to the point. If we take $\mathscr{G} = \mathbb{C}(y)$ in Theorem 6.7, we obtain

$$H^i(X, P_y) \simeq R^i p_{2*}\Big(P\big|_{X \times \{y\}}\Big) \simeq H^i\big(E^\bullet(y)\big),$$

and so the complex $E^\bullet$ does compute the correct cohomology groups.

Exercises.

Exercise 6.1. Let X be a compact complex manifold, and suppose that $\tau \in A^{0,1}(X)$ satisfies $d\tau = 0$. Prove that if a smooth function $f \in A^0(X)$ solves the equation $\bar{\partial} f + f\tau = 0$, then it also solves the equation $\partial f - f\bar{\tau} = 0$. (Hint: The pullback of τ to the universal covering space of X is exact.)

Exercise 6.2. Proof that P is, up to isomorphism, uniquely determined by the two properties in Proposition 6.5.

Exercise 6.3. Calculate the first Chern class $c_1(P)$ of the Poincaré bundle. (Hint: The first Chern class only depends on the underlying smooth line bundle.)

Exercise 6.4. Show that the Poincaré bundle on X is the pullback, via

$$\operatorname{alb} \times \operatorname{id} \colon X \times \operatorname{Pic}^0(X) \to \operatorname{Alb}(X) \times \operatorname{Pic}^0(X)$$

of the Poincaré bundle on $\operatorname{Alb}(X) \times \operatorname{Pic}^0(X)$.

Lecture 7

Infinitesimal study of cohomology support loci. Recall from last time that, locally on $\mathrm{Pic}^0(X)$, we can find a bounded complex of vector bundles that, at a point $L \in \mathrm{Pic}^0(X)$, computes the cohomology groups $H^i(X, L)$. In general, a bounded complex of vector bundles is a good model for a "family of cohomology groups"; our goal today is to study such families infinitesimally.

Let X be a complex manifold; we denote the sheaf of germs of holomorphic functions by the symbol $\mathscr{O}_X$. For a point $x \in X$, we denote by $\mathscr{O}_{X,x}$ be the local ring at the point, by $\mathfrak{m}_x \subseteq \mathscr{O}_{X,x}$ its maximal ideal, and by $\mathbb{C}(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ the residue field. Consider a bounded complex

$$E^\bullet = \Big[\cdots \longrightarrow E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \longrightarrow \cdots \Big]$$

of locally free sheaves on X, with $\operatorname{rk} E^i = e_i$. Given a point $x \in X$, we denote by

$$E^\bullet(x) = E^\bullet/\mathfrak{m}_x E^\bullet = E^\bullet \otimes_{\mathscr{O}_X} \mathbb{C}(x)$$

the complex of vector spaces determined by the fibers of $E^\bullet$. Our goal is to study how the cohomology groups of this complex of vector spaces depend on $x \in X$. In particular, we are interested in the *cohomology support loci*

$$S^i_m(E^\bullet) = \big\{\, x \in X \;\big|\; \dim H^i\big(E^\bullet(x)\big) \geq m \,\big\}$$

and in their infinitesimal structure.

Lemma 7.1. *Each $S^i_m(E^\bullet)$ is a closed analytic subset of X.*

Proof. In fact, it is not hard to write down a coherent sheaf of ideals in $\mathscr{O}_X$ whose zero locus is $S^i_m(E^\bullet)$. Observe first that

$$\begin{aligned} \dim H^i\big(E^\bullet(x)\big) \geq m &\Longleftrightarrow \dim\ker d^i(x) - \dim \operatorname{im} d^{i-1}(x) \geq m \\ &\Longleftrightarrow \operatorname{rk} d^{i-1}(x) + \operatorname{rk} d^i(x) \leq e_i - m. \end{aligned}$$

As sets, we therefore have

$$S^i_m(E^\bullet) = \bigcap_{a+b=e_i-m+1} \big\{\, x \in X \;\big|\; \operatorname{rk} d^{i-1}(x) \leq a-1 \text{ or } \operatorname{rk} d^i(x) \leq b-1 \,\big\},$$

and so we may take the ideal sheaf of $S^i_m(E^\bullet)$ to be

$$\mathscr{I}^i_m(E^\bullet) = \sum_{a+b=e_i-m+1} \mathscr{J}_a(d^{i-1}) \cdot \mathscr{J}_b(d^i). \tag{7.2}$$

Here $\mathscr{J}_b(d^i)$ denotes the ideal sheaf locally generated by all $b \times b$-minors of d^i; this makes sense because, after choosing local trivializations for the bundles E^i and E^{i+1}, the differential d^i is simply given by a matrix of holomorphic functions. ☐

We now begin the infinitesimal study of the loci $S^i_m(E^\bullet)$. Fix a point $x \in X$, and let $T = T_x X$ denote the holomorphic tangent space to X at x; note that, as a complex vector space, T is dual to $\mathfrak{m}_x/\mathfrak{m}_x^2$. Roughly speaking, what we want to do is keep only the first-order terms in each differential d^i, and see how much information about $S^i_m(E^\bullet)$ we can extract from them. This is the reason for using the word "infinitesimal"; algebraically, it means that we consider the complex

$$E^\bullet/\mathfrak{m}_x^2 E^\bullet.$$

Since $\mathfrak{m}_x E^\bullet/\mathfrak{m}_x^2 E^\bullet \simeq E^\bullet(x) \otimes T^*$, the short exact sequence of complexes

$$0 \to \mathfrak{m}_x E^\bullet/\mathfrak{m}_x^2 E^\bullet \to E^\bullet/\mathfrak{m}_x^2 E^\bullet \to E^\bullet/\mathfrak{m}_x E^\bullet \to 0.$$

gives rise to connecting homomorphisms

$$D(d^i, x)\colon H^i\big(E^\bullet(x)\big) \to H^{i+1}\big(E^\bullet(x)\big) \otimes T^*. \tag{7.3}$$

In particular, we obtain for every tangent vector $v \in T$ a linear mapping

$$D_v(d^i, x)\colon H^i\big(E^\bullet(x)\big) \to H^{i+1}\big(E^\bullet(x)\big).$$

A more concrete description is the following: After choosing local trivializations for E^i and E^{i+1}, the differential d^i is given by a matrix of holomorphic functions; its derivative in the direction of v gives a linear map $E^i(x) \to E^{i+1}(x)$ that passes to cohomology and induces $D_v(d^i, x)$. One can show that $D_v(d^i, x) \circ D_v(d^{i-1}, x) = 0$; it follows that we get a complex of vector spaces

$$\dotsb \to H^{i-1}\big(E^\bullet(x)\big) \to H^i\big(E^\bullet(x)\big) \to H^{i+1}\big(E^\bullet(x)\big) \to \dotsb$$

that we denote by the symbol $D_v(E^\bullet, x)$ and call the *derivative complex of $E^\bullet$ at x in the direction v*.

We can assemble all the above complexes into a single complex of locally free sheaves on the tangent space T. Recall that the symmetric algebra

$$\operatorname{Sym} T^* = \bigoplus_{\ell=0}^{\infty} \operatorname{Sym}^\ell T^* \simeq \bigoplus_{\ell=0}^{\infty} \mathfrak{m}_x^\ell / \mathfrak{m}_x^{\ell+1}$$

is the algebra of all polynomial functions on T; in particular, T^* is the space of linear functions on T. If we denote by $\mathscr{O}_T$ the sheaf of germs of holomorphic functions on T, then (7.3) defines a morphism

$$D(d^i, x)\colon H^i\big(E^\bullet(x)\big) \otimes \mathscr{O}_T \to H^{i+1}\big(E^\bullet(x)\big) \otimes \mathscr{O}_T$$

between two (trivial) holomorphic vector bundles on T.

Lemma 7.4. *We have $D(d^i, x) \circ D(d^{i-1}, x) = 0$.*

Proof. The short exact sequence of complexes

$$0 \to \mathfrak{m}_x E^\bullet / \mathfrak{m}_x^3 E^\bullet \to E^\bullet / \mathfrak{m}_x^3 E^\bullet \to E^\bullet / \mathfrak{m}_x E^\bullet \to 0$$

gives rise to a connecting homomorphism

$$H^{i-1}\big(E^\bullet(x)\big) \to H^i\big(\mathfrak{m}_x E^\bullet / \mathfrak{m}_x^3 E^\bullet\big);$$

it is clear that if we project to $H^i\big(\mathfrak{m}_x E^\bullet / \mathfrak{m}_x^2 E^\bullet\big)$, we obtain $D(d^{i-1}, x)$. This gives us a factorization of $D(d^i, x) \circ D(d^{i-1}, x)$, as shown it the following diagram:

$$\begin{tikzcd}
H^{i-1}\big(E^\bullet(x)\big) \arrow[d] \arrow[dr, bend left, "D(d^{i-1},x)"] & & \\
H^i\big(\mathfrak{m}_x E^\bullet / \mathfrak{m}_x^3 E^\bullet\big) \arrow[r] & H^i\big(\mathfrak{m}_x E^\bullet / \mathfrak{m}_x^2 E^\bullet\big) \arrow[r, "D(d^i,x)"] \arrow[d, equal] & H^{i+1}\big(\mathfrak{m}_x^2 E^\bullet / \mathfrak{m}_x^3 E^\bullet\big) \arrow[d, equal] \\
& H^i\big(E^\bullet(x)\big) \otimes T^* \arrow[r] & H^{i+1}\big(E^\bullet(x)\big) \otimes \operatorname{Sym}^2 T^*
\end{tikzcd}$$

In the middle row, we are using two consecutive morphisms in the long exact sequence coming from the short exact sequence of complexes

$$0 \to \mathfrak{m}_x^2 E^\bullet / \mathfrak{m}_x^3 E^\bullet \to \mathfrak{m}_x E^\bullet / \mathfrak{m}_x^3 E^\bullet \to \mathfrak{m}_x E^\bullet / \mathfrak{m}_x^2 E^\bullet \to 0.$$

Their composition is equal to zero, and so the assertion is proved. $\square$

The infinitesimal information about the complex $E^\bullet$ at the point x is therefore contained in the following complex on T.

Definition 7.5. The *derivative complex* $D(E^\bullet, x)$ is the complex

$$\dotsb \to H^{i-1}\big(E^\bullet(x)\big) \otimes \mathscr{O}_T \to H^i\big(E^\bullet(x)\big) \otimes \mathscr{O}_T \to H^{i+1}\big(E^\bullet(x)\big) \otimes \mathscr{O}_T \to \dotsb$$

on the tangent space $T = T_x X$, with differentials $D(d^i, x)$.

It is clear that, if we restrict $D(E^\bullet, x)$ to a point $v \in T$, we obtain the derivative complex $D_v(E^\bullet, x)$ in the direction v. Now the derivative complex is itself a complex of vector bundles on the tangent space T, and so we can consider its cohomology support loci

$$S^i_m\big(D(E^\bullet, x)\big) = \big\{\, v \in T \,\big\vert\, \dim H^i\big(D_v(E^\bullet, x)\big) \geq m \,\big\}.$$

Because all the bundles in $D(E^\bullet, x)$ are trivial, and all the differentials are matrices with entries in T^*, these loci are cones in T. The main result is that the cohomology support loci of the derivative complex control the tangent cone of $S^i_m(E^\bullet)$ at x.

Theorem 7.6. *Suppose that $x \in S^i_m(E^\bullet)$ for some $m \geq 1$. Then*

$$TC_x\big(S^i_m(E^\bullet)\big) \subseteq S^i_m\big(D(E^\bullet, x)\big),$$

where the left-hand side denotes the tangent cone of $S^i_m(E^\bullet)$ at the point x.

Proof of the tangent cone theorem. Before we begin the proof, recall that if $Z \subseteq X$ is an analytic subvariety, defined by a coherent sheaf of ideals $\mathcal{I}$, then the tangent cone at a point $x \in Z$ is defined by the graded ideal

$$\bigoplus_{\ell=0}^{\infty} \frac{\mathcal{I} \cap \mathfrak{m}_x^\ell}{\mathcal{I} \cap \mathfrak{m}_x^{\ell+1}} \subseteq \operatorname{Sym} T^*.$$

It is therefore a conical subscheme of the tangent space T.

Because we need to keep track of powers of the maximal ideal, Theorem 7.6 is much easier to prove when all differentials d^i in the complex have entries in the maximal ideal $\mathfrak{m}_x$. We first prove a small structure theorem for the complex $E^\bullet$ that will allow us to reduce to that case.

Lemma 7.7. *After restricting to a sufficiently small open neighborhood of the point $x \in X$, we can find direct sum decompositions*

$$E^i \simeq F^{i-1} \oplus E^i_0 \oplus F^i,$$

such that each $d^i \colon E^i \to E^{i+1}$ is represented by a matrix of the form

$$\begin{pmatrix} 0 & 0 & \mathrm{id} \\ 0 & d^i_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $d^i_0(E^i_0) \subseteq \mathfrak{m}_x E^{i+1}_0$.

Proof. We construct such decompositions by working from right to left. Suppose that $d^i = 0$ for $i > n$, meaning that

$$\cdots \longrightarrow E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \longrightarrow 0$$

are the right-most terms in the complex. After choosing suitable bases in the vector spaces $E^n(x)$ and $E^{n+1}(x)$, we can certainly arrange that

$$d^n(x) = \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}$$

In a sufficiently small neighborhood of x, the upper right-hand corner of d^n remains invertible; after performing additional row and column operations, we therefore obtain decompositions

$$E^{n+1} \simeq F^n \oplus E^{n+1}_0 \quad \text{and} \quad E^n \simeq E^n_1 \oplus F^n,$$

with the property that

$$d^n = \begin{pmatrix} 0 & \mathrm{id} \\ d^n_1 & 0 \end{pmatrix} \quad \text{and} \quad d^n_1(E^n_1) \subseteq \mathfrak{m}_x E^{n+1}_0.$$

The relation $d^n \circ d^{n-1} = 0$ implies that $\operatorname{im} d^{n-1} \subseteq E_1^n$. Repeating the construction, we can find a possibly smaller open neighborhood and decompositions

$$E_1^n \simeq F^{n-1} \oplus E_0^n \quad \text{and} \quad E^{n-1} \simeq E_1^{n-1} \oplus F^{n-1},$$

with the property that

$$d^{n-1} = \begin{pmatrix} 0 & \mathrm{id} \\ d_1^{n-1} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad d_1^{n-1}(E_1^{n-1}) \subseteq \mathfrak{m}_x E_0^n.$$

Since $d^n \circ d^{n-1} = 0$, we deduce that

$$d^n = \begin{pmatrix} 0 & 0 & \mathrm{id} \\ 0 & d_0^n & 0 \end{pmatrix} \quad \text{and} \quad d_0^n(E_0^n) \subseteq \mathfrak{m}_x E_0^{n+1}.$$

This is the desired result for the differential d^n (with $F^{n+1} = 0$). Now we simply repeat the above proceduce for each of the remaining differentials. □

Note that $d_0^i \circ d_0^{i-1} = 0$, and so $E_0^\bullet$ forms a subcomplex of $E^\bullet$; in fact, the inclusion of $E_0^\bullet$ into $E^\bullet$ is a homotopy equivalence. This means in particular that the cohomology sheaves of the two complexes are the same. In the case of a local ring, a complex of free modules whose differentials have entries in the maximal ideal is called a *minimal complex*; the argument we have just given can be used to show that, over any local ring, a bounded complex of free modules contains a minimal subcomplex with the same cohomology.

Since we are only interested in the behavior near the point x, the fact that $E_0^\bullet$ is only defined in a neighborhood of x does not matter. With respect to the decomposition in Lemma 7.7, $E^\bullet(x)$ has differentials

$$d^i(x) = \begin{pmatrix} 0 & 0 & \mathrm{id} \\ 0 & d_0^i(x) & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mathrm{id} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and so $H^i\big(E^\bullet(x)\big) \simeq E_0^i(x)$. Under this isomorphism,

$$D(d^i, x) \colon H^i\big(E^\bullet(x)\big) \to H^{i+1}\big(E^\bullet(x)\big) \otimes T^*$$

is clearly identified with the homomorphism

$$E_0^i(x) \to \mathfrak{m}_x E_0^{i+1} / \mathfrak{m}_x^2 E_0^{i+1} \simeq E_0^{i+1}(x) \otimes T^*$$

induced by d_0^i. The derivative complex $D(E^\bullet, x)$ is therefore basically the linear part of the complex $E_0^\bullet$. Now we observe that the subcomplex also has the same cohomology support loci.

Lemma 7.8. *We have $S_m^i(E^\bullet) = S_m^i(E_0^\bullet)$, and in fact, $\mathcal{I}_m^i(E^\bullet) = \mathcal{I}_m^i(E_0^\bullet)$.*

Proof. Recall from (7.2) that

$$\mathcal{I}_m^i(E^\bullet) = \sum_{a+b=e_i-m+1} \mathcal{J}_a(d^{i-1}) \cdot \mathcal{J}_b(d^i).$$

Consider again the decomposition $E^i \simeq F^{i-1} \oplus E_0^i \oplus F^i$, and let $f_i = \operatorname{rk} F^i$. With respect to this decomposition, the differential d^i is represented by the matrix

$$\begin{pmatrix} 0 & 0 & \mathrm{id} \\ 0 & d_0^i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and it is easy to see that $\mathcal{J}_b(d^i) = \mathcal{J}_{b-f_i}(d_0^i)$; in fact, every $b \times b$-minor of the large matrix is (up to a sign) a $k \times k$-minor of the smaller matrix for some $b - f_i \leq k \leq b$, and therefore in the ideal generated by the minors of size $b - f_i$. It follows that

$$\begin{aligned}
\mathcal{I}_m^i(E^\bullet) &= \sum_{a+b=e_i-m+1} \mathcal{J}_{a-f_{i-1}}(d_0^{i-1}) \cdot \mathcal{J}_{b-f_i}(d_0^i) \\
&= \sum_{c+d=h_i-m+1} \mathcal{J}_c(d_0^{i-1}) \cdot \mathcal{J}_d(d_0^i) \\
&= \mathcal{I}_m^i(E_0^\bullet),
\end{aligned}$$

because $h^i = \operatorname{rk} E_0^i = e_i - f_i - f_{i-1}$. □

For the purpose of proving Theorem 7.6, we can therefore replace $E^\bullet$ by the subcomplex $E_0^\bullet$; in other words, we can assume without loss of generality that all differentials of $E^\bullet$ have entries in the maximal ideal, meaning that

$$d^i(E^i) \subseteq \mathfrak{m}_x E^{i+1}.$$

In that situation, the complex of vector spaces $E^\bullet(x)$ has trivial differentials, and so $H^i\big(E^\bullet(x)\big) \simeq E^i(x)$. This makes everything very simple.

Proof of Theorem 7.6. The ideal sheaf of $S_m^i(E^\bullet)$ is

$$\mathcal{I}_m^i(E^\bullet) = \sum_{a+b=e_i-m+1} \mathcal{J}_a(d^{i-1}) \cdot \mathcal{J}_b(d^i) \subseteq \mathfrak{m}_x^{e_i-m+1}.$$

The tangent cone to $S_m^i(E^\bullet)$ at the point x is defined by the graded ideal

$$J = \bigoplus_{\ell \in \mathbb{N}} J_\ell \subseteq \operatorname{Sym} T^*,$$

whose graded piece in degree ℓ is given by the formula

$$J_\ell = \frac{\mathcal{I}_m^i(E^\bullet) \cap \mathfrak{m}_x^\ell}{\mathcal{I}_m^i(E^\bullet) \cap \mathfrak{m}_x^{\ell+1}}.$$

It follows that $J_\ell = 0$ for $\ell \leq e_i - m$. On the other hand, the ideal sheaf of $S_m^i\big(D(E^\bullet, x)\big)$ is given by

$$\mathcal{I}_m^i\big(D(E^\bullet, x)\big) = \sum_{a+b=e_i-m+1} \mathcal{J}_a\big(D(d^{i-1}, x)\big) \cdot \mathcal{J}_b\big(D(d^i, x)\big);$$

notice that it is generated by elements of $\operatorname{Sym}^{e_i-m+1} T^*$, because each matrix $D(d^i, x)$ has entries in T^*. To prove the theorem, it therefore suffices to show that

$$J_{e_i-m+1} = \mathcal{I}_m^i\big(D(E^\bullet, x)\big)_{e_i-m+1}. \tag{7.9}$$

This is straightforward. Recall that under the isomorphism $H^i\big(E^\bullet(x)\big) \simeq E^i(x)$, the differentials in the derivative complex are the morphisms

$$E^i(x) \to \mathfrak{m}_x E^i(x)/\mathfrak{m}_x^2 E^i(x) \simeq E^i(x) \otimes T^*$$

induced by d^i. If we project the right-hand side of

$$\mathcal{I}_m^i(E^\bullet) = \sum_{a+b=e_i-m+1} \mathcal{J}_a(d^{i-1}) \cdot \mathcal{J}_b(d^i)$$

into $\mathfrak{m}_x^{e_i-m+1}/\mathfrak{m}_x^{e_i-m+2}$, we find that

$$\sum_{a+b=e_i-m+1} \mathcal{J}_a\big(D(d^{i-1}, x)\big)_a \cdot \mathcal{J}_d\big(D(d^i, x)\big)_b = \mathcal{I}_m^i\big(D(E^\bullet, x)\big)_{e_i-m+1},$$

and so (7.9) is proved. □

Consequences for cohomology support loci. As a corollary of (7.9), we get a description of the Zariski tangent space.

Corollary 7.10. *If $m = \dim H^i\big(E^\bullet(x)\big)$, then $S^i_m\big(D(E^\bullet, x)\big)$ is the Zariski tangent space to $S^i_m(E^\bullet)$ at the point x.*

Proof. We have $e_i = m$, and according to the above proof, $\mathcal{I}^i_m\big(D(E^\bullet, x)\big)$ is therefore generated by J_1. But J_1 consists exactly of the (linear) equations for the Zariski tangent space. $\square$

We list a few other consequences of Theorem 7.6. What they have in common is that properties of the derivative complex are used to bound the dimension of $S^i_m(E^\bullet)$, or to decide whether $S^i_m(E^\bullet)$ is a proper subset of X. This is exactly the type of result that we need to prove the generic vanishing theorem.

Corollary 7.11. *Set $m = \dim H^i\big(E^\bullet(x)\big)$. Then*

$$\dim_x S^i_m(E^\bullet) \leq \dim\big\{\, v \in T \;\big|\; D_v(d^i, x) = 0 \text{ and } D_v(d^{i-1}, x) = 0 \,\big\}.$$

In particular, if either $D_v(d^i, x) \neq 0$ or $D_v(d^{i-1}, x) \neq 0$ for some tangent vector $v \in T_x X$, then $S^i_m(E^\bullet)$ is a proper subset of X.

Proof. Recall that the dimension of the tangent cone $TC_x\big(S^i_m(E^\bullet)\big)$ is equal to the local dimension of $S^i_m(E^\bullet)$ at the point x. From Theorem 7.6, we therefore get

$$\dim_x S^i_m(E^\bullet) \leq \dim S^i_m\big(D(E^\bullet, x)\big).$$

Since $m = \dim H^i\big(E^\bullet(x)\big)$ is equal to the rank of the i-th sheaf in the derivative complex $D(E^\bullet, x)$, the only way in which a nonzero vector $v \in T_x X$ can belong to the set $S^i_m\big(D(E^\bullet, x)\big)$ is if both differentials $D(d^i, x)$ and $D(d^{i-1}, x)$ vanish at v, which translates into the condition $D_v(d^i, x) = D_v(d^{i-1}, x) = 0$. $\square$

This leads to the following condition for $S^i_m(E^\bullet)$ to be a proper subset of X.

Corollary 7.12. *If $H^i\big(D_v(E^\bullet, x)\big) = 0$ for some point $x \in X$ and some tangent vector $v \in T_x X$, then $S^i_m(E^\bullet)$ is a proper subset of X for $m \geq 1$.*

At the other end of the spectrum, we can use the derivative complex to detect isolated points of $S^i_m(E^\bullet)$.

Corollary 7.13. *If $x \in S^i_m(E^\bullet)$ for some $m \geq 1$, and if $H^i\big(D_v(E^\bullet, x)\big) = 0$ for every nonzero tangent vector $v \in T_x X$, then x is an isolated point of $S^i_m(E^\bullet)$.*

Proof. If $H^i\big(D_v(E^\bullet, x)\big)$ is nonzero, it means that either the differential $D_v(d^{i-1}, x)$ or the differential $D_v(d^i, x)$ must be nonzero. We can now apply Corollary 7.11 to conclude that $\dim_x S^i_m(E^\bullet) = 0$. $\square$

Next time, we will use these criteria, together with an explicit description of the derivative complex in the case of topologically trivial line bundles, to prove the generic vanishing theorem.

Exercises.

Exercise 7.1. Show that, after choosing local trivializations for the sheaves in the complex, $D_v(d^i, x)$ is indeed given by differentiating the entries in the matrix d^i. Uses this to show that $D_v(d^i, x) \circ D_v(d^{i-1}, x) = 0$.

Exercise 7.2. Show that the inclusion of the subcomplex $E^\bullet_0$ into $E^\bullet$ is a homotopy equivalence.

Cohomology support loci on a compact Kähler manifold. Today, we shall apply the abstract results about cohomology support loci to the case of a compact Kähler manifold X. Recall that

$$S_m^i(X) = \left\{ L \in \operatorname{Pic}^0(X) \;\middle|\; \dim H^i(X, L) \geq m \right\};$$

to simplify the notation, we also set $S^i(X) = S_1^i(X)$.

We fix a point $L \in \operatorname{Pic}^0(X)$; according to Theorem 6.7, we can find an open neighborhood U of L, and a bounded complex of vector bundles $E^\bullet$ on U, such that

$$R^i p_{2*}\bigl(P \otimes p_2^* \mathscr{G}\bigr) \simeq \mathcal{H}^i\bigl(E^\bullet \otimes \mathscr{G}\bigr) \tag{8.1}$$

for every coherent sheaf $\mathscr{G}$ on U. In particular, the complex of vector spaces $E^\bullet(y)$ at a point $y \in U$ computes the cohomology of the line bundle corresponding to y; this shows that

$$S_m^i(X) \cap U = S_m^i(E^\bullet),$$

in the notation from last time. The cohomology support loci $S_m^i(X)$ are therefore closed analytic subschemes of $\operatorname{Pic}^0(X)$, with the scheme structure defined locally by the coherent sheaf of ideals in (7.2).

To go further, we need a description of the derivative complex in terms of data on X. From our construction of $\operatorname{Pic}^0(X)$, it is clear that $T_L \operatorname{Pic}^0(X) \simeq \mathcal{H}^{0,1}(X)$, and so a tangent vector $v \in T_L \operatorname{Pic}^0(X)$ corresponds to a harmonic $(0,1)$-form $v \in \mathcal{H}^{0,1}(X)$.

Lemma 8.2. *The derivative complex $D_v(E^\bullet, L)$ is isomorphic to*

$$H^0(X, L) \to H^1(X, L) \to \cdots \to H^n(X, L),$$

with differential given by wedge product with $v \in \mathcal{H}^{0,1}(X)$.

Proof. Recall that the differentials in the derivative complex are constructed by taking the short exact sequence

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathscr{O}/\mathfrak{m}^2 \to \mathscr{O}/\mathfrak{m} \to 0,$$

where $\mathfrak{m}$ is the ideal sheaf of the point L; tensoring it by $E^\bullet$; and then computing the connecting homomorphism in the resulting long exact sequence. Because the isomorphism in (8.1) is functorial in $\mathscr{G}$, we can compute the connecting homomorphism on $X \times \operatorname{Pic}^0(X)$. In fact, all we need is the restriction of the Poincaré bundle P to the first infinitesimal neighborhood (corresponding to $\mathfrak{m}^2$) of $X \times \{L\}$. We can therefore do the computation on $X \times V$, where $V = \mathcal{H}^{0,1}(X)$, because the quotient mapping $V \to \operatorname{Pic}^0(X)$ is a local isomorphism. Recall from Proposition 6.5 that P was descended from the holomorphic line bundle on $X \times V$ defined by the operator

$$\bar{\partial}_{X \times V} + \sum_{j=1}^{g} t_j \cdot p_1^* v_j,$$

where $v_1, \dots, v_g \in V$ are a basis, and $t_1, \dots, t_g \in V^*$ the dual basis, viewed as a linear coordinate system on V.

Let $\tau \in \mathcal{H}^{0,1}(X)$ be any point mapping to $L \in \operatorname{Pic}^0(X)$; this means that L is, up to isomorphism, the holomorphic line bundle defined by the operator $\bar{\partial} + \tau$. Take an arbitrary cohomology class in $H^i(X, L)$; it is represented by a smooth form $\alpha \in A^{0,i}(X)$ with $\bar{\partial}\alpha + \tau \wedge \alpha = 0$. To compute the effect of the connecting homomorphism, we need to lift α to the first infinitesimal neighborhood of $X \times \{\tau\}$

in $X \times V$; we use the most obvious choice, namely $p_1^* \alpha$. Now we apply the operator defining P; we get

$$\bar{\partial}_{X \times V}\left(p_1^* \alpha\right) + \sum_{j=1}^g t_j \cdot p_1^*(v_j \wedge \alpha) = p_1^*\left(\bar{\partial} \alpha\right) + \sum_{j=1}^g t_j \cdot p_1^*(v_j \wedge \alpha) = \sum_{j=1}^g \bigl(t_j - t_j(\tau)\bigr) \cdot p_1^*(v_j \wedge \alpha),$$

keeping in mind that $\tau = \sum t_j(\tau) v_j$. Modulo $\mathfrak{m}^2$, this becomes

$$\sum_{j=1}^g (v_j \wedge \alpha) \otimes \bigl(t_j - t_j(\tau) + \mathfrak{m}^2\bigr) \in A^{0,i+1}(X) \otimes \mathfrak{m}/\mathfrak{m}^2.$$

Applied to a tangent vector $v \in V$, the differential is therefore simply the mapping

$$H^i(X, L) \to H^{i+1}(X, L), \quad [\alpha] \mapsto [v \wedge \alpha].$$

This concludes the proof. $\square$

This description of the derivative complex is still not very convenient, because it involves cohomology groups. To get a description that is more closely related to the Albanese mapping and holomorphic one-forms, we now use some results from Hodge theory. Recall from Corollary 6.3 that we have an isomorphism of complex vector spaces

$$\overline{H^i(X, L)} \simeq H^0\bigl(X, \Omega_X^i \otimes L^{-1}\bigr);$$

here the bar denotes the conjugate vector space. (Concretely, take the harmonic $(i,0)$-form representing a given cohomology class for L, and conjugate to obtain a harmonic $(0,i)$-form that represents a cohomology class for $\Omega_X^i \otimes L^{-1}$.) We also clearly have a commutative diagram

$$\begin{tikzcd}
\overline{H^i(X, L)} \arrow[r, "\overline{v} \wedge"] \arrow[d] & \overline{H^{i+1}(X, L)} \arrow[d] \\
H^0\bigl(X, \Omega_X^i \otimes L^{-1}\bigr) \arrow[r, "\overline{v} \wedge"] & H^0\bigl(X, \Omega_X^{i+1} \otimes L^{-1}\bigr)
\end{tikzcd}$$

for every $v \in \mathcal{H}^{0,1}(X)$. This means that if we conjugate the derivative complex

$$H^0(X, L) \to H^1(X, L) \to \cdots \to H^n(X, L),$$

in Lemma 8.2, and use the isomorphisms above, we obtain the complex

$$H^0\bigl(X, L^{-1}\bigr) \to H^0\bigl(X, \Omega_X^1 \otimes L^{-1}\bigr) \to \cdots \to H^0\bigl(X, \Omega_X^n \otimes L^{-1}\bigr),$$

whose differential is given by wedge product with $\overline{v} \in \mathcal{H}^{1,0}(X)$; note that $\overline{v}$ is a holomorphic one-form. The advantage of the second complex is that it involves only sections of vector bundles.

We are now in a good position to apply our earlier results about tangent cones to cohomology support loci. Recall that we showed

$$TC_L S_m^i(E^\bullet) \subseteq S_m^i\bigl(D(E^\bullet, L)\bigr). \tag{8.3}$$

The dimension estimate in Corollary 7.11 now takes the following form.

Theorem 8.4. *Let X be a compact Kähler manifold. Fix a point $L \in \operatorname{Pic}^0(X)$, and set $m = \dim H^i(X, L)$. Then*

$$\dim_L S_m^i(X) \leq \dim \left\{ \omega \in H^0(X, \Omega_X^1) \,\middle|\, \begin{array}{l} \omega \wedge \alpha = 0 \textit{ for } \alpha \in H^0\bigl(X, \Omega_X^{i-1} \otimes L^{-1}\bigr) \\ \omega \wedge \beta = 0 \textit{ for } \beta \in H^0\bigl(X, \Omega_X^i \otimes L^{-1}\bigr) \end{array} \right\}.$$

Proof. This is just a restatement of Corollary 7.11. From (8.3), we get

$$\dim_L S^i_m(X) \leq \dim\bigl\{ v \in T_L \operatorname{Pic}^0(X) \bigm| \dim H^i\bigl(D_v(E^\bullet, L)\bigr) \geq m \bigr\}.$$

But because the i-th vector space in the derivative complex is $H^i(X, L)$, which is m-dimensional by assumption, we can only have $\dim H^i\bigl(D_v(E^\bullet, L)\bigr) \geq m$ if the two differentials next to $H^i(X, L)$ are zero. After conjugation, this becomes exactly the condition that $\overline{v}$ is in the displayed subspace of $H^0(X, \Omega^1_X)$. □

Similarly, Corollary 7.12 gives a simple criterion for showing that $S^i(X)$ is a proper subset of $\operatorname{Pic}^0(X)$. The proof is the same as above.

Corollary 8.5. *If the sequence*

$$H^0\bigl(X, \Omega^{i-1}_X \otimes L^{-1}\bigr) \xrightarrow{\omega\wedge} H^0\bigl(X, \Omega^i_X \otimes L^{-1}\bigr) \xrightarrow{\omega\wedge} H^0\bigl(X, \Omega^{i+1}_X \otimes L^{-1}\bigr) \tag{8.6}$$

is exact for some $\omega \in H^0(X, \Omega^1_X)$, then $S^i(X) \neq \operatorname{Pic}^0(X)$.

Finally, here is a version of Corollary 7.13, which gives a simple criterion for L to be an isolated point of $S^i(X)$.

Corollary 8.7. *If (8.6) is exact for every nonzero $\omega \in H^0(X, \Omega^1_X)$, then L is an isolated point of $S^i(X)$.*

In all three cases, the advantage is that we are now working with global sections of coherent sheaves; these are much easier to use than general cohomology classes.

Proof of the generic vanishing theorem, Part 2. We are now set to prove Theorem 6.6. Let X be a compact Kähler manifold, and $\operatorname{alb}\colon X \to \operatorname{Alb}(X)$ its Albanese mapping (for some choice of base point $x_0 \in X$). Remember that our goal is to prove the following precise version of the generic vanishing theorem:

$$\operatorname{codim}_{\operatorname{Pic}^0(X)} S^i(X) \geq \dim \operatorname{alb}(X) - i. \tag{8.8}$$

Proof. Let Z be any irreducible component of $S^i(X)$, and choose a point $L \in Z$ at which $\dim H^i(X, L)$ is as small as possible. If we set $m = \dim H^i(X, L) \geq 1$, then $Z \subseteq S^i_m(X)$, and so it suffices to show that

$$\dim_L S^i_m(X) \leq \dim \operatorname{Pic}^0(X) - \dim \operatorname{alb}(X) + i. \tag{8.9}$$

We are going to estimate the left-hand side in several steps. To begin with, choose a nonzero element $\beta \in H^0\bigl(X, \Omega^i_X \otimes L^{-1}\bigr)$; this is possible because this space has dimension $m \geq 1$. Then we have

$$\begin{aligned} \dim_L S^i_m(X) &\leq \dim\bigl\{ \omega \in H^0(X, \Omega^1_X) \bigm| \omega \wedge \beta = 0 \bigr\} \\ &\leq \dim \ker e(x) + \dim\bigl\{ \varphi \in T^*_x X \bigm| \varphi \wedge \beta(x) = 0 \bigr\}. \end{aligned}$$

The first inequality comes from Theorem 8.4, and the second by using the evaluation morphism $e(x)\colon H^0(X, \Omega^1_X) \to T^*_x X$ at a point $x \in X$.

Recall that $H^0(X, \Omega^1_X)$ is the cotangent space to $\operatorname{Alb}(X)$; according to Lemma 5.3, $e(x)$ is the codifferential of the Albanese morphism at the point x. At a sufficiently general point x, the rank of $e(x)$ is therefore equal to $\dim \operatorname{alb}(X)$, which means that

$$\dim \ker e(x) = \dim H^0(X, \Omega^1_X) - \dim \operatorname{im} e(x) = \dim \operatorname{Pic}^0(X) - \dim \operatorname{alb}(X).$$

The inequality above now becomes

$$\dim_L S^i_m(X) \leq \dim \operatorname{Pic}^0(X) - \dim \operatorname{alb}(X) + \dim\bigl\{ \varphi \in T^*_x X \bigm| \varphi \wedge \beta(x) = 0 \bigr\}.$$

After a brief look at (8.9), it is clear that we are reduced to proving that

$$\dim\bigl\{ \varphi \in T^*_x X \bigm| \varphi \wedge \beta(x) = 0 \bigr\} \leq i$$

for any $x \in X$ with $\beta(x) \neq 0$. But since

$$\beta(x) \in \bigwedge^i T_x^* X \otimes L_x^{-1} \simeq \bigwedge^i T_x^* X,$$

this inequality is a consequence of Lemma 8.10 below. We have therefore proved the generic vanishing theorem. Note that we could use a pointwise argument because we were working with holomorphic objects; this is the main reason why we needed Hodge theory. □

Lemma 8.10. *Let V be a finite-dimensional vector space, and let $\beta \in \bigwedge^i V$ be a nonzero element. Then* $\dim\{ v \in V \mid v \wedge \beta = 0 \} \leq i$.

As mentioned before, (8.8) leads to the following generic vanishing theorem (in the literal sense of the word).

Corollary 8.11. *For a general line bundle $L \in \operatorname{Pic}^0(X)$, one has*

$$H^i(X, L) = 0$$

for every $i < \dim \operatorname{alb}(X)$.

Proof. By Theorem 6.6, $S^i(X)$ has codimension at least $\dim \operatorname{alb}(X) - i > 0$ in $\operatorname{Pic}^0(X)$, and for any $L \notin S^i(X)$, one has $H^i(X, L) = 0$. □

Maximal Albanese dimension. The generic vanishing theorem is especially useful when X has *maximal Albanese dimension*, in the sense that $\dim \operatorname{alb}(X) = \dim X$. An equivalent way of putting this is that the Albanese mapping of X is generically finite over its image.

Example 8.12. A submanifold of a compact complex torus has maximal Albanese dimension. Similarly, any resolution of singularities of a subvariety of an abelian variaty has maximal Albanese dimension.

If X has maximal Albanese dimension, then $H^i(X, L) = 0$ for $L \in \operatorname{Pic}^0(X)$ general and $i < \dim X$. This has the following numerical consequence.

Corollary 8.13. *If X has maximal Albanese dimension, then $\chi(X, \omega_X) \geq 0$.*

Proof. Let $n = \dim X$. By the Riemann-Roch theorem, the Euler characteristic of $\omega_X \otimes L$ is independent of L (because $c_1(L) = 0$), and so we have

$$\begin{aligned} \chi(X, \omega_X) = \chi(X, \omega_X \otimes L) &= \sum_{i=0}^{n} (-1)^i \dim H^i(X, \omega_X \otimes L) \\ &= \sum_{i=0}^{n} (-1)^i \dim H^{n-i}(X, L^{-1}) \end{aligned}$$

for arbitrary $L \in \operatorname{Pic}^0(X)$. If L is sufficiently general, then all terms with $i > 0$ are zero, and we conclude that $\chi(X, \omega_X) = \dim H^0(X, \omega_X \otimes L) \geq 0$. □

Example 8.14. Since $\chi(\mathbb{P}^1, \omega_{\mathbb{P}^1}) = -1$, one can use this to show that every holomorphic mapping from $\mathbb{P}^1$ to a compact complex torus must be constant; of course, this can be proved more easily in other ways.

When X has maximal Albanese dimension, the codimension of the set $S^i(X)$ is at least $\dim X - i$. It turns out that the cohomology support loci are indeed getting smaller as i increases. Here is the precise result.

Proposition 8.15. *If X has maximal Albanese dimension, then*

$$\operatorname{Pic}^0(X) \supseteq S^n(X) \supseteq \cdots \supseteq S^1(X) \supseteq S^0(X) = \{\mathscr{O}_X\},$$

where $n = \dim X$.

Proof. Fix an index $i < n$, and suppose that $L \in S^i(X)$, so that $H^i(X, L) \neq 0$. Then there is a nonzero element

$$\alpha \in H^0\big(X, \Omega_X^i \otimes L^{-1}\big) \simeq \overline{H^i(X, L)}.$$

It will be enough to show that $\omega \wedge \alpha \neq 0$ for some choice of $\omega \in H^0(X, \Omega_X^1)$, because

$$\omega \wedge \alpha \in H^0\big(X, \Omega_X^{i+1} \otimes L^{-1}\big) \simeq \overline{H^{i+1}(X, L)}.$$

But this can be proved in the same way as above. Namely, take a sufficiently general point $x \in X$ such that $\alpha(x) \neq 0$ and such that alb is a covering map on some neighborhood of x. Then $e(x)\colon H^0(X, \Omega_X^1) \to T_x^* X$ is surjective, and so we can find n holomorphic one-forms $\omega_1, \dots, \omega_n$ with the property that $\omega_1(x), \dots, \omega_n(x)$ form a basis for the cotangent space $T_x^* X$. But because $i < n$, it is then immediate that $\omega_k(x) \wedge \alpha(x) \neq 0$ for some index k. □

Another illustrative example is the case of compact complex tori.

Corollary 8.16. *Let X be a compact complex torus of dimension n. For any pair of integers $0 \leq p, q \leq n$, and for any $L \in \operatorname{Pic}^0(X)$, one has*

$$H^q\big(X, \Omega_X^p \otimes L\big) \neq 0 \quad \Longleftrightarrow \quad L \simeq \mathscr{O}_X.$$

Proof. Since X is a group, Ω_X^p is a trivial bundle of rank $\binom{n}{p}$, and so it suffices to prove the assertion when $p = 0$. Also, by virtue of Proposition 8.15, it will be enough to show that $S^n(X) = \{\mathscr{O}_X\}$. But since $\omega_X \simeq \mathscr{O}_X$, we have

$$H^n(X, L) = \operatorname{Hom}\Big(H^0(X, L^{-1}), \mathbb{C}\Big)$$

by Serre duality, and this group is clearly nonzero if and only if $L \simeq \mathscr{O}_X$. □

Exercises.

Exercise 8.1. Prove the linear algebra fact in Lemma 8.10.

Exercise 8.2. Give an algebraic proof for the fact that, on a complex abelian variety A, one has $H^i(A, L) = 0$ for every nontrivial line bundle $L \in \operatorname{Pic}^0(A)$.

Lecture 9

Generic vanishing for holomorphic forms. So far, we have been discussing only cohomology groups of the form $H^i(X,L)$; you may be asking yourself whether similar results are true for $H^q\big(X,\Omega_X^p\otimes L\big)$. There are two different ways to phrase the question:

(1) Do the groups $H^q(X,\Omega_X^p\otimes L)$ satisfy a generic vanishing theorem?
(2) Are there codimension bounds for the sets

$$S^q(X,\Omega_X^p)=\big\{\,L\in\operatorname{Pic}^0(X)\;\big|\;H^q(X,\Omega_X^p\otimes L)\neq 0\,\big\},$$

similar to those in Theorem 6.6?

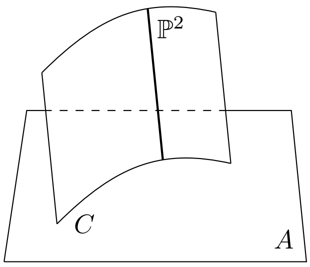
Note that, because of Serre duality,

$$H^i(X,L)\simeq\operatorname{Hom}\Big(H^{n-i}\big(X,\omega_X\otimes L^{-1}\big),\mathbb{C}\Big),$$

and so the answer to both questions is 'yes' when $p=0$ and $p=n$. The situation for intermediate values of p turns out to be much more complicated.

By analogy with Corollary 8.11, one might expect that $H^q(X,\Omega_X^p\otimes L)=0$ for general $L\in\operatorname{Pic}^0(X)$ and $p+q<\dim\operatorname{alb}(X)$. But this is not the case; in fact, Green and Lazarsfeld already gave the following counterexample in their first paper.

Example 9.1. Let A be a 4-dimensional abelian variety, and let $C\subseteq A$ be a smooth curve of genus at least two. We can of course always find such curves by taking complete intersections of sufficiently ample divisors. Define $f\colon X\to A$ to be the blowing up of A along C; the exceptional divisor $E\subseteq X$ is the projectivization of the normal bundle $N_{C|A}$, and therefore a $\mathbb{P}^2$-bundle over C.



Then f is quite obviously the Albanese mapping of X, and so every holomorphic line bundle on X with trivial first Chern class is the pullback of some $L\in\operatorname{Pic}^0(A)$. We are going to show that

$$H^2\big(X,\Omega_X^1\otimes f^*L\big)\neq 0 \tag{9.2}$$

for every $L\in\operatorname{Pic}^0(A)$; since $f(X)=A$ is 4-dimensional and $2+1<4$, this shows that the expected generalization of Corollary 8.11 does not hold.

To evaluate the above cohomology group, let $p\colon E\to C$ denote the restriction of f. A calculation in local coordinates shows that

$$R^ip_*\Omega_{E/C}^1=\begin{cases}\mathscr{O}_C & \text{if } i=1,\\ 0 & \text{if } i\neq 1;\end{cases} \tag{9.3}$$

From the short exact sequence

$$0\to f^*\Omega_A^1\to\Omega_X^1\to\Omega_{E/C}^1\to 0$$

we obtain the exactness of

$$H^2\big(X,f^*(\Omega_A^1\otimes L)\big)\to H^2\big(X,\Omega_X^1\otimes f^*L\big)\to H^2\big(X,\Omega_{E/C}^1\otimes f^*L\big)\to H^3\big(X,f^*(\Omega_A^1\otimes L)\big).$$

Because $f_* \mathscr{O}_X \simeq \mathscr{O}_A$ and $R^i f_* \mathscr{O}_X = 0$ for $i \geq 1$, we find that

$$H^i\bigl(X, f^*(\Omega_A^1 \otimes L)\bigr) \simeq H^i\bigl(A, \Omega_A^1 \otimes L\bigr) = 0$$

for $L \neq \mathscr{O}_A$. Consequently, using (9.3), we get

$$H^2\bigl(X, \Omega_X^1 \otimes f^*L\bigr) \simeq H^2\bigl(X, \Omega_{E/C}^1 \otimes f^*L\bigr) \simeq H^1\bigl(A, \mathscr{O}_C \otimes L\bigr) \simeq H^1\bigl(C, L\big|_C\bigr).$$

But since the genus of C is at least two, we know that this last group is nonzero for every $L \in \operatorname{Pic}^0(A)$. This proves (9.2).

The theorem of Green and Lazarsfeld. Green and Lazarsfeld proved the following generic vanishing theorem, in which the Albanese dimension $\dim \operatorname{alb}(X)$ is replaced by the codimension of the zero locus of a holomorphic one-form.

Theorem 9.4. *Suppose that there is a holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$ whose zero locus $Z(\omega)$ has codimension $\geq k$ in X. Then for generic $L \in \operatorname{Pic}^0(X)$,*

$$H^q\bigl(X, \Omega_X^p \otimes L\bigr) = 0 \quad \textit{whenever } p + q < k.$$

Our convention is that $k = \infty$ if $Z(\omega)$ is empty; in that case, a generic line bundle $L \in \operatorname{Pic}^0(X)$ satisfies $H^q\bigl(X, \Omega_X^p \otimes L\bigr) = 0$ for every $p, q \in \mathbb{Z}$.

To prove Theorem 9.4, we are going to use our deformation theory for cohomology groups to study the cohomology support loci

$$S^q(X, \Omega_X^p) = \bigl\{\, L \in \operatorname{Pic}^0(X) \bigm| H^q\bigl(X, \Omega_X^p \otimes L\bigr) \neq 0 \,\bigr\}$$

in a neighborhood of the origin in $\operatorname{Pic}^0(X)$. Just as before, they are related to the higher direct image sheaves

$$R^i p_{2*}\bigl(P \otimes p_1^* \Omega_X^p\bigr),$$

where P is the Poincaré line bundle on $X \times \operatorname{Pic}^0(X)$. In a neighborhood of the point $\mathscr{O}_X \in \operatorname{Pic}^0(X)$, we can find a bounded complex $E^\bullet$ of holomorphic vector bundles, with the property that

$$H^i\bigl(X, \Omega_X^p \otimes L\bigr) \simeq H^i\bigl(E^\bullet(L)\bigr).$$

By a similar argument as in Lemma 8.2, one shows that the derivative complex $D_v(E^\bullet, \mathscr{O}_X)$ is isomorphic to

$$H^0(X, \Omega_X^p) \to H^1(X, \Omega_X^p) \to \cdots \to H^n(X, \Omega_X^p),$$

with differential given by wedge product with $v \in \mathcal{H}^{0,1}(X)$. Using the isomorphisms

$$\overline{H^i(X, \Omega_X^p)} \simeq H^p(X, \Omega_X^i)$$

coming from Hodge theory, the conjugate of the derivative complex becomes

$$H^p(X, \mathscr{O}_X) \to H^p(X, \Omega_X^1) \to \cdots \to H^p(X, \Omega_X^n),$$

with differential given by wedge product with the holomorphic one-form $\bar{v}$. Note that when $0 < p < n$, we are dealing with actual cohomology groups; this is the place where the proof of Theorem 6.6 breaks down. In any case, the criterion in Corollary 7.12 reduces the proof of Theorem 9.4 to the following result.

Proposition 9.5. *Let $\omega \in H^0(X, \Omega_X^1)$ be a holomorphic one-form whose zero locus $Z(\omega)$ has codimension $\geq k$. Then the sequence*

$$H^q(X, \Omega_X^{p-1}) \xrightarrow{\omega \wedge} H^q(X, \Omega_X^p) \xrightarrow{\omega \wedge} H^q(X, \Omega_X^{p+1})$$

is exact whenever $p + q < k$.

To prove Proposition 9.5, we use a spectral sequence argument. The holomorphic one-form determines a Koszul complex $K^\bullet = K^\bullet(\Omega_X^1, \omega)$; concretely, this is the complex of holomorphic vector bundles

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\omega\wedge} \Omega_X^1 \xrightarrow{\omega\wedge} \Omega_X^2 \xrightarrow{\omega\wedge} \cdots \xrightarrow{\omega\wedge} \Omega_X^n \longrightarrow 0$$

If we denote by $\mathcal{H}^i K^\bullet$ the i-th cohomology sheaf of the Koszul complex, then we have two spectral sequences

$${}'E_1^{p,q} = H^q(X, K^p) \Longrightarrow \mathbf{H}^{p+q}(X, K^\bullet)$$

and

$${}''E_2^{p,q} = H^p\big(X, \mathcal{H}^q K^\bullet\big) \Longrightarrow \mathbf{H}^{p+q}(X, K^\bullet),$$

both converging to the hypercohomology of the complex $K^\bullet$. We shall prove below (using Hodge theory) that the first spectral sequence degenerates at E_2, meaning that ${}'E_2^{p,q} = {}'E_\infty^{p,q}$. The behavior of the second spectral sequence is controlled by the following lemma, which is basically a result in commutative algebra.

Lemma 9.6. *Let E be a holomorphic vector bundle of rank $n = \dim X$ on a complex manifold X, and let $s \in H^0(X, E)$ be a global section whose zero locus $Z(s)$ has codimension $\geq k$. Denote by $K^\bullet$ the Koszul complex constructed from $s\colon \mathscr{O}_X \to E$, indexed in such a way that $K^i = \bigwedge^i E$. Then $\mathcal{H}^q K^\bullet = 0$ for $q < k$.*

Proof. The sheaf $\mathcal{H}^q K^\bullet$ being supported on $Z = Z(s)$, it suffices to show that its stalk at any point $x \in Z$ vanishes if $q < k$. Let $\mathscr{O}_{X,x}$ be the local ring at such a point, and $\mathfrak{m}_x$ its maximal ideal. After choosing a local trivialization for E near x, the section s is given by n holomorphic functions $f_1, \dots, f_n \in \mathfrak{m}_x$ that generate the ideal of Z in $\mathscr{O}_{X,x}$. Since $\dim Z \leq n - k$, we may assume without loss of generality that $f_1, \dots, f_k$ cut out a subvariety of dimension $n - k$ in some neighborhood of $x \in Z$. Now $\mathscr{O}_{X,x}$ is a regular local ring, and so $f_1, \dots, f_k$ are then automatically a regular sequence. By a familiar result in commutative algebra (which you can find, for example, in Eisenbud's book), the Koszul complex $K(f_1, \dots, f_n)$ is therefore exact in every degree $q < k$. But

$$K^\bullet \otimes \mathscr{O}_{X,x} \simeq K(f_1, \dots, f_n),$$

and so we get the asserted vanishing. □

Granting the fact that the first spectral sequence degenerates at E_2, the proof of Proposition 9.5 is now easily concluded. By assumption, $Z(\omega)$ has codimension $\geq k$ in X, and so Lemma 9.6 shows that $\mathcal{H}^q K^\bullet = 0$ for $q < k$. Using the second spectral sequence, we therefore have

$$H^{p+q}\big(X, K^\bullet\big) = 0 \quad \text{for } p + q < k.$$

From the first spectral sequence, we now deduce that ${}'E_2^{p,q} = {}'E_\infty^{p,q} = 0$ for $p+q < k$; but this means exactly that the complex

$$H^q(X, \Omega_X^{p-1}) \xrightarrow{\omega\wedge} H^q(X, \Omega_X^p) \xrightarrow{\omega\wedge} H^q(X, \Omega_X^{p+1})$$

is exact whenever $p + q < k$.

To finish the proof of Proposition 9.5, and therefore of Theorem 9.4, it remains to show prove the degeneration of the first spectral sequence. This holds for any holomorphic one-form ω, independently of its zero locus.

Proposition 9.7. *Let X be a compact Kähler manifold, and let $\omega \in H^0(X, \Omega_X^1)$ be a holomorphic one-form. Let $K^\bullet = K^\bullet(\Omega_X^1, \omega)$ be the complex*

$$\mathscr{O}_X \xrightarrow{\omega\wedge} \Omega_X^1 \xrightarrow{\omega\wedge} \Omega_X^2 \xrightarrow{\omega\wedge} \cdots \xrightarrow{\omega\wedge} \Omega_X^n.$$

Then the first hypercohomology spectral sequence

$$'E_1^{p,q} = H^q(X, K^p) \Longrightarrow \mathbf{H}^{p+q}(X, K^\bullet)$$

degenerates at E_2.

The proof is a typical application of Hodge theory – indeed, it can be said that the two main applications of Hodge theory to algebraic geometry are the vanishing of certain cohomology groups, and the degeneration of certain spectral sequences. More precisely, it is based on representing cohomology classes by harmonic forms, and on the principle of two types (see Proposition 3.17).

Proof of Proposition 9.7. Let us briefly recall the construction of the spectral sequence. The complex $K^\bullet$ is resolved by the double complex with terms $\mathcal{A}_X^{p,q}$ and differentials $\omega\wedge$ and $\bar\partial$. All sheaves $\mathcal{A}_X^{p,q}$ are acyclic for the global sections functor, and so the hypercohomology of $K^\bullet$ is computed by the double complex

$$\begin{array}{ccccccccc}
A^{0,n}(X) & \xrightarrow{\omega\wedge} & A^{1,n}(X) & \xrightarrow{\omega\wedge} & A^{2,n}(X) & \xrightarrow{\omega\wedge} & \cdots & \xrightarrow{\omega\wedge} & A^{n,n}(X) \\
{\scriptstyle\bar\partial}\big\uparrow & & {\scriptstyle\bar\partial}\big\uparrow & & {\scriptstyle\bar\partial}\big\uparrow & & & & {\scriptstyle\bar\partial}\big\uparrow \\
\vdots & & \vdots & & \vdots & & \iddots & & \vdots \\
{\scriptstyle\bar\partial}\big\uparrow & & {\scriptstyle\bar\partial}\big\uparrow & & {\scriptstyle\bar\partial}\big\uparrow & & & & {\scriptstyle\bar\partial}\big\uparrow \\
A^{0,2}(X) & \xrightarrow{\omega\wedge} & A^{1,2}(X) & \xrightarrow{\omega\wedge} & A^{2,2}(X) & \xrightarrow{\omega\wedge} & \cdots & \xrightarrow{\omega\wedge} & A^{n,2}(X) \\
{\scriptstyle\bar\partial}\big\uparrow & & {\scriptstyle\bar\partial}\big\uparrow & & {\scriptstyle\bar\partial}\big\uparrow & & & & {\scriptstyle\bar\partial}\big\uparrow \\
A^{0,1}(X) & \xrightarrow{\omega\wedge} & A^{1,1}(X) & \xrightarrow{\omega\wedge} & A^{2,1}(X) & \xrightarrow{\omega\wedge} & \cdots & \xrightarrow{\omega\wedge} & A^{n,1}(X) \\
{\scriptstyle\bar\partial}\big\uparrow & & {\scriptstyle\bar\partial}\big\uparrow & & {\scriptstyle\bar\partial}\big\uparrow & & & & {\scriptstyle\bar\partial}\big\uparrow \\
A^{0,0}(X) & \xrightarrow{\omega\wedge} & A^{1,0}(X) & \xrightarrow{\omega\wedge} & A^{2,0}(X) & \xrightarrow{\omega\wedge} & \cdots & \xrightarrow{\omega\wedge} & A^{n,0}(X)
\end{array}$$

Now obviously $'E_1^{p,q} \simeq H^{p,q}(X)$, and the differential $d_1^{p,q} \colon {'E_1^{p,q}} \to {'E_1^{p+1,q}}$ is nothing but wedge product with the $(1,0)$-form ω.

To prove that the spectral sequence degenerates, we first show that $d_2 = 0$. Take an arbitrary element of $'E_2^{p,q}$; it is represented by a cohomology class in $H^{p,q}(X)$ whose wedge product with ω is zero in $H^{p+1,q}(X)$. Let $\alpha \in \mathcal{H}^{p,q}(X)$ be the unique harmonic representative; by assumption, $\omega\wedge\alpha$ is $\bar\partial$-exact. Both α and ω being harmonic, we have

$$\partial(\omega\wedge\alpha) = \partial\omega\wedge\alpha - \omega\wedge\partial\alpha = 0.$$

By the $\partial\bar\partial$-Lemma, there exists $\beta \in A^{p-1,q-1}(X)$ such that $\omega\wedge\alpha = \bar\partial\partial\beta$, and then

$$d_2^{p,q}[\alpha] = [\omega\wedge\partial\beta] \in {'E_2^{p+2,q-1}}.$$

To show that this class is zero, it suffices to prove that $\omega\wedge\partial\beta$ is $\bar\partial$-exact. But it is clearly $\bar\partial$-closed and, on account of $\omega\wedge\partial\beta = -\partial(\omega\wedge\beta)$, also ∂-exact; by the $\partial\bar\partial$-Lemma, we therefore have $\omega\wedge\partial\beta = \bar\partial\partial\gamma$ for some $\gamma \in A^{p+1,q-2}(X)$. This proves that $d_2^{p,q}[\alpha] = 0$, and hence that $d_2 = 0$.

In the above notation, $d_3^{p,q}[\alpha]$ is then equal to $[\omega\wedge\partial\gamma] \in {'E_3^{p+3,q-2}}$, and similar reasoning shows that this class is zero; etc. The details are left as an exercise. □

Codimension bounds. Now let us return to the other question from the beginning of class, namely whether one has codimension bounds for the sets

$$S^q(X, \Omega_X^p) = \left\{\, L \in \operatorname{Pic}^0(X) \;\middle|\; H^q(X, \Omega_X^p \otimes L) \neq 0 \,\right\}.$$

Despite the counterexample from above, there is in fact an analogue of Theorem 6.6; it was discovered only recently, by Mihnea Popa and myself, and independently by Thomas Krämer and Rainer Weissauer.

Theorem 9.8. *Let X be a smooth projective variety. Then one has*

$$\operatorname{codim} S^q(X, \Omega_X^p) \geq \left| p + q - \dim X \right| - \delta(\mathrm{alb})$$

where $\delta(\mathrm{alb})$ is the so-called defect of semismallness *of the Albanese morphism.*

In general, the *defect of semismallness* of a proper morphism $f \colon X \to Y$ between two complex manifolds is the quantity

$$\delta(f) = \dim X \times_Y X - \dim X = \max_{\ell \in \mathbb{N}} \Bigl(2\ell - \dim X + \dim Y_\ell \Bigr),$$

where $Y_\ell = \bigl\{\, y \in Y \bigm| \dim f^{-1}(y) \geq \ell \,\bigr\}$ for $\ell \in \mathbb{N}$. This invariant measures, in a rather subtle way, how complicated the morphism f is. The name comes from the fact that $\delta(f) = 0$ if and only if f is semismall; this notion plays a role in the theory of perverse sheaves.

Example 9.9. In Example 9.1, where X was the blowing up of an abelian fourfold along a curve, you can easily calculate that $\delta(f) = 1$; this explains why the naive generalization of the generic vanishing theorem does not hold.

One can show that the result in Theorem 9.8 is optimal, in the sense that, on every X, the inequality becomes an equality for some choice of $p, q \in \mathbb{Z}$. On the other hand, the inequality

$$\operatorname{codim} \bigl\{\, L \in \operatorname{Pic}^0(X) \bigm| H^q(X, L) \neq 0 \,\bigr\} \geq \dim X - \delta(\mathrm{alb}) - q$$

that we get for $p = 0$ is weaker than the one in the original generic vanishing theorem, which reads

$$\operatorname{codim} \bigl\{\, L \in \operatorname{Pic}^0(X) \bigm| H^q(X, L) \neq 0 \,\bigr\} \geq \dim \mathrm{alb}(X) - q.$$

The proof of Theorem 9.8 is considerably more difficult than that of the generic vanishing theorem; the main tool is Morihiko Saito's theory of Hodge modules.

Exercises.

Exercise 9.1. Complete the proof of Proposition 9.7 by showing that $d_r = 0$ for every $r \geq 3$.

Exercise 9.2. Prove that $\dim X - \delta(f) \leq \dim f(X)$.

Lecture 10

Beauville's theorem. So far, we have been studying properties of cohomology support loci on arbitrary compact Kähler manifolds. To get a better feeling for the general result, we shall now look at a special case: our goal will be to understand the structure of the set

$$S^1(X) = \left\{ L \in \operatorname{Pic}^0(X) \;\middle|\; H^1(X, L) \neq 0 \right\}$$

when X is a compact Kähler manifold. The following example shows how fibrations of X over curves of genus at least two lead to positive-dimensional components in $S^1(X)$.

Example 10.1. Let C be a smooth compact curve, and suppose that there is a holomorphic mapping $f\colon X \to C$ with connected fibers. Take any line bundle $L \in \operatorname{Pic}^0(C)$, and consider the cohomology of its pullback f^*L. Because f has connected fibers, $f_* f^* L \simeq f_* \mathscr{O}_X \otimes L \simeq L$; from the Leray spectral sequence

$$E_2^{p,q} = H^p\big(C, R^q f_* \mathscr{O}_X \otimes L\big) \Longrightarrow H^{p+q}\big(X, f^*L\big),$$

we therefore get an exact sequence

$$0 \to H^1(C, L) \to H^1\big(X, f^*L\big) \to H^0\big(C, R^1 f_* \mathscr{O}_X \otimes L\big) \to 0.$$

If the genus of C is at least two, then $H^1(C, L) \neq 0$, which means that f^* embeds $\operatorname{Pic}^0(C)$ into $S^1(X)$. If the genus of C is equal to one, then $H^1(C, L) = 0$ unless L is trivial; we shall see below that f may nevertheless be responsible for a component of $S^1(X)$, but that this case is more complicated.

In their first paper, Green and Lazarsfeld conjectured that basically every positive-dimensional component of $S^1(X)$ is of this form. This conjecture was proved shortly afterwards by Arnaud Beauville. Before we can state Beauville's theorem, we have to introduce some notation. Denote by $\operatorname{Pic}^\tau(X)$ the set of holomorphic line bundles L on X whose first Chern class $c_1(L) \in H^2(X, \mathbb{Z})$ is torsion. Note that we have an exact sequence

$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}^\tau(X) \to \ker\Big(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})\Big) \to 0.$$

Given a morphism $f\colon X \to C$ with connected fibers, we define

$$\operatorname{Pic}^\tau(X, f) = \left\{ L \in \operatorname{Pic}^\tau(X) \;\middle|\; L \text{ is trivial on every smooth fiber of } f \right\}.$$

and set $\operatorname{Pic}^0(X, f) = \operatorname{Pic}^\tau(X, f) \cap \operatorname{Pic}^0(X)$. We shall see below that $\operatorname{Pic}^0(X, f)$ is almost the same as the image of $f^*\colon \operatorname{Pic}^0(C) \to \operatorname{Pic}^0(X)$; the difference has to do with the singular fibers of f. With that in mind, here is Beauville's result.

Theorem 10.2. *Let X be a compact Kähler manifold. Let $\{f_i\colon X \to C_i\}_{i \in I}$ be the collection of all fibrations of X onto curves of genus ≥ 1. Then $S^1(X)$ is the union of the following subsets:*

(1) $\operatorname{Pic}^0(X, f_i)$ for every $i \in I$ with $g(C_i) \geq 2$;
(2) $\operatorname{Pic}^0(X, f_i) \setminus f_i^* \operatorname{Pic}^0(C_i)$ for every $i \in I$ with $g(C_i) = 1$;
(3) finitely many isolated points.

Note. Frédéric Campana later completed the analysis of $S^1(X)$ by showing that every isolated point has finite order. More recently, Thomas Delzant gave a different proof for Theorem 10.2, based on ideas from geometric group theory.

The proof divides itself into two parts: (a) Every fibration $f_i\colon X \to C_i$ leads to a positive-dimensional component of $S^1(X)$. (b) Conversely, every positive-dimensional component of $S^1(X)$ comes from some $f_i\colon X \to C_i$.

From fibrations to positive-dimensional components. We begin the proof of the first part by studying more carefully the case of a fibration $f\colon X \to C$; as above, we assume that f has connected fibers, and so $f_* \mathscr{O}_X \simeq \mathscr{O}_C$. Our first goal is to have a good description of the line bundles in $\operatorname{Pic}^\tau(X, f)$.

For every point $y \in C$, the fiber $f^* y$ can be written in the form

$$f^* y = \sum_{i=1}^{d} n_i E_i,$$

where the E_i are irreducible and reduced; moreover, the union of all the E_i is connected (by our assumptions on f).

Lemma 10.3. *The classes $[E_1], \dots, [E_d]$ are linearly independent in $H^2(X, \mathbb{C})$.*

Proof. The proof of this fact is based on the Hodge index theorem. Fix a Kähler metric on X, with Kähler form ω. Recall that we have the intersection pairing

$$\alpha \cdot \beta = \int_X \alpha \wedge \beta \wedge \omega^{n-2}$$

on the vector space $H^2(X, \mathbb{C})$; according to the Hodge index theorem, its signature is $(+1, -1, \dots, -1)$ on the subspace $H^{1,1}(X)$. Since $F = f^* y$ is a fiber of f, we have $[F] \cdot [F] = 0$; moreover, the restriction of the pairing to the subspace

$$[F]^\perp = \left\{ \alpha \in H^{1,1}(X) \;\middle|\; \alpha \cdot [F] = 0 \right\} \tag{10.4}$$

is negative semi-definite, and one has $\alpha \cdot \alpha = 0$ if and only if α is a multiple of $[F]$. (Exercise: Deduce these assertions about $[F]^\perp$ from the Hodge index theorem.) Note that we have $[E_i] \cdot [F] = 0$ for every $i = 1, \dots, d$, and so $[E_1], \dots, [E_d] \in [F]^\perp$.

Before we prove the linear independence, we shall first show that if

$$\alpha = \sum_{i=1}^{d} c_i [E_i]$$

is any element such that $\alpha \cdot [E_j] = 0$ for every $j = 1, \dots, d$, then the coefficient vector $(c_1, \dots, c_d) \in \mathbb{Q}^d$ must be a multiple of $(n_1, \dots, n_d)$. Suppose that this was not the case. By adding a suitable multiple of $[F]$, we can then arrange that the c_i are nonzero and do not all have the same sign. The signs define a partition $\{1, \dots, d\} = I_1 \sqcup I_2$ with $c_i > 0$ for $i \in I_1$ and $c_i < 0$ for $i \in I_2$, and we obtain

$$\sum_{i \in I_1} c_i [E_i] \cdot \sum_{j \in I_1} c_j [E_j] = \sum_{i \in I_1} c_i [E_i] \cdot \sum_{j \in I_2} (-c_j) [E_j].$$

Because the union of all the E_i is connected, we must have $[E_i] \cdot [E_j]$ for at least one pair $(i, j) \in I_1 \times I_2$; it follows that the right-hand side is > 0. But this contradicts the Hodge index theorem, because the left-hand side is the square of an element of $[F]^\perp$. The conclusion is that $(c_1, \dots, c_d)$ must be a multiple of $(n_1, \dots, n_d)$.

To finish the proof, suppose that there is a linear relation $c_1[E_1] + \dots + c_d[E_d] = 0$. By the above, $(c_1, \dots, c_d) = (cn_1, \dots, cn_d)$, and so $c[F] = 0$. But this can only be true if $c = 0$, because F is an effective divisor. □

Now suppose that $f^* y = \sum n_i E_i$, and let m be the greatest common divisor of the coefficients n_i. If $m > 1$, we say that $f^* y$ is a *multiple fiber* of f; in that case, we can write $f^* y = mF$ for some effective divisor F. In the following, we shall denote by $m_1 F_1, \dots, m_s F_s$ the multiple fibers of f.

Lemma 10.5. *Let D be a divisor on X whose irreducible components are all contained in the fibers of f. Then the class of D in $H^2(X, \mathbb{C})$ is equal to zero if and* *only if* $D = f^*\delta + \sum k_i F_i$, *where* $k_i \in \mathbb{Z}$ *and* δ *is a divisor on* C *such that*

$$\deg\delta + \sum_{i=1}^{s} \frac{k_i}{m_i} = 0.$$

Proof. Let F be a general smooth fiber of f. The divisor D is contained in finitely many fibers of f, and can be decomposed accordingly as $D = D_1 + \cdots + D_r$; note that $[D_i]\cdot[F] = 0$. Now if $[D] = 0$, it follows that $[D_i]\cdot[D_i] = 0$, and the Hodge index theorem implies that $[D_i]$ is a multiple of $[F]$. By Lemma 10.3, D_i is therefore a rational multiple of the fiber in which it is contained. It follows that

$$D = f^*\delta + \sum_{i=1}^{s} k_i F_i.$$

Because F is linearly equivalent to $m_i F_i$, we see that D is linearly equivalent to

$$\left(\deg\delta + \sum_{i=1}^{s} \frac{k_i}{m_i}\right)\cdot F;$$

in particular, $[D]$ is zero in $H^2(X,\mathbb{C})$ iff the number in parentheses is zero. □

After these preliminaries, we can now describe the set $\operatorname{Pic}^\tau(X,f)$ more concretely.

Proposition 10.6. *For* $L \in \operatorname{Pic}^\tau(X)$, *the following three conditions are equivalent:*

(a) The sheaf f_*L *is nontrivial.*
(b) The restriction of L *to every smooth fiber of* f *is trivial.*
(c) There is $L_0 \in \operatorname{Pic}^0(C)$ *and integers* $k_1,\dots,k_s$ *with* $\sum k_i/m_i = 0$, *such that*

$$L \simeq f^*L_0 \otimes \mathscr{O}_X\Big(\sum k_i F_i\Big).$$

Proof. Observe that the sheaf f_*L is torsion-free, and therefore locally free (because C is a smooth curve). This fact makes the equivalence between (a) and (b) easy to prove. Indeed, if the restriction of L to every smooth fiber of f is trivial, then f_*L has generic rank one, and is therefore a line bundle on C. Conversely, $f_*L \neq 0$ implies by base change that $H^0(F, L|_F) \neq 0$ for every smooth fiber F; since the restriction of L is an element of $\operatorname{Pic}^\tau(F)$, and F is a compact smooth curve, it follows that $L|_F$ must be trivial.

Since (c) obviously implies (b), all that remains is to prove the other implication. Let $U \subseteq C$ be the maximal open subset over which f is smooth. Then f_*L is torsion-free and its restriction to U has rank one; therefore it must be a line bundle on C. Now $f^*f_*L \to L$ is surjective on $f^{-1}(U)$; after choosing a nontrivial meromorphic section of f_*L, this implies that L is isomorphic to a line bundle of the form $\mathscr{O}_X(D)$, where D is a divisor on X whose irreducible components are contained in the fibers of f. Because $[D] = c_1(L)$ is zero in $H^2(X,\mathbb{C})$, we can apply Lemma 10.5 to get

$$D = f^*\delta + \sum_{i=1}^{s} k_i F_i,$$

where δ is a divisor on C of degree $\deg\delta = -\sum k_i/m_i$. By adding a suitable multiple of m_i to one of the integers k_i, we can arrange that $\deg\delta = 0$; if we now set $L_0 = \mathscr{O}_C(\delta)$, we have

$$L \simeq \mathscr{O}_X(D) = f^*L_0 \otimes \mathscr{O}_X\Big(\sum k_i F_i\Big). \qquad \square$$

Our next goal is to describe the connected components of $\operatorname{Pic}^\tau(X,f)$. To that end, we introduce the subgroup

$$\Gamma^\tau(f) \subseteq \bigoplus_{i=1}^{s} \mathbb{Z}/m_i\mathbb{Z},$$

consisting of all tuples $(\bar{k}_1, \dots, \bar{k}_s)$ such that $\sum k_i/m_i$ is an integer. The following proposition shows that the connected components of $\operatorname{Pic}^\tau(X, f)$ are in bijection with $\Gamma^\tau(f)$, and that each connected component is isomorphic to $\operatorname{Pic}^0(C)$.

Proposition 10.7. *We have an exact sequence*

$$0 \longrightarrow \operatorname{Pic}^0(C) \xrightarrow{f^*} \operatorname{Pic}^\tau(X, f) \xrightarrow{\varphi} \Gamma^\tau(f) \longrightarrow 0$$

Proof. Note that $f^* \operatorname{Pic}^0(C) \to \operatorname{Pic}^\tau(X, f)$ is injective because $f_* \mathscr{O}_X \simeq \mathscr{O}_C$. Based on the description of $\operatorname{Pic}^\tau(X, f)$ in Proposition 10.6, we would like to define

$$\varphi\colon \operatorname{Pic}^\tau(X, f) \to \Gamma^\tau(f), \quad f^* L_0 \otimes \mathscr{O}_X\Big(\sum k_i F_i\Big) \mapsto (\bar{k}_1, \dots, \bar{k}_s).$$

With this definition, it is clear that $\varphi \circ f^* = 0$; but it is not clear that φ is independent of the representation of the line bundle. We can prove both the exactness of the sequence, and the fact that φ is well-defined, if we manage to show that

$$\psi\colon \Gamma^\tau(f) \to \frac{\operatorname{Pic}^\tau(X, f)}{f^* \operatorname{Pic}^0(C)}, \quad (\bar{k}_1, \dots, \bar{k}_s) \mapsto \mathscr{O}_X\Big(\sum k_i F_i\Big)$$

is an isomorphism. Proposition 10.6 says that ψ is surjective. To prove that ψ is also injective, suppose that we have integers $k_1, \dots, k_s$ such that the line bundle $L = \mathscr{O}_X\big(k_1 F_1 + \dots + k_s F_s\big)$ belongs to $f^* \operatorname{Pic}^0(C)$. The restriction of L to F_i is then trivial; on the other hand, it equals $\mathscr{O}_{F_i}(k_i F_i)$. One can show that the line bundle $\mathscr{O}_{F_i}(F_i)$ has order exactly m_i in $\operatorname{Pic}(F_i)$; this implies that that $m_i \mid k_i$, which proves our claim. ∎

The next step is to calculate $H^1(X, L)$ for $L \in \operatorname{Pic}^\tau(X, f)$. Since we are going to use the Leray spectral sequence for this, we need to know $f_* L$.

Proposition 10.8. *Suppose that $L \in \operatorname{Pic}^\tau(X)$ can be written in the form*

$$L = f^* L_0 \otimes \mathscr{O}_X\Big(\sum k_i F_i\Big)$$

for $L_0 \in \operatorname{Pic}(C)$ and integers $k_1, \dots, k_s$ with $0 \leq k_i < m_i$ and $\sum k_i/m_i \in \mathbb{Z}$. Then $f_* L$ is isomorphic to L_0.

Proof. Let $N = \mathscr{O}_X(k_1 F_1 + \dots + k_s F_s)$; then it suffices to show that $f_* N \simeq \mathscr{O}_C$. Consider the exact sequence

$$0 \to N(-F_i) \to N \to N\big|_{F_i} \to 0.$$

By induction on the integer $k_1 + \dots + k_s$, it suffices to show that if $k_i > 0$, then the restriction of N to F_i has no nontrivial global sections. But this restriction is isomorphic to $\mathscr{O}_{F_i}(k_i F_i)$, and it is known than $\mathscr{O}_{F_i}(F_i)$ has order exactly m_i in $\operatorname{Pic}(F_i)$. In particular, $\mathscr{O}_{F_i}(k_i F_i)$ is a nontrivial torsion element of $\operatorname{Pic}(F_i)$. With some additional work, one can then show that $\mathscr{O}_{F_i}(k_i F_i)$ cannot have a nontrivial global section. ∎

The following result generalizes the example from the beginning of class.

Proposition 10.9. *Let S be a connected component of $\operatorname{Pic}^\tau(X, f)$, and let $k_1, \dots, k_s$ be integers such that $0 \leq k_i < m_i$ and such that the line bundles in S are of the form $f^* L_0 \otimes \mathscr{O}_X(k_1 F_1 + \dots + k_s F_s)$ for $L_0 \in \operatorname{Pic}(C)$. Then one has*

$$\dim H^1(X, L) \geq g(C) - 1 + \sum_{i=1}^{s} \frac{k_i}{m_i} \tag{10.10}$$

for every $L \in S$.

Proof. As before, the Leray spectral sequence gives rise to an exact sequence

$$0 \to H^1(C, L_0) \to H^1(X, L) \to H^0\big(C, R^1 f_* L\big) \to 0,$$

using that $f_* L \simeq L_0$. By the Riemann-Roch formula,

$$\dim H^1(C, L_0) = \dim H^0(C, L_0) + g(C) - 1 - \deg L_0 \geq g(C) - 1 - \deg L_0;$$

note that $H^0(C, L_0) \simeq H^0(X, L)$ vanishes unless $L \simeq \mathscr{O}_X$. To compute the degree of L_0, let F be any smooth fiber of f; then $m_i F_i$ is linearly equivalent to F, and so

$$0 = c_1(L) = \deg L_0 \cdot [F] + \sum_{i=1}^{s} \frac{k_i}{m_i}[F],$$

which implies that $\deg L_0 = -\sum k_i/m_i$. With some additional work, one can show that the inequality is actually an equality except for finitely many $L \in S$; because we do not need this fact to prove Theorem 10.2, we shall not dwell on this point. □

The inequality in Proposition 10.9 tells us whether a fibration $f\colon X \to C$ leads to a positive-dimensional subset of $S^1(X)$. This proves one half of Theorem 10.2.

Corollary 10.11. *Let $f\colon X \to C$ be a morphism with connected fibers, and let S be a connected component of $\operatorname{Pic}^0(X, f)$. Then $S \subseteq S^1(X)$, except possibly in the case when $g(C) = 1$ and $S = f^* \operatorname{Pic}^0(C)$.*

Proof. If $g(C) \geq 2$, then the right-hand side of (10.10) is positive, and so $L \in S^1(X)$ for every $L \in S$. If $g(C) = 1$, we can draw the same conclusion, except in the case when $k_1 = \cdots = k_s = 0$, which corresponds to having $S = f^* \operatorname{Pic}^0(C)$. Using results about vector bundles on elliptic curves, Beauville shows that $H^1(X, L) = 0$ for general $L \in f^* \operatorname{Pic}^0(C)$, and so $f^* \operatorname{Pic}^0(C)$ never contributes a component of positive dimension when $g(C) = 1$; in the second part of the proof, we will give a different argument for this fact. □

From positive-dimensional components to fibrations. We now turn to the second part of the proof of Theorem 10.2. Suppose that $L \in S^1(X)$ is a non-isolated point; we have to show that it lies on some $\operatorname{Pic}^0(X, f_i)$, but not on $f_i^* \operatorname{Pic}^0(C)$ in case $g(C_i) = 1$. Because L is not an isolated point, the criterion in Corollary 8.7 shows that there exists a nonzero holomorphic one-form ω for which the sequence

$$H^0\big(X, L^{-1}\big) \xrightarrow{\omega\wedge} H^0\big(X, \Omega_X^1 \otimes L^{-1}\big) \xrightarrow{\omega\wedge} H^0\big(X, \Omega_X^2 \otimes L^{-1}\big)$$

is not exact. Because $L \in \operatorname{Pic}^0(X, f)$ if and only if $L^{-1} \in \operatorname{Pic}^0(X, f)$, this reduces the proof of Theorem 10.2 to the following generalization of the classical Castelnuovo-de Franchis lemma (which is the case $L = \mathscr{O}_X$).

Proposition 10.12. *Let $L \in \operatorname{Pic}^0(X)$, and suppose that there is a holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$ for which the sequence*

$$H^0(X, L) \xrightarrow{\omega\wedge} H^0(X, \Omega_X^1 \otimes L) \xrightarrow{\omega\wedge} H^0(X, \Omega_X^2 \otimes L)$$

is not exact. Then there is a morphism $f\colon X \to C$ to a curve of genus at least one, such that $\omega \in f^* H^0(C, \Omega_C^1)$ and

$$L \in \operatorname{Pic}^0(X, f) \text{ in case } g(C) \geq 2,$$
$$L \in \operatorname{Pic}^0(X, f) \setminus f^* \operatorname{Pic}^0(C) \text{ in case } g(C) = 1.$$

Proof. By assumption, there is a nonzero holomorphic section $\alpha \in H^0(X, \Omega_X^1 \otimes L)$ with the property that $\alpha \wedge \omega = 0$. When L is nontrivial, α is automatically not exact (because $H^0(X, L) = 0$); when L is trivial, we may assume that α is not a multiple of ω. The idea is to use α to construct a meromorphic function g on X

such that $dg \wedge \omega = 0$. This will give us a morphism to $\mathbb{P}^1$, and by taking the Stein factorization, we will obtain the desired fibration of X over a curve.

To construct the meromorphic function, recall two facts from Hodge theory. The first is that the line bundle L admits a Hermitian metric whose Chern connection $\nabla \colon L \to A^1(L)$ is flat. Concretely, L can be described by an operator of the form $\bar{\partial} + \tau$, for $\tau \in \mathcal{H}^{0,1}(X)$, and then $\nabla = d + \tau - \bar{\tau}$. The second fact is that every holomorphic section of $\Omega_X^p \otimes L$ is ∇-harmonic, and therefore automatically ∇-closed; for the details, refer back to Theorem 6.2, whose proof was based on the Kähler identities for the operators $\bar{\partial} + \tau$ and $\partial - \bar{\tau}$.

Now back to the construction of the meromorphic function g. The relation $\alpha \wedge \omega = 0$ implies that $\alpha = \omega \otimes s$ for a meromorphic section of L; this section is holomorphic at all points where ω is not zero. (See Lemma 10.13 for the proof.) Hodge theory implies that $\nabla \alpha = 0$ and $d\omega = 0$, and so we get $\omega \wedge \nabla s = 0$ (as a meromorphic section of $\Omega_X^2 \otimes L$). As before, this means that there is a meromorphic function g on X with the property that $\nabla s = g\omega \otimes s$; note that g is holomorphic on the set where $\omega \otimes s$ is holomorphic and nonzero. If we apply the flat connection ∇ one more time, we obtain

$$0 = \nabla(\nabla s) = \nabla\big(g\omega \otimes s\big) = dg \wedge \omega \otimes s + g d\omega \otimes s - g\omega \wedge \nabla s = dg \wedge \omega \otimes s,$$

which leads to the desired relation $dg \wedge \omega = 0$.

We now use g to construct a fibration $f \colon X \to C$ with $g(C) \geq 1$. By our choice of α, the meromorphic section s cannot be holomorphic, and therefore has poles; a local calculation shows that ∇s must have poles of higher order, which means that the one-form $g\omega = s^{-1}\nabla s$ cannot be holomorphic. Equivalently, g is not constant, and therefore defines a nontrivial meromorphic mapping from X to $\mathbb{P}^1$. After resolving indeterminacies by blowing up along complex submanifolds, we obtain holomorphic mappings $p \colon \tilde{X} \to X$ and $\tilde{g} \colon \tilde{X} \to \mathbb{P}^1$ such that $\tilde{g}$ is an extension of $g \circ p$. Now consider the Stein factorization

$$\begin{tikzcd}
\tilde{X} \arrow[r, "f"] \arrow[rr, bend left, "\tilde{g}"] \arrow[d, "p"] & C \arrow[r] & \mathbb{P}^1 \\
X
\end{tikzcd}$$

of $\tilde{g}$; by definition, $f \colon \tilde{X} \to C$ has connected fibers, and $C \to \mathbb{P}^1$ is finite. Because $dg \wedge \omega = 0$, the restriction of $p^*\omega$ to a general fiber of f is zero; consequently, $p^*\omega$ is the pullback of a meromorphic one-form ω_0 on C. Because $f^*\omega_0$ is holomorphic, ω_0 is necessarily holomorphic as well; it follows that $g(C) \geq 1$. Now all the exceptional divisors of $p \colon \tilde{X} \to X$ have to map to points in C, and so we conclude, after the fact, that g was actually defined everywhere. In particular, we may assume without loss of generality that $\tilde{X} = X$ and $\omega = f^*\omega_0$. This gives us the desired fibration $f \colon X \to C$ over a curve of genus at least one.

To prove that $L \in \operatorname{Pic}^0(X, f)$, we return to the relation $\alpha = f^*\omega_0 \otimes s$. Because α is holomorphic, there is an effective divisor δ on C such that pole divisor of s is contained in $f^*\delta$; for example, we can take the divisor of zeros of ω_0. Now s is a nontrivial holomorphic section of $L \otimes \mathscr{O}_X(f^*\delta)$, and so

$$H^0\big(X, L \otimes \mathscr{O}_X(f^*\delta)\big) \simeq H^0\big(C, f_*L \otimes \mathscr{O}_C(\delta)\big) \neq 0.$$

This shows that f_*L is nontrivial, and therefore $L \in \operatorname{Pic}^0(X, f)$ by Proposition 10.6.

To conclude the proof, it remains to eliminate the possibility that $g(C) = 1$ and $L = f^*L_0$ for some $L_0 \in \operatorname{Pic}^0(C)$. But if this was the case, then ω_0 would be a holomorphic one-form on an elliptic curve, and therefore without zeros. As above, the relation $\alpha = f^*\omega_0 \otimes s$ shows that s is a holomorphic section of L, which is only possible if $L \simeq \mathscr{O}_X$ and s is constant; but then α would be proportional to ω, contradicting our inital choice. $\square$

During the proof, we used the following elementary lemma.

Lemma 10.13. *If $\alpha \in H^0(X, \Omega_X^1 \otimes L)$ satisfies $\omega \wedge \alpha = 0$ for some nonzero $\omega \in H^0(X, \Omega_X^1)$, then $\alpha = \omega \otimes s$ for a meromorphic section s of L.*

Proof. The best way to see this is as follows: Choose a local trivialization of L, say by a nonvanishing section $\sigma_U \in H^0(U, L)$, and write $\omega\big|_U = f_1 dz_1 + f_2 dz_2$ in local holomorphic coordinates z_1, z_2 on U. Then

$$\alpha\big|_U = (u_1 dz_1 + u_2 dz_2) \otimes \sigma,$$

for holomorphic functions $u_1, u_2 \in H^0(U, \mathscr{O}_X)$, and $\omega \wedge \alpha = 0$ is equivalent to $u_1 f_2 - u_2 f_1 = 0$. But then

$$\alpha\big|_U = \omega \otimes \frac{u_1}{f_1}\sigma_U = \omega \otimes \frac{u_2}{f_2}\sigma_U,$$

and so $s_U = u_1/f_1 \sigma_U = u_2/f_2 \sigma_U$ is a meromorphic section of L on U, holomorphic outside the zero locus of ω. Because α is a global section of $\Omega_X^1 \otimes L$, it is easy to see that these sections glue together into a global meromorphic section of L on X. $\square$

Exercises.

Exercise 10.1. Prove the assertions about the subspace in (10.4). (*Hint:* If ω is a Kähler form on X, then the Hodge index theorem says that the intersection pairing is negative definite on the subspace $[\omega]^\perp$.)

Exercise 10.2. Let mF be a multiple fiber of $f \colon X \to C$. Show that $\mathscr{O}_F(kF)$ has no nontrivial global sections unless $m \mid k$. (Hint: Use Lemma 10.5.)

Exercise 10.3. Show that $\operatorname{Pic}^\tau(X, f) \neq f^* \operatorname{Pic}^0(C)$ exactly when the fibration $f \colon X \to C$ has two multiple fibers $m_i F_i$ and $m_j F_j$ that satisfy $\gcd(m_i, m_j) \neq 1$.

Exercise 10.4. Prove the classical Castelnuovo-de Franchis lemma: Suppose that $\alpha, \beta \in H^0(X, \Omega_X^1)$ are two linearly independent holomorphic one-forms on a compact Kähler manifold, with $\alpha \wedge \beta = 0$. Then there is a fibration $f \colon X \to C$ to a curve of genus at least two such that α and β lie in the image of $H^0(C, \Omega_C^1)$.

Exercise 10.5. Let $f_1 \colon X \to C_1$ and $f_2 \colon X \to C_2$ be two fibrations over curves of genus at least two. Show that if $f_1^* \operatorname{Pic}^0(C_1) \subseteq f_2^* \operatorname{Pic}^0(C_2)$, then $C_1 = C_2$.

Exercise 10.6. Prove that on a given compact Kähler manifold, there are only finitely many fibrations over curves of genus at least two. Find an example with infinitely many fibrations over an elliptic curve.

Lecture 11

The structure of cohomology support loci. Our topic today is the second paper of Green and Lazarsfeld, which describes in more detail the structure of the cohomology support loci

$$S^i_m(X) = \left\{ L \in \operatorname{Pic}^0(X) \;\middle|\; \dim H^i(X, L) \geq m \right\}.$$

Here is the main result, suggested by Beauville's theorem about $S^1(X)$.

Theorem 11.1. *Let $Z \subseteq S^i_m(X)$ be an irreducible component. Then Z is a translate of a subtorus of* $\operatorname{Pic}^0(X)$.

The idea of the proof is the following. Recall that we constructed $\operatorname{Pic}^0(X)$ as the quotient V/Λ, where $V = \mathcal{H}^{0,1}(X)$ is the space of harmonic $(0,1)$-forms on X. Translates of subtori are characterized by the fact that their preimage in V is the translate of a linear subspace. We will prove a much stronger statement, namely that the higher direct image sheaves of the Poincaré bundle are locally computed by a *linear complex* (= a complex of trivial vector bundles whose differentials are matrices of linear forms).

Green and Lazarsfeld originally thought of this as the vanishing of higher obstructions. Recall that infinitesimal deformations of cohomology, and therefore the infinitesimal structure of the locus $S^i_m(X)$, are described by the derivative complex. In the case of a linear complex, no information is lost when taking the derivative complex; this means that there are no obstructions to extending infinitesimal deformations. Indeed, a linear space has the property that every tangent vector can be continued to an actual line inside the space. We will see later how this fact leads to improvements in the infinitesimal results, too.

The linearity theorem. During the proof of the generic vanishing theorem, we appealed to the general result in Theorem 6.7 to find a bounded complex of vector bundles that computes the higher direct image sheaves of the Poincaré bundle. This time around, we shall use Hodge theory to write down such a complex explicitly.

We begin by recalling the definition of the Poincaré bundle. Choose a base point $x_0 \in X$, and let P denote the Poincaré bundle on $X \times \operatorname{Pic}^0(X)$; it is a holomorphic line bundle whose defining property is that

$$P\big|_{X \times \{L\}} \simeq L \quad \text{and} \quad P\big|_{\{x_0\} \times \operatorname{Pic}^0(X)} \simeq \mathscr{O}_{\operatorname{Pic}^0(X)}.$$

From the construction of the Poincaré bundle in Proposition 6.5, the pullback of P to $X \times V$ can be described as follows: the underlying smooth vector bundle is the trivial bundle $X \times V \times \mathbb{C}$, and the complex structure is given by the operator

$$\bar{\partial}_{\tilde{P}} = \bar{\partial}_{X \times V} + \sum_{j=1}^{g} t_j \cdot p_1^* v_j.$$

Here $v_1, \dots, v_g$ are a basis of $V = \mathcal{H}^{0,1}(X)$, and $t_1, \dots, t_g \in V^*$ is the dual basis, which we view as a linear coordinate system on V (centered at the origin).

In the notation of the commutative diagram

$$\begin{tikzcd} X \times V \arrow[r, "\operatorname{id} \times \pi"] \arrow[d, "p_2"] & X \times \operatorname{Pic}^0(X) \arrow[d, "p_2"] \\ V \arrow[r, "\pi"] & \operatorname{Pic}^0(X) \end{tikzcd}$$

we have $\pi^* R^i p_{2*} P \simeq R^i p_{2*} \tilde{P}$, where $\tilde{P} = (\operatorname{id} \times \pi)^* P$. Because the projection from V to $\operatorname{Pic}^0(X)$ is a local biholomorphism, we can therefore work on V.

The first step is to construct a good resolution for the line bundle $\tilde{P}$ to compute the higher direct images. To that end, we decompose the $\bar{\partial}$-operator on $X \times V$ as

$$\bar{\partial}_{X \times V} = \bar{\partial}_X + \bar{\partial}_V,$$

where $\bar{\partial}_X$ is differentiation in the X-direction, and $\bar{\partial}_V$ in the V-direction. We can then introduce sheaves $\mathscr{K}^q$, defined by

$$\Gamma\big(U, \mathscr{K}^q\big) = \big\{\, \alpha \in A^{0,q}_{X \times V/V}(U) \;\big|\; \bar{\partial}_V \alpha = 0 \,\big\}$$

for any open set $U \subseteq X \times V$. To make this more concrete, let $z_1, \dots, z_n$ be local holomorphic coordinates on X, and recall that also have a coordinate system $t_1, \dots, t_g$ on V. Then sections of $\mathscr{K}^q$ are smooth $(0,q)$-forms that can be locally written in the form

$$\sum_{j_1, \dots, j_q} f_{j_1, \dots, j_q}(z_1, \dots, z_n, t_1, \dots, t_g) \cdot d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q},$$

with $f_{j_1,\dots,j_q}$ smooth functions in the coordinates $z_1, \dots, z_n, t_1, \dots, t_g$ that are holomorphic in $t_1, \dots, t_g$. We have the following variant of Lemma 5.5.

Lemma 11.2. *Let $n = \dim X$. Then the complex of sheaves*

$$\mathscr{K}^0 \to \mathscr{K}^1 \to \dots \to \mathscr{K}^n,$$

with differential $\bar{\partial}_X + \sum_{j=1}^g t_j \cdot p_1^* v_j$, is a resolution of $\tilde{P}$.

Proof. With a little bit of care, this follows from the usual holomorphic Poincaré lemma. Locally on X, we can certainly find smooth functions $f_1, \dots, f_n$ with the property that $v_j = \bar{\partial} f_j$. Now it is easy to check that the function

$$e^f = \exp \sum_{j=1}^g t_j f_j$$

is a nowhere vanishing local solution of the differential equation

$$\left(\bar{\partial}_X - \sum_{j=1}^g t_j \cdot p_1^* v_j \right) e^f = 0.$$

Now suppose that we have a smooth form $\alpha \in \mathscr{K}^q(U)$ with

$$\Big(\bar{\partial}_X + \sum t_j p_1^* v_j \Big)\, \alpha = 0.$$

Then the properties of e^f imply that $\bar{\partial}_X\big(e^f \alpha\big) = 0$; note that $e^f \alpha$ is still a local section of $\mathscr{K}^q$. According to the usual holomorphic Poincaré lemma, we can find $\beta \in A^{0,q-1}_{X \times V/X}(U)$ with $e^f \alpha = \bar{\partial}_X \beta$. But then

$$\alpha = \Big(\bar{\partial}_{X \times V} + \sum t_j \cdot p_1^* v_j \Big)\, (e^{-f} \beta),$$

which almost proves the exactness of the complex. The only question is whether we can take $\beta \in \mathscr{K}^{q-1}(U)$, meaning subject to the condition $\bar{\partial}_V \beta = 0$. This actually follows from the proof of the holomorphic Poincaré lemma: it involves integrating $e^f \alpha$, and by differentiating under the integral sign, one can show that β is again holomorphic in $t_1, \dots, t_g$. □

This resolution is good for computing the higher direct images of $\tilde{P}$, because of the following fact.

Lemma 11.3. *For each $q = 0, \dots, n$, we have $R^i p_{2*} \mathscr{K}^q = 0$ for $i > 0$, which means that the sheaf $\mathscr{K}^q$ is acyclic for the functor p_{2*}.*

Proof. This follows by using a partition of unity on X; the point is that we can multiply by smooth functions on X without affecting the condition $\bar{\partial}_V \alpha = 0$. □

Because each of the sheaves $\mathscr{K}^q$ in the resolution is acyclic for p_{2*}, the complex of sheaves $K^\bullet = p_{2*}\mathscr{K}^\bullet$ computes the higher direct image sheaves of $\tilde{P}$. Concretely, we have

$$\Gamma(U, K^q) = \Gamma\big(U, p_{2*}\mathscr{K}^q\big) = \big\{ \alpha \in A^{0,q}_{X\times V/V}(X \times U) \bigm| \bar\partial_V \alpha = 0 \big\},$$

and so K^q is a sheaf of $\mathscr{O}_V$-modules. Even though the individual sheaves in the complex are far from being coherent, we know from Grauert's direct image theorem that the cohomology sheaves of $K^\bullet$ are coherent sheaves on V.

Now we will use Hodge theory to define a subcomplex $H^\bullet \subseteq K^\bullet$ that is linear. Fix a line bundle $L \in \mathrm{Pic}^0(X)$, and choose a harmonic $(0,1)$-form τ such that L is isomorphic to the holomorphic line bundle defined by the operator $\bar\partial + \tau$. For convenience, we recenter the coordinate system $t_1, \dots, t_g$ on V at the point τ; then

$$\bar\partial_{\tilde{P}} = \big(\bar\partial_{X\times V} + p_1^*\tau\big) + \sum_{j=1}^{g} t_j \cdot p_1^* v_j.$$

Let $\nabla = d + \tau - \bar\tau$ be the Chern connection for the Hermitian metric on L (induced from the isomorphism of smooth vector bundles $L \simeq X \times \mathbb{C}$), and consider the space of ∇-harmonic forms

$$\mathcal{H}^{0,q}(X, L) = \big\{ \alpha \in A^{0,q}(X) \bigm| \alpha \text{ is } \nabla\text{-harmonic} \big\}.$$

Recall from Theorem 6.2 that ∇-harmonic forms are automatically in the kernel of $\bar\partial + \tau$ and of $\partial - \bar\tau$; moreover, every cohomology class contains a unique harmonic representative, and so $\mathcal{H}^{0,q}(X, L) \simeq H^q(X, L)$. We can then define locally free sheaves

$$H^q = \mathcal{H}^{0,q}(X, L) \otimes \mathscr{O}_V;$$

sections of H^q are naturally also sections of the sheaf $p_{2*}\mathscr{K}^q$, which means that $H^q \subseteq K^q$. If we apply the differential $\bar\partial_{\tilde{P}}$ to a section $\alpha \otimes f$ of H^q, we find that

$$\bar\partial_{\tilde{P}}(\alpha \otimes f) = (\bar\partial + \tau)\alpha \otimes f + \sum_{j=1}^{g} (v_j \wedge \alpha) \otimes t_j f = \sum_{j=1}^{g} (v_j \wedge \alpha) \otimes t_j f,$$

because $(\bar\partial + \tau)\alpha = 0$. This expression is linear in $t_1, \dots, t_g$. To prove that $H^\bullet$ is indeed a subcomplex, we have to show that $\bar\partial_{\tilde{P}}(\alpha \otimes f)$ is again section of H^{q+1}.

Lemma 11.4. *If $\alpha \in \mathcal{H}^{0,q}(X, L)$ and $v \in \mathcal{H}^{0,1}(X)$, then $v \wedge \alpha \in \mathcal{H}^{0,q+1}(X, L)$.*

Proof. Because of the Kähler identities, we have

$$\frac{1}{2}\big(\nabla\nabla^* + \nabla^*\nabla\big) = (\partial - \bar\tau)(\partial - \bar\tau)^* + (\partial - \bar\tau)^*(\partial - \bar\tau),$$

and so it suffices to show that

$$(\partial - \bar\tau)(v \wedge \alpha) = 0 \quad \text{and} \quad (\partial - \bar\tau)^*(v \wedge \alpha) = 0.$$

The second equality holds by default, and for the first one, we compute that

$$(\partial - \bar\tau)(v \wedge \alpha) = \partial(v \wedge \alpha) - \bar\tau \wedge v \wedge \alpha = -v \wedge (\partial\alpha) + v \wedge \bar\tau \wedge \alpha = 0,$$

using that $\partial v = 0$ and $\partial\alpha = \bar\tau \wedge \alpha$. We conclude that $v \wedge \alpha$ is again ∇-harmonic. $\square$

The lemma shows that the complex of locally free sheaves

$$H^0 \to H^1 \to \cdots \to H^n,$$

with differential

$$\alpha \otimes f \mapsto \sum_{j=1}^{g} (v_j \wedge \alpha) \otimes t_j f,$$

is a subcomplex of $K^0 \to K^1 \to \cdots \to K^n$. Now all that we have to do is prove that the cohomology sheaves of the subcomplex are still isomorphic to $R^i p_{2*}\tilde{P}$. This is true at least in a neighborhood of the point $\tau \in V$; because the subcomplex is defined using the cohomology of L, a local result is the best we can hope for.

Lemma 11.5. *The inclusion $H^\bullet \hookrightarrow K^\bullet$ is a quasi-isomorphism on an open neighborhood of the point $\tau \in V$.*

Proof. Let $R = \mathscr{O}_{V,\tau}$ be the local ring at the point τ, and let $\mathfrak{m}$ be its maximal ideal; $\mathfrak{m}$ is generated by the holomorphic functions $t_1, \dots, t_g$. Since the cohomology sheaves of both complexes are coherent, it suffices to prove that $H^\bullet \otimes R \hookrightarrow K^\bullet \otimes R$ is a quasi-isomorphism. We shall prove this by using a spectral sequence calculation.

To simplify the notation, we continue to denote the two complexes of R-modules by $H^\bullet$ and $K^\bullet$. On $K^\bullet$, we define a decreasing filtration by setting

$$F^p K^\bullet = \mathfrak{m}^p K^\bullet,$$

and consider the associated spectral sequence. We have

$$E_0^{p,q}(K^\bullet) = \frac{F^p K^{p+q}}{F^{p+1} K^{p+q}} \simeq A^{0,p+q}(X) \otimes_{\mathbb{C}} \frac{\mathfrak{m}^p}{\mathfrak{m}^{p+1}},$$

because $K^{p+q}(\tau) = p_{2*}\mathscr{K}^{p+q} \otimes \mathbb{C}(\tau) \simeq A^{0,p+q}(X)$. The differentials $d_0^{p,q}$ are given by the formula $(\bar{\partial} + \tau) \otimes \mathrm{id}$, and so Lemma 5.5 shows that

$$E_1^{p,q}(K^\bullet) \simeq H^{p+q}(X, L) \otimes_{\mathbb{C}} \frac{\mathfrak{m}^p}{\mathfrak{m}^{p+1}}.$$

Because the R-modules K^i are not finitely generated, it is not immediately clear that the spectral sequence converges. According to the standard convergence criterion (which is proved for example in Theorem 3.3 of McLeary's book on spectral sequences), the spectral sequence of a filtered complex converges provided that

$$\bigcup_{p \in \mathbb{Z}} F^p K^i = K^i \quad \text{and} \quad \bigcap_{p \in \mathbb{Z}} F^p K^i = \{0\}.$$

In our case, the first condition is obviously true ($F^0 K^i = K^i$), while the second,

$$\bigcap_{p=1}^{\infty} F^p K^i = \bigcap_{p=1}^{\infty} \mathfrak{m}^p K^i = \{0\},$$

follows from the fact that sections of $p_{2*}\mathscr{K}^i$ are holomorphic in $t_1, \dots, t_g$. Consequently, the spectral sequence converges to the cohomology of $K^\bullet$, or in other words, to the stalk at the point τ of the coherent sheaves $R^i p_{2*}\tilde{P}$.

Similarly, we have a decreasing filtration

$$F^p H^\bullet = \mathfrak{m}^p H^\bullet,$$

and the associated spectral sequence satisfies

$$E_0^{p,q}(H^\bullet) = \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}} \simeq \mathcal{H}^{0,p+q}(X, L) \otimes_{\mathbb{C}} \frac{\mathfrak{m}^p}{\mathfrak{m}^{p+1}}.$$

Because the differentials in the complex $H^\bullet$ are linear in $t_1, \dots, t_g$, we have $d_0^{p,q} = 0$, and so the E_1-page of the spectral sequence is also given by

$$E_1^{p,q}(H^\bullet) \simeq \mathcal{H}^{0,p+q}(X, L) \otimes_{\mathbb{C}} \frac{\mathfrak{m}^p}{\mathfrak{m}^{p+1}}.$$

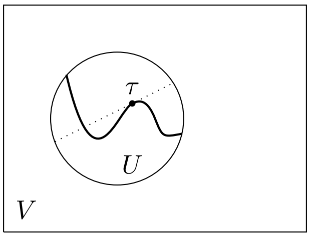
It converges to the cohomology of $H^\bullet$.

Now observe that the inclusion $H^\bullet \hookrightarrow K^\bullet$ induces a morphism between the two spectral sequences. To prove that the resulting morphism between their limits is an isomorphism, it suffices to show that $E_\infty^{p,q}(H^\bullet) \simeq E_\infty^{p,q}(K^\bullet)$; this is because both spectral sequences are convergent, and because their limits are finitely generated R-modules. But in fact, we already have $E_1^{p,q}(H^\bullet) \simeq E_1^{p,q}(K^\bullet)$ because $\mathcal{H}^{0,p+q}(X,L) \simeq H^{p+q}(X,L)$ by Hodge theory. This proves that the two complexes are quasi-isomorphic on the level of stalks, and therefore in a small open neighborhood of the point $\tau \in V$. □

The structure theorem and its consequences. We can now prove the structure theorem of Green and Lazarsfeld for cohomology support loci.

Theorem. *Let $Z \subseteq S_m^i(X)$ be an irreducible component. Then Z is a translate of a subtorus of* $\mathrm{Pic}^0(X)$.

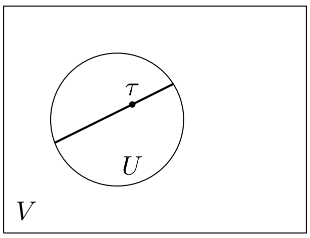
Proof. The result being trivial for $m = 0$, we may assume that $m \geq 1$. After increasing the value of m, if necessary, we can arrange that $\dim H^i(X,L) = m$ for all L in a Zariski-open subset $Z_0 \subseteq Z$; without loss of generality, Z_0 is contained in the smooth locus of Z. Take any line bundle $L \in Z_0$ and choose a harmonic $(0,1)$-form $\tau \in V$ such that the operator $\bar\partial + \tau$ represents L. By our choice of Z_0, the intersection $\pi^{-1}(Z_0) \cap U$ is a complex submanifold of any sufficiently small open set U containing the point τ.



We will use the linearity theorem to show that this submanifold must be the trace of an affine subspace (namely, the tangent space to $\pi^{-1}(Z_0)$ at the point τ). In a neighborhood U of the point τ, the sheaves $R^i p_{2*}\tilde{P}$ are computed by the complex

$$\mathcal{H}^{0,0}(X,L) \otimes \mathscr{O}_V \to \mathcal{H}^{0,1}(X,L) \otimes \mathscr{O}_V \to \cdots \to \mathcal{H}^{0,n}(X,L) \otimes \mathscr{O}_V$$

with differential $\alpha \otimes f \mapsto \sum (v_j \wedge \alpha) \otimes t_j f$. The intersection between U and $\pi^{-1}(Z_0)$ consists of those points where the i-th cohomology of the complex is m-dimensional. But $\dim \mathcal{H}^{0,i}(X,L) = \dim H^i(X,L) = m$, and so this happens precisely at those points where the two differentials next to $\mathcal{H}^{0,i}(X,L)$ are zero. Because both are matrices of linear forms in $t_1, \dots, t_g$, it follows that $\pi^{-1}(Z_0) \cap U$ is an affine subspace of V; this affine subspace is of course nothing but the holomorphic tangent space to $\pi^{-1}(Z_0)$ at the point τ.



We have now arrived at the following situation: $\pi^{-1}(Z_0)$ is covered by open subsets of V, each of which intersects $\pi^{-1}(Z_0)$ in an affine subspace of V. This means that the function that assigns to a point of $\pi^{-1}(Z_0)$ its holomorphic tangent space, is locally constant. But this function is clearly holomorphic, and because $\pi^{-1}(Z_0)$ is connected, it must be constant. Consequently, $\pi^{-1}(Z_0)$ is a dense open subset of an affine subspace, and $\pi^{-1}(Z)$ an affine subspace. You can prove as an exercise that this can only happen when Z is a translate of a subtorus. □

Exercises.

Exercise 11.1. Show that if Z is an analytic subvariety of a compact complex torus V/Γ, and if $\pi^{-1}(Z)$ is equal to a linear subspace of V, then Z must be a subtorus.

Lecture 12

Consequences for the infinitesimal results. Last time, we showed that, locally on $\operatorname{Pic}^0(X)$, the higher direct image sheaves $R^i p_{2*} P$ are computed by a linear complex. This fact leads to several improvements in our infinitesimal results. The point is that a linear complex is its own derivative complex (up to identifying a neighborhood of the given point with a neighborhood in the tangent space). In particular, the inclusion in Theorem 7.6 now becomes

$$TC_x\big(S^i_m(E^\bullet)\big) = S^i_m\big(D(E^\bullet, x)\big). \tag{12.1}$$

All the consequences of this result also become stronger: for example, Corollary 7.11 now gives us the following formula for the dimension of cohomology support loci.

Corollary 12.2. *Set $m = \dim H^i\big(E^\bullet(x)\big)$. Then*

$$\dim_x S^i_m(E^\bullet) = \dim\big\{\, v \in T \;\big|\; D_v(d^i, x) = 0 \text{ and } D_v(d^{i-1}, x) = 0 \,\big\}.$$

Another useful improvement is a necessary and sufficient condition for isolated points, strengthening Corollary 7.13.

Corollary 12.3. *Set $m = \dim H^i\big(E^\bullet(x)\big)$. Then x is an isolated point of $S^i_m(E^\bullet)$ if and only if $H^i\big(D_v(E^\bullet, x)\big) = 0$ for every nonzero $v \in T_x X$.*

Of course, all of those results hold for any linear complex $E^\bullet$.

Generalization of Beauville's theorem. Returning to geometric consequences of Theorem 11.1, we can now also prove a generalization of Beauville's result about $S^1(X)$.

Theorem 12.4. *Let $Z \subseteq S^i_m(X)$ be an irreducible component for some $m \geq 1$. Then there exists a normal analytic space Y with $\dim Y \leq i$, and a surjective holomorphic mapping $f\colon X \to Y$ with connected fibers, such that Z is contained in a translate of $f^* \operatorname{Pic}^0(Y)$. Moreover, any resolution of singularities of Y has maximal Albanese dimension.*

Note that the result is weaker than Beauville's theorem, because we are not claiming that $f^*\operatorname{Pic}^0(Y) \subseteq S^i_m(X)$; in fact, there are examples where Z is strictly smaller than the translate of $f^*\operatorname{Pic}^0(X)$ that it is contained in. Also note that varieties of maximal Albanese dimension are one possible generalization of curves of genus ≥ 1.

Proof. According to Theorem 11.1, there is a subtorus $T \subseteq \operatorname{Pic}^0(X)$ such that Z is equal to a translate of T. Let $\hat{T} = \operatorname{Pic}^0(T)$ be the dual torus; because $\operatorname{Alb}(X)$ is dual to $\operatorname{Pic}^0(X)$, the inclusion $T \hookrightarrow \operatorname{Pic}^0(X)$ gives rise to a surjective holomorphic mapping $\operatorname{Alb}(X) \to \hat{T}$. By composing it with the Albanese mapping, we obtain a holomorphic mapping $h\colon X \to \hat{T}$; consider its Stein factorization

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "h"] & Y \arrow[r, "g"] & \hat{T} \end{tikzcd} \tag{12.5}$$

Then Y is a normal analytic space, $f\colon X \to Y$ has connected fibers, and Y is finite over its image in $\hat{T}$ (which implies in particular that any resolution of singularities of Y has maximal Albanese dimension). It remains to show that $\dim Y \leq i$, and that Z is contained in a translate of $f^*\operatorname{Pic}^0(Y)$.

The second assertion is easy. After dualizing again, the mapping $\operatorname{Alb}(X) \to \hat{T}$ induces an inclusion $\operatorname{Pic}^0(\hat{T}) \hookrightarrow \operatorname{Pic}^0(X)$ that coincides with the original embedding of $T \simeq \operatorname{Pic}^0(\hat{T})$ into $\operatorname{Alb}(X) \simeq \operatorname{Pic}^0\big(\operatorname{Pic}^0(X)\big)$. Because of the factorization in

(12.5), the inclusion $T \hookrightarrow \mathrm{Pic}^0(X)$ factors through $\mathrm{Pic}^0(Y)$, which means precisely that Z is contained in a translate of $f^* \mathrm{Pic}^0(Y)$.

To prove that $\dim Y \leq i$, let L be a sufficiently general point of Z, and let $v_1, \dots, v_d \in \mathcal{H}^{0,1}(X)$ be a basis for the tangent space to Z at the point L. By assumption, $\dim H^i(X, L) \geq 1$, and so there is a nonzero $\alpha \in \mathcal{H}^{0,i}(X, L)$. Because v_j belongs to the tangent space, the derivative complex in the direction of v_j must have zero differentials, which means that $v_j \wedge \alpha = 0$ for $j = 1, \dots, d$. If we set $\omega_j = \overline{v_j} \in H^0(X, \Omega_X^1)$, and $\beta = \bar{\alpha} \in H^0\big(X, \Omega_X^i \otimes L^{-1}\big)$, then equivalently $\omega_j \wedge \beta = 0$ for $j = 1, \dots, d$.

Now it is not hard to show that

$$\langle \omega_1, \dots, \omega_d \rangle = \mathrm{im}\Big(h^* \colon H^0\big(\hat{T}, \Omega_{\hat{T}}^1\big) \to H^0(X, \Omega_X^1)\Big).$$

At a general point $x \in X$, the rank of the mapping $h \colon X \to \hat{T}$ is therefore equal to the dimension of the span of $\omega_1(x), \dots, \omega_d(x)$. On the other hand, the rank is equal to the dimension of $h(X) = g(Y)$; because Y is finite over $g(Y)$, we get

$$\dim Y = \dim \big\langle \omega_1(x), \dots, \omega_d(x) \big\rangle$$

for general $x \in X$. After choosing a local trivialization for L^{-1} near the point x, we may consider $\beta(x)$ as a nonzero element of $\bigwedge^i T_x^* X$; since $\omega_j(x) \wedge \beta(x) = 0$ for $j = 1, \dots, d$, we conclude from Lemma 8.10 that

$$\dim Y = \dim \big\langle \omega_1(x), \dots, \omega_d(x) \big\rangle \leq i.$$

This completes the proof. ∎

As a matter of fact, Theorem 12.4 immediately gives us another proof of the generic vanishing theorem.

Corollary 12.6. *One has* $\operatorname{codim} S^i(X) \geq \dim \mathrm{alb}(X) - i$.

Proof. Let $Z \subseteq S^i(X)$ be any irreducible component, and let $f \colon X \to Y$ be a fibration with the properties described in Theorem 12.4.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{alb}"] & Y \arrow[d, "g"] \\ \mathrm{Alb}(X) \arrow[r] & \hat{T} \end{tikzcd}$$

In the notation of the preceding proof, $\mathrm{alb}(X)$ maps onto $g(Y)$, and the fibers are contained in the fibers of the surjective mapping $\mathrm{Alb}(X) \to \hat{T}$; this leads to

$$\dim \mathrm{Alb}(X) - \dim \hat{T} \geq \dim \mathrm{alb}(X) - \dim g(Y).$$

Because $\dim g(Y) = \dim Y$ and $\dim Z = \dim \hat{T}$, we can rewrite this as

$$\operatorname{codim} Z \geq \dim \mathrm{alb}(X) - \dim Y \geq \dim \mathrm{alb}(X) - i,$$

using that $\dim Y \leq i$. This proves the asserted inequality. ∎

The advantage of this method – namely, of first proving a structure theorem for cohomology support loci, and then deducing inequalities for their codimension – is that it generalizes to other situations. For example, the results about holonomic $\mathscr{D}$-modules on abelian varieties that I talked about in last week's seminar are proved by the same method.

Kodaira dimension. The results of Green and Lazarsfeld have many surprising applications to algebraic geometry. The first one is a very simple proof for a theorem of Kawamata about varieties of Kodaira dimension zero. In order to appreciate those results better, let us first understand the problem.

In fact, we shall begin with a brief review of Kodaira dimension. (If you need more details, please see Section 2.1 in Lazarsfeld's book *Positivity in algebraic geometry*.) Let X be a smooth projective variety – in fact, it would be enough to assume that X is normal – and let L be a line bundle on X. If $L^{\otimes m}$ has nontrivial global sections, then it defines a rational mapping

$$\phi_m \colon X \dashrightarrow \mathbb{P}\big(H^0(X, L^{\otimes m})\big)$$

and we let $Y_m = \phi_m(X)$ denote the closure of its image. The *Iitaka dimension* of the line bundle L is defined to be

$$\kappa(X, L) = \max_{m \in \mathbb{N}} \big(\dim Y_m\big)$$

if $H^0(X, L^{\otimes m}) \neq 0$ for at least one $m \geq 1$; if not, we set $\kappa(X, L) = -\infty$. In fact, one has $\dim Y_m = \kappa(X, L)$ once m is sufficiently large and divisible; moreover, there are positive constants C_1, C_2 with the property that

$$C_1 \cdot m^{\kappa(X,L)} \leq \dim H^0(X, L^{\otimes m}) \leq C_2 \cdot m^{\kappa(X,L)},$$

where m is again assumed to be sufficiently large and divisible.

One can show that, for m sufficiently large, the rational mappings ϕ_m stabilize in the following sense: there is a morphism $\phi_\infty \colon X_\infty \to Y_\infty$ between two smooth projective varieties, such that ϕ_m is birationally equivalent to ϕ_∞. This morphism is unique up to birational equivalence, and is called the *Iitaka fibration* of the line bundle L. By construction, $\dim Y_\infty = \kappa(X, L)$; moreover, ϕ_∞ is an algebraic fiber space, meaning that it has connected fibers. It is also known that the restriction of L to a very general fiber of ϕ_∞ has Iitaka dimension equal to zero.

The most interesting case of the above considerations is when $L = \omega_X$ is the canonical bundle of a smooth projective variety. In that case, $\kappa(X) = \kappa(X, \omega_X)$ is called the *Kodaira dimension* of X, and $\phi_\infty \colon X_\infty \to Y_\infty$ is called the *Iitaka fibration* of X. Note that X_∞ is birational to X, and that a very general fiber of ϕ_∞ has Kodaira dimension zero. If we define the *m-th plurigenus* of X as

$$P_m(X) = \dim H^0(X, \omega_X^{\otimes m}),$$

then we have $C_1 \cdot m^{\kappa(X)} \leq P_m(X) \leq C_2 \cdot m^{\kappa(X)}$ for m sufficiently large and divisible, and so the Kodaira dimension tells us the rate of growth of the plurigenera.

Varieties of Kodaira dimension zero. Because of the Iitaka fibration, an important problem in birational geometry is to understand the structure of varieties of Kodaira dimension zero. There is the following precise conjecture, due to Kenji Ueno; it is part of a general set of conjectures about the behavior of the Kodaira dimension in algebraic fiber spaces.

Conjecture 12.7 (Ueno's Conjecture K)**.** *Let X be a smooth projective variety with $\kappa(X) = 0$, and let $\mathrm{alb} \colon X \to \mathrm{Alb}(X)$ denote its Albanese mapping. Then*

(i) alb *is surjective with connected fibers;*
(ii) if F is a general fiber of alb*, then $\kappa(F) = 0$;*
(iii) after passing to a finite étale cover, X becomes birational to $F \times \mathrm{Alb}(X)$.

The status of this conjecture is the following. (i) was proved by Yujiro Kawamata, using rather difficult arguments from Hodge theory. Subsequently, Lawrence Ein and Robert Lazarsfeld found a very simple proof based on Theorem 11.1, and we shall discuss at least the first half of it today. (ii) has been proved by Junyan

Cao and Mihai Păun; an earlier claimed proof by Jungkai Chen and Christopher Hacon turned out to have a gap. (iii) is wide open.

Ueno's conjecture is important because it is a test case of a more general conjecture, known as *Iitaka's Conjecture* $C_{m,n}$, which predicts that if $f\colon X \to Y$ is an algebraic fiber space with general fiber F, then $\kappa(X) \geq \kappa(Y) + \kappa(F)$. This conjecture is only known in special cases, most importantly when the fiber F is of general type (in the sense that $\kappa(F) = \dim F$).

We will spend the remainder of today's class by going through the very pretty proof of the following theorem by Ein and Lazarsfeld.

Theorem 12.8. *Let X be a smooth projective variety of Kodaira dimension zero. Then the Albanese mapping* $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ *is surjective.*

The proof is based on two surprisingly simple observations. But first, a word about the meaning of the condition $\kappa(X) = 0$. By definition, the sequence of plurigenera $P_m(X)$ is bounded; actually, we even have $P_m(X) \leq 1$ for all m. Indeed, if $P_m(X) \geq 2$ for some m, then we could find two linearly independent sections of $\omega_X^{\otimes m}$, and by multiplying these together, we would get $P_{km}(X) \geq k+1$, contradicting $\kappa(X) = 0$. Thus we can say that if X has Kodaira dimension zero, then $P_m(X) = 1$ for m sufficiently large and divisible.

For the time being, we shall assume that $P_1(X) = P_2(X) = 1$. The first observation is that this condition has an effect on the locus $S^n(X)$, where $n = \dim X$.

Proposition 12.9. *If $P_1(X) = P_2(X) = 1$, then $\mathscr{O}_X$ is an isolated point of $S^n(X)$.*

Proof. Since $P_1(X) \neq 0$, we have

$$H^n(X, \mathscr{O}_X) \simeq \mathrm{Hom}\big(H^0(X, \omega_X), \mathbb{C}\big) \neq 0,$$

and so $\mathscr{O}_X \in S^n(X)$. Suppose that it is not an isolated point. Then by Theorem 11.1, $S^n(X)$ contains a subtorus T of positive dimension. In particular, T is a subgroup, and so if $L \in T$, then also $L^{-1} \in T$. This means that the image of the multiplication map

$$H^0(X, \omega_X \otimes L) \otimes H^0(X, \omega_X \otimes L^{-1}) \to H^0(X, \omega_X^{\otimes 2})$$

is nonzero for every $L \in T$. Now $\omega_X^{\otimes 2}$ only has one global section because $P_2(X) = 1$; let D be the corresponding effective divisor on X. By the above, the divisor of any global section of $\omega_X \otimes L$ has to be contained in D; but because D has only finitely many irreducible components, we can find two distinct points $L_1, L_2 \in T$, and nontrivial sections $s_1 \in H^0(X, \omega_X \otimes L_1)$ and $s_2 \in H^0(X, \omega_X \otimes L_2)$, such that $\mathrm{div}\, s_1 = \mathrm{div}\, s_2$. But then $\omega_X \otimes L_1 \simeq \omega_X \otimes L_2$, which contradicts the fact that L_1 and L_2 are distinct points of T. □

The second observation of Ein and Lazarsfeld is that $S^n(X)$ is closely related to the geometry of the Albanese mapping.

Proposition 12.10. *If the origin is an isolated point of $S^n(X)$, then the Albanese mapping* $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ *is surjective.*

Proof. Since $\mathscr{O}_X$ lies in $S^n(X)$, we have $H^0(X, \omega_X) \geq 1$. Let $s \in H^0(X, \omega_X)$ be any nontrivial section. Because $\mathscr{O}_X$ is an isolated point of $S^n(X)$, the criterion in Corollary 12.3 shows (after conjugating) that the mapping

$$H^0(X, \Omega_X^{n-1}) \xrightarrow{\omega \wedge} H^0(X, \Omega_X^n)$$

is surjective for every nonzero $\omega \in H^0(X, \Omega_X^1)$. In particular, we have $s(x) = 0$ at every point $x \in X$ where $\omega(x) = 0$. From this, we can deduce without much difficulty that alb must be surjective.

Indeed, suppose that alb was not surjective. Take an arbitary point $x \in X$. The differential $T_x X \to T_{\mathrm{alb}(x)} \operatorname{Alb}(X)$ of the Albanese mapping is obviously not surjective; after dualizing and using Lemma 5.3, we find that the evaluation mapping $H^0(X, \Omega_X^1) \to T_x^* X$, $\omega \mapsto \omega(x)$, is not injective. Thus, there is at least one nonzero holomorphic one-form with $\omega(x) = 0$. By the above, we then have $s(x) = 0$; but because x was an arbitrary point of X, this contradicts the fact that $s \neq 0$. □

Together, the two propositions prove Theorem 12.8 in the case when $P_1(X) = 1$. The general case requires only a little bit of extra work.

Proof of Theorem 12.8. If $P_1(X) = 1$, then $\kappa(X) = 0$ forces $P_2(X) = 1$, and so we are done by the above. If not, one can find a smooth projective variety Y of Kodaira dimension zero, and a generically finite morphism $f \colon Y \to X$, such that $P_1(Y) = 1$. (This is *Fujita's lemma*; it is proved by taking a resolution of the branched covering defined by a nontrivial section of $\omega_X^{\otimes m}$.) Then alb_Y is surjective, and we can use this to show that alb_X is also surjective. First, observe that $f^* \colon \operatorname{Pic}^0(X) \to \operatorname{Pic}^0(Y)$ has finite kernel: if $f^* L$ is trivial, then we get $L \otimes f_* \mathscr{O}_Y \simeq f_* \mathscr{O}_Y$ from the projection formula; setting $r = \deg f$, it follows that $L^{\otimes r} \otimes \det(f_* \mathscr{O}_Y) \simeq \det(f_* \mathscr{O}_Y)$, which shows that the r-th power of L is trivial. Dually, this means that $\operatorname{Alb}(Y) \to \operatorname{Alb}(X)$ is surjective. In the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\mathrm{alb}_Y"] & X \arrow[d, "\mathrm{alb}_X"] \\ \operatorname{Alb}(Y) \arrow[r] & \operatorname{Alb}(X) \end{tikzcd}$$

the composition $Y \to \operatorname{Alb}(Y) \to \operatorname{Alb}(X)$ is therefore surjective; the conclusion is that alb_X must be surjective as well. □

Exercises.

Exercise 12.1. Prove the following assertion about $X \to \hat{T}$ that was used during the proof of Theorem 12.4: if $v_1, \dots, v_d \in \mathcal{H}^{0,1}(X)$ are a basis for the tangent space to a subtorus $T \subseteq \operatorname{Pic}^0(X)$, and if $\omega_j = \overline{v_j} \in H^0(X, \Omega_X^1)$, then

$$\langle \omega_1, \dots, \omega_d \rangle = \operatorname{im}\Bigl(H^0\bigl(\hat{T}, \Omega_{\hat{T}}^1\bigr) \to H^0(X, \Omega_X^1) \Bigr).$$

Exercise 12.2. Let $T = V/\Lambda$ be a compact complex torus.

(a) Show that the dual torus $\hat{T} = \operatorname{Pic}^0(T)$ is isomorphic to the quotient $\hat{V}/\hat{\Lambda}$, where $\hat{V} = \operatorname{Hom}_{\mathbb{C}}\bigl(\overline{V}, \mathbb{C}\bigr)$ is the space of conjugate-linear functionals on V, and $\hat{\Lambda} = \operatorname{Hom}_{\mathbb{Z}}\bigl(\Lambda, \mathbb{Z}(1)\bigr)$.

(b) Prove that $T \simeq \operatorname{Pic}^0(\hat{T})$.

Exercise 12.3. Show that when X is a compact Kähler manifold, one has

$$\operatorname{Pic}^0\bigl(\operatorname{Pic}^0(X)\bigr) \simeq \operatorname{Alb}(X).$$

Lecture 13

Maximal Albanese dimension and holomorphic Euler characteristic. Let X be a compact Kähler manifold that is of maximal Albanese dimension; recall that this means that the Albanese mapping $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ is generically finite over its image. We proved earlier on (in Corollary 8.13) that $\chi(X, \omega_X) \geq 0$; this holds because, for general $L \in \mathrm{Pic}^0(X)$, one has

$$\chi(X, \omega_X) = \chi(X, \omega_X \otimes L) = \dim H^0(X, \omega_X \otimes L),$$

the higher cohomology groups being zero because of the generic vanishing theorem. We now want to study the boundary case $\chi(X, \omega_X) = 0$. There are two reasons for being interested in this case: (1) It will be important for our next application, namely singularities of theta divisors. (2) Varieties that lie on the boundary of some geometric inequality often have interesting properties; quite frequently, one can even classify all such cases.

Theorem 13.1. *Let X be a compact Kähler manifold of maximal Albanese dimension. If $\chi(X, \omega_X) = 0$, then the Albanese image $\mathrm{alb}(X)$ is fibered by tori.*

The converse is only partially true: If X itself is fibered by tori, then clearly $\chi(X, \omega_X) = 0$ because the holomorphic Euler characteristic of a compact complex torus is zero. On the other hand, the Albanese image of X might be fibered by tori without $\chi(X, \omega_X)$ being zero; this can happen for instance when the Albanese mapping is surjective but of degree greater than one. The problem of classifying all smooth projective varieties of maximal Albanese dimension with $\chi(X, \omega_X) = 0$ is not solved. But in the case when X is also of general type, there is at least a conjecture about what the answer might be; if you are interested, have a look at the articles by Jungkai Chen, Olivier Debarre, and Zhi Jiang.

Now let us prove Theorem 13.1. As in the case of Theorem 12.4, we are going to use an irreducible component of some cohomology support locus to construct the desired fibration. An important ingredient is Proposition 8.15: it tells us that, because X is of maximal Albanese dimension,

$$\mathrm{Pic}^0(X) \supseteq S^n(X) \supseteq \cdots \supseteq S^1(X) \supseteq S^0(X) = \{\mathscr{O}_X\}, \tag{13.2}$$

where $n = \dim X$. According to the generic vanishing theorem,

$$\operatorname{codim} S^i(X) \geq \dim \mathrm{alb}(X) - i = n - i;$$

in particular, $S^i(X) \neq \mathrm{Pic}^0(X)$ for every $i \leq n-1$. If $\chi(X, \omega_X) = 0$, then we have $H^0(X, \omega_X \otimes L) = 0$ for general $L \in \mathrm{Pic}^0(X)$; using Serre duality, we conclude that the biggest set $S^n(X)$ is also a proper subset of $\mathrm{Pic}^0(X)$.

Now fix an irreducible component $Z \subseteq S^n(X)$, and let $k \geq 1$ be its codimension in $\mathrm{Pic}^0(X)$. It is clear that Z cannot be contained in any $S^i(X)$ with $i < n-k$, because $\operatorname{codim} S^i(X) \geq n - i$. Our first task is to show that Z is necessarily an irreducible component of $S^{n-k}(X)$.

To that end, take a sufficiently general point $L \in Z$, with the property that $\dim H^i(X, L)$ is as small as possible for every $0 \leq i \leq n$. Of course, we have $H^i(X, L) = 0$ for $i < n-k$; our goal is to prove that $H^{n-k}(X, L) \neq 0$. Observe that if $H^i(X, L) \neq 0$ for some $n-k \leq i \leq n$, then $Z \subseteq S^i(X)$, and because of the containment in (13.2), Z is then actually an irreducible component of $S^i(X)$.

Lemma 13.3. *In this situation, the derivative complex*

$$0 \longrightarrow H^{n-k}(X, L) \xrightarrow{v\cup} H^{n-k+1}(X, L) \xrightarrow{v\cup} \cdots \xrightarrow{v\cup} H^n(X, L) \longrightarrow 0$$

in the direction of a vector $v \in H^1(X, \mathscr{O}_X)$ is exact if and only if $v \notin T_L Z$.

Proof. If $v \in T_L Z$, then we have already seen before that all the differentials in the derivative complex have to vanish; because $H^n(X, L) \neq 0$, this means in particular that the complex is not exact.

Conversely, suppose that for a certain $v \in H^1(X, \mathscr{O}_X)$, the complex is not exact in some degree $n-k \leq i \leq n$. Then $H^i(X, L) \neq 0$, and by the above, Z is an irreducible component of $S^i(X)$. Because $L \in Z$ is a general point, this implies that $T_L S^i(X) = T_L Z$. Now recall from (12.1) that, as a consequence of the linearity theorem, $T_L S^i(X)$ is equal to the i-th cohomology support locus of the derivative complex. Because v lies in this locus by assumption, we see that $v \in T_L Z$. □

We can now prove that Z has to be an irreducible component of $S^{n-k}(X)$.

Lemma 13.4. *In the situation described above, we have $Z \subseteq S^{n-k}(X)$.*

Proof. Let $V \subseteq T_L \operatorname{Pic}^0(X)$ be a k-dimensional subspace such that $V \cap T_L Z = \{0\}$. By Lemma 13.3, the derivative complex in the direction of v is exact for every nonzero $v \in V$. To exploit this fact, let $\mathbb{P} = \mathbb{P}(V^*)$ denote the $(k-1)$-dimensional projective space of lines in V. If we restrict the derivative complex in Definition 7.5 from $T_L \operatorname{Pic}^0(X)$ to V, projectivize, and then tensor by $\mathscr{O}_{\mathbb{P}}(-n)$, we obtain a complex of vector bundles

$$0 \to H^{n-k}(X, L) \otimes \mathscr{O}_{\mathbb{P}}(-k) \to H^{n-k+1}(X, L) \otimes \mathscr{O}_{\mathbb{P}}(-k+1) \to \cdots$$
$$\cdots \to H^n(X, L) \otimes \mathscr{O}_{\mathbb{P}} \to 0.$$

It is exact as a complex of sheaves, because the pointwise complexes are exact by our choice of V. Now consider the hypercohomology spectral sequence

$$E_1^{p,q} = H^{n+p}(X, L) \otimes H^q\big(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(p)\big);$$

it converges to the hypercohomology of the complex, which is zero (because the complex is exact). Since we are on projective space, $E_1^{p,q} = 0$ for $-k < p < 0$; this means that $d_1 = \cdots = d_{k-1} = 0$. In order for the limit to be zero, the final differential d_k has to be an isomorphism. In particular,

$$d_k^{-k,k-1} \colon H^{n-k}(X, L) \otimes H^{k-1}\big(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-k)\big) \to H^n(X, L) \otimes H^0\big(\mathbb{P}, \mathscr{O}_{\mathbb{P}}\big)$$

must be an isomorphism. The conclusion is that $H^{n-k}(X, L) \simeq H^n(X, L) \neq 0$; because $L \in Z$ was a general point, this proves that $Z \subseteq S^{n-k}(X)$. □

We now have an irreducible component $Z \subseteq S^{n-k}(X)$ of codimension k. Recall from Theorem 11.1 that Z is a translate of a subtorus $T \subseteq \operatorname{Pic}^0(X)$. From here on, the proof is essentially the same as that of Theorem 12.4. Let $\hat{T} = \operatorname{Pic}^0(T)$ be the dual torus; we obtain a surjective holomorphic mapping $\pi \colon \operatorname{Alb}(X) \to \hat{T}$ whose fibers are finite unions of translates of a k-dimensional subtorus $K \subseteq \operatorname{Alb}(X)$.

To conclude the proof, we are going to show that $\operatorname{alb}(X)$ is fibered by translates of K. To that end, consider the composition

$$f \colon \operatorname{alb}(X) \to \hat{T}.$$

As in the proof of Theorem 12.4, the fact that a translate of T is contained in $S^{n-k}(X)$ implies that $\dim f\big(\operatorname{alb}(X)\big) \leq n-k$; because $\operatorname{alb}(X)$ itself is n-dimensional, all fibers of f have dimension at least k. On the other hand, the fibers of f are contained in the fibers of π, which are finite unions of translates of K. Because $\dim K = k$, this can only happen if every fiber of f is equal to a finite union of translates of K. After replacing f by its Stein factorization, we have a holomorphic mapping from $\operatorname{alb}(X)$ to an analytic space of dimension $(n-k)$, all of whose fibers are translates of K. This proves that $\operatorname{alb}(X)$ is fibered by tori.

Principally polarized abelian varieties. Our next topic is perhaps the most spectacular application of Green-Lazarsfeld theory: a theorem about the singularities of the theta divisor on an arbitrary principally polarized abelian variety. To motivate the result, let us first review some definitions, and then discuss a concrete example, namely Jacobians of compact Riemann surfaces.

Throughout, we let A be a g-dimensional complex abelian variety. As a complex manifold, A is a compact complex torus; to fix the notation, let us say that $A = V/\Gamma$, where V is a g-dimensional complex vector space, and $\Gamma \subseteq V$ a lattice of rank $2g$. If we let $0 \in A$ denote the image of the origin, then clearly $V \simeq T_0 A$ and $\Gamma \simeq \pi_1(A, 0)$. By saying that A is an abelian variety, we are assuming that A can be embedded as a submanifold into complex projective space. This is basically a condition on the lattice Γ; here is the precise criterion for V/Γ to be projective.

Theorem 13.5. *A compact complex torus $T = V/\Gamma$ is projective if and only if there exists a positive definite Hermitian bilinear form $h \colon V \times V \to \mathbb{C}$ whose imaginary part $E = -\operatorname{Im} h$ takes integral values on $\Gamma \times \Gamma$.*

Proof. A compact complex manifold X is projective iff it has a positive line bundle L; this is a consequence of the Kodaira embedding theorem, which says that the sections of a sufficiently high power of L separate points and tangent vectors. The first Chern class $c_1(L)$ is represented by a closed positive $(1,1)$-form whose cohomology class in $H^2(X, \mathbb{C})$ is integral; conversely, any such form is the first Chern class of a positive line bundle. In order to prove that X is projective, it is therefore enough to find a closed positive $(1,1)$-form whose cohomology class is integral.

The two conditions in the proposition are saying precisely that T carries such a form, although some translation is needed to see this. To begin with, choose a basis $v_1, \dots, v_g \in V$, and let $z_1, \dots, z_g \in V^*$ be the corresponding linear coordinate system on V. The Hermitian form h is given by a $g \times g$-matrix with entries

$$h_{j,k} = h(v_j, v_k);$$

the matrix is Hermitian symmetric and positive definite. The associated $(1,1)$-form

$$\omega = \frac{i}{2} \sum_{j,k=1}^{g} h_{j,k} dz_j \wedge d\bar{z}_k$$

is therefore positive and, obviously, closed. To see that its class in $H^2(T, \mathbb{C})$ belongs to the image of $H^2(T, \mathbb{Z})$, we compute its integrals over a collection of 2-cycles that generate $H_2(T, \mathbb{Z})$. We can use the images in T of

$$[0,1] \times [0,1] \to V, \quad (x, y) \mapsto x\gamma + y\delta,$$

where $\gamma, \delta \in \Gamma$ are two arbitrary elements. Then

$$dz_j \wedge d\bar{z}_k = (\gamma_j \, dx + \delta_j \, dy) \wedge (\overline{\gamma_k} \, dx + \overline{\delta_k} \, dy) = \big(\gamma_j \overline{\delta_k} - \delta_j \overline{\gamma_j}\big) \, dx \wedge \, dy$$

The integral in question thus becomes

$$\frac{i}{2} \sum_{j,k=1}^{g} \int_0^1 \int_0^1 h_{j,k} dz_j \wedge d\bar{z}_k = \frac{i}{2} \sum_{j,k=1}^{g} h_{j,k} \big(\gamma_j \overline{\delta_k} - \delta_j \overline{\gamma_j}\big) = \frac{i}{2} \Big(h(\gamma, \delta) - h(\delta, \gamma) \Big).$$

This is easily seen to equal $E(\gamma, \delta) = -\operatorname{Im} h(\gamma, \delta)$, and so we get the result. □

Note. If we denote by $J \colon V \to V$ the homomorphism given by multiplication by i, then the fact that h is a Hermitian form implies $h(Jv, Jw) = h(v, w)$. It follows that $E(Jv, Jw) = E(v, w)$; moreover,

$$h(v, w) = E(v, Jw) - iE(v, w),$$

and so E uniquely determines h.

A Hermitian form $h\colon V\times V\to\mathbb{C}$ with the above properties is called a *polarization* of the abelian variety A; the proof shows that it corresponds uniquely to a positive (and, therefore, ample) holomorphic line bundle L on A. The abelian variety is *principally polarized* if $H^0(A,L)$ is one-dimensional; recall from the first lecture that every ample line bundle on an abelian variety has sections. In that case, there is a well-defined effective divisor $\Theta\subseteq A$ such that $L\simeq\mathscr{O}_A(\Theta)$; it is called the *theta divisor* of the principal polarization. Using the Riemann-Roch formula and Kodaira vanishing,

$$\dim H^0(A,L)=\chi(A,L)=\frac{1}{g!}\Theta^g=\frac{1}{g!}\int_X c_1(L)^{\wedge g},$$

and so the polarization is principal iff $g!=\Theta^g$. Let us restate this condition is terms of the Hermitian form h. Since $E=-\operatorname{Im}h$ takes integer values on $\Gamma\times\Gamma$, it gives rise to a skew-symmetric $2g\times 2g$-matrix with integer entries. The square root of the determinant of this matrix is equal to $\Theta^g/g!$ – this can be proved for example by integrating the form $\omega^{\wedge g}/g!$ over a $2g$-dimensional cube in V spanned by the vectors in a basis of Γ.

From now on, we assume that h is a principal polarization. By elementary linear algebra, we can then find a basis of $\Gamma\simeq\mathbb{Z}^{2g}$ in which $E=-\operatorname{Im}h$ takes the form

$$\begin{pmatrix}0 & \mathrm{id}_g\\ -\mathrm{id}_g & 0\end{pmatrix};$$

here id_g denotes the identity matrix of size $g\times g$. By using the first g vectors $v_1,\dots,v_g$ as a basis for V, we can represent the embedding of the lattice $\Gamma\subseteq V$ by a $g\times 2g$-matrix of complex numbers of the form (id_g,Ω); concretely, this means that the lattice is spanned by the $2g$ vectors

$$v_1,\dots,v_g \quad\text{and}\quad \sum_{j=1}^{g}\Omega_{1,j}v_j,\dots,\sum_{j=1}^{g}\Omega_{g,j}v_j.$$

Exercise 13.1. Use the fact that $E=-\operatorname{Im}h$ to prove that, with respect to the basis $v_1,\dots,v_g$ of V, the Hermitian form h is represented by the matrix $(\operatorname{Im}\Omega)^{-1}$. Deduce that $\Omega=\Omega^T$ and that $\operatorname{Im}\Omega$ is positive definite.

The above exercise shows that Ω is a point of the *Siegel upper half space*

$$\mathcal{H}_g=\bigl\{\,\Omega\in\mathrm{Mat}_{g\times g}(\mathbb{C})\bigm|\Omega=\Omega^T\text{ and }\operatorname{Im}\Omega>0\,\bigr\}.$$

One can show that two such matrices determine isomorphic principally polarized abelian varieties iff they are conjugate by an element of the symplectic group $Sp_{2g}(\mathbb{Z})$; the action by $Sp_{2g}(\mathbb{Z})$ amounts to choosing a different basis for the lattice. This means that the quotient complex manifold

$$\mathcal{A}_g=\mathcal{H}_g/\,Sp_{2g}(\mathbb{Z})$$

is the moduli space of g-dimensional principally polarized abelian varieties. It is an open subset of the space of symmetric matrices, and therefore of dimension $g(g+1)/2$. In summary, we have the following result.

Theorem 13.6. *For every g-dimensional principally polarized abelian variety A, there is a matrix $\Omega\in\mathcal{H}_g$, unique up to conjugation by $Sp_{2g}(\mathbb{Z})$, such that*

$$A\simeq\mathbb{C}^g/\bigl(\mathbb{Z}^g+\Omega\mathbb{Z}^g\bigr);$$

in the standard basis on $\mathbb{C}^g$, the polarization is represented by the matrix $(\operatorname{Im}\Omega)^{-1}$.

You have definitely seen this result before in the case of elliptic curves. We can also describe the theta divisor from this point of view. To begin with, the line bundle $\mathscr{O}_A(\Theta)$ can be constructed directly from Ω, as seen by the following exercise.

Exercise 13.2. Define an action of $\mathbb{Z}^g + \Omega\mathbb{Z}^g$ on the product $\mathbb{C}^g \times \mathbb{C}$ by

$$\gamma \cdot (z,t) = (z+\gamma, t) \quad \text{and} \quad \Omega\gamma \cdot (z,t) = \Big(z + \Omega\gamma, t \cdot e^{\pi i \cdot \gamma^T \Omega \gamma + 2\pi i \cdot \gamma^T z}\Big).$$

Show that the quotient is a holomorphic line bundle on $\mathbb{C}^g / \big(\mathbb{Z}^g + \Omega\mathbb{Z}^g\big)$ whose first Chern class is represented by the $(1,1)$-form

$$\omega = \frac{i}{2} \sum_{j,k=1}^{g} h_{j,k} dz_j \wedge d\bar{z}_k,$$

where $h = (\operatorname{Im}\Omega)^{-1}$.

Since $\operatorname{Im}\Omega$ is positive definite, we can then write down a global section of $\mathscr{O}_A(\Theta)$, namely the so-called *Riemann theta function*

$$\theta(z) = \sum_{\gamma \in \mathbb{Z}^g} \exp\Big(\pi i \cdot \gamma^T \Omega \gamma + 2\pi i \cdot \gamma^T z\Big).$$

The series converges absolutely and uniformly for every $z \in \mathbb{C}^g$, making θ a holomorphic function.

Exercise 13.3. Prove that $\theta(z)$ converges absolutely and uniformly, and that the resulting holomorphic function on $\mathbb{C}^g$ satisfies the two functional equations

$$\theta(z+\gamma) = \theta(z) \quad \text{and} \quad \theta(z+\Omega\gamma) = e^{-\pi i \cdot \gamma^T \Omega \gamma - 2\pi i \gamma^T z} \cdot \theta(z)$$

for every $\gamma \in \mathbb{Z}^g$. Deduce that θ defines a global section of $\mathscr{O}_A(\Theta)$.

In particular, the divisor of zeros of θ descends to a well-defined effective divisor Θ on the quotient $A = \mathbb{C}^g / \big(\mathbb{Z}^g + \Omega\mathbb{Z}^g\big)$. Unfortunately, this description of the theta divisor is of limited use for understanding its geometric properties (such as singularities). Next time, we will see how one can use the generic vanishing theorem (and its consequences) to obtain a very surprising general result about the singularities of theta divisors.

Lecture 14

The example of Jacobians. Now let us discuss an important class of principally polarized abelian varieties, namely Jacobians of compact Riemann surfaces. Let C be a compact Riemann surface of genus $g \geq 1$, and let $J(C)$ be its Jacobian variety. Recall that $J(C)$ is equal to $\operatorname{Pic}^0(C)$, the space of degree zero line bundles on C; after choosing a base point $x_0 \in C$, we have a holomorphic mapping

$$C \to J(C), \quad x \mapsto \mathscr{O}_C(x - x_0). \tag{14.1}$$

Alternatively, we can think of the points of $J(C)$ as being divisors of degree zero on C (up to linear equivalence). Because C is one-dimensional, $J(C)$ is also isomorphic to the Albanese variety

$$\operatorname{Alb}(C) = \frac{\operatorname{Hom}\bigl(H^0(C, \Omega_C^1), \mathbb{C}\bigr)}{H_1(C, \mathbb{Z})},$$

and the mapping from C to $J(C)$ may be identified with the Albanese mapping. One can show that the intersection pairing on $H_1(C, \mathbb{Z})$ defines a principal polarization on $\operatorname{Alb}(C) \simeq J(C)$, making the Jacobian into a principally polarized abelian variety. Corresponding to this polarization, there is a well-defined theta divisor $\Theta \subseteq J(C)$, which has been studied at least since the days of Riemann. Perhaps the most famous result about Θ is the *Torelli theorem*, which says that C is uniquely determined by the pair $\bigl(J(C), \Theta\bigr)$.

Most of what we know about the theta divisor comes from the study of linear series on curves. For any integer $d \geq 1$, let C_d denote the d-fold symmetric product of C; it is a smooth projective variety of dimension d. We may identify the points of C_d with effective divisors of degree d of the form $D = x_1 + \dots + x_d$. Using the base point $x_0 \in C$, we have a well-defined mapping

$$f_d \colon C_d \to J(C), \quad x_1 + \dots + x_d \mapsto \mathscr{O}_C(x_1 + \dots + x_d - dx_0),$$

specializing to (14.1) when $d = 1$. What are the fibers of f_d? Two cycles D and D' map to the same point in $J(C)$ exactly when $\mathscr{O}_C(D) \simeq \mathscr{O}_C(D')$. The fiber over the point $f_d(D)$ is therefore in bijection with the linear system $|D|$, which is a projective space of dimension $h^0(D) - 1$. It can be shown that the scheme-theoretic fiber is also isomorphic to $|D|$. For sufficiently large values of d – in fact, for $d \geq 2g - 1$ – f_d is a projective bundle; but for smaller values of d, the fiber dimension can jump around a lot.

Now consider the case $d = g - 1$. The image of $f_{g-1} \colon C_{g-1} \to J(C)$ is an irreducible divisor; Riemann showed that, up to a translate, this divisor is equal to the theta divisor Θ. (The translate is unavoidable because f_{g-1} depends on the choice of base point $x_0 \in C$.) After suitably translating the image, we thus obtain a resolution of singularities $f \colon C_{g-1} \to \Theta$ of the theta divisor. To see that f is birational, it suffices to find a fiber that consists of a single point; or, in other words, a line bundle of degree $g-1$ with a one-dimensional space of global sections. Here we observe that the canonical bundle ω_C has degree $2g - 2$ and g global sections; after subtracting $g - 1$ general points, we obtain a line bundle of degree $g - 1$ with only one global section. Using this resolution of singularities, George Kempf proved the following theorem.

Theorem 14.2 (Kempf)**.** *The theta divisor is irreducible and normal, with rational singularities.*

Recall that a variety X has *rational singularities* if there is a resolution of singularities $f \colon X' \to X$ with $f_* \mathscr{O}_{X'} \simeq \mathscr{O}_X$ and $R^i f_* \mathscr{O}_{X'} = 0$ for $i \geq 1$. (Once there is one such resolution, every other one has the same property.) Intuitively, this means that on all the fibers of f, the higher cohomology groups of the structure sheaf are zero. In the case of the theta divisor, $f\colon C_{g-1} \to \Theta$ is such a resolution: in fact, all fibers of f are projective spaces.

More precise information about the singularities of Θ comes from the so-called Riemann singularity theorem. Let us denote by

$$\Sigma_k(\Theta) = \left\{ x \in \Theta \;\middle|\; \operatorname{mult}_x \Theta \geq k \right\}$$

the set of points where the multiplicity of Θ is at least k. Since Θ is reduced, $\Sigma_1(\Theta) = \Theta$, whereas $\Sigma_2(\Theta)$ is equal to the singular locus of Θ.

Theorem 14.3 (Riemann). *Let $D \in C_{g-1}$ be an effective divisor of degree $g-1$. Then the multiplicity of Θ at the point $f(D)$ is equal to $h^0(D)$.*

Riemann's theorem shows that understanding the singularities of the theta divisor is equivalent to understanding linear series on C. Let us denote by $W_d^r \subseteq \operatorname{Pic}^d(C)$ the set of all line bundles of degree d with at least $r+1$ global sections. Several classical theorems give information about it; you can find more details in the book by Enrico Arbarello, Maurizio Cornalba, Phillip Griffiths, and Joe Harris.

(1) Clifford's theorem says that if D is an effective divisor of degree $d \leq 2g-1$, then one has $h^0(D) - 1 \leq d/2$ (and the inequality is strict unless C is hyperelliptic, or D is a canonical divisor).
(2) Martens' theorem says that for $2 \leq d \leq g-1$ and $2 \leq 2r \leq d$, one has $\dim W_d^r \leq d - 2r$ (and the inequality is strict unless C is hyperelliptic).

Riemann's theorem shows that $\Sigma_k(\Theta)$ is, up to a translate, equal to the set W_{g-1}^{k-1}. From Clifford's theorem, we deduce that $\Sigma_k(\Theta)$ is empty unless $k-1 \leq (g-1)/2$ or $2k-1 \leq g$; from Martens' theorem, we deduce that, in the remaining cases,

$$\dim \Sigma_k(\Theta) \leq (g-1) - 2(k-1) = g - 2k + 1.$$

We can summarize both statements in the inequality

$$\operatorname{codim}_{J(C)} \Sigma_k(\Theta) \geq 2k - 1,$$

valid for every $k \geq 1$. It gives another proof for Kempf's result that Θ is normal: the singular locus $\Sigma_2(\Theta)$ has codimension at least two in Θ; for a hypersurface in a smooth variety, this property is equivalent to being normal.

Note. It is conjectured that the theta divisor on any principally polarized abelian variety (A, Θ) satisfies the same inequalities

$$\operatorname{codim}_A \Sigma_k(\Theta) \geq 2k - 1,$$

provided it is irreducible. But because we lack a good resolution of singularities for Θ as in the case of Jacobians, this looks like a very hard problem.

The point to take away from the example is that the geometry of Jacobians (and their theta divisors) is very well understood, because it can be described in terms of linear series on curves.

Singularities of theta divisors. Now the question is whether we can say something about the singularities of the theta divisor on an arbitrary principally polarized abelian variety (A, Θ). This is difficult, because we do not have a good model for Θ as in the case of Jacobians.

One thing to remember is that theta divisors do not have to be irreducible.

Example 14.4. The product $A_1 \times A_2$ of two principally polarized abelian varieties is again principally polarized; the theta divisor on the product is $\Theta_1 \times A_2 + A_1 \times \Theta_2$, which is reducible.

On the other hand, Θ is always reduced. To see why, let us write

$$\Theta = \sum_{k=1}^{n} m_k D_k$$

with $m_k \geq 1$ and D_k irreducible. Since $\mathscr{O}_A(\Theta)$ has only one section, the same is clearly true for $\mathscr{O}_A(m_k D_k)$. Now it is a general fact that, on an abelian variety, a line bundle of the form $\mathscr{O}_A(mD)$ with $m \geq 2$ has at least two linearly independent sections. In our situation, this is not possible, and so Θ must be reduced.

The first general result about the singularities of theta divisors was proved by Kollár. He observed that one can use multiplier ideals and the Kawamata-Viehweg vanishing theorem to get a handle on the singularities of Θ. As his result logically precedes that of Ein and Lazarsfeld, we shall have a look at it first.

Fix a principally polarized abelian variety (A, Θ); for the time being, we think of A as being a smooth projective algebraic variety. The line bundle $\mathscr{O}_A(\Theta)$ is then ample, and $\dim H^0\big(A, \mathscr{O}_A(\Theta)\big) = 1$. For any real number $\alpha \geq 0$, we can consider the *multiplier ideal* $\mathcal{J}(A, \alpha\Theta)$. We briefly recall the definition; for more details, see the second volume of Lazarsfeld's book *Positivity in Algebraic Geometry*. In general, let X be a smooth projective variety, and D an effective divisor on X. Take a log resolution of the pair (X, D), meaning a birational morphism

$$\mu \colon X' \to X$$

such that both the proper transform $\mu^* D$ and the relative canonical divisor $K_{X'/X}$ have simple normal crossing support. Then

$$\mathcal{J}(X, \alpha D) = \mu_* \mathscr{O}_{X'}\big(K_{X'/X} - \lfloor \alpha \cdot \mu^* D \rfloor\big).$$

Since $\mu_* \mathscr{O}_{X'}(K_{X'/X}) \simeq \mathscr{O}_X$, this defines an ideal sheaf in $\mathscr{O}_X$; one can show that it is independent of the choice of log resolution. The multiplier ideals satisfy the following version of the Kawamata-Viehweg vanishing theorem: If L is any divisor such that $L - \alpha D$ is big and nef (as an $\mathbb{R}$-divisor), then

$$H^i\big(X, \mathscr{O}_X(K_X + L) \otimes \mathcal{J}(X, \alpha D)\big) = 0$$

for every $i \geq 1$. Kollár's idea is to apply this vanishing theorem to the case of (A, Θ). Here is his main result.

Theorem 14.5. *Let (A, Θ) be a principally polarized abelian variety.*

(a) For $0 < \varepsilon < 1$, one has $\mathcal{J}\big(A, (1-\varepsilon)\Theta\big) = \mathscr{O}_A$.

(b) One has $\operatorname{codim}_A \Sigma_k(\Theta) \geq k$ for every $k \geq 0$.

Proof. In the language of the minimal model program, the assertion about the multiplier ideal means that the pair (A, Θ) is log canonical. Suppose to the contrary that $\mathcal{J}\big(A, (1-\varepsilon)\Theta\big) \neq \mathscr{O}_A$ for some $0 < \varepsilon < 1$. Let Z denote the closed subscheme defined by the multiplier ideal in question, and consider the exact sequence

$$0 \to \mathscr{O}_A(\Theta) \otimes \mathcal{J}\big(A, (1-\varepsilon)\Theta\big) \to \mathscr{O}_A(\Theta) \to \mathscr{O}_Z(\Theta) \to 0.$$

Since $\Theta - (1-\varepsilon)\Theta = \varepsilon\Theta$ is ample, the Kawamata-Viehweg vanishing theorem shows that the sheaf on the left has vanishing higher cohomology groups; in particular, this means that the restriction morphism

$$H^0\big(A, \mathscr{O}_A(\Theta)\big) \to H^0\big(Z, \mathscr{O}_Z(\Theta)\big)$$

is surjective. But since $Z \subseteq \Theta$, the unique (!) section of $\mathscr{O}_A(\Theta)$ vanishes along Z, and so we get $H^0\big(Z, \mathscr{O}_Z(\Theta)\big) = 0$. Now the simple Lemma 14.6 below gives us the desired contradiction.

The result about $\Sigma_k(\Theta)$ follows from the triviality of the multiplier ideal. To see why, take any irreducible component $Z \subseteq \Sigma_k(\Theta)$ and set $\ell = \operatorname{codim}_A Z$. We can construct a log resolution $\mu \colon X' \to X$ by first blowing up Z and then resolving singularities afterwards. (Recall that the multiplier ideal is independent of the choice of resolution.) Let $E \subseteq X'$ denote the proper transform of the exceptional divisor of the initial blowup. Since Z has codimension ℓ, the relative canonical divisor $K_{X'/X}$ contains the divisor $(\ell-1)E$. On the other hand, Θ has multiplicity at least k at every point of Z, and so $\mu^*\Theta$ contains the divisor kE. Consequently,

$$K_{X'/X} - \lfloor (1-\varepsilon)\mu\Theta \rfloor \text{ contains the divisor } (\ell-1)E - \lfloor (1-\varepsilon)k \rfloor E.$$

The coefficient at E has to be nonnegative for every $0 < \varepsilon < 1$, otherwise the pushforward would be contained in the ideal sheaf of Z. We conclude that

$$(\ell - 1) - \lfloor (1-\varepsilon)k \rfloor \geq 0;$$

if we make $\varepsilon > 0$ small enough, this gives $\ell \geq k$. □

Lemma 14.6. *If $Z \subseteq A$ is a nonempty subscheme, then $H^0\big(Z, \mathscr{O}_Z(\Theta)\big) \neq 0$.*

Proof. For any point $a \in A$, denote by $\Theta_a = \Theta + a$ the translate of Θ by a. If we choose the point a sufficiently general, then Θ_a intersects Z transversely, and so $H^0\big(Z, \mathscr{O}_Z(\Theta_a)\big) \neq 0$. Letting $a \to 0$, the result now follows by semicontinuity. □

In particular, Θ contains no points of multiplicity greater than g; Roy Smith and Robert Varley later showed that if Θ contains a g-fold point, then (A, Θ) splits into a product of g elliptic curves.

The theorem of Ein and Lazarsfeld. After this historical excursion, let us now turn to the theorem of Ein and Lazarsfeld, which is to date the strongest general result about theta divisors. You can get an idea for how surprising this was at the time by reading Kollár's review of the paper on `MathSciNet`.

Theorem 14.7. *Let (A, Θ) be a principally polarized abelian variety. If Θ is irreducible, then it is normal and has rational singularities.*

The theta divisor is a hypersurface in a smooth variety, and therefore Gorenstein; its dualizing sheaf is given by $\omega_\Theta \simeq \mathscr{O}_\Theta(\Theta)$. By duality, the condition on rational singularities is equivalent to having $\omega_\Theta \simeq f_*\omega_X$ for every (or equivalently, one) resolution of singularities $f \colon X \to \Theta$. The idea of the proof is to measure the difference between the two sides by an ideal sheaf, and then to use the results of Green and Lazarsfeld to prove that this ideal sheaf must be trivial.

The ideal sheaf in question is the so-called adjoint ideal $\operatorname{ad}(\Theta)$; it is defined more generally for any reduced effective divisor in a smooth projective variety.

Proposition 14.8. *Let D be a reduced effective divisor in a smooth projective variety M, and let $f \colon X \to D$ be any resolution of singularities. Then one has an exact sequence*

$$0 \to \mathscr{O}_M(K_M) \to \mathscr{O}_M(K_M + D) \otimes \operatorname{adj}(D) \to f_*\omega_X \to 0$$

in which $\operatorname{adj}(D) \subseteq \mathscr{O}_M$ *is the* adjoint ideal *of D. Its cosupport is contained in the singular locus of D, and* $\operatorname{adj}(D) = \mathscr{O}_M$ *if and only if D is normal and has rational singularities.*

Proof. We begin by defining the adjoint ideal. Choose a log resolution $\mu \colon M' \to M$ of the pair (M, D), and let D' be the proper transform of the divisor D; note that D' will be a disjoint union of smooth subvarieties in case D is reducible. We can then write $\mu^*D = D' + F$ for an effective μ-exceptional divisor F, and define

$$\operatorname{adj}(D) = \mu_*\mathscr{O}_{M'}\big(K_{M'/M} - F\big).$$

For the same reason as before, this is an ideal sheaf in $\mathscr{O}_M$. From our choice of F, it is obvious that $K_{M'} + D' = \mu^*(K_M + D) + (K_{M'/M} - F)$, and therefore that

$$\mu_*\mathscr{O}_{M'}(K_{M'} + D') \simeq \mathscr{O}_M(K_M + D) \otimes \operatorname{adj}(D).$$

By adjunction (applied to the smooth divisor D'), we have an exact sequence

$$0 \to \mathscr{O}_{M'}(K_{M'}) \to \mathscr{O}_{M'}(K_{M'} + D') \to \mathscr{O}_{D'}(K_{D'}) \to 0.$$

Pushing forward and using that $\mu_* \mathscr{O}_{M'}(K_{M'}) \simeq \mathscr{O}_M(K_M)$ and $R^1 \mu_* \mathscr{O}_{M'}(K_{M'}) = 0$ (by the Grauert-Riemenschneider theorem), we obtain the exact sequence

$$0 \to \mathscr{O}_M(K_M) \to \mathscr{O}_M(K_M + D) \otimes \operatorname{adj}(D) \to \nu_* \mathscr{O}_{D'}(K_{D'}) \to 0.$$

Here ν denotes the restriction of μ to D'; by construction, this is a resolution of singularities of D. To obtain the asserted exact sequence, we only need to observe that the sheaf $f_* \omega_X$ is independent of the choice of resolution: indeed, any two resolutions are dominated by a third, and so the claim follows again from the Grauert-Riemenschneider theorem.

To prove the remaining assertions, note that D is a hypersurface in the smooth variety M, and so its dualizing sheaf satisfies $\omega_D \simeq \mathscr{O}_D(K_M + D)$. From the exact sequence above, we see that $\operatorname{adj}(D)$ is trivial iff $f_* \omega_X \simeq \omega_D$. By duality, this is equivalent to the condition that $f_* \mathscr{O}_X \simeq \mathscr{O}_D$ and $R^i f_* \mathscr{O}_X = 0$ for $i > 0$, which is in turn equivalent to D being normal and having rational singularities. ☐

We are now ready to prove the theorem of Ein and Lazarsfeld.

Proof of Theorem 14.7. According to Proposition 14.8, it is enough to show that $\operatorname{adj}(\Theta) = \mathscr{O}_A$. To that end, let $f \colon X \to \Theta$ be a resolution of singularities; note that X is of maximal Albanese dimension, and therefore $\chi(X, \omega_X) \geq 0$ by Corollary 8.13. In fact, we can say more: since Θ is irreducible, we must have $\chi(X, \omega_X) \geq 1$. This is a consequence of Theorem 13.1: the Albanese image of X is equal to Θ, and Θ is not fibered by tori. (If it was fibered by translates of a subtorus $B \subseteq A$, then every translate of B would either be contained in Θ or disjoint from it; but this cannot happen because Θ is an ample divisor.)

As in the proof of Kollár's theorem, our starting point is the exact sequence

$$0 \to \mathscr{O}_A \to \mathscr{O}_A(\Theta) \otimes \operatorname{adj}(\Theta) \to f_* \omega_X \to 0. \tag{14.9}$$

For $a \in A$, set $\Theta_a = \Theta + a$. Then the line bundle $L_a = \mathscr{O}_A(\Theta_a - \Theta)$ belongs to $\operatorname{Pic}^0(A)$, and the resulting mapping

$$A \to \operatorname{Pic}^0(A), \quad a \mapsto L_a,$$

is an isomorphism of abelian varieties. (It is injective, and therefore bijective, because no two translates of Θ can be linearly equivalent to each other, due to the fact that $\dim H^0\bigl(A, \mathscr{O}_A(\Theta)\bigr) = 1$.)

After tensoring (14.9) by the line bundle $L_a = \mathscr{O}_A(\Theta_a - \Theta) \in \operatorname{Pic}^0(A)$, we obtain

$$0 \to L_a \to \mathscr{O}_A(\Theta_a) \otimes \operatorname{adj}(\Theta) \to L_a \otimes f_* \omega_X \to 0.$$

For $a \neq 0$, we have $H^0(A, L_a) = H^1(A, L_a) = 0$, and therefore

$$\dim H^0\bigl(A, \mathscr{O}_A(\Theta_a) \otimes \operatorname{adj}(\Theta)\bigr) = \dim H^0\bigl(X, \omega_X \otimes f^* L_a\bigr).$$

Now comes the crucial point. Since X is birational to Θ, it is of maximal Albanese dimension; for general $a \in A$, the generic vanishing theorem implies that

$$\dim H^0\bigl(X, \omega_X \otimes f^* L_a\bigr) = \chi(X, \omega_X \otimes f^* L_a) = \chi(X, \omega_X) \geq 1.$$

This means that the subscheme defined by $\operatorname{adj}(\Theta)$ is contained in a general translate of Θ. Of course, this can only happen if the subscheme in question is empty. We conclude that $\operatorname{adj}(\Theta) = \mathscr{O}_A$. ☐

As in the case of multiplier ideals, one can deduce a bound on the codimension of the sets $\Sigma_k(\Theta)$ from the triviality of the adjoint ideal.

Corollary 14.10. *If Θ is irreducible, then $\operatorname{codim}_A \Sigma_k(\Theta) \geq k+1$ for every $k \geq 2$.*

Note that we have to exclude the case $k = 1$, because $\Sigma_1(\Theta) = \Theta$ is a divisor.

Exercises.

Exercise 14.1. Show that the linear mapping

$$\mathcal{H}^{0,1}(C) \to \operatorname{Hom}\bigl(\mathcal{H}^{1,0}(C), \mathbb{C}\bigr), \quad \alpha \mapsto \frac{1}{2\pi i} \int_C \alpha \wedge -$$

induces an isomorphism between $\operatorname{Pic}^0(C)$ and $\operatorname{Alb}(C)$. Show that under this isomorphism, (14.1) is identified with the Albanese mapping of C.

Exercise 14.2. Show that the intersection pairing $E(\gamma, \delta) = \gamma \cdot \delta$ on the integral homology group $H_1(C, \mathbb{Z})$ defines a principal polarization of $\operatorname{Alb}(C)$. The corresponding Hermitian form on $\operatorname{Hom}\bigl(H^0(C, \Omega_C^1), \mathbb{C}\bigr)$ induces a Hermitian form on $H^0(C, \Omega_C^1)$; prove that it is given by the formula

$$(\omega_1, \omega_2) \mapsto i \int_C \omega_1 \wedge \overline{\omega_2}.$$

Exercise 14.3. Let $D \subseteq M$ be a reduced divisor. Show that if $\operatorname{adj}(D) = \mathscr{O}_M$, then for $k \geq 2$, every irreducible component of the locus $\Sigma_k(D)$ has codimension at least $(k + 1)$ in M.

Lecture 15

Products of principally polarized abelian varieties. To finish up the discussion about principally polarized abelian varieties and their theta divisors, we shall study the case when $\operatorname{codim} \Sigma_k(\Theta) = k$ for some integer $k \geq 2$. Recall that Smith and Varley proved that when Θ has a point of multiplicity $\dim A$ (= the maximal possible value), then (A, Θ) decomposes completely into a product of elliptic curves. Another application of Theorem 14.7 is the following generalization of this fact.

Corollary 15.1. *Let (A, Θ) be a principally polarized abelian variety. For $k \geq 2$, the set $\Sigma_k(\Theta)$ contains an irreducible component of codimension k if and only if (A, Θ) splits into a product of k principally polarized abelian varieties.*

Proof. Suppose that for some integer $k \geq 2$, the set $\Sigma_k(\Theta)$ has an irreducible component Z of codimension k. According to the remark above, this can only happen if $\operatorname{adj}(\Theta) \neq \mathscr{O}_A$; now Theorem 14.7 shows that Θ has to be reducible. By general theory, (A, Θ) therefore splits into a product of several principally polarized abelian varieties. (See for example the book *Abelian varieties* by Debarre.) Let

$$(A, \Theta) \simeq (A_1, \Theta_1) \times \cdots \times (A_r, \Theta_r)$$

be the decomposition into irreducible principally polarized abelian varieties; our goal is to prove that $r \geq k$. The product decomposition implies that

$$\Theta = \bigcup_{i=1}^{r} A_1 \times \cdots \times \Theta_i \times \cdots \times A_r.$$

Since $Z \subseteq \Sigma_k(\Theta)$, there are integers $k_1, \dots, k_r \geq 0$ with $k_1 + \cdots + k_r \geq k$, such that

$$Z \subseteq \Sigma_{k_1}(\Theta_1) \times \cdots \times \Sigma_{k_r}(\Theta_r).$$

We already know from Kollár's theorem that $\operatorname{codim}_{A_i} \Sigma_{k_i}(\Theta_i) \geq k_i$, and so

$$k = \operatorname{codim}_A Z \geq \sum_{i=1}^{r} \operatorname{codim}_{A_i} \Sigma_{k_i}(\Theta_i) \geq \sum_{i=1}^{r} k_i \geq k.$$

It follows that $\operatorname{codim}_{A_i} \Sigma_{k_i} = k_i$; because Θ_i is irreducible, the theorem of Ein and Lazarsfeld implies that we must have $k_i \leq 1$. But now we get $k \leq k_1 + \cdots + k_r \leq r$, which proves that (A, Θ) is the product of at least k principally polarized abelian varieties. The converse is left as an exercise. $\square$

Simpson's theorem. Let X be a smooth projective variety. According to the structure theorem for cohomology support loci, every irreducible component of $S^i_m(X)$ is a translate of a subtorus of $\operatorname{Pic}^0(X)$. Around the time Green and Lazarsfeld proved this, Arnaud Beauville and Fabrizio Catanese conjectured – based, in part, on Beauville's results about the positive-dimensional components of $S^1(X)$ – that the translates are always by points of *finite order.* This conjecture was proved shortly afterwards by Carlos Simpson.

Theorem 15.2. *Let X be a projective complex manifold. Every irreducible component of $S^q_m(X)$ is a translate of a subtorus of $\operatorname{Pic}^0(X)$ by a point of finite order.*

This may seem like a minor improvement of Theorem 11.1, but in fact, it is crucial for many applications. We will see several examples later in the course.

There are now three completely different proofs: (1) Simpson's original argument, which is based on a result from transcendental number theory; (2) a proof by reduction to positive characteristic, due to Richard Pink and Damian Roessler; (3) a proof based on $\mathscr{D}$-modules and the decomposition theorem that I found recently. The first two are more elementary, but need the assumption that X is projective (because they go through varieties defined over number fields); the third one is less elementary, but can be modified to prove the same result on compact Kähler manifolds. (This was recently done by Botong Wang.) The plan for the next few lectures is to understand Simpson's proof. As a preparation, we shall revisit $\operatorname{Pic}^0(X)$ and $\operatorname{Alb}(X)$, from the point of view of algebraic geometry.

The Picard variety is projective. Earlier in the semester, we discussed at length how to define the Albanese and Picard variety of a compact Kähler manifold. In that setting, $\operatorname{Alb}(X)$ and $\operatorname{Pic}^0(X)$ are compact complex tori that are dual to each other; they also satisfy certain universal properties. Now let X be a smooth projective variety; our goal is to show that $\operatorname{Alb}(X)$ and $\operatorname{Pic}^0(X)$ are then also projective – in other words, abelian varieties.

The easiest way is to use the criterion from Theorem 13.5, which gives a necessary and sufficient condition for a compact complex torus to be projective. Recall that we defined

$$\operatorname{Pic}^0(X) = \frac{\mathcal{H}^{0,1}(X)}{\left\{ \tau \in \mathcal{H}^{0,1}(X) \mid \bar{\tau} - \tau \text{ has periods in } \mathbb{Z}(1) \right\}},$$

based on the description of holomorphic line bundles with trivial first Chern class by operators of the form $\bar{\partial} + \tau$. To show that $\operatorname{Pic}^0(X)$ is projective, all we have to do is find a Hermitian inner product

$$h \colon \mathcal{H}^{0,1}(X) \times \mathcal{H}^{0,1}(X) \to \mathbb{C}$$

with the property that $E = -\operatorname{Im} h$ takes integer values on the lattice.

To do this, we need an additional result from Hodge theory. Let X be a compact Kähler manifold, with Kähler form ω; recall that the inner product on forms is

$$(\alpha, \beta) \mapsto \int_X \alpha \wedge *\overline{\beta}.$$

Here the $*$-operator is defined in terms of the metric, and satisfies

$$*\colon A^{p,q}(X) \to A^{n-q,n-p}(X),$$

where $n = \dim X$. The point is that one can describe the $*$-operator – and therefore the inner product – more explicitly on a Kähler manifold. As in the case of the Kähler identities, the key is the operator

$$\Lambda_\omega \colon A^{p,q}(X) \to A^{p-1,q-1}(X),$$

which is defined as the adjoint of the Lefschetz operator $L_\omega(\alpha) = \omega \wedge \alpha$. A form $\alpha \in A^k(M)$ is called *primitive* if it satisfies $\Lambda_\omega \alpha = 0$. One can show that nonzero primitive forms only exist for $k \leq n$; in fact, all the forms

$$\alpha, L_\omega \alpha, \dots, L_\omega^{n-k} \alpha$$

are then nonzero, whereas $L_\omega^{n-k+1}\alpha = 0$. This may remind you of the representation theory for the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, and in fact, the two operators L_ω and Λ_ω are part of such a representation. From this point of view, the primitive forms are exactly the vectors of highest weight.

Lemma 15.3. *Let $\alpha \in A^{p,q}(M)$ be a primitive form, meaning that $\Lambda_\omega \alpha = 0$. Then*

$$*\alpha = (-1)^{\frac{k(k+1)}{2}} i^{p-q} \frac{L_\omega^{n-k}}{(n-k)!} \alpha,$$

where $k = p + q \leq n$.

Observe that $L_\omega^{n-k}\alpha$ is a form of type $(p+n-k, q+n-k) = (n-q, n-p)$, and that the same is true for $*\alpha$; this makes it somewhat plausible that the two should be the same up to a constant factor. As usual, the proof can be done on $\mathbb{C}^n$ with the standard metric (because the identity does not involve any derivatives of the metric). Let me show you the calculation in the case $k = 1$, which is the one we need below.

Proof of Lemma 15.3 (for $k = 1$). One-forms are automatically primitive; the lemma is therefore asserting that

$$*\alpha = -i \cdot \frac{L_\omega^{n-1}}{(n-1)!}\alpha$$

for every $\alpha \in A^{1,0}(X)$. (The other formula follows by conjugation.) We may work on $\mathbb{C}^n$ with the standard metric. Let $z_1, \dots, z_n$ be the usual coordinates, and write $z_j = x_j + iy_j$; then the Kähler form is given by

$$\omega = \sum_{j=1}^n dx_j \wedge dy_j = \sum_{j=1}^n \omega_j.$$

Because $dx_1, dy_1, \dots, dx_n, dy_n$ are a positively oriented orthonormal basis,

$$\begin{aligned} *dx_k &= dx_1 \wedge dy_1 \wedge \dots \wedge \widehat{dx_k} \wedge dy_k \wedge \dots \wedge dx_n \wedge dy_n \\ &= \omega_1 \wedge \dots \wedge \widehat{\omega_k} \wedge \dots \wedge \omega_n \wedge dy_k \end{aligned}$$

and

$$\begin{aligned} *dy_k &= -dx_1 \wedge dy_1 \wedge \dots \wedge dx_k \wedge \widehat{dy_k} \wedge \dots \wedge dx_n \wedge dy_n \\ &= -\omega_1 \wedge \dots \wedge \widehat{\omega_k} \wedge \dots \wedge \omega_n \wedge dx_k. \end{aligned}$$

Putting the two together, we get

$$\begin{aligned} *dz_k = *dx_k + i * dy_k &= \omega_1 \wedge \dots \wedge \widehat{\omega_k} \wedge \dots \wedge \omega_n \wedge (dy_k - idx_k) \\ &= -i \cdot \omega_1 \wedge \dots \wedge \widehat{\omega_k} \wedge \dots \wedge \omega_n \wedge dz_k. \end{aligned}$$

On the other hand, it is easy to see that

$$\frac{\omega^{\wedge(n-1)}}{(n-1)!} = \sum_{j=1}^n \omega_1 \wedge \dots \wedge \widehat{\omega_j} \wedge \dots \wedge \omega_n.$$

This implies the desired formula for $*dz_j$; because the $*$-operator is linear over $A^0(X)$, we get the result. □

Now back to the problem of showing that $\mathrm{Pic}^0(X)$ is an abelian variety when X is projective. Fix an embedding of X into projective space, and let ω be the Kähler form of the induced Kähler metric; note that $[\omega] \in H^2(X, \mathbb{Z})$. We can define a Hermitian inner product on the space $\mathcal{H}^{0,1}(X)$ by setting

$$h(\tau_1, \tau_2) = C(n-1)! \int_X \tau_1 \wedge *\overline{\tau_2} = -i \cdot C \int_X \tau_1 \wedge \overline{\tau_2} \wedge \omega^{n-1}$$

with $C > 0$. It remains to show we can choose the constant C in such a way that $E = -\operatorname{Im} h$ takes integer values on the lattice. Suppose that $\overline{\tau_j} - \tau_j = 2\pi i \cdot \alpha_j$, where all periods of $\alpha_j \in \mathcal{H}^1(X)$ are integers. We have

$$\begin{aligned} E(\tau_1, \tau_2) = -\operatorname{Im} h(\tau_1, \tau_2) &= \frac{C}{2} \int_X \left(\overline{\tau_1} \wedge \tau_2 + \tau_1 \wedge \overline{\tau_2} \right) \wedge \omega^{n-1} \\ &= -\frac{C}{2}(2\pi i)^2 \int_X \alpha_1 \wedge \alpha_2 \wedge \omega^{n-1} = 2\pi^2 C \int_X \alpha_1 \wedge \alpha_2 \wedge \omega^{n-1}. \end{aligned}$$

This evaluates to an integer if we take $C = (2\pi^2)^{-1}$. The conclusion is that $\mathrm{Pic}^0(X)$ is an abelian variety; because $\mathrm{Alb}(X)$ is isomorphic to $\mathrm{Pic}^0\big(\mathrm{Pic}^0(X)\big)$, the same is true for the Albanese variety of X.

Algebraic definition of Albanese and Picard variety. For the purpose of proving Theorem 15.2, it is enough to know that $\mathrm{Pic}^0(X)$ and $\mathrm{Alb}(X)$ have the structure of algebraic varieties. Although we have just shown this, we shall spend some more time discussing the algebraic definition of the Picard and the Albanese variety. This is partly to compare it with the analytic definition, and partly because it involves some interesting results about abelian varieties that we will need later on anyway.

Let me begin by saying something about $\mathrm{Pic}^0(X)$. As one of the first applications of scheme theory, Grothendieck gave an algebraic definition of the Picard scheme. The idea is that one first defines a Picard functor, which associates to an arbitrary scheme S all invertible sheaves (of a given type) on the product $X \times S$, modulo the equivalence relation defined by invertible sheaves coming from S. One then shows that this functor is represented by a scheme, in the sense that invertible sheaves on $S \times X$ as above are in one-to-one correspondence with morphisms from S to the Picard scheme. The actual construction is not easy; if you are interested, you can find a nice treatment of the surface case in Mumford's book *Lectures on curves on an algebraic surface*.

With the benefit of this general theory, we can define $\mathrm{Pic}^0(X)$ as the component of the Picard scheme parametrizing invertible sheaves of the same type as $\mathscr{O}_X$. Because we are working over the complex numbers, $\mathrm{Pic}^0(X)$ is a reduced projective group scheme, and therefore a complex abelian variety. The theory works just as well over any field of characteristic zero; in particular, if X is defined over a subfield $k \subseteq \mathbb{C}$, the same is true for $\mathrm{Pic}^0(X)$. By definition, $\mathrm{Pic}^0(X)$ represents the Picard functor, and so there is a universal line bundle P_X on the product $X \times \mathrm{Pic}^0(X)$, unique up to line bundles coming from $\mathrm{Pic}^0(X)$. If we choose a base point $x_0 \in X$, then we can eliminate this ambiguity by requiring that the restriction of P_X to $\{x_0\} \times \mathrm{Pic}^0(X)$ is trivial; we shall assume this from now on. (When X is defined over a subfield $k \subseteq \mathbb{C}$, we should of course take x_0 to be a k-rational point.)

Here is how the universal property of $\mathrm{Pic}^0(X)$ works. Given an arbitrary morphism $S \to \mathrm{Pic}^0(X)$, we can pull back the universal line bundle to $X \times S$ to obtain a family of line bundles on X, parametrized by the points of S. Conversely, any line bundle on $X \times S$, whose restriction to the fibers has the same type as $\mathscr{O}_X$, is obtained from a unique morphism $S \to \mathrm{Pic}^0(X)$ in this way.

In principle, we could also define the Albanese variety by a universal property; but then it is not so easy to show that $\mathrm{Alb}(X)$ and $\mathrm{Pic}^0(X)$ are related. In the analytic treatment, the compact complex tori $\mathrm{Pic}^0(X)$ and $\mathrm{Alb}(X)$ are dual to each other; this suggest defining

$$\mathrm{Alb}(X) = \mathrm{Pic}^0\bigl(\mathrm{Pic}^0(X)\bigr)$$

as the abelian variety dual to $\mathrm{Pic}^0(X)$. Let $P_{\mathrm{Pic}^0(X)}$ denote the universal line bundle on the product $\mathrm{Alb}(X) \times \mathrm{Pic}^0(X)$, normalized by the condition that its restriction to $\mathrm{Alb}(X) \times \{0\}$ is trivial. (Here and below, we use 0 to denote the zero element in an abelian variety.)

Of course, this definition is only meaningful if we manage to construct an Albanese mapping $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ with the usual properties. Here the trick is to consider P_X as a family of line bundles on $\mathrm{Pic}^0(X)$, parametrized by the points of the scheme X. The universal property of the Picard scheme provides us with a unique morphism

$$\mathrm{alb}\colon X \to \mathrm{Alb}(X)$$

with the property that $(\mathrm{alb} \times \mathrm{id})^* P_{\mathrm{Pic}^0(X)} \simeq P_X$. We call $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ the *Albanese morphism* of X; remember that it depends on the choice of base point.

Lemma 15.4. *For the base point $x_0 \in X$, one has* $\mathrm{alb}(x_0) = 0$.

Proof. The restriction of P_X to $\{x_0\} \times \operatorname{Pic}^0(X)$ is trivial, and so the result follows from the universal property of the Picard scheme. □

To show that this definition of $\operatorname{Alb}(X)$ makes sense, we should prove that it has the same properties as on compact Kähler manifolds. This will take us some time.

Theorem 15.5. *The morphism* $\operatorname{alb}\colon X \to \operatorname{Alb}(X)$ *satisfies:*

(a) Every morphism $f\colon X \to A$ *to an abelian variety factors through* alb*.*
(b) The morphism $\operatorname{alb}^*\colon \operatorname{Pic}^0\bigl(\operatorname{Alb}(X)\bigr) \to \operatorname{Pic}^0(X)$ *is an isomorphism.*

To see what the issue is, let us consider for a moment the universal property. From the given morphism $f\colon X \to A$, we obtain $f^*\colon \operatorname{Pic}^0(A) \to \operatorname{Pic}^0(X)$, which is a morphism of abelian varieties. The construction of $\operatorname{Alb}(X)$ now gives us a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\operatorname{alb}_X"] \arrow[d, "f"] & \operatorname{Alb}(X) \arrow[d, "\varphi"] \\ A \arrow[r, "\operatorname{alb}_A"] & \operatorname{Alb}(A) \end{tikzcd} \tag{15.6}$$

in which φ is a morphism of abelian varieties. To get the desired factorization, we clearly need to know that $A \to \operatorname{Alb}(A)$ is an isomorphism.

Theorem 15.7. *Let A be a complex abelian variety, with base point $a_0 = 0$. Then the morphism* $\operatorname{alb}_A\colon A \to \operatorname{Alb}(A)$ *is an isomorphism.*

We will deal with this next time; in the process, we will prove several useful results about line bundles on abelian varieties.

Exercises.

Exercise 15.1. Show that if (A, Θ) splits into a product of k principally polarized abelian varieties, then $\Sigma_k(\Theta)$ contains an irreducible component of codimension k.

Exercise 15.2. Find a geometric interpretation for isolated points of $S^1(X)$. (Hint: If $L \in S^1(X)$ is an isolated point, consider a finite étale covering on which L becomes trivial.)

Exercise 15.3. Use the universal property of $\operatorname{Pic}^0(X)$ to show that every morphism $f\colon X \to Y$ induces a morphism of abelian varieties $\hat{f}\colon \operatorname{Pic}^0(Y) \to \operatorname{Pic}^0(X)$. Show that it takes the closed point corresponding to a line bundle L to the closed point corresponding to the line bundle f^*L.

Lecture 16

The Albanese variety of an abelian variety. Let A be an abelian variety (defined over the complex numbers, although what I am going to say applies to any algebraically closed field of characteristic zero). Recall that $\operatorname{Pic}^0(A)$ denotes the connected component of the Picard scheme containing $\mathscr{O}_A$; it is again an abelian variety. There is a universal line bundle P_A on the product $A \times \operatorname{Pic}^0(A)$, normalized by the condition that its restriction to $\{0\} \times \operatorname{Pic}^0(A)$ is trivial. We defined

$$\operatorname{Alb}(A) = \operatorname{Pic}^0\bigl(\operatorname{Pic}^0(A)\bigr),$$

and (after swapping the two factors) have another universal bundle $P_{\operatorname{Pic}^0(A)}$ on the product $\operatorname{Alb}(A) \times \operatorname{Pic}^0(A)$. The universal property of the Picard scheme gives us a unique morphism (that we shall call the *Albanese morphism*)

$$\operatorname{alb} \colon A \to \operatorname{Alb}(A)$$

with the property that $(\operatorname{alb} \times \operatorname{id})^* P_{\operatorname{Pic}^0(A)} \simeq P_A$. Last time, we showed that $\operatorname{alb}(0) = 0$; our goal today is to prove that alb is an isomorphism of abelian varieties (see Theorem 15.7 from last time). We will prove this by purely algebraic arguments, following the treatment in §8 of Mumford's book *Abelian varieties*.

To warm up, let us first show that sending A to $\operatorname{Pic}^0(A)$ is a contravariant functor on the category of abelian varieties. This is an easy consequence of the universal property; nevertheless, I will give a careful proof, because the same argument will appear again later on.

Lemma 16.1. *Let $f \colon A \to B$ be a morphism of abelian varieties. It induces a morphism $\hat{f} \colon \operatorname{Pic}^0(B) \to \operatorname{Pic}^0(A)$, with the property that the closed point corresponding to the line bundle L is mapped to the closed point corresponding to $f^* L$.*

Proof. By the universal property of the Picard scheme, the line bundle $(f \times \operatorname{id})^* P_B$ is of the form $(\operatorname{id} \times \hat{f})^* P_A$ for a unique morphism $\hat{f} \colon \operatorname{Pic}^0(B) \to \operatorname{Pic}^0(A)$.

$$\begin{tikzcd}
A \times \operatorname{Pic}^0(B) \arrow[r, "\operatorname{id} \times \hat{f}"] \arrow[d, "f \times \operatorname{id}"] & A \times \operatorname{Pic}^0(A) \\
B \times \operatorname{Pic}^0(B)
\end{tikzcd}$$

If we restrict the isomorphism to $A \times \{L\}$, we obtain $\hat{f}(L) \simeq f^* L$. $\square$

From the universal property, you can easily show that $(f \circ g)\hat{} = \hat{g} \circ \hat{f}$, and so the association $A \mapsto \operatorname{Pic}^0(A)$ is indeed a contravariant functor. The content of Theorem 15.7 is that it behaves like a duality operation on the category of abelian varieties. More on this later.

Before we begin the proof of Theorem 15.7, let me recall some basic properties of line bundles in $\operatorname{Pic}^0(A)$. Let $m \colon A \times A \to A$ denote the group operation on A, and let $\iota \colon A \to A$ denote the inverse, which can also be viewed as multiplication by -1. The following result explains how line bundles in $\operatorname{Pic}^0(A)$ behave under these operations.

Lemma 16.2. *Every $L \in \operatorname{Pic}^0(A)$ satisfies*

$$m^* L \simeq p_1^* L \otimes p_2^* L \quad \text{and} \quad \iota^* L \simeq L^{-1}.$$

In particular, if $t_a \colon A \to A$ denotes translation by a point $a \in A$, we have $t_a^* L \simeq L$.

Proof. Roughly speaking, the point is that both identities are true for $\mathscr{O}_A$; from there, they extend to arbitrary $L \in \operatorname{Pic}^0(A)$ because we are dealing with an algebraic family of line bundles. To prove the first identity, we will show that

$$(m \times \mathrm{id})^* P_A \simeq p_{13}^* P_A \otimes p_{23}^* P_A \tag{16.3}$$

as line bundles on $A \times A \times \hat{A}$. Consider the line bundle

$$(m \times \mathrm{id})^* P_A \otimes p_{13}^* P_A^{-1} \otimes p_{23}^* P_A^{-1}.$$

Its restrictions to $\{0\} \times A \times \mathrm{Pic}^0(A)$, to $A \times \{0\} \times \mathrm{Pic}^0(A)$, and to $A \times A \times \{0\}$ are easily seen to be trivial. Because A and $\mathrm{Pic}^0(A)$ are projective, we can now apply the theorem of the cube (Theorem II.6 in *Abelian varieties*) to conclude that the line bundle itself must be trivial. We get the first identity by restricting to $A \times A \times \{L\}$. Because $t_a(x) = m(a,x)$, it follows that $t_a^* L \simeq L$ for every $a \in A$.

To prove the second identity, we pull back (16.3) along the morphism

$$(\mathrm{id}, \iota) \times \mathrm{id} \colon A \times \hat{A} \to (A \times A) \times \hat{A}.$$

We find that $(\iota \times \mathrm{id})^* P_A \otimes P_A$ is the trivial bundle: the reason is our normalization of P_A, and the fact that $m \circ (\mathrm{id}, \iota)$ maps the whole of A into $\{0\}$. Consequently,

$$(\iota \times \mathrm{id})^* P_A \simeq P_A^{-1};$$

now the second identity follows by restricting to $A \times \{L\}$. □

We can use these observations to give an algebraic proof for the following result (which we had proved earlier with the help of the generic vanishing theorem).

Lemma 16.4. *If $L \in \mathrm{Pic}^0(A)$ is nontrivial, then $H^k(A, L) = 0$ for all $k \geq 0$.*

Proof. This is almost obvious for $k = 0$: if s is a nontrivial section of L, then $\iota^* s$ is a nontrivial section of $\iota^* L \simeq L^{-1}$; of course this can only happen if $L \simeq \mathscr{O}_A$. For the remaining cases, we use induction; we may assume that $H^i(A, L) = 0$ for every $0 \leq i < k$. Using the group operation $m \colon A \times A \to A$, we factor $\mathrm{id} \colon A \to A$ as

$$A \to A \times A \to A, \quad a \mapsto (a, 0) \mapsto a.$$

On the level of cohomology, we therefore obtain a factorization

$$H^k(A, L) \to H^k\bigl(A \times A, m^* L\bigr) \to H^k(A, L)$$

of the identity morphism. Because $m^* L \simeq p_1^* L \otimes p_2^* L$, we can use the Künneth formula to compute the group in the middle as

$$H^k\bigl(A \times A, m^* L\bigr) \simeq \bigoplus_{i+j=k} H^i(A, L) \otimes H^j(A, L).$$

It is zero by the inductive hypothesis, and so $H^k(A, L) = 0$ as well. □

We are now ready to start the proof of Theorem 15.7. The first step is to show that A and $\mathrm{Pic}^0(A)$ always have the same dimension. For a point $a \in A$, we have the automorphism $t_a \colon A \to A$ defined by $t_a(x) = a + x$. Now fix an arbitrary line bundle L on A, not necessarily in $\mathrm{Pic}^0(A)$, and consider the morphism

$$\varphi_L \colon A \to \mathrm{Pic}^0(A), \quad a \mapsto t_a^* L \otimes L^{-1}.$$

We have $\varphi_L(a) \in \mathrm{Pic}^0(A)$ becase $\varphi_L(0) = \mathscr{O}_A$ and A is connected. According to the theorem of the square (*Abelian varieties*, Section II.6),

$$L \otimes t_{a+b}^* L \simeq t_a^* L \otimes t_b^* L,$$

and so φ_L is actually a group homomorphism. (The proof is similar to that of the identity $m^* L \simeq p_1^* L \otimes p_2^* L$, and so we shall skip it.) For $L \in \mathrm{Pic}^0(A)$, the morphism φ_L is identically zero; this is because L is translation invariant. In general, the dimension of the image is related to the positivity of the line bundle.

Proposition 16.5. *Let L be an ample line bundle. Then $\varphi_L\colon A \to \operatorname{Pic}^0(A)$ is surjective, and its kernel $K(L) \subseteq A$ is a finite subgroup.*

In fact, the order of $K(L)$ is equal to the square of $\dim H^0(A, L)$; as an exercise, you can try to deduce this from the proof below. In any event, the lemma clearly implies that $\dim \operatorname{Pic}^0(A) = \dim A$.

Proof. The first observation is that on $A \times A$, one has

$$m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \simeq (\mathrm{id} \times \varphi_L)^* P_A. \tag{16.6}$$

To see why, consider again their difference

$$m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \otimes (\mathrm{id} \times \varphi_L)^* P_A^{-1}.$$

From the definition of φ_L, it is clear that the restriction of this line bundle to $A \times \{a\}$ is trivial for every $a \in A$; consequently, it comes from a line bundle on A by $p_1\colon A \times A \to A$. Because the restriction to $\{0\} \times A$ is also trivial by our normalization of P_A, the line bundle in question must be trivial.

If we now invert the identity in (16.6), and pull back along $(\iota, \mathrm{id})\colon A \to A \times A$, we get

$$(\iota^* L) \otimes L \simeq (\iota, \mathrm{id})^* (\mathrm{id} \times \varphi_L)^* P_A^{-1} \simeq (\iota, \varphi_L)^* P_A^{-1};$$

notice that this is still an ample line bundle. Because $(\iota, \varphi_L)^* P_A^{-1}$ restricts to the trivial line bundle on $K(L)$, the restriction of this ample line bundle to the subgroup $K(L)$ is trivial; this is possibly only when $\dim K(L) = 0$. This gives us the finiteness part.

It remains to show that φ_L is surjective. Fix any line bundle $M \in \operatorname{Pic}^0(A)$; we have to prove that $M \simeq t_a^* L \otimes L^{-1}$ for some $a \in A$. The idea is to consider the cohomology of the line bundle

$$K = m^* L \otimes p_1^* L^{-1} \otimes p_2^* (L^{-1} \otimes M^{-1}) \simeq (\mathrm{id} \otimes \varphi_L)^* P_A \otimes p_2^* M^{-1}$$

on $A \times A$. If M is not in the image of φ_L, then the restriction of K to every fiber $\{a\} \times A$ is nontrivial; we will show that this assumption leads to a contradiction.

According to Lemma 16.4, we have

$$H^i\big(A, K|_{\{a\} \times A}\big) = 0$$

for every $i \geq 0$. Because K is flat over A, the base change theorem shows that the direct image sheaves $R^i p_{1*} K$ are all trivial. If we now apply the Leray spectral sequence to the morphism $p_1\colon A \times A \to A$, we find that $H^k(A \times A, K) = 0$ for every $k \geq 0$.

Now we use the same argument for the second projection $p_2\colon A \times A \to A$. The restriction of K to $A \times \{a\}$ is trivial if and only if $a \in K(L)$; by base change, the support of the higher direct image sheaves $R^i p_{2*} K$ is therefore contained in the finite set $K(L)$. For dimension reasons, the Leray spectral sequence for p_2 degenerates, and we get

$$H^0(A, R^k p_{2*} K) \simeq H^k(A \times A, K) = 0.$$

This implies that $R^k p_{2*} K = 0$ for every $k \geq 0$. Another application of the base change theorem shows that the restriction of K to every fiber $A \times \{a\}$ must have vanishing cohomology; but this contradicts the fact that the line bundle in question is the trivial bundle when $a = 0$. We conclude that $\varphi_L\colon A \to \operatorname{Pic}^0(A)$ is indeed surjective. ∎

As a consequence of the proof, we obtain the following alternative description of the line bundles in $\operatorname{Pic}^0(A)$. (This is used as the definition of $\operatorname{Pic}^0(A)$ by Mumford.)

Corollary 16.7. *Let L be a line bundle on A. Then $L \in \operatorname{Pic}^0(A)$ if and only if it is translation invariant, meaning that $t_a^* L \simeq L$ for every $a \in A$.*

Proof. The argument above in fact works for any translation invariant line bundle M; because A is connected, it follows that those line bundles form a connected subgroup of $\mathrm{Pic}(A)$. By definition, this says that every translation invariant line bundle belongs to $\mathrm{Pic}^0(A)$; the converse has already been shown earlier. $\square$

The second step in the proof of Theorem 15.7 is to show that $\mathrm{alb}\colon A \to \mathrm{Alb}(A)$ is injective; since both abelian varieties have the same dimension, this is enough to conclude that they must be isomorphic.

Proposition 16.8. *The Albanese morphism* $\mathrm{alb}\colon A \to \mathrm{Alb}(A)$ *is injective.*

Proof. Let $K \subseteq A$ be any finite subgroup of the kernel; what we have to prove is that $K = \{0\}$. Let $\pi\colon A \to A/K$ denote the quotient morphism, and $\hat{\pi}\colon \mathrm{Pic}^0(A/K) \to \mathrm{Pic}^0(A)$ the induced morphism between the Picard schemes; its defining property is that $(\pi \times \mathrm{id})^* P_{A/K} \simeq (\mathrm{id} \times \hat{\pi})^* P_A$. Because K lies in the kernel of alb, we have a factorization

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[rr, bend right, "\mathrm{alb}"'] & A/K \arrow[r, "f"] & \mathrm{Alb}(A) \end{tikzcd}$$

If we define $Q = (f \times \mathrm{id})^* P_{\mathrm{Pic}^0(A)}$, then $P_A \simeq (\mathrm{alb} \times \mathrm{id})^* P_{\mathrm{Pic}^0(A)} \simeq (\pi \times \mathrm{id})^* Q$. By the universal property of the Picard scheme, the line bundle Q defines a morphism

$$q\colon \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(A/K)$$

with $(\mathrm{id} \times q)^* P_{A/K} \simeq Q$. The relation $P_A \simeq (\pi \times \mathrm{id})^* Q$ gives us

$$P_A \simeq (\mathrm{id} \times q)^* (\pi \times \mathrm{id})^* P_{A/K} \simeq (\mathrm{id} \times q)^* (\mathrm{id} \times \hat{\pi})^* P_A,$$

which means exactly that the composition $\hat{\pi} \circ q$ is the identity:

$$\begin{tikzcd} \mathrm{Pic}^0(A) \arrow[r, "q"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{Pic}^0(A/K) \arrow[r, "\hat{\pi}"] & \mathrm{Pic}^0(A) \end{tikzcd}$$

Because all three abelian varieties have the same dimension, it follows that q and $\hat{\pi}$ are isomorphisms. We can now apply the following lemma to show that $K = \{0\}$, and hence that $\mathrm{alb}\colon A \to \mathrm{Alb}(A)$ must be injective. $\square$

Lemma 16.9. *Let* $\pi\colon A \to B$ *be a surjective morphism between two abelian varieties of the same dimension. Then the kernel of* $\hat{\pi}\colon \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(A)$ *is a finite group of the same order as* $\ker \pi$.

Proof. $K = \ker \pi$ is an abelian group of some finite order d; it acts on the locally free sheaf $\pi_* \mathscr{O}_A$, which therefore splits up as

$$\pi_* \mathscr{O}_A \simeq \bigoplus_{i=1}^{d} L_i$$

into a sum of d invertible sheaves. This holds because we are working over $\mathbb{C}$; more generally, any field containing all d-th roots of unity would be okay. The line bundles $L_1, \dots, L_d$ are translation invariant, and therefore belong to $\mathrm{Pic}^0(B)$. Note that because $H^0(B, \pi_* \mathscr{O}_A) \simeq H^0(A, \mathscr{O}_A)$ is one-dimensional, exactly one of the line bundles L_i has to be trivial.

To prove the lemma, we are going to show that $L_1, \dots, L_d$ are pairwise non-isomorphic, and that

$$\ker \hat{\pi} = \{L_1, \dots, L_d\}.$$

The first step is to show that $\pi^* L_i \simeq \mathscr{O}_A$ for every $i = 1, \dots, d$. Consider the following commutative diagram:

$$\begin{tikzcd} A \times_B A \arrow[r, "p_2"] \arrow[d, "p_1"] & A \arrow[d, "\pi"] \\ A \arrow[r, "\pi"] & B \end{tikzcd}$$

It is easy to see that $A \times_B A \simeq A \times K$, and that the above diagram is just

$$\begin{tikzcd} A \times K \arrow[r, "m"] \arrow[d, "p_1"] & A \arrow[d, "\pi"] \\ A \arrow[r, "\pi"] & B \end{tikzcd}$$

which implies (by flat base change) that $\pi^* \pi_* \mathscr{O}_A$ is isomorphic to $\mathscr{O}_A^{\oplus d}$. But then $\pi^* L_i \simeq \mathscr{O}_A$, as claimed. This argument also proves that $L_i \simeq L_j$ can only happen when $i = j$: the reason is that

$$H^0\big(B, L_i^{-1} \otimes \pi_* \mathscr{O}_A\big) \simeq H^0\big(A, \pi^* L_i^{-1}\big) \simeq H^0(A, \mathscr{O}_A).$$

At this point, we know that $\ker \hat{\pi} \supseteq \{L_1, \dots, L_d\}$ contains at least d elements. To finish the proof, we have to show that the two sets are equal.

Consider a line bundle $L \in \operatorname{Pic}^0(B)$ that belongs to $\ker \hat{\pi}$, or in other words, such that $\pi^* L$ is trivial. By the projection formula,

$$\pi_* \mathscr{O}_A \simeq \pi_* \pi^* L \simeq L \otimes \pi_* \mathscr{O}_A.$$

If we tensor by L^{-1} and take global sections, we see that $L^{-1} \otimes L_i$ must be trivial for some value of i, and hence that $L \simeq L_i$. This proves that $\ker \hat{\pi}$ is a subgroup of $\operatorname{Pic}^0(B)$ of order d. □

From now on, we shall identify A with its Albanese variety $\operatorname{Alb}(A)$. Under this identification, we have $P_A \simeq P_{\operatorname{Pic}^0(A)}$, which means that the universal line bundle on A is also (after switching the two factors) the universal line bundle on $\operatorname{Pic}^0(A) \times A$. In the future, we shall denote this line bundle simply by P, and refer to it as the *normalized Poincaré bundle* for the abelian variety A.

Exercises.

Exercise 16.1. Let L be an ample line bundle. Show that the size of $\ker \varphi_L$ is equal to the square of $\dim H^0(A, L)$. (Hint: Use the same argument as in Proposition 16.5, but with $M = L$.)

Exercise 16.2. Use the universal property of the Picard scheme to show that the diagram in (15.6) is commutative.

Exercise 16.3. Prove that $\operatorname{alb}^* \colon \operatorname{Pic}^0\big(\operatorname{Alb}(X)\big) \to \operatorname{Pic}^0(X)$ is an isomorphism.

Lecture 17

The Albanese mapping. Let X be a smooth projective variety, and

$$\mathrm{alb}\colon X \to \mathrm{Alb}(X)$$

its Albanese mapping; recall that it is defined by the condition $(\mathrm{alb} \times \mathrm{id})^* P_{\mathrm{Pic}^0(X)} \simeq P_X$. To wrap up the discussion from last time, let me quickly explain how one proves the universal property: every morphism from X to an abelian variety factors uniquely through the Albanese mapping.

Suppose we are given such a morphism $f\colon X \to A$. It induces a morphism of abelian varieties $\hat{f}\colon \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$, and therefore also $\varphi\colon \mathrm{Alb}(X) \to \mathrm{Alb}(A)$, and in fact, the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{alb}_X"] \arrow[d, "f"] & \mathrm{Alb}(X) \arrow[d, "\varphi"] \\ A \arrow[r, "\mathrm{alb}_A"] & \mathrm{Alb}(A) \end{tikzcd}$$

is commutative. To prove this claim, recall from Lemma 16.1 that $\hat{f}$ is uniquely determined by the condition that

$$(f \times \mathrm{id})^* P_A \simeq (\mathrm{id} \times \hat{f})^* P_X.$$

Likewise, φ is determined by the condition that

$$(\mathrm{id} \times \hat{f})^* P_{\mathrm{Pic}^0(X)} \simeq (\varphi \times \mathrm{id})^* P_{\mathrm{Pic}^0(A)}.$$

By the universal property of the Picard scheme, the identity $\varphi \circ \mathrm{alb}_X = \mathrm{alb}_A \circ f$ is equivalent to

$$(\mathrm{alb}_X \times \mathrm{id})^* (\varphi \times \mathrm{id})^* P_{\mathrm{Pic}^0(A)} \simeq (f \times \mathrm{id})^* (\mathrm{alb}_A \times \mathrm{id})^* P_{\mathrm{Pic}^0(A)}.$$

This follows without difficulty from the relations above:

$$\begin{aligned} (\mathrm{alb}_X \times \mathrm{id})^* (\varphi \times \mathrm{id})^* P_{\mathrm{Pic}^0(A)} &\simeq (\mathrm{alb}_X \times \mathrm{id})^* (\mathrm{id} \times \hat{f})^* P_{\mathrm{Pic}^0(X)} \\ &\simeq (\mathrm{id} \times \hat{f})^* (\mathrm{alb}_X \times \mathrm{id})^* P_{\mathrm{Pic}^0(X)} \\ &\simeq (\mathrm{id} \times \hat{f})^* P_X \\ &\simeq (f \times \mathrm{id})^* P_A \\ &\simeq (f \times \mathrm{id})^* (\mathrm{alb}_A \times \mathrm{id})^* P_{\mathrm{Pic}^0(A)}. \end{aligned}$$

We know from Theorem 15.7 that $\mathrm{alb}_A\colon A \to \mathrm{Alb}(A)$ is an isomorphism; this gives us the desired factorization of f. I will leave it as an exercise to show the uniqueness; and also to prove that

$$\mathrm{alb}^*\colon \mathrm{Pic}^0\bigl(\mathrm{Alb}(X)\bigr) \to \mathrm{Pic}^0(X)$$

is an isomorphism.

Line bundles with flat connection. Now let us return to our original topic, namely Simpson's theorem about the structure of cohomology support loci (see Theorem 15.2 above). The proof is based on a detailed study of line bundles with flat connection. We have already seen that, on a compact Kähler manifold, every holomorphic line bundle with trivial first Chern class (represented by an operator of the form $\bar{\partial} + \tau$) has a canonical flat connection (represented by the operator $d + \tau - \bar{\tau}$). The connections we obtain in this way are always unitary; in fact, they are compatible with a Hermitian metric on the line bundle by construction. It turns out that, by going to the larger class of all flat (but not necessarily unitary) connections, some additional structure becomes visible; it is this additional structure that Simpson uses in his proof.

Let me start with some general remarks about line bundles with flat connection; as usual, there is also a very concrete description that I shall explain afterwards. Let X be a compact Kähler manifold, and let L be a holomorphic line bundle on X. A (holomorphic) *connection* on L is an operator

$$\nabla \colon L \to \Omega_X^1 \otimes_{\mathscr{O}_X} L$$

that satisfies the Leibniz rule $\nabla(fs) = df \otimes s + f\nabla s$; the connection is called *flat* if its curvature $\nabla \circ \nabla$ is equal to zero. (Here we extend the connection to an operator $\nabla \colon \Omega_X^1 \otimes L \to \Omega_X^2 \otimes L$ by enforcing the Leibniz rule.) It is not hard to show that L admits such a connection if and only if its first Chern class $c_1(L) \in H^2\big(X, \mathbb{Z}(1)\big)$ is torsion; this is equivalent to saying that the image of $c_1(L)$ in $H^2(X, \mathbb{C})$ is zero.

Definition 17.1. We denote by $M_{\mathrm{DR}}(X)$ the moduli space of all pairs (L, ∇), where L is a holomorphic line bundle and $\nabla \colon L \to \Omega_X^1 \otimes L$ a flat connection.

This notation was invented by Simpson; the subscript stands for "de Rham". One can show that $M_{\mathrm{DR}}(X)$ is a complex manifold, and in fact a complex Lie group: the group operation is defined by the tensor product

$$(L_1, \nabla_1) \otimes (L_2, \nabla_2) = (L_1 \otimes L_2, \nabla_1 \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_2),$$

and the unit element is the pair $(\mathscr{O}_X, d)$. We will see below why this is the case.

Given (L, ∇), we get a locally constant sheaf $\ker \nabla$ of one-dimensional $\mathbb{C}$-vector spaces on X by taking the sheaf of local holomorphic solutions of the equation $\nabla s = 0$; the integrability condition $\nabla \circ \nabla = 0$ ensures that this equation always has nowhere vanishing local solutions. We can recover the holomorphic line bundle from the locally constant sheaf because $L \simeq \mathscr{O}_X \otimes_{\mathbb{C}} (\ker \nabla)$; this also determines the connection because

$$\nabla(f \otimes s) = df \otimes s$$

for s a local section of $\ker \nabla$. Thus holomorphic line bundles with flat connection, and locally constant sheaves of one-dimensional $\mathbb{C}$-vector spaces, are the same thing. This is of course true also for vector bundles of higher rank.

Now a locally constant sheaf is equivalent to a representation of the fundamental group. In fact, once we fix a reference point $x_0 \in X$, we obtain a representation of $\pi_1(X, x_0)$ on the fiber of L at the point x_0, by analytic continutation of solutions to the equation $\nabla s = 0$. Since L is a line bundle, this gives us a homomorphism

$$\rho \colon \pi_1(X, x_0) \to \mathbb{C}^\times,$$

or, in other words, a character of the fundamental group. Conversely, a character ρ determines a locally constant sheaf $\mathbb{C}_\rho$, and therefore a line bundle with flat connection. (More on this later.)

Definition 17.2. We denote by

$$M_{\mathrm{B}}(X) = \mathrm{Hom}\big(\pi_1(X, x_0), \mathbb{C}^\times\big)$$

the space of all characters of the fundamental group.

This notation is also due to Simpson; this time, the subscript stands for "Betti". $M_{\mathrm{B}}(X)$ is a complex Lie group, too, with group operation given by the usual product. In fact, we can easily describe its structure. By the Hurewicz theorem, $M_{\mathrm{B}}(X) \simeq H^1(X, \mathbb{C}^\times)$, and so it fits into an exact sequence

$$0 \to \frac{H^1(X, \mathbb{C})}{H^1\big(X, \mathbb{Z}(1)\big)} \to M_{\mathrm{B}}(X) \to H^2\big(X, \mathbb{Z}(1)\big)_{tors} \to 0.$$

This shows that the connected component $M_{\mathrm{B}}^0(X)$ of the trivial character is isomorphic to $(\mathbb{C}^\times)^{2g}$, where $g = \dim H^0(X, \Omega_X^1)$; the quotient is a finite group, of the same order as the torsion subgroup of $H_1(X, \mathbb{Z})$. According to the discussion above, we have an isomorphism of complex Lie groups

$$M_{\mathrm{DR}}(X) \to M_{\mathrm{B}}(X)$$

that takes a pair (L, ∇) to the unique character ρ such that $\mathbb{C}_\rho \simeq \ker \nabla$ is isomorphic to the local system of ∇-flat sections of L. The isomorphism has the property that the image of ρ in the torsion subgroup of $H^2\big(X, \mathbb{Z}(1)\big)$ is precisely the first Chern class $c_1(L)$.

Concrete description. We now restrict our attention to the connected component $M^0_{\mathrm{DR}}(X)$ containing the point $(\mathscr{O}_X, d)$. This means that we only consider pairs (L, ∇) with $c_1(L) = 0$; just as in the case of $\mathrm{Pic}^0(X)$, it is then possible to describe all the objects very concretely in terms of harmonic one-forms.

Let $\varepsilon \in \mathcal{H}^1(X)$ be a harmonic one-form; note that this is equivalent to both $\varepsilon^{1,0}$ and $\varepsilon^{0,1}$ being d-closed. Then the operator $d + \varepsilon$ determines a holomorphic line bundle L with flat connection ∇. The holomorphic line bundle is the one corresponding to the operator $\bar\partial + \varepsilon^{0,1}$; to describe the connection, note that locally, $\varepsilon^{0,1} = \bar\partial f$, and then e^{-f} is a local holomorphic section of L because

$$(\bar\partial + \varepsilon^{0,1})e^{-f} = 0.$$

If we apply the operator $d + \varepsilon$ to this section, we get

$$(d + \varepsilon)e^{-f} = (\varepsilon^{1,0} - \partial f) \otimes e^{-f},$$

which is a local holomorphic section of $\Omega^1_X \otimes L$ because

$$\bar\partial(\varepsilon^{1,0} - \partial f) = \partial\bar\partial f = \partial\varepsilon^{0,1} = 0.$$

What we get in this way is a well-defined connection ∇ on L. Using the analytic description of $\mathrm{Pic}^0(X)$ that we developed earlier in the semester, you can check that every point of $M^0_{\mathrm{DR}}(X)$ is obtained in this way.

Note that $d+\varepsilon$ gives the trivial pair precisely when there is a nowhere vanishing global section $f \in A^0(X)$ that is both holomorphic and flat; this translates into the condition that $df + \varepsilon f = 0$. If we impose the condition that $f(x_0) = 1$, the solution is necessarily given by the integral

$$f = \exp\left(-\int_{x_0}^{x} \varepsilon\right). \tag{17.3}$$

This is well-defined on X if and only if all periods of ε belong to $\mathbb{Z}(1) = 2\pi i \cdot \mathbb{Z}$. In other words:

Proposition 17.4. *Let X be a compact Kähler manifold. Then one has*

$$M^0_{\mathrm{DR}}(X) \simeq \frac{\mathcal{H}^1(X)}{\left\{\, \varepsilon \in \mathcal{H}^1(X) \;\middle|\; \varepsilon \text{ has periods in } \mathbb{Z}(1) \,\right\}},$$

with $\varepsilon \in \mathcal{H}^1(X)$ corresponding to the line bundle with flat connection defined by the operator $d + \varepsilon$.

When ε does not have periods in $\mathbb{Z}(1)$, the formula in (17.3) only makes sense by analytic continuation (because the integral depends on the choice of path). It follows that the character corresponding to $d + \varepsilon$ must be

$$\pi_1(X, x_0) \to \mathbb{C}^\times, \quad \gamma \mapsto \exp\left(-\int_\gamma \varepsilon\right).$$

This rule defines an injective homomorphism of complex Lie groups

$$M^0_{\mathrm{DR}}(X) \to M^0_{\mathrm{B}}(X);$$

because both sides have dimension $\dim H^1(X, \mathbb{C})$, it must be an isomorphism.

Example 17.5. Our earlier discussion of $\operatorname{Pic}^0(X)$ fits into this framework as well. Recall that the holomorphic line bundle defined by $\bar\partial + \tau$ has a flat connection $d+\tau-\bar\tau$ (from the Hermitian metric induced by the smooth trivialization). Because the condition for triviality is the same, we get an embedding

$$\operatorname{Pic}^0(X) \to M^0_{\mathrm{DR}}(X), \quad [\tau] \mapsto [\tau - \bar\tau].$$

Its image in $M^0_{\mathrm{B}}(X)$ is a connected component of the space of all *unitary characters* $\operatorname{Hom}\bigl(\pi_1(X, x_0), U(1)\bigr)$. Note that the image is not a complex submanifold.

Hodge theory for line bundles with flat connection. Some of the results from Hodge theory carry over to the case of line bundles with (not necessarily unitary) flat connection. The most important one is the Kähler identities. Recall that in the case of a holomorphic line bundle given by $\bar\partial + \tau$, the two operators $\partial - \bar\tau$ and $\bar\partial + \tau$ satisfied the Kähler identities. The correct generalization of this fact to the case $d + \varepsilon$ was found by Simpson.

Let us fix a harmonic one-form $\varepsilon \in \mathcal{H}^1(X)$. Instead of the obvious decomposition $d+\varepsilon = (\partial+\varepsilon^{1,0})+(\bar\partial+\varepsilon^{0,1})$ – for which the Kähler identities are not true in general – we use the non-obvious decomposition

$$d + \varepsilon = \left(\partial + \frac{\varepsilon^{1,0} - \overline{\varepsilon^{0,1}}}{2} + \frac{\varepsilon^{0,1} + \overline{\varepsilon^{1,0}}}{2}\right) + \left(\bar\partial + \frac{\varepsilon^{0,1} - \overline{\varepsilon^{1,0}}}{2} + \frac{\varepsilon^{1,0} + \overline{\varepsilon^{0,1}}}{2}\right).$$

We can write this somewhat more compactly as

$$d + \varepsilon = \bigl(\partial - \bar\tau + \bar\theta\bigr) + (\bar\partial + \tau + \theta),$$

by defining two auxiliary harmonic forms

$$\theta = \frac{\varepsilon^{1,0} + \overline{\varepsilon^{0,1}}}{2} \in \mathcal{H}^{1,0}(X) \quad \text{and} \quad \tau = \frac{\varepsilon^{0,1} - \overline{\varepsilon^{1,0}}}{2} \in \mathcal{H}^{0,1}(X).$$

The point of the funny decomposition above is that the two operators in parentheses satisfy the Kähler identities:

$$\begin{aligned}(\partial - \bar\tau + \bar\theta)^* &= i[\Lambda_\omega, \bar\partial + \tau + \theta] \\ (\bar\partial + \tau + \theta)^* &= -i[\Lambda_\omega, \partial - \bar\tau + \bar\theta]\end{aligned}$$

The reason is that $\tau^* = i[\Lambda_\omega, \bar\tau]$ and $\theta^* = -i[\Lambda_\omega, \bar\theta]$; this is something we have already proved (in Lemma 6.1).

Now suppose that $[\varepsilon] \in M^0_{\mathrm{DR}}(X)$ is the unit element, which means that ε has periods in $\mathbb{Z}(1)$. Then ε is purely imaginary, which means that $\varepsilon^{0,1} = \overline{\varepsilon^{1,0}}$, and hence $\theta = 0$ and $\varepsilon = \tau - \bar\tau$. In particular, $[\tau] \in \operatorname{Pic}^0(X)$ is also trivial. We therefore obtain a well-defined mapping

$$M^0_{\mathrm{DR}}(X) \to \operatorname{Pic}^0(X) \times H^0(X, \Omega^1_X), \quad [\varepsilon] \mapsto \bigl([\tau], \theta\bigr),$$

which is an isomorphism of real (but not complex) Lie groups (because the formulas for τ and θ involve complex conjugation). It associates to every line bundle with flat connection in $M^0_{\mathrm{DR}}(X)$ another line bundle together with a holomorphic one-form. Note that the new line bundle is *different* from the original one (unless $\varepsilon = \tau - \bar\tau$, or equivalently, unless $\theta = 0$).

Definition 17.6. A *Higgs bundle* of rank one is a pair (L, θ), where $L \in \operatorname{Pic}^\tau(X)$ is a holomorphic line bundle with $c_1(L) \in H^2\bigl(X, \mathbb{Z}(1)\bigr)$ torsion, and $\theta \in H^0(X, \Omega^1_X)$ a holomorphic one-form.

In higher rank, a Higgs bundle is a holomorphic vector bundle E with torsion Chern classes, together with a morphism $\theta \colon E \to \Omega^1_X \otimes E$ that satisfies $\theta \wedge \theta = 0$; for line bundles, this specializes to the definition from above.

Definition 17.7. We denote by

$$M_{\mathrm{Dol}}(X) = \mathrm{Pic}^{\tau}(X) \times H^0(X, \Omega_X^1)$$

the moduli space of all Higgs bundles of rank one.

This is the third (and last) moduli space that we will use; the subscript stands for "Dolbeault". One can show that the isomorphism between $M_{\mathrm{DR}}^0(X)$ and $M_{\mathrm{Dol}}^0(X)$ extends to an isomorphism between $M_{\mathrm{DR}}(X)$ and $M_{\mathrm{Dol}}(X)$. To summarize, we have three different models for the space of characters of the fundamental group:

$$M_{\mathrm{DR}}(X) \quad M_{\mathrm{B}}(X) \quad M_{\mathrm{Dol}}(X)$$

All are isomorphic as real Lie groups, but their complex – and, when X is projective, also their algebraic – structures are very different.

Example 17.8. By taking $\theta = 0$, we can realize $\mathrm{Pic}^0(X)$ as a complex submanifold of $M_{\mathrm{Dol}}^0(X)$. But its image in $M_{\mathrm{B}}^0(X)$ is the space of unitary characters, which is no longer a complex submanifold.

The idea behind Simpson's proof of Theorem 15.2 is to exploit the different structures to obtain information about cohomology support loci. To get results about $\mathrm{Pic}^0(X)$, all we need to do is restrict to $\theta = 0$ at the end.

Cohomology and harmonic forms. Now we turn to the computation of cohomology. Fix a harmonic one-form $\varepsilon \in \mathcal{H}^1(X)$, and let $\rho \in M_{\mathrm{B}}^0(X)$ denote the associated character, and $(L, \theta) \in M_{\mathrm{Dol}}^0(X)$ the associated Higgs bundle. We define the cohomology groups of the Higgs bundle (L, θ) as

$$H^{p,q}(X, L, \theta) = \frac{\ker\bigl(\theta \colon H^q(X, \Omega_X^p \otimes L) \to H^q(X, \Omega_X^{p+1} \otimes L)\bigr)}{\mathrm{im}\bigl(\theta \colon H^q(X, \Omega_X^{p-1} \otimes L) \to H^q(X, \Omega_X^p \otimes L)\bigr)}.$$

Note that $H^{p,q}(X, L, \theta)$ is the cohomology in degree p of the complex

$$0 \longrightarrow H^q(X, L) \xrightarrow{\theta} H^q(X, \Omega_X^1 \otimes L) \xrightarrow{\theta} \cdots \xrightarrow{\theta} H^q(X, \Omega_X^n \otimes L) \longrightarrow 0;$$

this complex already occured as (the conjugate of) the derivative complex during our study of cohomology support loci for sheaves of holomorphic forms. Another reason for introducing these cohomology groups is that they appear in the following version of the Hodge decomposition.

Theorem 17.9. *For every $k = 0, \dots, 2\dim X$, one has*

$$H^k(X, \mathbb{C}_\rho) \simeq \bigoplus_{p+q=k} H^{p,q}(X, L, \theta).$$

Proof. Because $\mathbb{C}_\rho$ is the sheaf of local solutions to the equation $df + \varepsilon f = 0$, it is not hard to show that the complex of soft sheaves

$$\mathcal{A}_X^0 \xrightarrow{d+\varepsilon} \mathcal{A}_X^1 \xrightarrow{d+\varepsilon} \cdots \xrightarrow{d+\varepsilon} \mathcal{A}_X^n$$

is a resolution of $\mathbb{C}_\rho$. As usual, this means that the cohomology groups of $\mathbb{C}_\rho$ are computed by the complex $A^\bullet(X)$ with differential $d + \varepsilon$. By the same argument as in Hodge theory, every cohomology class in $H^k(X, \mathbb{C}_\rho)$ is uniquely represented by a harmonic k-form, meaning a k-form that lies in the kernel of $d + \varepsilon$ and $(d + \varepsilon)^*$.

Just as in Theorem 3.10, the Kähler identities lead to the relation

$$\begin{aligned}(d+\varepsilon)(d+\varepsilon)^* &+ (d+\varepsilon)^*(d+\varepsilon) \\ &= 2\Bigl((\bar\partial + \tau + \theta)(\bar\partial + \tau + \theta)^* + (\bar\partial + \tau + \theta)^*(\bar\partial + \tau + \theta)\Bigr).\end{aligned}$$

between two Laplace operators; this shows that a k-form is harmonic if and only if it is in the kernel of $\partial + \tau + \theta$ and $(\partial + \tau + \theta)^*$. It follows that the complex

$$A^\bullet(X), \text{ with differential } \bar\partial + \tau + \theta,$$

also computes the cohomology of $\mathbb{C}_\rho$. This complex has the great advantage that we have removed the differential operator ∂ from the picture.

To make the connection with the Higgs bundle (L, θ), we observe that our complex is the single complex associated with the double complex

$$A^{\bullet,\bullet}(X), \text{ with differentials } \theta \text{ and } \bar\partial + \tau.$$

Now let us consider the spectral sequence of this double complex:

$$E_0^{p,q} = A^{p,q}(X) \Longrightarrow H^{p+q}(X, \mathbb{C}_\rho)$$

The differential d_0 is induced by $\bar\partial + \tau$; because the complex $A^{p,\bullet}(X)$ with differential $\bar\partial + \tau$ resolves $\Omega_X^p \otimes L$ (by Lemma 5.5), we obtain

$$E_1^{p,q} \simeq H^q(X, \Omega_X^p \otimes L).$$

The differential d_1 is induced by θ, and so

$$E_2^{p,q} \simeq H^{p,q}(X, L, \theta)$$

is exactly the cohomology of the Higgs bundle (L, θ). The same argument as in the proof of Proposition 9.7 (based on the principle of two types) shows that the spectral sequence degenerates at E_2; this gives us the desired decomposition of $H^k(X, \mathbb{C}_\rho)$ into the sum of $E_2^{p,q}$ with $p + q = k$. □

When ρ is unitary (which is equivalent to $\theta = 0$), we obtain

$$H^k(X, \mathbb{C}_\rho) \simeq \bigoplus_{p+q=k} H^q(X, \Omega_X^p \otimes L)$$

for the associated holomorphic line bundle. We are of course interested in the cohomology groups $H^q(X, \Omega_X^p \otimes L)$; the isomorphism above relates them with the purely topological cohomology groups $H^k(X, \mathbb{C}_\rho)$.

Exercises.

Exercise 17.1. Let L be a holomorphic line bundle on a compact Kähler manifold. Show that L admits a flat connection $\nabla \colon L \to \Omega_X^1 \otimes L$ if and only if the first Chern class $c_1(L)$ is zero in $H^2(X, \mathbb{C})$.

Exercise 17.2. Show that the composition $M_{\mathrm{DR}}(X) \to M_{\mathrm{B}}(X) \to H^2\big(X, \mathbb{Z}(1)\big)$ takes a pair (L, ∇) to $c_1(L)$.

Exercise 17.3. Show that every point of $M_{\mathrm{DR}}^0(X)$ can be realized as $d + \varepsilon$ for some $\varepsilon \in \mathcal{H}^1(X)$.

Lecture 18

Cohomology jump loci. Recall from last time that when $\rho \in M_{\mathrm{B}}^0(X)$ is a unitary character (which means that the corresponding Higgs bundle (L, θ) satisfies $\theta = 0$), one has a decomposition

$$H^k(X, \mathbb{C}_\rho) \simeq \bigoplus_{p+q=k} H^q(X, \Omega_X^p \otimes L).$$

This suggests that in order to understand the cohomology support loci

$$S_m^q(X, \Omega_X^p) = \left\{ L \in \operatorname{Pic}^0(X) \;\middle|\; \dim H^q(X, \Omega_X^p \otimes L) \geq m \right\},$$

it should be enough to know the structure of the sets

$$\Sigma_m^k(X) = \left\{ \rho \in M_{\mathrm{B}}(X) \;\middle|\; \dim H^k(X, \mathbb{C}_\rho) \geq m \right\}.$$

In the literature, these sets are often called *cohomology jump loci*. Note that they only depend on the underlying topological space of X.

Before we can say anything about $\Sigma_m^k(X)$, we first have to be more precise about what the sheaf $\mathbb{C}_\rho$ is. By the Hurewicz theorem, every character of $\pi_1(X, x_0)$ factors through $H_1(X, \mathbb{Z})$; to simplify the notation, we put $\Lambda = H_1(X, \mathbb{Z})$. Now take $\pi \colon Y \to X$ to be the covering space of X corresponding to the normal subgroup

$$\ker\bigl(\pi_1(X, x_0) \to \Lambda\bigr);$$

then Λ is isomorphic to the group of deck transformations and acts transitively on the fibers of π. The covering space has the property that the composition of ρ with $\pi_1(Y, y_0) \to \pi_1(X, x_0)$ becomes trivial. (Here $y_0 \in \pi^{-1}(x_0)$ is a base point on Y.)

Lemma 18.1. *For every open subset $U \subseteq X$, one has*

$$H^0(U, \mathbb{C}_\rho) = \left\{ s \colon \pi^{-1}(U) \to \mathbb{C} \;\middle|\; \begin{array}{l} s(y) \textit{ is locally constant, and} \\ s(\lambda y) = \rho(\lambda)^{-1} s(y) \textit{ for all } \lambda \in \Lambda \end{array} \right\}.$$

Proof. Note that because Λ acts transitively on $\pi^{-1}(x)$, every stalk of the sheaf defined by the right-hand side is indeed isomorphic to $\mathbb{C}$. To simplify the argument, we shall only do the case where ρ is the character corresponding to $d + \varepsilon$; here $\mathbb{C}_\rho$ is the sheaf of solutions to the equation $df + \varepsilon f = 0$. Because we defined the covering space in terms of $H_1(X, \mathbb{Z})$, the pullback $\pi^*\varepsilon$ no longer has any periods, and so

$$F(y) = \exp\left(- \int_{y_0}^{y} \pi^* \varepsilon \right)$$

is a well-defined function on Y. Now consider some $f \in H^0(U, \mathbb{C}_\rho)$. It solves the equation $df + \varepsilon f = 0$, and so $\pi^* f = sF$ for a unique function $s \colon \pi^{-1}(U) \to \mathbb{C}$ that satisfies $ds = 0$, and must therefore be locally constant. Because $F(\lambda y) = \rho(\lambda) F(y)$, we see that $s(\lambda y) = \rho(\lambda)^{-1} s(y)$, as claimed above. Conversely, for any such function s, the product sF is invariant under the Λ-action, and therefore descends to a solution of $df + \varepsilon f = 0$ on U. $\square$

We have already seen that the space of characters $M_{\mathrm{B}}(X)$ is an affine algebraic variety of dimension $\dim H^1(X, \mathbb{C})$. Let us show that the cohomology jump loci are closed algebraic subsets of this affine variety.

Proposition 18.2. *Let X be a compact Kähler manifold. For every $m, k \in \mathbb{Z}$,*

$$\Sigma_m^k(X) \subseteq M_{\mathrm{B}}(X)$$

is a closed algebraic subset.

Proof. It will be useful to have a more invariant description of the affine variety $M_{\mathrm{B}}(X)$. Recall that $\Lambda = H_1(X, \mathbb{Z})$ is a quotient of $\pi_1(X, x_0)$; it is of course the direct sum of a free abelian group of rank $\dim H^1(X, \mathbb{C})$ and a finite abelian group. Now denote by

$$R = \mathbb{C}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{C} \cdot e_\lambda$$

the group ring of Λ; the ring structure is defined by setting $e_\lambda \cdot e_\mu = e_{\lambda+\mu}$. Then

$$M_{\mathrm{B}}(X) \simeq \operatorname{Spec} R$$

is isomorphic to (the complex manifold defined by) the affine variety $\operatorname{Spec} R$. In fact, every character $\rho \in M_{\mathrm{B}}(X)$ gives rise to a maximal ideal

$$\mathfrak{m}_\rho = R \cdot \left\{ e_\lambda - \rho(\lambda) \,\middle|\, \lambda \in \Lambda \right\} \subseteq R,$$

and therefore to a closed point of $\operatorname{Spec} R$. Conversely, a maximal ideal $\mathfrak{m} \subseteq R$ determines a character by taking the composition

$$\pi_1(X, x_0) \to \Lambda \to \operatorname{Aut}(R/\mathfrak{m}),$$

remembering that $\lambda \in \Lambda$ acts on R as left multiplication by e_λ. It is easy to see that the two constructions are inverse to each other.

In order to show that the sets $\Sigma_m^k(X)$ are closed algebraic, we shall construct a bounded complex of free R-modules that computes them. The main point is to have a uniform way of computing the cohomology of $\mathbb{C}_\rho$ for $\rho \in M_{\mathrm{B}}(X)$. One possibility is to use Čech cohomology. Since X is in particular a compact Riemannian manifold, we can (with some work) find an open covering

$$X = U_1 \cup \cdots \cup U_d,$$

with the property that any intersection of open sets in the covering is contractible.[1] For any $\rho \in M_{\mathrm{B}}(X)$, the cohomology groups of the locally constant sheaf $\mathbb{C}_\rho$ are therefore computed by the Čech complex

$$C^\bullet(U_1, \dots, U_d, \mathbb{C}_\rho) = \left[\bigoplus_{i_0} H^0(U_{i_0}, \mathbb{C}_\rho) \to \bigoplus_{i_0 < i_1} H^0(U_{i_0} \cap U_{i_1}, \mathbb{C}_\rho) \to \cdots \right]$$

Note that on each intersection of open sets, the space of sections of $\mathbb{C}_\rho$ is non-canonically isomorphic to $\mathbb{C}$, and so we have a bounded complex of finite-dimensional vector spaces. The advantage is that this way of calculating cohomology works the same way for every $\rho \in M_{\mathrm{B}}(X)$.

To exploit this fact, we define a locally constant sheaf of R-modules $\mathcal{L}_R$ on X that acts much like a "universal object"; concretely,

$$H^0(U, \mathcal{L}_R) = \left\{ s\colon \pi^{-1}(U) \to R \;\middle|\; \begin{array}{l} s(y) \text{ is locally constant, and} \\ s(\lambda y) = e_\lambda^{-1} s(y) \text{ for all } \lambda \in \Lambda \end{array} \right\}.$$

In the quotient $R/\mathfrak{m}_\rho$, the element e_λ^{-1} gets identified with $\rho(\lambda)^{-1}$, and so we have $\mathcal{L}_R \otimes_R R/\mathfrak{m}_\rho \simeq \mathbb{C}_\rho$. Now consider the Čech complex of the locally constant sheaf of R-modules $\mathcal{L}_R$, which is

$$C^\bullet(U_1, \dots, U_d, \mathcal{L}_R) = \left[\bigoplus_{i_0} H^0(U_{i_0}, \mathcal{L}_R) \to \bigoplus_{i_0 < i_1} H^0(U_{i_0} \cap U_{i_1}, \mathcal{L}_R) \to \cdots \right].$$

For the same reason as above, it is a bounded complex of free R-modules; it also has the property that

$$C^\bullet(U_1, \dots, U_d, \mathcal{L}_R) \otimes_R R/\mathfrak{m}_\rho \simeq C^\bullet(U_1, \dots, U_d, \mathbb{C}_\rho)$$

[1] One can take each U_i to be a small geodesic ball; in suitable local coordinates, such sets (and their intersections) are convex, hence contractible. Thanks to Paola Frediani for explaining this.

for every $\rho \in M_{\mathrm{B}}(X)$. The sets $\Sigma_m^k(X)$ are therefore nothing but the cohomology support loci of this complex; in particular, they are closed algebraic subsets of $\operatorname{Spec} R$. $\square$

Note. Each $\Sigma_m^k(X)$ is actually defined over $\mathbb{Q}$. In fact, the proof above works just as well for the ring $R = \mathbb{Q}[\Lambda]$: the complex manifold $M_{\mathrm{B}}(X)$ is the set of $\mathbb{C}$-rational points of the affine variety $\operatorname{Spec} R$; each $\Sigma_m^k(X)$ is the set of $\mathbb{C}$-rational points of an algebraic subset defined by a bounded complex of free R-modules. This observation will be useful later on.

The structure of cohomology jump loci. We shall now prove the first part of Simpson's Theorem 15.2, namely that the cohomology support loci $S_m^q(X, \Omega_X^p)$ are finite unions of translates of subtori in $\operatorname{Pic}^0(X)$. Here it is not necessary to assume that X is projective; we shall therefore present the proof for an arbitrary compact Kähler manifold. The idea, which goes back to the work of Beauville, is to establish the desired result first for the sets $\Sigma_m^k(X)$. In this context, the correct generalization of translates of subtori is the following.

Definition 18.3. A *linear subvariety* of $M_{\mathrm{B}}(X)$ is a subset of the form

$$\rho \cdot \operatorname{im}\Big(f^* \colon M_{\mathrm{B}}(T) \to M_{\mathrm{B}}(X)\Big),$$

where $f \colon X \to T$ is a holomorphic mapping to a compact complex torus, and $\rho \in M_{\mathrm{B}}(X)$ is a character.

The same definition applies to $M_{\mathrm{DR}}(X)$ and $M_{\mathrm{Dol}}(X)$.

Example 18.4. The structure of linear subvarieties is easiest to understand in the model $M_{\mathrm{Dol}}(X)$. Here we just get

$$\operatorname{im}\Big(\operatorname{Pic}^0(T) \times H^0(T, \Omega_T^1) \to \operatorname{Pic}^0(X) \times H^0(X, \Omega_X^1)\Big),$$

and so a linear subvariety is (up to translation by some character) the product of a subtorus of $\operatorname{Pic}^0(X)$ and the corresponding space of holomorphic one-forms.

Note that the Albanese mapping $\operatorname{alb} \colon X \to \operatorname{Alb}(X)$ induces an isomorphism

$$\operatorname{alb}^* \colon M_{\mathrm{B}}\big(\operatorname{Alb}(X)\big) \to M_{\mathrm{B}}^0(X);$$

because every morphism from X to a compact complex torus factors through $\operatorname{Alb}(X)$, we could also define linear subvarieties in terms of surjective mappings from $\operatorname{Alb}(X)$ to compact complex tori.

Here is the main result for today. As far as I can tell, it was first proved in this form by Donu Arapura; the proof below is due to Simpson.

Theorem 18.5. *Every $\Sigma_m^k(X)$ is a finite union of linear subvarieties of $M_{\mathrm{B}}(X)$.*

Before we prove this, let us see how it implies the structure theorem for cohomology support loci in $\operatorname{Pic}^0(X)$. We define the auxiliary sets

$$S_m^{p,q}(X) = \big\{\, (L, \theta) \in M_{\mathrm{Dol}}(X) \;\big|\; \dim H^{p,q}(X, L, \theta) \geq m \,\big\}$$

in the moduli space of Higgs bundles.

Corollary 18.6. *Every $S_m^{p,q}(X)$ is a finite union of linear subvarieties of $M_{\mathrm{Dol}}(X)$.*

Proof. This follows from the Hodge decomposition

$$H^k(X, \mathbb{C}_\rho) \simeq \bigoplus_{p+q=k} H^{p,q}(X, L, \theta).$$

Suppose that Z is an irreducible component of some $S_m^{p,q}(X)$. Let $k = p + q$, and for every $0 \leq i \leq k$, define $m(i)$ as the generic value of $\dim H^{i,k-i}(X, L, \theta)$ on Z.

The image of Z in $M_{\mathrm{B}}(X)$ is an irreducible real-analytic subvariety; with our choice of $m(0), m(1), \dots, m(k)$, it is an irreducible component of

$$\Sigma^k_{m(0)+m(1)+\cdots+m(k)}(X),$$

According to the theorem, every such irreducible component is a linear subvariety of $M_{\mathrm{B}}(X)$; but then Z itself must also be a linear subvariety of $M_{\mathrm{Dol}}(X)$. $\square$

Note that $S^q_m(X, \Omega^p_X) = S^{p,q}_m(X) \cap \mathrm{Pic}^0(X)$, because when $\theta = 0$, we have

$$H^{p,q}(X, L, 0) = H^q(X, \Omega^p_X \otimes L).$$

Since the intersection of a linear subvariety with $\mathrm{Pic}^0(X)$ is a translate of a subtorus, we have proved the following generalization of Theorem 11.1.

Corollary 18.7. *Let X be a compact Kähler manifold. Then every irreducible component of $S^q_m(X, \Omega^p_X)$ is a translate of a subtorus of $\mathrm{Pic}^0(X)$.*

Proof of the theorem. To prove Theorem 18.5, we are going to exploit the different structures of the two moduli spaces $M_{\mathrm{B}}(X)$ and $M_{\mathrm{Dol}}(X)$. The general idea is that any set that is sufficiently nice in at least two of the three spaces

$$M_{\mathrm{DR}}(X) \quad M_{\mathrm{B}}(X) \quad M_{\mathrm{Dol}}(X)$$

has to be a finite union of linear subvarieties. Simpson's paper contains several other instances of this principle.

Let $Z \subseteq \Sigma^k_m(X)$ be an irreducible component; to simplify the argument, we shall assume that $Z \subseteq M^0_{\mathrm{B}}(X)$. By Proposition 18.2, Z is a closed algebraic subvariety of $M^0_{\mathrm{B}}(X)$. On the other hand, we have the decomposition

$$H^k(X, \mathbb{C}_\rho) \simeq \bigoplus_{p+q=k} H^{p,q}(X, L, \theta).$$

Because the cohomology groups on the right-hand side do not change when we multiply θ by a nonzero complex number, it follows that the image of Z in $M^0_{\mathrm{Dol}}(X)$ is stable under the natural $\mathbb{C}^\times$-action

$$\mathbb{C}^\times \times M_{\mathrm{Dol}}(X) \to M_{\mathrm{Dol}}(X), \quad \lambda \cdot (L, \theta) = (L, \lambda\theta).$$

This reduces the problem to the following general result.

Theorem 18.8. *Let $Z_{\mathrm{B}} \subseteq M_{\mathrm{B}}(X)$ be an irreducible analytic subvariety, and denote by Z_{Dol} its image in $M_{\mathrm{Dol}}(X)$. If Z_{Dol} is stable under the $\mathbb{C}^\times$-action, then Z_{B} is a linear subvariety.*

An interesting point is that we are originally interested in subsets of $\mathrm{Pic}^0(X)$; in order to get information about them, we go to the larger space $M^0_{\mathrm{Dol}}(X)$ and exploit the additional $\mathbb{C}^\times$-action there ... which is of course not visible on the subset $\theta = 0$.

For the proof, we choose a smooth point of Z_{B}; without any essential loss of generality, we can assume that this point is the one corresponding to the trivial character. (In fact, this can always be arranged by suitably translating Z_{B}.) Then Z_{Dol} contains the origin $(\mathscr{O}_X, 0)$ in $M^0_{\mathrm{Dol}}(X)$ as a smooth point. From our construction of the two moduli spaces last time, it is clear that the holomorphic tangent space to $M^0_{\mathrm{B}}(X)$ at the origin is $\mathcal{H}^1(X)$, while the one to $M^0_{\mathrm{Dol}}(X)$ is $\mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X)$. They are isomorphic as $\mathbb{R}$-vector spaces, and the isomorphism between them is given by

$$h \colon \mathcal{H}^1(X) \to \mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X), \quad h(\varepsilon) = \left(\frac{\varepsilon^{0,1} - \overline{\varepsilon^{1,0}}}{2}, \frac{\varepsilon^{1,0} + \overline{\varepsilon^{0,1}}}{2} \right).$$

The formula for the inverse is $h^{-1}(\tau, \theta) = (\theta - \bar{\tau}) + (\tau + \bar{\theta})$; note that both h and h^{-1} are only $\mathbb{R}$-linear. Let us denote by J the $\mathbb{R}$-linear operator on $\mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X)$ induced by multiplication by i on $\mathcal{H}^1(X)$. It is easy to see that

$$J(\tau, \theta) = h\Big(i(\theta - \bar{\tau}) + i(\tau + \bar{\theta})\Big) = \big(i\bar{\theta}, -i\bar{\tau}\big),$$

and so this is what the complex structure on $\mathcal{H}^1(X)$ looks like when transported to $\mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X)$.

Now let $W \subseteq \mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X)$ denote the tangent space to Z_{Dol} at the origin; it is of course only an $\mathbb{R}$-linear subspace. Because Z_{B} is a complex submanifold in a neighborhood of the origin, we do know that $h^{-1}(W)$ is a $\mathbb{C}$-linear subspace of $\mathcal{H}^1(X)$; this means that $J(W) \subseteq W$. It is also clear that W is stable under the $\mathbb{C}^\times$-action on the second factor. This puts strong restrictions on what W can be.

Lemma 18.9. *Let $W \subseteq \mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X)$ be an $\mathbb{R}$-linear subspace that is stable under the $\mathbb{C}^\times$-action on $\mathcal{H}^{1,0}(X)$ and satisfies $J(W) \subseteq W$. Then*

$$W = \overline{V} \times V$$

for a $\mathbb{C}$-linear subspace $V \subseteq \mathcal{H}^{1,0}(X)$.

Proof. Let $w = (\tau, \theta) \in W$ be an arbitrary element. By assumption, $(\tau, \lambda\theta) \in W$ for every $\lambda \in \mathbb{C}^\times$; this clearly implies that

$$W = W^{0,1} \times W^{1,0},$$

where $W^{1,0} = W \cap \mathcal{H}^{1,0}(X)$ is a $\mathbb{C}$-linear subspace, and $W^{0,1} = W \cap \mathcal{H}^{0,1}(X)$ is an $\mathbb{R}$-linear subspace. But now the formula for J shows that

$$W^{0,1} = \overline{W^{1,0}},$$

and so it is in fact also $\mathbb{C}$-linear. $\square$

The remainder of the argument consists in showing that Z_{Dol} is in fact equal to the image of this linear subspace in $M^0_{\mathrm{Dol}}(X)$. Recall from the construction of the two moduli spaces that we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{H}^1(X) & \xrightarrow{\;h\;} & \mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X) \\
\downarrow & & \downarrow{\scriptstyle \pi \times \mathrm{id}} \\
M^0_{\mathrm{B}}(X) & \longrightarrow & \mathrm{Pic}^0(X) \times H^0(X, \Omega^1_X);
\end{array}$$

here $\pi \colon \mathcal{H}^{0,1}(X) \to \mathrm{Pic}^0(X)$ is our usual quotient map.

Consider the preimage $(\pi \times \mathrm{id})^{-1}(Z_{\mathrm{Dol}})$. We already know that its tangent space at the point $(0,0)$ is equal to $\overline{V} \times V$; the fact that the subset is closed and preserved by the $\mathbb{C}^\times$-action now implies that the entire subspace $\{0\} \times V$ must be contained in $(\pi \times \mathrm{id})^{-1}(Z_{\mathrm{Dol}})$.

Lemma 18.10. *The subspace $\{0\} \times V$ is contained in $(\pi \times \mathrm{id})^{-1}(Z_{\mathrm{Dol}})$.*

Proof. Take an arbitrary nonzero vector $\theta \in V$. Because $(0, \theta)$ belongs to the tangent space, we can find a sequence of points $(\tau_n, \theta_n) \in (\pi \times \mathrm{id})^{-1}(Z_{\mathrm{Dol}})$ that converges to $(0,0)$ and has the property that

$$\mathbb{C} \cdot (\tau_n, \theta_n) \to \mathbb{C} \cdot (0, \theta).$$

For some choice of $\lambda_n \in \mathbb{C}^\times$, the sequence $\lambda_n \theta_n$ converges to θ; because $(\pi \times \mathrm{id})^{-1}(Z_{\mathrm{Dol}})$ is closed and invariant under the $\mathbb{C}^\times$-action, it must contain $(0, \theta)$. $\square$

We can use the fact that Z_{B} is complex-analytic to show that $(\pi \times \mathrm{id})^{-1}(Z_{\mathrm{Dol}})$ contains its tangent space at the point $(0,0)$.

Lemma 18.11. *The subspace $\overline{V} \times V$ is contained in $(\pi \times \mathrm{id})^{-1}(Z_{\mathrm{Dol}})$.*

Proof. Let $\theta_1, \dots, \theta_r \in V$ be a basis. According to the previous lemma,

$$\left(0, \sum_{j=1}^{r} (a_j + ib_j)\theta_j\right) \in (\pi \times \mathrm{id})^{-1}(Z_{\mathrm{Dol}})$$

for every choice of $a_1, \dots, a_r, b_1, \dots, b_r \in \mathbb{R}$. If we apply h^{-1}, we find that the point

$$\sum_{j=1}^{r} (a_j + ib_j)\theta_j + \sum_{j=1}^{r} (a_j - ib_j)\bar{\theta}_j \in \mathcal{H}^1(X) \tag{18.12}$$

belongs to the preimage of Z_{B}. This preimage is a complex-analytic subset of $\mathcal{H}^1(X)$, and therefore defined by holomorphic functions. Because a holomorphic function on $\mathbb{C}^{2r}$ that vanishes along $\mathbb{R}^{2r}$ has to be identically zero, it follows that (18.12) still belongs to the preimage of Z_{B} even when $a_1, \dots, a_r, b_1, \dots, b_r \in \mathbb{C}$. After relabeling the coefficients, the same is true for

$$\sum_{j=1}^{r} a_j\theta_j + \sum_{j=1}^{r} b_j\bar{\theta}_j.$$

If we transform this back to $\mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X)$, we get

$$\left(\sum_{j=1}^{r} \frac{b_j - \overline{a_j}}{2}\bar{\theta}_j, \sum_{j=1}^{r} \frac{a_j + \overline{b_j}}{2}\theta_j\right) \in (\pi \times \mathrm{id})^{-1}(Z_{\mathrm{Dol}}).$$

But every vector in $\overline{V} \times V$ is of this form for some $a_1, \dots, a_r, b_1, \dots, b_r \in \mathbb{C}$ $\square$

The lemma shows that $\pi(\overline{V}) \times V$ is contained in Z_{Dol}. Both have the same dimension (because $\overline{V} \times V$ is the tangent space), and so this can only happen if

$$Z_{\mathrm{Dol}} = \pi(\overline{V}) \times V.$$

As in the proof of the Theorem 12.4, $T = \pi(\overline{V})$ is therefore a subtorus of $\mathrm{Pic}^0(X)$; and V is isomorphic to the space of holomorphic one-forms on the dual torus. If we define $f\colon X \to \hat{T}$ by composing the Albanese mapping of X with the induced mapping $\mathrm{Alb}(X) \to \hat{T}$, this says exactly that

$$Z_{\mathrm{Dol}} = \mathrm{im}\Big(f^*\colon M_{\mathrm{Dol}}(\hat{T}) \to M_{\mathrm{Dol}}(X)\Big)$$

is a linear subvariety. But then Z_{B} must be a linear subvariety of $M_{\mathrm{B}}(X)$, too.

Exercises.

Exercise 18.1. In class, we only proved Theorem 18.5 for irreducible components of $\Sigma_m^k(X)$ that lie in $M_{\mathrm{B}}^0(X)$. The purpose of this exercise is to prove it in general.

(a) Choose a decomposition $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus \dim H^1(X,\mathbb{C})} \oplus G$, where G is a finite abelian group, and let $\pi\colon Y \to X$ denote the covering space corresponding to the kernel of $\pi_1(X, x_0) \to G$. Show that $M_{\mathrm{B}}(Y)$ is connected.

(b) Show that $\pi^*\colon M_{\mathrm{B}}(X) \to M_{\mathrm{B}}(Y)$ is surjective, and that its kernel is a finite subgroup of $M_{\mathrm{B}}(X)$ that contains exactly one character from each connected component.

(c) Now take a character $\sigma \in M_{\mathrm{B}}(Y)$, and consider the locally constant sheaf $\pi_*\mathbb{C}_\sigma$ on X. Show that the natural G-action decomposes it as

$$\pi_*\mathbb{C}_\sigma \simeq \bigoplus_{\pi^*(\rho)=\sigma} \mathbb{C}_\rho.$$

(d) Deduce from this decomposition that Theorem 18.5 is true for every irreducible component of $\Sigma_m^k(X)$.

Lecture 19

Points of finite order. From now on, we suppose that X is a smooth projective variety. Recall that the cohomology jump loci

$$\Sigma_m^k(X) = \left\{ \rho \in M_{\mathrm{B}}(X) \;\middle|\; \dim H^k(X, \mathbb{C}_\rho) \geq m \right\}$$

are finite unions of linear subvarieties of $M_{\mathrm{B}}(X)$. Because X is projective, $\mathrm{Pic}^0(X)$ is an abelian variety, and so every subtorus of $\mathrm{Pic}^0(X)$ is also an abelian variety. Every linear subvariety is therefore of the form

$$\rho \cdot \mathrm{im}\big(f^* \colon M_{\mathrm{B}}(A) \to M_{\mathrm{B}}(X)\big),$$

where $f \colon X \to A$ is a morphism to an abelian variety, and $\rho \in M_{\mathrm{B}}(X)$ is some character. In fact, the following stronger result is true.

Theorem 19.1. *Every irreducible component of $\Sigma_m^k(X)$ contains a point of finite order.*

This means that $\Sigma_k^k(X)$ is a finite union of subsets of the form

$$\rho \cdot \mathrm{im}\big(f^* \colon M_{\mathrm{B}}(A) \to M_{\mathrm{B}}(X)\big),$$

where $\rho \in M_{\mathrm{B}}(X)$ is a point of finite order. By the same argument as in the proof of Corollary 18.7, we can deduce from this result about $\Sigma_m^k(X)$ the following result about cohomology support loci.

Corollary 19.2. *Let X be a smooth projective variety. Then every irreducible component of $S_m^q(X, \Omega_X^p)$ is a translate of an abelian subvariety of $\mathrm{Pic}^0(X)$ by a point of finite order.*

The de Rham moduli space. To prove Theorem 19.1, we shall exploit the relationship between the two moduli spaces $M_{\mathrm{B}}(X)$ and $M_{\mathrm{DR}}(X)$. We begin with a few remarks about their algebraic structure. Recall that $M_{\mathrm{B}}(X)$ is always an affine algebraic variety, defined over $\mathbb{Q}$. Since X is projective, $\mathrm{Pic}^\tau(X)$ is projective, and

$$M_{\mathrm{Dol}}(X) = \mathrm{Pic}^\tau(X) \times H^0(X, \Omega_X^1)$$

is also a quasi-projective algebraic variety; the algebraic description of the Picard scheme implies that when X is defined over an algebraically closed subfield $k \subseteq \mathbb{C}$, then so are $\mathrm{Pic}^\tau(X)$ and $M_{\mathrm{Dol}}(X)$.

One can show that the same is true for $M_{\mathrm{DR}}(X)$; in fact, there is an algebraic construction for the moduli space of line bundles with flat connection. We are not going to discuss this construction in general; instead, I will show a simple proof for why $M_{\mathrm{DR}}^0(X)$ is quasi-projective.

Proposition 19.3. *Let X be a smooth projective variety. Then $M_{\mathrm{DR}}^0(X)$ is quasi-projective.*

The idea is to embed $M_{\mathrm{DR}}^0(X)$ into a larger space of so-called *flat λ-connections*; this larger space will then turn out to be a vector bundle over $\mathrm{Pic}^0(X)$.

Definition 19.4. Let $\lambda \in \mathbb{C}$ be a complex number. A *λ-connection* on a holomorphic line bundle L is a $\mathbb{C}$-linear morphism $\nabla \colon L \to \Omega_X^1 \otimes L$ that satisfies the following version of the Leibniz rule:

$$\nabla(fs) = f\nabla(s) + \lambda df \otimes s$$

The λ-connection is called *flat* if (after the usual extension) $\nabla \circ \nabla = 0$.

Example 19.5. A (flat) 1-connection on L is a (flat) connection in the usual sense. A 0-connection on L is the same thing as an $\mathscr{O}_X$-linear morphism from L to $\Omega_X^1 \otimes L$, or in other words, a global holomorphic one-form; it is automatically flat.

As in the case of usual connections, we can describe λ-connections on line bundles with trivial first Chern class by operators of the form

$$\lambda\partial + \bar{\partial} + \varepsilon,$$

where $\varepsilon \in \mathcal{H}^1(X)$ is a harmonic one-form. Let me briefly review how this works. The holomorphic line bundle L is the one corresponding to the operator $\bar{\partial} + \varepsilon^{0,1}$. Locally, we have $\varepsilon^{0,1} = \bar{\partial}\varphi$, and so $e^{-\varphi}$ is a nowhere vanishing local holomorphic section of L. In

$$(\lambda\partial + \bar{\partial} + \varepsilon)e^{-\varphi} = (\varepsilon^{1,0} - \lambda\partial\varphi) \otimes e^{-\varphi},$$

the $(1,0)$-form in parentheses is holomorphic; now the λ-connection is defined by setting

$$\nabla(fe^{-\varphi}) = f(\varepsilon^{1,0} - \lambda\partial\varphi) \otimes e^{-\varphi} + \lambda df \otimes e^{-\varphi}$$

for every local holomorphic function f. One can show that these local expressions patch together to give a well-defined $\nabla\colon L \to \Omega_X^1 \otimes L$.

It is also not hard to figure out when $\lambda\partial + \bar{\partial} + \varepsilon$ defines the trivial λ-connection λd on the trivial line bundle $\mathscr{O}_X$. The condition for this is that there should be a nowhere vanishing smooth function $f \in A^0(X)$ such that

$$(\lambda\partial + \bar{\partial} + \varepsilon)f = 0.$$

Now there are two cases:

(1) When $\lambda \neq 0$, we can rewrite this condition as
$$(d + \lambda^{-1}\varepsilon^{1,0} + \varepsilon^{0,1})f = 0;$$
this happens if and only if $\lambda^{-1}\varepsilon^{1,0} + \varepsilon^{0,1}$ has periods in $\mathbb{Z}(1)$.

(2) When $\lambda = 0$, the condition becomes $\varepsilon^{1,0} = 0$ and $(\bar{\partial} + \varepsilon^{0,1})f = 0$; this happens when $\varepsilon^{0,1} - \overline{\varepsilon^{0,1}}$ has periods in $\mathbb{Z}(1)$.

To write this in a more symmetric form, let $\Gamma \subseteq \mathcal{H}^1(X)$ denote the space of all harmonic one-forms that have periods in $\mathbb{Z}(1)$. Then $\lambda\partial + \bar{\partial} + \varepsilon$ defines the trivial λ-connection if and only if $\varepsilon = \lambda\gamma^{1,0} + \gamma^{0,1}$ for some $\gamma \in \Gamma$.

Definition 19.6. We denote by $\tilde{M}^0_{\mathrm{DR}}(X)$ the space of all pairs (L, ∇), where $L \in \mathrm{Pic}^0(X)$ and ∇ is a flat λ-connection on L for some $\lambda \in \mathbb{C}$.

The discussion above leads to the following analytic description of the moduli space. We have an embedding

$$\Gamma \times \mathbb{C} \to \mathcal{H}^1(X) \times \mathbb{C}, \quad (\gamma, \lambda) \mapsto (\lambda\gamma^{1,0} + \gamma^{0,1}, \lambda),$$

and $\tilde{M}^0_{\mathrm{DR}}(X)$ is the quotient of $\mathcal{H}^1(X) \times \mathbb{C}$ by the equivalence relation

$$(\varepsilon_1, \lambda_1) \sim (\varepsilon_2, \lambda_2) \quad \Longleftrightarrow \quad \lambda_1 = \lambda_2 \text{ and } \varepsilon_1 = \varepsilon_2 + (\lambda\gamma^{1,0} + \gamma^{0,1}) \text{ for some } \gamma \in \Gamma.$$

The advantage is that $\tilde{M}^0_{\mathrm{DR}}(X)$ has a much simpler structure that $M^0_{\mathrm{DR}}(X)$.

Lemma 19.7. *The projection $\tilde{M}^0_{\mathrm{DR}}(X) \to \mathrm{Pic}^0(X)$ is a holomorphic vector bundle.*

Proof. It suffices to prove this after pulling back to the universal covering space $\mathcal{H}^{0,1}(X)$ of $\mathrm{Pic}^0(X)$. With the help of the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^1(X) \times \mathbb{C} & \xrightarrow{p} & \mathcal{H}^{0,1}(X) \\ \downarrow & & \downarrow \\ \tilde{M}^0_{\mathrm{DR}}(X) & \longrightarrow & \mathrm{Pic}^0(X), \end{array}$$

one can easily show that

$$\mathcal{H}^1(X) \times \mathbb{C} \simeq \tilde{M}^0_{\mathrm{DR}}(X) \times_{\mathrm{Pic}^0(X)} \mathcal{H}^{0,1}(X).$$

Now the projection p takes (ε, λ) to $\varepsilon^{0,1}$, and so the pullback is isomorphic to the trivial bundle with fiber $H^0(X, \Omega_X^1) \times \mathbb{C}$. $\square$

When X is a smooth projective variety, $\operatorname{Pic}^0(X)$ is an abelian variety; according to Serre's theorem, every holomorphic vector bundle is therefore automatically algebraic. It follows that $\tilde{M}_{\mathrm{DR}}^0(X)$ is a quasi-projective algebraic variety. We also have a morphism of vector bundles

$$\tilde{M}_{\mathrm{DR}}^0(X) \to \operatorname{Pic}^0(X) \times \mathbb{C}$$

that takes a flat λ-connection (L, ∇) to the pair (L, λ); this morphism is also automatically algebraic. If we denote by $\lambda \colon \tilde{M}_{\mathrm{DR}}^0(X) \to \mathbb{C}$ the induced morphism, then

$$M_{\mathrm{DR}}^0(X) \simeq \lambda^{-1}(1),$$

because a flat 1-connection is the same thing as a flat connection. This shows that $M_{\mathrm{DR}}^0(X)$ is a quasi-projective algebraic variety, too.

In fact, we can do slightly better and show that when X is defined over an algebraically closed subfield $k \subseteq \mathbb{C}$, the same is true for $\tilde{M}_{\mathrm{DR}}^0(X)$ and $M_{\mathrm{DR}}^0(X)$. To do that, let $\mathscr{E}_X$ denote the coherent sheaf of holomorphic sections of the vector bundle $\tilde{M}_{\mathrm{DR}}^0(X) \to \operatorname{Pic}^0(X)$. By construction, it sits in an exact sequence

$$0 \to H^0(X, \Omega_X^1) \otimes \mathscr{O}_{\operatorname{Pic}^0(X)} \to \mathscr{E}_X \to \mathscr{O}_{\operatorname{Pic}^0(X)} \to 0. \tag{19.8}$$

From our description above, we can easily determine the class of this extension.

Lemma 19.9. *The extension class of* (19.8) *is*

$$-\operatorname{id} \in \operatorname{Hom}\bigl(H^0(X, \Omega_X^1), H^0(X, \Omega_X^1)\bigr) \simeq \operatorname{Hom}\bigl(H^0(X, \Omega_X^1), \mathbb{C}\bigr) \otimes H^0(X, \Omega_X^1).$$

Proof. The extension class is an element of

$$\begin{aligned} \operatorname{Ext}^1\bigl(\mathscr{O}_{\operatorname{Pic}^0(X)}, \mathscr{O}_{\operatorname{Pic}^0(X)} \otimes H^0(X, \Omega_X^1)\bigr) &\simeq H^1\bigl(\operatorname{Pic}^0(X), \mathscr{O}_{\operatorname{Pic}^0(X)}\bigr) \otimes H^0(X, \Omega_X^1) \\ &\simeq \operatorname{Hom}\bigl(H^0(X, \Omega_X^1), \mathbb{C}\bigr) \otimes H^0(X, \Omega_X^1). \end{aligned}$$

The reason for the second isomorphism is that $H^1\bigl(\operatorname{Pic}^0(X), \mathscr{O}_{\operatorname{Pic}^0(X)}\bigr)$ is the tangent space to $\operatorname{Pic}^0\bigl(\operatorname{Pic}^0(X)\bigr) \simeq \operatorname{Alb}(X)$. To compute the extension class, we have to lift the constant section 1 of $\mathscr{O}_{\operatorname{Pic}^0(X)}$ to a smooth section of $\mathscr{E}_X$, apply the $\bar{\partial}$-operator to it to obtain a $(0,1)$-form with coefficients in $H^0(X, \Omega_X^1)$, and then take its cohomology class.

The lifting comes from the fact that every $L \in \operatorname{Pic}^0(X)$ has a canonical unitary connection: if L is given by the operator $\bar{\partial} + \tau$, this connection is $d + \tau - \bar{\tau}$. To compute the necessary derivative, we can work on the universal covering space $\mathcal{H}^{0,1}(X)$. Here the lifting is

$$s \colon \mathcal{H}^{0,1}(X) \to \mathcal{H}^1(X) \times \mathbb{C}, \quad \tau \mapsto (\tau - \bar{\tau}, 1).$$

If we let $\tau_1, \dots, \tau_g \in \mathcal{H}^{0,1}(X)$ be a basis, and denote by $z_1, \dots, z_g$ the dual basis, then

$$s(z_1, \dots, z_g) = \left(\sum_{j=1}^g \bigl(z_j \tau_j - \bar{z}_j \bar{\tau}_j\bigr), 1 \right) \quad \text{and therefore} \quad \bar{\partial} s = -\sum_{j=1}^g d\bar{z}_j \otimes \bar{\tau}_j.$$

Now $\bar{\tau}_1, \dots, \bar{\tau}_g$ is a basis of $H^0(X, \Omega_X^1)$, and $\bar{z}_1, \dots, \bar{z}_g$ is the dual basis; under the isomorphism above, this element therefore corresponds to

$$-\operatorname{id} \in \operatorname{Hom}\bigl(H^0(X, \Omega_X^1), \mathbb{C}\bigr) \otimes H^0(X, \Omega_X^1). \qquad \square$$

This formula for the extension class makes it clear that when X, and therefore $\operatorname{Pic}^0(X)$, are defined over k, the same is true for the exact sequence in (19.8). But this means exactly that $\tilde{M}_{\mathrm{DR}}^0(X)$ and $M_{\mathrm{DR}}^0(X) = \lambda^{-1}(1)$ are also defined over k.

Exercises. The algebraic construction of the space $M_{\mathrm{DR}}(X)$ is based on a different interpretation of connections, due to Grothendieck. Let X be a smooth algebraic variety. Let $\mathcal{I}_\Delta$ denote the ideal sheaf of the diagonal in $X \times X$, and consider the closed subscheme $X^{(1)} \subseteq X \times X$ defined by $\mathcal{I}_\Delta^2$. Denote by p_1 and p_2 the two projections from $X^{(1)}$ to X.

Exercise 19.1. Show that a flat connection on a line bundle L is the same thing as an isomorphism $p_1^* L \simeq p_2^* L$ whose restriction to $X \subseteq X^{(1)}$ is the identity.

We know that $M_{\mathrm{DR}}\big(\mathrm{Alb}(X)\big) \simeq M_{\mathrm{DR}}^0(X)$; when X is smooth projective, we are therefore dealing with the moduli space of line bundles with flat connection on the abelian variety $\mathrm{Alb}(X)$.

Exercise 19.2. Let A be an abelian variety, and let $L \in \mathrm{Pic}^0(A)$ be a translation invariant line bundle on A. Show that a flat connection on L is the same thing as a splitting of the short exact sequence

$$0 \to \mathfrak{m}L/\mathfrak{m}^2 L \to L/\mathfrak{m}^2 L \to L/\mathfrak{m}L \to 0,$$

where $\mathfrak{m} \subseteq \mathscr{O}_A$ stands for the ideal sheaf of the point $0 \in A$. (Hint: Use the diagram

$$\begin{tikzcd}
 & A \times A \arrow[d, "f"] & \\
A \arrow[ur, bend left, "{(\mathrm{id},0)}"] \arrow[r, "\Delta"] & A \times A \arrow[r, "p_2"] \arrow[d, "p_1"] & A \\
 & A &
\end{tikzcd}$$

where $f(a,b) = (a, a+b)$, and the fact that $m^* L \simeq p_1^* L \otimes p_2^* L$ on $A \times A$.)

Lecture 20

Reduction to number fields. The proof of Theorem 19.1 will be carried out in two steps:

(1) We reduce the problem to the case where everything is defined over $\bar{\mathbb{Q}}$.
(2) In that case, we use a result from transendence theory to prove that every irreducible component contains a points of finite order.

Let us begin with the first step. Here the main point is that $M_{\mathrm{B}}(X)$, as well as the cohomology jump loci $\Sigma_m^k(X)$, only depend on the underlying topological space of X. This leaves us free to deform the complex structure on X – basically by varying the coefficients of the equations for X in a projective embedding – to produce a smooth projective variety, with the same underlying topological space, that is defined over $\bar{\mathbb{Q}}$.

In general, one can try to reduce questions about arbitrary complex algebraic varieties to those that are defined over $\bar{\mathbb{Q}}$ with the help of a technique called *spreading out*. We observe that X is (isomorphic to) a subvariety of projective space $\mathbb{P}^N_{\mathbb{C}}$, and therefore defined by finitely many homogeneous polynomial equations. By taking the coefficients of all those equations, we obtain a finitely generated subfield $k \subseteq \mathbb{C}$; clearly X is defined over k. Of course, k is the fraction field of a finitely generated $\mathbb{Q}$-algebra R (= the subalgebra of $\mathbb{C}$ generated by the coefficients of the equations), and therefore the function field of the affine algebraic variety $\operatorname{Spec} R$ over $\mathbb{Q}$. By construction, the homogeneous polynomials defining X have coefficients in R, and therefore define a closed subscheme of $\mathbb{P}^N_R$. After throwing away certain bad points (corresponding to those values of the coefficients where the subscheme defined by the equations has the wrong dimension, is not smooth, and so on), we obtain a smooth affine variety B with function field k, a smooth projective morphism

$$f\colon \mathscr{X} \to B$$

defined over $\mathbb{Q}$, and a point $b_0 \in B(\mathbb{C})$, such that $f^{-1}(b_0)$ is isomorphic to X. Because f is smooth, all fibers have the same underlying topological space. Now the fiber over any point $b \in B(\bar{\mathbb{Q}})$ is a smooth projective variety defined over $\bar{\mathbb{Q}}$.

Of course, we can apply the construction more generally to a finite collection of quasi-projective varieties; and, by considering graphs, also to morphisms between them. In our situation, we shall choose a finitely generated subfield $k \subseteq \mathbb{C}$ that contains the coefficients of the defining equations for the following objects:

(i) The smooth projective variety X.
(ii) The Abelian varieties $\operatorname{Pic}^0(X)$ and $\operatorname{Alb}(X)$.
(iii) The Albanese morphism $\operatorname{alb}\colon X \to \operatorname{Alb}(X)$.
(iv) The moduli space $M_{\mathrm{DR}}(X)$.
(v) The images in $M_{\mathrm{DR}}(X)$ of all the cohomology jump loci $\Sigma_m^k(X)$.

Note that the fifth condition makes sense because the image of $\Sigma_m^k(X)$ in $M_{\mathrm{DR}}(X)$ is a finite union of linear subvarieties, and therefore algebraic. (In fact, only this last condition is essential, because we know from the algebraic construction of $\operatorname{Pic}^0(X)$ that if X and the base point $x_0 \in X$ are defined over $\bar{\mathbb{Q}}$, then so are the other three.) We can then construct a morphism $f\colon \mathscr{X} \to B$ as above, with the additional property that over every point $b \in B(\bar{\mathbb{Q}})$, all the objects in the list above are defined over $\bar{\mathbb{Q}}$. Because $M_{\mathrm{B}}(X)$ and $\Sigma_m^k(X)$ are the same for every fiber of f, this reduces Theorem 19.1 to the case when everything is defined over $\bar{\mathbb{Q}}$.

The criterion of Schneider-Lang. Next, let me introduce the result from transcendence theory that will be used in the proof of Theorem 19.1. Recall that we have an isomorphism of complex Lie groups

$$\Phi\colon M_{\mathrm{DR}}(X) \to M_{\mathrm{B}}(X),$$

that takes the line bundle with connection defined by $d + \varepsilon$ to the character

$$\rho\colon \pi_1(X, x_0) \to \mathbb{C}^\times, \quad \rho(\gamma) = \exp\left(-\int_\gamma \varepsilon\right).$$

Because of the reductions from above, we may assume that X is a smooth projective variety defined over $\bar{\mathbb{Q}}$. We can also arrange that the abelian varieties $\operatorname{Pic}^0(X)$ and $\operatorname{Alb}(X)$, the Albanese morphism $\operatorname{alb}\colon X \to \operatorname{Alb}(X)$, the moduli space $M_{\mathrm{DR}}(X)$, and the sets $\Phi^{-1}\big(\Sigma_m^k(X)\big)$, are defined over $\bar{\mathbb{Q}}$. The same is then true for every irreducible component.

The main tool in the proof is the *criterion of Schneider-Lang.* It is a generalization of the classical result, due to Lindemann and Weierstraß, that at least one of the two numbers α and e^α is always transcendental. This type of result is relevant in our situation because, as you can see from the formula above, the mapping Φ involves an exponential function. Here is the precise statement.

Theorem 20.1 (Schneider-Lang)**.** *Let G be a connected and commutative algebraic group defined over $\bar{\mathbb{Q}}$, and let $\psi\colon \mathbb{C}^n \to G(\mathbb{C})$ be an analytic homomorphism whose differential $d\psi(0)$ is defined over $\bar{\mathbb{Q}}$. Let $\Gamma \subseteq \mathbb{C}^n$ be a subgroup that contains at least n linearly independent elements. If $\psi(\Gamma) \subseteq G(\bar{\mathbb{Q}})$, then the dimension of the Zariski closure of $\psi(\mathbb{C}^n)$ is at most n.*

Example 20.2. To get a feeling for this theorem, let us see how it implies the classical result that α and e^α cannot both be algebraic numbers. Let $G = \mathbb{C} \times \mathbb{C}^\times$, with the obvious group structure, and consider the homomorphism

$$\psi\colon \mathbb{C} \to \mathbb{C} \times \mathbb{C}^\times, \quad \psi(z) = (z, e^{\alpha z}).$$

For the subgroup $\Gamma \subseteq \mathbb{C}$, we take $\Gamma = \mathbb{Z}$. The differential $d\psi(0)$ is equal to $(1, \alpha)$, and $\psi(k) = (k, e^{k\alpha})$ for $k \in \mathbb{Z}$. If $\alpha \in \bar{\mathbb{Q}}$ and $e^\alpha \in \bar{\mathbb{Q}}$, then all the assumptions are satisfied, and we conclude that $\psi(\mathbb{C})$ is contained in an algebraic curve. In other words, there is a nontrivial Laurent polynomial $f(z,t) \in \mathbb{C}[z, t, t^{-1}]$ such that

$$f(z, e^{\alpha z}) = 0 \quad \text{for every } z \in \mathbb{C}.$$

By evaluating at the points $z = 2\pi i/\alpha \cdot k$, for $k \in \mathbb{Z}$, we find that $f(z, 1)$ has infinitely many zeros, and is therefore identically zero. This means that $(t-1)$ divides $f(z,t)$. From this, we easily obtain a contradiction.

Points over number fields. Let A be an abelian variety defined over $\bar{\mathbb{Q}}$, and suppose that $\rho \in M_{\mathrm{B}}(A)$ is a $\bar{\mathbb{Q}}$-rational point, with the property that the corresponding point $(L, \nabla) \in M_{\mathrm{DR}}(A)$ is also defined over $\bar{\mathbb{Q}}$. Concretely, this means that character ρ takes values in $\bar{\mathbb{Q}}$; and that both L and ∇ are defined over $\bar{\mathbb{Q}}$. In the application to Simpson's theorem below, A will be a subvariety of the Albanese variety $\operatorname{Alb}(X)$.

Proposition 20.3. *In this situation, $\rho \in M_{\mathrm{B}}(A)$ is a point of finite order.*

To apply the criterion of Schneider-Lang, we let G be the group consisting of all pairs (a, φ), where $a \in A$, and $\varphi\colon t_a^* L \to L$ is an isomorphism. (Recall that the line bundle $L \in \operatorname{Pic}^0(A)$ is translation invariant.) Since $L \in \operatorname{Pic}^0(A)$ is defined over $\bar{\mathbb{Q}}$, this is clearly a commutative algebraic group defined over $\bar{\mathbb{Q}}$. Because $\operatorname{Aut}(L) \simeq \mathbb{G}_m$, it sits in an exact sequence

$$1 \to \mathbb{G}_m \to G \to A \to 0.$$

As a complex manifold, $A = V/\Gamma$, where $V = T_0 A$ and $\Gamma = \pi_1(A, 0)$. The connection ∇ defines an analytic homomorphism

$$\psi\colon V \to G(\mathbb{C})$$

that lifts the quotient mapping $\pi\colon V \to A$. To see why, note that $\pi^* L$ has a nowhere vanishing holomorphic section s with $(\pi^*\nabla)(s) = 0$; we can normalize it by specifying the value $s(0)$. We then obtain

$$\psi\colon V \to G, \quad \psi(v) = \big(\pi(v), s(v)/s(0)\big).$$

Since (L, ∇) is defined over $\bar{\mathbb{Q}}$, the differential $d\psi(0)$ is defined over $\bar{\mathbb{Q}}$. In fact, we have $\nabla s(0) = \omega \otimes s(0)$ for an element $\omega \in T_0^* A$ that is defined over $\bar{\mathbb{Q}}$, and then

$$d\psi(0)\colon T_0 A \to T_0 A \times \mathbb{C}, \quad v \mapsto \big(v, \omega(v)\big).$$

Finally, we observe that the restriction of ψ to the lattice Γ gives a homomorphism

$$\Gamma \to A \times \mathbb{C}^\times, \quad \gamma \mapsto \big(0, s(\gamma)/s(0)\big);$$

this is of course nothing but the character ρ that we started from. It takes values in $\bar{\mathbb{Q}}$, because the corresponding point of $M_{\mathrm{B}}(A)$ is defined over $\bar{\mathbb{Q}}$. All the assumptions in the criterion of Schneider-Lang are therefore satisfied.

The conclusion is that $\psi(V)$ is contained in an algebraic subgroup $G' \subseteq G$ of dimension at most n. In fact, $\dim G' = n$ because the projection $G' \to A$ must be surjective; therefore $G' \to A$ is finite, and so the intersection $G'(\mathbb{C}) \cap \mathbb{C}^\times$ is a finite group. But this means exactly that the character $\rho\colon \Gamma \to \mathbb{C}^\times$ takes values in a finite group, and is therefore a point of finite order.

The proof of Simpson's theorem. We can now do the general case.

Proposition 20.4. *In the situation above, let $Z_{\mathrm{B}} \subseteq M_{\mathrm{B}}(X)$ be a linear subvariety. If Z_{B} and Z_{DR} are both defined over $\bar{\mathbb{Q}}$, then Z_{B} contains a point of finite order.*

Proof. Because every connected component of $M_{\mathrm{B}}(X)$ contains points of finite order, we can assume without loss of generality that $Z_{\mathrm{B}} \subseteq M_{\mathrm{B}}^0(X)$. Moreover, we can replace X by the abelian variety $\mathrm{Alb}(X)$, because

$$M_{\mathrm{B}}\big(\mathrm{Alb}(X)\big) \simeq M_{\mathrm{B}}^0(X).$$

By assumption, there is a morphism f from $\mathrm{Alb}(X)$ to an abelian variety such that Z_{B} is a translate of $\operatorname{im} f^*$ by a point $\rho \in M_{\mathrm{B}}\big(\mathrm{Alb}(X)\big)$. Note that the abelian variety and the morphism are uniquely determined by Z_{DR}, and so they are defined over $\bar{\mathbb{Q}}$ as well. Now let $A \subseteq \ker f$ denote the connected component of the kernel containing the unit element; it is an abelian variety defined over $\bar{\mathbb{Q}}$. Restriction to A defines a surjective morphism

$$i^*\colon M_{\mathrm{B}}\big(\mathrm{Alb}(X)\big) \to M_{\mathrm{B}}(A),$$

and to prove that Z_{B} contains a point of finite order, it is enough to show that its image under i^* does. Now every character in the image of f^* is sent to the trivial character by i^*, and so the image of Z_{B} is equal to the point $i^*(\rho) \in M_{\mathrm{B}}(A)$. This point, as well as its image in $M_{\mathrm{DR}}(A)$, are both defined over $\bar{\mathbb{Q}}$. We can therefore apply Proposition 20.3 to conclude that it must be of finite order. But then Z_{B} also contains a point of finite order. □

Together with the reductions above, this is enough to prove Theorem 19.1.

Exercises.

Exercise 20.1. Let X be a smooth projective variety defined over $\bar{\mathbb{Q}}$. Show that if $(L, \nabla) \in M_{\mathrm{DR}}^0(X)$ is a $\bar{\mathbb{Q}}$-rational point if and only if the line bundle L and the connection ∇ are defined over $\bar{\mathbb{Q}}$.

Lecture 21

Note. The second half of the course, which begins today, is devoted to an algebraic treatment of the generic vanishing theorem on a smooth projective variety X. We will only use two results from the first half of the course:

(a) The algebraic definition of $\operatorname{Pic}^0(X)$ and $\operatorname{Alb}(X)$, and some results about line bundles on abelian varieties.

(b) The structure theorem for cohomology support loci by Green-Lazarsfeld and Simpson: every irreducible component of

$$S^i(X) = \big\{\, L \in \operatorname{Pic}^0(X) \,\big|\, H^i(X, L) \neq 0 \,\big\}$$

is a translate of an abelian subvariety of $\operatorname{Pic}^0(X)$ by a point of finite order.

If you would like to go back to those results, the notes from Lecture 15 are a good place to start.

Hacon's proof of the generic vanishing theorem. Several years after the original papers of Green and Lazarsfeld had appeared, Christopher Hacon discovered another proof of Theorem 6.6 in the case when X is a smooth projective complex algebraic variety. His proof uses tools from algebraic geometry – most notably, vanishing theorems and the derived category – and provides a very different view of the original result.

Let me begin by restating the generic vanishing theorem in the form Hacon uses. Let X be a smooth projective complex algebraic variety, say of dimension n. According to Theorem 6.6, we then have

$$\operatorname{codim}\big\{\, L \in \operatorname{Pic}^0(X) \,\big|\, H^i(X, L) \neq 0 \,\big\} \geq \dim \operatorname{alb}(X) - i.$$

By Serre duality, $H^i(X, \omega_X \otimes L)$ is dual to $H^{n-i}(X, L^{-1})$; we thus arrive at the following equivalent formulation of the generic vanishing theorem.

Theorem 21.1. *Let X be a smooth projective complex algebraic variety. Then*

$$\operatorname{codim}\big\{\, L \in \operatorname{Pic}^0(X) \,\big|\, H^i(X, \omega_X \otimes L) \neq 0 \,\big\} \geq i - \big(\dim X - \dim \operatorname{alb}(X)\big)$$

for every integer $i \geq 0$.

Note that the term in parentheses is the dimension of the general fiber of the Albanese mapping; we shall see later where this comes from.

In his proof, Hacon uses the following three tools:

(1) Derived categories, more precisely, the so-called *bounded derived category* $\mathrm{D}^b_{coh}(\mathscr{O}_X)$; here X can be any algebraic variety. For now, you can think of the objects of this category as being bounded complexes of coherent sheaves on X; for practical reasons, a more complicated definition is better. The derived category makes it very easy to work with derived functors such as $\mathbf{R}f_*$, and many useful results such as the projection formula, Grothendieck duality, or the base change theorem, work best in the derived category.

(2) The *Fourier-Mukai transform*, discovered by Shigeru Mukai. This is an equivalence $\mathbf{R}\Phi_P \colon \mathrm{D}^b_{coh}(\mathscr{O}_A) \to \mathrm{D}^b_{coh}(\mathscr{O}_{\hat{A}})$ between the derived category of an abelian variety A and that of the dual abelian variety $\hat{A} = \operatorname{Pic}^0(A)$. If $\mathscr{F}$ is a coherent sheaf, then the complex $\mathbf{R}\Phi_P(\mathscr{F})$ contains the information about all the cohomology groups

$$H^i(A, \mathscr{F} \otimes L), \quad \text{for } L \in \operatorname{Pic}^0(A).$$

More precisely, if we tensor $\mathbf{R}\Phi_P(\mathscr{F})$ by the structure sheaf of a point $L \in \operatorname{Pic}^0(A)$, in the derived category, then the cohomology of the resulting complex of vector spaces computes $H^i(A, \mathscr{F} \otimes L)$. The Fourier-Mukai transform is another reason for using the derived category.

(3) Results by Kollár about higher direct images of dualizing sheaves. Kollár showed that for a surjective projective morphism $f\colon X \to Y$ with X smooth, the higher direct image sheaves $R^\ell f_*\omega_X$ have very similar properties to ω_X itself: they are torsion-free, and satisfy a Kodaira-type vanishing theorem. Moreover, the complex $\mathbf{R}f_*\omega_X$ splits, in the derived category, meaning that in $\mathrm{D}^b_{coh}(\mathscr{O}_Y)$, one has
$$\mathbf{R}f_*\omega_X \simeq \bigoplus_\ell R^\ell f_*\omega_X[-\ell].$$
You can think of this as being a stronger form of E_2-degeneration of the Leray spectral sequence. Kollár's theorem is the reason for restating the generic vanishing theorem in terms of ω_X.

We shall discuss all of these in more detail later on, as a preparation for understanding Hacon's proof. In particular, we shall spend some time on reviewing derived categories and derived functors, as well as the main technical results such as Grothendieck duality and base change.

The derived category. Derived categories were introduced to have a better foundation for the theory of derived functors. When we calculate derived functors such as Tor or Ext, we typically find a (locally free, or flat, or injective) resolution of our given module/sheaf, apply the functor in question to each term of the resolution, and then take cohomology. The main idea behind the derived category is to keep not just the cohomology modules/sheaves, but the complexes themselves. Because the same module/sheaf can be resolved in many different ways, keeping the complex only makes sense if we declare different complexes obtained in this way to be isomorphic. This leads to the notion of a *quasi-isomorphism*: a morphism between two complexes that induces isomorphisms on cohomology.

Example 21.2. Consider the case of modules over a ring. Every module M has a (typically infinite) free resolution
$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$
and in the derived category, we want to consider the complex $F_\bullet$ as being isomorphic to M. If $G_\bullet$ is another free resolution of M, then a basic result in homological algebra says that there is a morphism of complexes $f\colon F_\bullet \to G_\bullet$ making the diagram
$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "d"] \arrow[d, "f"] & F_1 \arrow[r, "d"] \arrow[d, "f"] & F_0 \arrow[r] \arrow[d, "f"] & M \arrow[d, "\mathrm{id}"] \\
\cdots \arrow[r] & G_2 \arrow[r, "d"] & G_1 \arrow[r, "d"] & G_0 \arrow[r] & M
\end{tikzcd}$$
commute. This morphism is only unique up to homotopy: for any other choice $f'\colon F_\bullet \to G_\bullet$, there are homomorphisms $s\colon F_n \to G_{n+1}$ such that $f' - f = ds + sd$.
$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, dashed] & F_1 \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, dashed, "s"'] & F_0 \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "s"'] & M \arrow[d, "\mathrm{id}"] \\
\cdots \arrow[r] & G_2 \arrow[r, "d"] & G_1 \arrow[r, "d"] & G_0 \arrow[r] & M
\end{tikzcd}$$
If we want to consider M, $F_\bullet$, and $G_\bullet$ as being isomorphic to each other, the two liftings of $\mathrm{id}\colon M \to M$ should be equal, and so we are forced to consider morphisms of complexes up to homotopy.

Example 21.3. In other cases, say for computing Ext, we might want to replace M by an injective resolution of the form
$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots,$$

Now an injective resolution and a free resolution do not have much in common; the only thing we can say is that we have a morphism of complexes

$$\begin{tikzcd} \cdots \arrow[r] & F_2 \arrow[r, "d"] \arrow[d] & F_1 \arrow[r, "d"] \arrow[d] & F_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & I^0 \arrow[r, "d"] & I^1 \arrow[r, "d"] & I^2 \arrow[r] & \cdots \end{tikzcd}$$

that is an isomorphism on the level of cohomology—being resolutions of M, both complexes have cohomology only in degree zero. If we want both complexes to be isomorphic as objects of the derived category, we need to make sure that such quasi-isomorphisms have inverses.

Quasi-isomorphisms also arise naturally if we consider resolutions of complexes.

Example 21.4. An injective resolution of a complex $M^\bullet$ of modules is a complex $I^\bullet$ of injective modules, and a morphism of complexes $M^\bullet \to I^\bullet$ that induces isomorphisms on cohomology. This generalizes the usual definition for a single module to complexes.

Unfortunately, not every quasi-isomorphism has an inverse. The following example (in the category of $\mathbb{Z}$-modules) shows one way in which this can happen.

Example 21.5. In the category of $\mathbb{Z}$-modules, the morphism

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

is a quasi-isomorphism; but it clearly has no inverse, not even up to homotopy, because there are no nontrivial homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$.

Let me now explain the classical construction of the derived category. Let $\mathfrak{A}$ be an arbitrary abelian category (such as modules over a ring, or coherent sheaves on a scheme). Depending on what kind of complexes we want to consider, there are several derived categories: the unbounded derived category $\mathrm{D}(\mathfrak{A})$, whose objects are arbitrary complexes of objects in $\mathfrak{A}$; the categories $\mathrm{D}^+(\mathfrak{A})$ and $\mathrm{D}^-(\mathfrak{A})$, whose objects are semi-infinite complexes that are allowed to be infinite in the positive respectively negative direction; and finally the bounded derived category $\mathrm{D}^b(\mathfrak{A})$, whose objects are bounded complexes of objects in $\mathfrak{A}$. All of these categories are constructed in two stages; we explain this in the case of $\mathrm{D}^b(\mathfrak{A})$.

(1) Starting from the category of bounded complexes $\mathrm{K}^b(\mathfrak{A})$, form the so-called homotopy category $\mathrm{H}^b(\mathfrak{A})$. It has exactly the same objects, but the morphisms between two complexes are taken up to homotopy; in other words,
$$\mathrm{Hom}_{\mathrm{H}^b(\mathfrak{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{K}^b(\mathfrak{A})}(A^\bullet, B^\bullet) \big/ \mathrm{Hom}^0_{\mathrm{K}^b(\mathfrak{A})}(A^\bullet, B^\bullet),$$
where $\mathrm{Hom}^0_{\mathrm{K}^b(\mathfrak{A})}(A^\bullet, B^\bullet)$ denotes the subgroup of those morphisms that are homotopic to zero.

(2) Now form the derived category $\mathrm{D}^b(\mathfrak{A})$ by inverting quasi-isomorphisms; this can be done by a formal construction similar to the passage from $\mathbb{Z}$ to $\mathbb{Q}$. That is to say, in $\mathrm{D}^b(\mathfrak{A})$, a morphism between two complexes $A^\bullet$ and $B^\bullet$ is represented by a fraction f/h, which stands for the diagram

$$\begin{tikzcd} & C^\bullet \arrow[dl, "h"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $f\colon C^\bullet \to B^\bullet$ is a morphism of complexes and $h\colon C^\bullet \to A^\bullet$ is a quasi-isomorphism. As with ordinary fractions, there is an equivalence relation that we shall not dwell on; it is also not entirely trivial to show that the composition of two morphisms is again a morphism.

In other words, the objects of the derived category are still just complexes; but the set of morphisms between two complexes has become more complicated (especially because a morphism may involve an additional complex).

Example 21.6. For us, the most interesting case is when the abelian category is $\mathrm{Coh}(X)$, the category of coherent sheaves on a scheme X. By applying the above construction, we get the bounded derived category $\mathrm{D}^b\big(\mathrm{Coh}(X)\big)$; once again, the objects of this category are just bounded complexes of coherent sheaves. For practical purposes, a broader definition of the derived category is more useful. Inside the unbounded derived category $\mathrm{D}(\mathscr{O}_X)$ of complexes of sheaves of $\mathscr{O}_X$-modules, consider the full subcategory $\mathrm{D}^b_{coh}(\mathscr{O}_X)$; by definition, a complex

$$\cdots \to \mathscr{F}^{-1} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \mathscr{F}^2 \to \cdots$$

belongs to this subcategory if its cohomology sheaves $\mathcal{H}^i(\mathscr{F}^\bullet)$ are coherent, and nonzero for only finitely many values of i. Clearly,

$$\mathrm{D}^b\big(\mathrm{Coh}(X)\big) \subseteq \mathrm{D}^b_{coh}(\mathscr{O}_X),$$

and under some mild assumptions on X, this inclusion is actually an equivalence of categories. The larger category has the advantage of being more flexible: for example, an injective resolution of a coherent sheaf is an object of $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ but not of $\mathrm{D}^b\big(\mathrm{Coh}(X)\big)$.

Morphisms in the derived category. The definition of the derived category leads to several questions. The first one is whether one can describe the space of morphisms between two complexes in more basic terms. At least in the case of complexes with only one nonzero cohomology object, this is possible.

We first define the following *shift functor*. Given a complex $A^\bullet \in \mathrm{K}(\mathfrak{A})$ and an integer $n \in \mathbb{Z}$, we obtain a new complex $A^\bullet[n]$ by setting

$$A^\bullet[n] = A^{\bullet+n};$$

we also multiply all the differentials in the original complex by the factor $(-1)^n$. (This convention makes it easier to remember certain formulas.) For example, if $A^\bullet$ is the complex

$$\cdots \longrightarrow A^{-1} \xrightarrow{d} A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \longrightarrow \cdots$$

then $A^\bullet[1]$ is the same complex shifted to the left by one step,

$$\cdots \longrightarrow A^{0} \xrightarrow{-d} A^1 \xrightarrow{-d} A^2 \xrightarrow{-d} A^3 \longrightarrow \cdots$$

and with the sign of all differentials changed. This operation passes to the derived category, and defines a collection of functors $[n]\colon D(\mathfrak{A}) \to D(\mathfrak{A})$.

Example 21.7. Morphisms in $\mathrm{D}^b(\mathfrak{A})$ are related to Ext-groups in the sense of Yoneda. If A and B are two objects of the abelian category $\mathfrak{A}$, then one has

$$\mathrm{Hom}_{\mathrm{D}^b(\mathfrak{A})}\big(A, B[n]\big) \simeq \mathrm{Ext}^n(A, B);$$

in particular, this group is trivial for $n < 0$. Let us consider the case $n = 1$. An element of $\mathrm{Ext}^1(A, B)$ is represented by a short exact sequence of the form

$$0 \to B \to E \to A \to 0.$$

Now the morphism of complexes

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

is obviously a quasi-isomorphism; on the other hand, we have

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{id}"] & E \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

and together, they determine a morphism in $\mathrm{D}^b(\mathfrak{A})$ from A (viewed as a complex in degree 0) to $B[1]$ (viewed as a complex in degree -1).

Exercise 21.1. Show that, conversely, every element of $\mathrm{Hom}_{\mathrm{D}^b(\mathfrak{A})}\big(A, B[1]\big)$ gives rise to an extension of A by B, and that the two constructions are inverse to each other.

Other models for the derived category. Recall that the objects of the bounded derived category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ are complexes of sheaves of $\mathscr{O}_X$-modules whose cohomology sheaves are coherent and vanish outside some bounded interval. I already mentioned that, under some mild assumptions on X, this category is equivalent to the much smaller category $\mathrm{D}^b\big(\mathrm{Coh}(X)\big)$, whose objects are bounded complexes of coherent sheaves on X. There are various other models for the derived category, each based on a certain class of sheaves (such as injective sheaves or flat sheaves). Let me illustrate this principle with the following example.

Example 21.8. Let $\mathrm{Inj}(\mathscr{O}_X)$ denote the (additive, but not abelian) category of injective sheaves of $\mathscr{O}_X$-modules. Every $\mathscr{O}_X$-module has a semi-infinite resolution by injectives; using the Cartan-Eilenberg construction, every semi-infinite complex of $\mathscr{O}_X$-modules is quasi-isomorphic to a semi-infinite complex of injectives. This means that the inclusion

$$\mathrm{D}^+\big(\mathrm{Inj}(\mathscr{O}_X)\big) \subseteq \mathrm{D}^+(\mathscr{O}_X)$$

is an equivalence of categories. By restricting to complexes with bounded and coherent cohomology sheaves, we also obtain an equivalence of categories

$$\mathrm{D}^b_{coh}\big(\mathrm{Inj}(\mathscr{O}_X)\big) \simeq \mathrm{D}^b_{coh}(\mathscr{O}_X).$$

The advantage of using injectives is that we do not need to worry about inverses for quasi-isomorphisms. Indeed, suppose that $f\colon I_1^\bullet \to I_2^\bullet$ is a quasi-isomorphism between two complexes of injective $\mathscr{O}_X$-modules. The universal mapping property of injectives implies that there is a morphism of complexes $g\colon I_2^\bullet \to I_1^\bullet$ such that both $f \circ g$ and $g \circ f$ are homotopic to the identity. Thus

$$\mathrm{D}^+\big(\mathrm{Inj}(\mathscr{O}_X)\big) \simeq H^+\big(\mathrm{Inj}(\mathscr{O}_X)\big)$$

and, extending our earlier notation in the obvious way, also

$$H^b_{coh}\big(\mathrm{Inj}(\mathscr{O}_X)\big) \simeq \mathrm{D}^b_{coh}(\mathscr{O}_X).$$

The same construction works for sheaves of flat $\mathscr{O}_X$-modules; under certain additional assumptions on the scheme X, one can also use locally free sheaves.

In this model for the derived category, the morphisms are much easier to describe. Nevertheless, it is better to work with the category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$, because it gives us more flexibility: we can choose injective or flat or locally free resolutions as the occasion demands.

Lecture 22

Triangulated categories. The derived category is no longer an abelian category, because the kernel and cokernel of a morphism do not make sense. (This is due to all the additional morphisms that we have introduced when adding inverses for quasi-isomorphisms.) But there is a replacement for short exact sequences, the so-called distinguished triangles, and $D^b(\mathfrak{A})$ is an example of a *triangulated category.*

A triangulated category is given by specifying a class of triangles. The motivation for introducing triangles lies in the mapping cone construction from homological algebra; let us briefly review this construction, and explain in what sense it acts as a substitute for short exact sequences. Given a morphism of complexes $f\colon A^\bullet \to B^\bullet$, the *mapping cone* of f is the complex

$$C_f^\bullet = B^\bullet \oplus A^\bullet[1] = B^\bullet \oplus A^{\bullet+1}$$

with differential $d(b,a) = (db + fa, -da)$. (As explained below, the terminology comes from the mapping cone in algebraic topology.) Since we defined $A^\bullet[1]$ by changing the sign of all differentials, this makes the sequence of complexes

$$0 \to B^\bullet \to C_f^\bullet \to A^\bullet[1] \to 0$$

short exact. In total, we have a sequence of four morphisms

$$A^\bullet \to B^\bullet \to C_f^\bullet \to A^\bullet[1], \tag{22.1}$$

and the composition of any two adjacent morphisms is zero *up to homotopy.*

Exercise 22.1. Verify that the composite morphisms

$$A^\bullet \to B^\bullet \to C_f^\bullet \qquad \text{and} \qquad C_f^\bullet \to A^\bullet[1] \to B^\bullet[1]$$

are both homotopic to zero.

A sequence of four morphisms as in (22.1) is called a *triangle*; this is because we can arrange it into the shape of a triangle, with the convention that the arrow marked [1] really goes from $C_f^\bullet$ to $A^\bullet[1]$:

$$\begin{tikzcd} & A^\bullet \arrow[dr] & \\ C_f^\bullet \arrow[ur, "{[1]}"] & & B^\bullet \arrow[ll] \end{tikzcd}$$

The short exact sequence of complexes gives rise to a long exact sequence

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C_f^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

for the cohomology of the complexes. In order to write down this long exact sequence, all we need is the four morphisms in (22.1). Taking this example as a model, we say that any sequence of four morphisms of complexes

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

is a *distiguished triangle* if it is isomorphic, in the derived category, to a triangle coming from the mapping cone construction. (In particular, the composition of two adjacent morphisms in the triangle is then actually homotopic to zero.) This definition endows the derived category with the structure of a triangulated category.

Here are two basic properties of distinguished triangles that you should try to verify as an exercise. There are many others, and by abstracting from this example, Verdier arrived at the concept of a triangulated category; since the precise definition is not relevant for our purposes, we shall not dwell on the details.

Exercise 22.2. Suppose that $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle. Show that $B^\bullet \to C^\bullet \to A^\bullet[1] \to B^\bullet[1]$ and $C^\bullet[-1] \to A^\bullet \to B^\bullet \to C^\bullet$ are again distinguished triangles. This means that distinguished triangles can be "rotated" in both directions.

Exercise 22.3. Show that a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ gives rise to a long exact sequence

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

in the abelian category $\mathfrak{A}$.

I already mentioned that distinguished triangles are a replacement for short exact sequences; let me elaborate on this point a bit. On the one hand, the prototypical example of a distinguished triangle in (22.1) came from the short exact sequence of the mapping cone. On the other hand, once we look at complexes up to quasi-isomorphism, every short exact sequence of complexes is actually that of a mapping cone (under some conditions on $\mathfrak{A}$). Let me illustrate this claim with the example of modules over a ring.

Example 22.2. Suppose we have a short exact sequence of complexes of R-modules

$$0 \to B^\bullet \to C^\bullet \to A^\bullet[1] \to 0.$$

Up to quasi-isomorphism, we can replace any complex by a free resolution, and so we may assume that $A^\bullet$ is a complex of free R-modules. We can then choose splittings

$$C^n \simeq B^n \oplus A^{n+1}.$$

With respect to this decomposition, the differential $d\colon C^n \to C^{n+1}$ is represented by a matrix

$$\begin{pmatrix} d & f \\ 0 & -d \end{pmatrix}$$

for some homomorphism $f\colon A^n \to B^n$; the identity $d \circ d = 0$ implies that f defines a morphism of complexes from $A^\bullet$ to $B^\bullet$, and our exact sequence of complexes is the one for the mapping cone of f.

In closing, let me mention one other general fact that is frequently useful. Namely, suppose that $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle in $\mathrm{D}^b(\mathfrak{A})$. Then for every $E^\bullet \in \mathrm{D}^b(\mathfrak{A})$, one gets two long exact sequences of abelian groups

$$\cdots \to \operatorname{Hom}(E^\bullet, A^\bullet) \to \operatorname{Hom}(E^\bullet, B^\bullet) \to \operatorname{Hom}(E^\bullet, C^\bullet) \to \operatorname{Hom}\bigl(E^\bullet, A^\bullet[1]\bigr) \to \cdots$$

$$\cdots \to \operatorname{Hom}\bigl(A^\bullet[1], E^\bullet\bigr) \to \operatorname{Hom}(C^\bullet, E^\bullet) \to \operatorname{Hom}(B^\bullet, E^\bullet) \to \operatorname{Hom}(A^\bullet, E^\bullet) \to \cdots$$

where $\operatorname{Hom}(-,-)$ means the set of morphisms in $\mathrm{D}^b(\mathfrak{A})$.

An analogy with algebraic topology. There is a useful analogy with algebraic topology; it is explained in greater depth in Richard Thomas' lectures notes *Derived categories for the working mathematician.* Suppose that X and Y are two simply connected simplicial complexes. If they are homotopy equivalent, then their singular homology groups $H_i(X,\mathbb{Z})$ and $H_i(Y,\mathbb{Z})$ are isomorphic; the converse is not true. But it turns out that the simplicial chain complex $C_\bullet(X)$ contains enough information to detect homotopy equivalence. In fact, there is the following theorem by Whitehead:

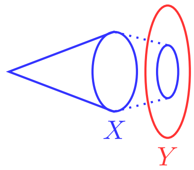
Theorem. *Two (simply connected) simplicial complexes X and Y are homotopy equivalent iff there is a third simplicial complex Z and two morphisms of complexes*

$$h\colon C_\bullet(Z) \to C_\bullet(X) \quad \textit{and} \quad f\colon C_\bullet(Z) \to C_\bullet(Y)$$

that induce isomorphisms on homology.

In this way, the category of (simply connected) simplicial complexes up to homotopy equivalence embeds into the derived category of complexes of abelian groups.

Algebraic topology is also where the term “mapping cone” originated. Let $f\colon X \to Y$ be a continuous mapping between two topological spaces. The mapping cone C_f is formed by taking the disjoint union of $X \times [0,1]$ and Y, collapsing $X \times \{0\}$ to a point, and identifying $X \times \{1\}$ with the image $f(X) \subseteq Y$. Schematically,

$C_f =$ 

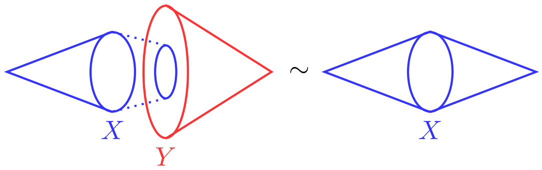
The resulting space fits into a sequence of mappings

$$X \xrightarrow{f} Y \xrightarrow{i} C_f$$

It can serve as a sort of kernel and cokernel for f; for example, if f is injective, then C_f is homotopy-equivalent to the quotient space Y/X. We can apply the cone construction to the natural inclusion $i\colon Y \to C_f$, and extend the above sequence of mappings to

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \longrightarrow C_i \tag{22.3}$$

Since $Y \subseteq C_i$ can be retracted to one of the two vertices, the space C_i is homotopy-equivalent to ΣX, the suspension of X; in a schematic drawing,

$C_i =$  $= \Sigma X$

Taking homology, and using the suspension isomorphism $H_i(\Sigma X) \simeq H_{i-1}(X)$, we then obtain a long exact sequence

$$\cdots \to H_i(X) \to H_i(Y) \to H_i(Y,X) \to H_{i-1}(X) \to H_{i-1}(Y) \to \cdots$$

It is a pleasant exercise to check that $H_i(Y,X) \to H_{i-1}(X)$ agrees with the usual boundary map in singular homology. Thus (22.3) is, in a sense, a distinguished triangle of topological spaces (up to homotopy equivalence).

Here is one final point. Suppose that $f\colon X \to Y$ is a simplicial mapping between two simplicial complexes. Then C_f is again a simplicial complex: its i-simplices are the i-simplices in Y, together with the cones on the $(i-1)$-simplices in X. One can use this to show that the simplicial chain complex of C_f is given by

$$C_\bullet(C_f) = C_\bullet(Y) \oplus C_{\bullet-1}(X),$$

with differential $\partial(\eta,\xi) = \bigl(\partial\eta + f\xi, -\partial\xi\bigr)$. As you can see, this is precisely the mapping cone of the morphism of complexes $C_\bullet(X) \to C_\bullet(Y)$ induced by f.

Derived functors. From now on, we shall concentrate on the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$, where X is a scheme. Here is a very useful fact:

Example 22.4. If X is nonsingular and quasi-compact, so that every coherent sheaf on X has a finite resolution by locally free sheaves, then every complex in $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ is quasi-isomorphic to a bounded complex of locally free sheaves.

Our goal is to define derived functors for the commonly used functors in algebraic geometry, such as $\otimes$, $\mathcal{H}om$, or pushforwards and pullbacks. The original functors are either left or right exact, and in classical homological algebra, the higher derived functors correct the lack of exactness. In the setting of triangulated categories, the relevant definition is the following.

Definition 22.5. An additive functor between two triangulated categories is *exact* if it takes distinguished triangles to distinguished triangles.

If we have an exact functor $F\colon \mathrm{D}^b(\mathfrak{A}) \to \mathrm{D}^b(\mathfrak{B})$ between the derived categories of two abelian categories, we get a long exact sequence in cohomology: if $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle in $\mathrm{D}^b(\mathfrak{A})$, then $F(A^\bullet) \to F(B^\bullet) \to F(C^\bullet) \to F(A^\bullet)[1]$ is a distinguished triangle in $\mathrm{D}^b(\mathfrak{B})$, and so

$$\cdots \to H^i F(A^\bullet) \to H^i F(B^\bullet) \to H^i F(C^\bullet) \to H^{i+1} F(A^\bullet) \to \cdots$$

is a long exact sequence in the abelian category $\mathfrak{B}$. This explains the terminology.

When defining a derived functor, we have two choices:

(1) Use a definition that works only for certain complexes, such as complexes of injective sheaves or flat sheaves. Then show that the subcategory consisting of such complexes is equivalent to the entire derived category. In this way, we obtain a non-constructive definition of the functor.
(2) Use a definition that works for arbitrary complexes. This may require more effort, but seems better from a mathematical point of view.

Example 22.6. Let $f\colon X \to Y$ be a morphism of schemes, say quasi-compact and separated (in order for f_* to preserve quasi-coherence). We want to define the derived functor $\mathbf{R}f_*\colon \mathrm{D}^+\big(\mathrm{QCoh}(X)\big) \to \mathrm{D}^+\big(\mathrm{QCoh}(Y)\big)$. Since we already know that injective sheaves are acyclic, we should obviously define

$$\mathbf{R}f_* I^\bullet = f_* I^\bullet$$

if $I^\bullet$ is a complex of injective sheaves. Since the subcategory $\mathrm{D}^+\big(\mathrm{Inj}(X)\big)$ is equivalent to $\mathrm{D}^+\big(\mathrm{QCoh}(X)\big)$, we can choose an inverse functor to the inclusion – this basically amounts to choosing an injective resolution for every complex of quasi-coherent sheaves – and compose the two. In this way, we obtain a functor

$$\mathbf{R}f_*\colon \mathrm{D}^+\big(\mathrm{QCoh}(X)\big) \to \mathrm{D}^+\big(\mathrm{QCoh}(Y)\big).$$

If f is proper, then f_* preserves coherence, and $\mathbf{R}f_*$ restricts to a functor

$$\mathbf{R}f_*\colon \mathrm{D}^b_{coh}(\mathscr{O}_X) \to \mathrm{D}^b_{coh}(\mathscr{O}_Y).$$

It remains to verify that $\mathbf{R}f_*$ is an exact functor.

Exercise 22.4. Show that $\mathbf{R}f_*$ takes distinguished triangles to distinguished triangles. (Hint: It is enough to prove this for a triangle of the form

$$I_1^\bullet \to I_2^\bullet \to C_\varphi^\bullet \to I_1^\bullet[1],$$

for $\varphi\colon I_1^\bullet \to I_2^\bullet$ a morphism between two complexes of injective sheaves.)

Example 22.7. If the above definition of $\mathbf{R}f_*$ involves too many choices for your taste, here is another possibility. Flasque sheaves are also acyclic for f_*, and have the advantage that there is a canonical resolution by flasque sheaves, the so-called *Godement resolution.* Given a sheaf of abelian groups $\mathscr{F}$, let $G^0(\mathscr{F})$ denote the sheaf of discontinuous sections: for any open subscheme $U \subseteq X$,

$$G^0(\mathscr{F})(U) = \prod_{x \in U} \mathscr{F}_x.$$

This sheaf is flasque and contains $\mathscr{F}$ as a subsheaf. Now we define $G^1(\mathscr{F})$ by applying the same construction to the cokernel of $\mathscr{F} \hookrightarrow G^0(\mathscr{F})$; in general, we set $G^{n+1}(\mathscr{F}) = G^0\big(G^n(\mathscr{F})/G^{n-1}(\mathscr{F})\big)$. The resulting complex of sheaves

$$0 \to \mathscr{F} \to G^0(\mathscr{F}) \to G^1(\mathscr{F}) \to G^2(\mathscr{F}) \to \cdots$$

is exact; this is the Godement resolution $G^\bullet(\mathscr{F})$. The same construction produces canonical flasque resolutions for complexes of sheaves: apply the construction to each sheaf in the complex to get a double complex, and then take the associated single complex. This allows us to define $\mathbf{R}f_*$ by setting

$$\mathbf{R}f_* F = f_* G^\bullet(F)$$

for any $F \in \mathrm{D}^+(\mathscr{O}_X)$. One can show that $\mathbf{R}f_*\mathscr{F}$ is canonically isomorphic to $f_*\mathscr{F}$ when $\mathscr{F}$ is a flasque sheaf; up to isomorphism, the two constructions of $\mathbf{R}f_*$ are therefore the same.

By one of those methods, one can also define the derived functors $\overset{\mathbf{L}}{\otimes}$, $\mathbf{R}\mathcal{H}om$, $\mathbf{R}\Gamma$, $\mathbf{R}\mathrm{Hom}$, as well as $\mathbf{L}f^*$ for morphisms $f\colon X \to Y$. All of the properties of the underived functors carry over to this setting: for example, $\mathbf{L}f^*$ is the left adjoint of $\mathbf{R}f_*$. In classical homological algebra, the composition of two functors leads to a spectral sequence (such as the Grothendieck spectral sequence); in the derived category, this simply becomes an identity between two derived functors.

Example 22.8. For two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, one has $\mathbf{R}g_* \circ \mathbf{R}f_* \simeq \mathbf{R}(g \circ f)_*$. This can be proved by observing that the pushforward of an injective sheaf is again injective: for a complex of injective sheaves,

$$(g \circ f)_* I^\bullet = g_*\big(f_* I^\bullet\big).$$

A special case of this is the formula $\mathbf{R}\Gamma(Y, -) \circ \mathbf{R}f_* \simeq \mathbf{R}\Gamma(X, -)$, which is the derived category version of the Leray spectral sequence.

Example 22.9. Similar reasoning proves the formula $\mathbf{R}\Gamma \circ \mathbf{R}\mathcal{H}om \simeq \mathbf{R}\mathrm{Hom}$.

The big advantage of working in the derived category is that many relations among the underived functors that are true only for locally free sheaves, now hold in general. Technically, this is true on nonsingular varieties, because every complex in $\mathrm{D}^b_{coh}(\mathscr{O}_X)$ is then quasi-isomorphic to a bounded complex of locally free sheaves.

As a case in point, let us consider the projection formula. The version in Hartshorne says that if $f\colon X \to Y$ is a morphism of schemes, and if $\mathscr{E}$ is a locally free $\mathscr{O}_Y$-module of finite rank, then $f_*\big(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E}\big) \simeq f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{E}$. In the derived category, we have the following generalization.

Proposition 22.10. *Let $f\colon X \to Y$ be a morphism of schemes, with Y nonsingular and quasi-compact. Then one has*

$$\mathbf{R}f_*\big(F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} \mathbf{L}f^* G\big) \simeq \mathbf{R}f_* F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_Y} G$$

for every $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and every $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$.

Proof. We may assume without loss of generality that G is a bounded complex of locally free sheaves and that F is a complex of injective sheaves. In that case,

$$\mathbf{R}f_*\big(F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} \mathbf{L}f^* G\big) = f_*\big(F \otimes_{\mathscr{O}_X} f^* G\big),$$

and by the usual projection formula, this is isomorphic to

$$f_* F \otimes_{\mathscr{O}_Y} G = \mathbf{R}f_* F \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_Y} G. \qquad \square$$

Grothendieck duality. The purpose of today's class is to introduce three basic tools for working with derived categories. One result that we shall use frequently is Grothendieck duality. The general theory is fairly complicated, and so we shall only discuss a special case that is sufficient for the purposes of this course.

Let me begin by recalling Serre's duality theorem. It says that if $\mathscr{F}$ is a coherent sheaf on a smooth projective variety X, then

$$\operatorname{Ext}^{n-i}(\mathscr{F}, \omega_X) \simeq \operatorname{Hom}_{\mathbb{C}}\bigl(H^i(X, \mathscr{F}), \mathbb{C}\bigr),$$

where $n = \dim X$ and ω_X denotes the canonical bundle of X. We can reformulate this using the derived category. Because of the relationship between Ext-groups and morphisms in the derived category, we have

$$\begin{aligned} H^i(X, \mathscr{F}) &\simeq \operatorname{Ext}^i(\mathscr{O}_X, \mathscr{F}) \simeq \operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}\bigl(\mathscr{O}_X, \mathscr{F}[i]\bigr) \\ \operatorname{Ext}^{n-i}(\mathscr{F}, \omega_X) &\simeq \operatorname{Hom}\bigl(\mathscr{F}[i], \omega_X[n]\bigr). \end{aligned}$$

Serre duality can therefore be rewritten in the form

$$\operatorname{Hom}\bigl(F, \omega_X[n]\bigr) \simeq \operatorname{Hom}\bigl(\operatorname{Hom}(\mathscr{O}_X, F), \mathbb{C}\bigr),$$

where $F = \mathscr{F}[i]$. Using suitable resolutions, this can be improved to the following general result in the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_X)$.

Theorem 23.1. *Let X be a smooth projective variety, and let F and G be two objects of $\mathrm{D}^b_{coh}(\mathscr{O}_X)$. Then one has an isomorphism of $\mathbb{C}$-vector spaces*

$$\operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}\Bigl(F, G \otimes \omega_X[n]\Bigr) \simeq \operatorname{Hom}\Bigl(\operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}(G, F), \mathbb{C}\Bigr)$$

that is functorial in F and G.

Grothendieck duality is a relative version of Serre duality, where instead of a single variety, one has a proper morphism $f\colon X \to Y$. In Grothendieck's formulation, duality becomes a statement about certain functors: we have the derived pushforward functor $\mathbf{R}f_*\colon \mathrm{D}^b_{coh}(\mathscr{O}_X) \to \mathrm{D}^b_{coh}(\mathscr{O}_Y)$, and the problem is to construct a right adjoint $f^!\colon \mathrm{D}^b_{coh}(\mathscr{O}_Y) \to \mathrm{D}^b_{coh}(\mathscr{O}_X)$, pronounced "$f$-shriek". In other words, we would like to define $f^!$ in such a way that we have functorial isomorphisms

$$\operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_Y)}\bigl(\mathbf{R}f_* F, G\bigr) \simeq \operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_X)}\bigl(F, f^! G\bigr)$$

for $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$. For arbitrary proper morphisms, the construction requires considerable technical effort; it is explained in Hartshorne's book *Residues and Duality.* (There is also a modern treatment by Amnon Neeman, based on the Brown' representability theorem.) But in the special case that both X and Y are smooth projective, there is a much simpler construction due to Alexei Bondal and Mikhail Kapranov.

Theorem 23.2. *If $f\colon X \to Y$ is a morphism between two smooth projective varieties, then*

$$f^! G = \omega_X[\dim X] \otimes \mathbf{L}f^*\bigl(G \otimes \omega_Y^{-1}[-\dim Y]\bigr)$$

for any $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$.

Proof. This follows very easily from the fact that $\mathbf{L}f^*$ is the left adjoint of $\mathbf{R}f_*$ – if we use Serre duality to interchange left and right. Fix two objects $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$. Applying Serre duality on Y, we get

$$\operatorname{Hom}\bigl(\mathbf{R}f_* F, G \otimes \omega_Y[\dim Y]\bigr) \simeq \operatorname{Hom}\Bigl(\operatorname{Hom}\bigl(G, \mathbf{R}f_* F\bigr), \mathbb{C}\Bigr).$$

Because $\mathbf{L}f^*$ is the left adjoint of $\mathbf{R}f_*$, we have

$$\operatorname{Hom}\bigl(G, \mathbf{R}f_* F\bigr) \simeq \operatorname{Hom}\bigl(\mathbf{L}f^* G, F\bigr).$$

If we now apply Serre duality on X, we get back to

$$\operatorname{Hom}\Big(\operatorname{Hom}\big(\mathbf{L}f^*G, F\big), \mathbb{C}\Big) \simeq \operatorname{Hom}\big(F, \mathbf{L}f^*G \otimes \omega_X[\dim X]\big).$$

Putting all three isomorphisms together, we obtain the desired formula for $f^!G$. $\square$

For a more concise statement, let $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$ denote the relative canonical bundle; then the formula in Theorem 23.2 becomes

$$f^! = \omega_{X/Y}[\dim X - \dim Y] \otimes \mathbf{L}f^*.$$

Note that $\dim X - \dim Y$ is simply the relative dimension of the morphism f. To summarize, we have a functorial isomorphism

$$\operatorname{Hom}\Big(\mathbf{R}f_*F, G\Big) \simeq \operatorname{Hom}\Big(F, \omega_{X/Y}[\dim X - \dim Y] \otimes \mathbf{L}f^*G\Big)$$

for $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$. In this form, Grothendieck duality will appear frequently in the derived category calculations below.

Base change. Another technical result that we shall use below is the base change theorem. As in the case of Grothendieck duality, there is a very general statement (in the derived category); for our purposes, however, two special cases are enough, and so we shall restrict our attention to those.

The general problem addressed by the base change theorem is the following. Suppose we have a cartesian diagram of schemes:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

We would like to compare the two functors g^*f_* and $f'_*g'^*$; more generally, on the level of the derived category, the two functors $\mathbf{L}g^*\mathbf{R}f_*$ and $\mathbf{R}f'_*\mathbf{L}g'^*$. Using the adjointness of pullback and pushforward, we always have morphisms of functors

$$g^*f_* \to f'_*g'^* \quad \text{and} \quad \mathbf{L}g^*\mathbf{R}f_* \to \mathbf{R}f'_*\mathbf{L}g'^*,$$

but without some assumptions on f or g – or on the sheaves or complexes to which we apply the functors – they are not isomorphisms.

The simplest case where the two functors are isomorphic is when g (and hence also g') is flat. We begin by looking at the case of sheaves.

Lemma 23.3. *Suppose that g is flat, and that f is separated and quasi-compact. Then the base change morphism*

$$g^*f_*\mathscr{F} \to f'_*g'^*\mathscr{F}$$

is an isomorphism for every quasi-coherent sheaf $\mathscr{F}$ on X.

Proof. The statement is local on Y and Y', and so we may assume without loss of generality that $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$ are affine, with A' flat over A. Let $\mathscr{F}' = g'^*\mathscr{F}$; then all sheaves involved are quasi-coherent on Y', and so it suffices to show that

$$\mathscr{F}(X) \otimes_A A' \to \mathscr{F}'(X')$$

is an isomorphism.

We first consider the case when $X = \operatorname{Spec} B$ is also affine; in that case, $X' = \operatorname{Spec} A' \otimes_A B$. We have $\mathscr{F} = \tilde{M}$ for some B-module M; then $g^*f_*\mathscr{F}$ is the quasi-coherent sheaf corresponding to the A'-module

$$A' \otimes_A M_A,$$

while $f'_* g'^* \mathscr{F}$ is the quasi-coherent sheaf corresponding to

$$(A' \otimes_A B) \otimes_B M.$$

The two are evidently isomorphic, which proves the assertion in case X is affine. In general, cover X by finitely many affine open subsets $U_1, \dots, U_n$. Because $\mathscr{F}$ is a sheaf, the complex of A-modules

$$0 \to \mathscr{F}(X) \to \bigoplus_{i=1}^{n} \mathscr{F}(U_i) \to \bigoplus_{i,j=1}^{n} \mathscr{F}(U_i \cap U_j)$$

is exact. Now A' is flat over A, and so

$$0 \to \mathscr{F}(X) \otimes_A A' \to \bigoplus_{i=1}^{n} \mathscr{F}(U_i) \otimes_A A' \to \bigoplus_{i,j=1}^{n} \mathscr{F}(U_i \cap U_j) \otimes_A A'$$

remains exact. We conclude from the affine case above that the kernel is isomorphic to $\mathscr{F}'(X')$, which is the result we were after. ☐

In the derived category, we have the following version.

Proposition 23.4. *Suppose that g is flat, and the f is separated and quasi-compact. Then for any $F \in \mathrm{D}^+\big(\mathrm{QCoh}(X)\big)$, the base change morphism*

$$\mathbf{L}g^* \mathbf{R}f_* F \to \mathbf{R}f'_* \mathbf{L}g'^* F$$

is an isomorphism.

Proof. After replacing F by an injective resolution, we may assume without loss of generality that F is a complex of injective quasi-coherent sheaves. The result now follows by applying Lemma 23.3 termwise. ☐

Another special case that we shall use below is that $f \colon X \to Y$ is a proper morphism, and $\mathscr{F}$ is a coherent sheaf on X that is flat over Y. (For example, this situation arises if f is proper and smooth, and $\mathscr{F}$ a locally free sheaf on X.) In that case, we are interested in comparing the higher direct image sheaves $R^i f_* \mathscr{F}$ with the fiberwise cohomology groups $H^i\big(X_y, \mathscr{F}_y\big)$, where $X_y = f^{-1}(y)$ denotes the scheme-theoretic fiber over a point $y \in Y$, and $\mathscr{F}_y$ the restriction of $\mathscr{F}$ to X_y. This case is discussed at depth in the section on the semicontinuity theorem (III.12) in Hartshorne's book.

The essential point is the following. Since the problem is local on Y, it may be assumed that $Y = \operatorname{Spec} A$ is affine. The higher direct image sheaves are coherent (because f is proper), and correspond to the finitely generated A-modules $H^i(X, \mathscr{F})$. The first step is to calculate these cohomology groups in a good way. Here Hartshorne constructs a bounded complex

$$\dots \to E^{i-1} \to E^i \to E^{i+1} \to \dots$$

of free A-modules of finite rank, such that one has functorial isomorphisms

$$H^i\big(E^\bullet \otimes_A M\big) \simeq H^i\big(X, \mathscr{F} \otimes_A M\big)$$

for all A-modules M. By taking $M = A$, one gets

$$H^i(E^\bullet) \simeq H^i(X, \mathscr{F}),$$

and by taking $M = A/P$ for a point $P \in \operatorname{Spec} A$, one gets

$$H^i\big(E^\bullet \otimes_A A/P\big) \simeq H^i\big(X_P, \mathscr{F}_P\big).$$

The second step is to solve the following purely algebraic problem about bounded complexes of free A-modules of finite rank: to compare $H^i(E^\bullet) \otimes_A A/P$ and $H^i\big(E^\bullet \otimes_A A/P\big)$. This is very similar to what we did in Lecture 7 when we studied cohomology support loci for complexes of vector bundles. You can find many results about this relationship in Hartshorne's book; here we shall only cite the one that will be used below.

Proposition 23.5. *Let $f\colon X \to Y$ be proper, and let $\mathscr{F}$ be a coherent sheaf on X that is flat over Y. Then*

$$\bigcup_{i\geq n} \operatorname{Supp} R^i f_* \mathscr{F} = \bigcup_{i\geq n} \left\{\, y \in Y \;\middle|\; H^i\bigl(X_y, \mathscr{F}_y\bigr) \neq 0 \,\right\}$$

for every $n \in \mathbb{Z}$.

Proof. The statement is local on Y, and so we may assume that $Y = \operatorname{Spec} A$ is affine. According to the discussion above, the higher direct image sheaves are computed by a bounded complex $E^\bullet$ of free A-modules of finite rank; what we need to prove is the following equality between subsets of $\operatorname{Spec} A$:

$$\bigcup_{i\geq n} \left\{\, P \;\middle|\; H^i(E^\bullet) \otimes_A A/P \neq 0 \,\right\} = \bigcup_{i\geq n} \left\{\, P \;\middle|\; H^i\bigl(E^\bullet \otimes_A A/P\bigr) \neq 0 \,\right\}$$

It is technically easier to prove this for the complements of the two sets:

$$\bigcap_{i\geq n} \left\{\, P \;\middle|\; H^i(E^\bullet) \otimes_A A/P = 0 \,\right\} = \bigcap_{i\geq n} \left\{\, P \;\middle|\; H^i\bigl(E^\bullet \otimes_A A/P\bigr) = 0 \,\right\}$$

Suppose P is an element of the left-hand side. After localizing at P, we may assume that (A, P) is a local ring. Nakayama's lemma shows that $H^i(E^\bullet) = 0$ for $i \geq n$; by a simple spectral sequence argument, it follows that $H^i\bigl(E^\bullet \otimes_A A/P\bigr) = 0$ in the same range. Conversely, suppose that P is an element of the right-hand side. After localising, we may again assume that (A, P) is a local ring. We have $H^i\bigl(E^\bullet \otimes_A A/P\bigr) = 0$ for $i \geq n$; now the construction in Lemma 7.7 shows that we must have $H^i(E^\bullet) = 0$ for $i \geq n$. This concludes the proof. □

This result will be very useful to us when we study cohomology support loci.

Kollár's theorem. The third important result that I would like to discuss is Kollár's theorem about higher direct images of dualizing sheaves. The precise result is the following.

Theorem 23.6. *Let $f\colon X \to Y$ be a surjective morphism between projective complex algebraic varieties. If X is smooth, then:*

(i) The sheaves $R^i f_* \omega_X$ are torsion-free sheaves on Y.

(ii) One has a non-canonical isomorphism

$$\mathbf{R} f_* \omega_X \simeq \bigoplus_i R^i f_* \omega_X[-i]$$

in the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_Y)$.

(iii) If L is an ample line bundle on Y, then $H^j\bigl(Y, R^i f_* \omega_X \otimes L\bigr) = 0$ for $j > 0$.

Informally stated, Kollár's result is that the higher direct image sheaves of ω_X behave much like ω_X itself: they are torsion-free and satisfy the Kodaira vanishing theorem. We have already proved (iii) in Theorem 4.6; the proof of (i) and (ii) uses more advanced results from Hodge theory – in particular, some results about polarized variations of Hodge structure – and so I can only give a brief outline here.

What we have to do is to construct a sequence of morphisms

$$R^i f_* \omega_X[-i] \to \mathbf{R} f_* \omega_X$$

in the derived category, in such a way that the induced morphism between the i-th cohomology sheaves is the identity. We can then take the direct sum to obtain the required splitting in $\mathrm{D}^b_{coh}(\mathscr{O}_Y)$. After a shift, this is equivalent to constructing a collection of morphisms

$$R^i f_* \omega_X \to \mathbf{R} f_* \omega_X[i].$$

This comes for free in the case $i = 0$, because from the definition of the derived functor, we automatically have a morphism $f_* \omega_X \to \mathbf{R} f_* \omega_X$. Indeed, let

$$0 \to \omega_X \to I^0 \to I^1 \to I^2 \to \cdots$$

be an injective resolution; then $\mathbf{R} f_* \omega_X = f_* I^\bullet$, and because f_* is left-exact, the resulting morphism $f_* \omega_X \to f_* I^\bullet$ is an isomorphism in degree 0. We also observe that $f_* \omega_X$ is the pushforward of a locally free sheaf, and therefore torsion-free.

Proof of Theorem 23.6. Set $k = \dim X - \dim Y$; we proceed by induction on $k \geq 0$. In the case $k = 0$, the Grauert-Riemenschneider theorem shows that $R^i f_* \omega_X = 0$ for $i > 0$; the natural morphism $f_* \omega_X \to \mathbf{R} f_* \omega_X$ is therefore a quasi-isomorphism, and so everything is proved in that case.

When $k \geq 1$, pick a sufficiently ample smooth hypersurface $H \subseteq X$ with the property that $f(H) = Y$ and that $R^i f_* \omega_X(H) = 0$ for every $i > 0$. Let $g \colon H \to Y$ denote the restriction of f. Adjunction gives us a short exact sequence

$$0 \to \omega_X \to \omega_X(H) \to \omega_H \to 0.$$

After pushing forward to Y, we have an exact sequence

$$0 \to f_* \omega_X \to f_* \omega_X(H) \to g_* \omega_H \to R^1 f_* \omega_X \to 0 \tag{23.7}$$

as well as isomorphisms

$$R^{i-1} g_* \omega_H \simeq R^i f_* \omega_X$$

for $i \geq 2$. By the inductive hypothesis (applied to $g \colon H \to Y$), the sheaves $R^i f_* \omega_X$ are therefore torsion-free for every $i \neq 1$.

The key result that Kollár proves – by appealing to the theory of variations of Hodge structure – is that the morphism $g_* \omega_H \to R^1 f_* \omega_X$ in (23.7) has a section. This implies that $R^1 f_* \omega_X$ is a direct summand of the torsion-free sheaf $g_* \omega_H$, and hence torsion-free. We have therefore proved (i).

On the other hand, we can consider the direct image functor in the derived category; it gives us a distinguished triangle

$$\mathbf{R} f_* \omega_X \to \mathbf{R} f_* \omega_X(H) \to \mathbf{R} g_* \omega_H \to \mathbf{R} f_* \omega_X[1].$$

According to the inductive hypothesis (applied to $g \colon H \to Y$) we already have a collection of morphisms

$$R^{i-1} g_* \omega_H \to \mathbf{R} g_* \omega_H[i-1].$$

By composing with the morphism in the distinguished triangle, we therefore obtain the required morphisms

$$R^i f_* \omega_X \simeq R^{i-1} g_* \omega_H \to \mathbf{R} g_* \omega_H[i-1] \to \mathbf{R} f_* \omega_X[i]$$

for $i \geq 2$. To deal with the remaining case $i = 1$, we take the composition

$$R^1 f_* \omega_X \to g_* \omega_H \to \mathbf{R} g_* \omega_H \to \mathbf{R} f_* \omega_X[1]$$

with the section coming from (23.7). It remains to verify that, in each case, the induced morphism between the cohomology sheaves in degree 0 is the identity; this is an easy exercise. □

Corollary 23.8. *Under the assumptions of the theorem, the Leray spectral sequence for the cohomology of ω_X degenerates at E_2, and with $k = \dim X - \dim Y$, one has*

$$H^j(X, \omega_X) \simeq \bigoplus_{i=0}^{k} H^{j-i}\big(Y, R^i f_* \omega_X\big).$$

Proof. Note that k is equal to the dimension of the general fiber of f. By base change, the sheaf $R^i f_* \omega_X$ is supported on a proper subset of Y for $i > k$; being torsion-free, it must be zero. We now obtain the result for the cohomology of ω_X by using the identity $\mathbf{R}\Gamma(X, -) \simeq \mathbf{R}\Gamma(Y, -) \circ \mathbf{R}f_*$ and the splitting in (ii). $\square$

Lecture 24

The Fourier-Mukai transform. We now come to a very important result in the theory of abelian varieties: Shigeru Mukai's version of the Fourier transform. Let A be a complex abelian variety, and let $\hat{A} = \operatorname{Pic}^0(A)$ denote the dual abelian variety. We denote the normalized Poincaré bundle on $A \times \hat{A}$ by the letter P. For a point $\alpha \in \hat{A}$, we denote by P_α the corresponding line bundle on A; for a point $a \in A$, we denote by $\hat{P}_a$ the corresponding line bundle on $\hat{A}$. In other words,

$$P\big|_{A\times\{\alpha\}} = P_\alpha \quad \text{and} \quad P\big|_{\{a\}\times\hat{A}} = \hat{P}_a.$$

Mukai's idea is to use P as the "kernel of an integral transform" – in the same way that the Fourier transform from functions on a vector space to functions on the dual vector space is defined by integration against a kernel function on the product. In our situation, we consider the following product:

$$\begin{tikzcd} A \times \hat{A} \arrow[r, "p_2"] \arrow[d, "p_1"] & \hat{A} \\ A & \end{tikzcd}$$

Given a coherent sheaf $\mathscr{F}$ on A, we can pull it back to $A \times \hat{A}$, tensor by P, and then push forward to $\hat{A}$; the resulting sheaf

$$p_{2*}\big(p_1^*\mathscr{F} \otimes P\big)$$

is again coherent because p_2 is a proper morphism. This is of course not an exact functor, and so we should really perform these operations in the derived category. For $F \in \mathrm{D}^b_{coh}(\mathscr{O}_A)$, we therefore define

$$\mathbf{R}\Phi_P F = \mathbf{R}p_{2*}\big(\mathbf{L}p_1^*F \overset{\mathbf{L}}{\otimes} P\big) = \mathbf{R}p_{2*}\big(p_1^*F \otimes P\big)$$

which is an object of $\mathrm{D}^b_{coh}(\mathscr{O}_{\hat{A}})$ because p_2 is proper. (Note that the functors p_1^* and $\otimes P$ are already exact.) In this way, we obtain an exact functor

$$\mathbf{R}\Phi_P \colon \mathrm{D}^b_{coh}(\mathscr{O}_A) \to \mathrm{D}^b_{coh}(\mathscr{O}_{\hat{A}}),$$

called the *Fourier-Mukai transform.* In analogy with the Fourier transform in analyis, we can think of $\mathbf{R}\Phi_P$ as decomposing a coherent sheaf (or complex of coherent sheaves) with respect to the basic line bundles P_α; the support of the Fourier-Mukai transform, inside $\hat{A}$, is something like the "spectrum" of the original sheaf.

Two examples that show how the Fourier-Mukai transform works in practice:

Example 24.1. Let $\mathscr{O}_a$ be the structure sheaf of a closed point $a \in A$. The restriction of P to $\{a\} \times \hat{A}$ is the line bundle $\hat{P}_a$, and so we have

$$\mathbf{R}\Phi_P \mathscr{O}_a \simeq \hat{P}_a.$$

In other words, the Fourier-Mukai transform takes a point of A to the corresponding line bundle on $\hat{A}$. This means that the spectrum of a point is everything – do you see the analogy with Fourier analysis?

Example 24.2. What is the Fourier-Mukai transform of the structure sheaf $\mathscr{O}_A$? From the definition, we immediately get

$$\mathbf{R}\Phi_P \mathscr{O}_A \simeq \mathbf{R}p_{2*}P,$$

and Proposition 24.3 shows that this equals $\mathscr{O}_0[-g]$.

The following proposition is not only needed in the example above, but it also plays a very important role in the general theory. (For example, we shall use it later to prove that $\mathbf{R}\Phi_P$ is an equivalence of categories.)

Proposition 24.3. *Let $g = \dim A$. Then we have*

$$R^i p_{2*} P \simeq \begin{cases} \mathscr{O}_0 & \text{for } i = g, \\ 0 & \text{for } i \neq g. \end{cases}$$

Equivalently, $\mathbf{R}p_{2*}P \simeq \mathscr{O}_0[-g]$ as objects of $\mathrm{D}^b_{coh}(\mathscr{O}_{\hat{A}})$.

The proof takes longer than you might expect, but is very clever. To get started, recall from Lemma 16.4 that all cohomology groups of a nontrivial line bundle in $\mathrm{Pic}^0(A)$ vanish: if $\alpha \neq 0$, then one has

$$H^i(A, P_\alpha) = 0$$

for every $i = 0, 1, \dots, g$. Because P is flat over $\hat{A}$, we can now use the base change theorem to conclude that the sheaves $R^i p_{2*} P$ are supported at the point $0 \in \hat{A}$. In particular, they all have finite length; we also note that, for dimension reasons, $R^i p_{2*} P = 0$ unless $0 \leq i \leq g = \dim A$.

In the next step, we use Serre duality to show that $R^i p_{2*} P = 0$ for $i < g$. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p\big(\hat{A}, R^q p_{2*} P\big) \Longrightarrow H^{p+q}\big(A \times \hat{A}, P\big).$$

We have $E_2^{p,q} = 0$ for $p > 0$; the spectral sequence therefore degenerates at E_2 and gives us isomorphisms

$$H^i\big(A \times \hat{A}, P\big) \simeq H^0\big(\hat{A}, R^i p_{2*} P\big). \tag{24.4}$$

In particular, this group vanishes for $i > g$. Because the canonical bundle of $A \times \hat{A}$ is trivial, Serre duality shows that

$$H^i\big(A \times \hat{A}, P\big)^* \simeq H^{2g-i}\big(A \times \hat{A}, P^{-1}\big) \simeq H^{2g-i}\big(A \times \hat{A}, P\big).$$

The group on the right vanishes for $2g - i > g$, and so (24.4) is zero for $i < g$. Because the sheaf $R^i p_{2*} P$ has finite length, it follows that $R^i p_{2*} P = 0$ for $i < g$.

Now we have to show that the remaining sheaf $R^g p_{2*} P$ is isomorphic to $\mathscr{O}_0$. The base change theorem gives us

$$R^g p_{2*} P \otimes \mathscr{O}_0 \simeq H^g(A, \mathscr{O}_A) \simeq \mathbb{C}.$$

By Nakayama's lemma, it follows that $R^g p_{2*} P \simeq \mathscr{O}_{\hat{A}} / \mathscr{J}$ for a certain ideal sheaf $\mathscr{J} \subseteq \mathscr{O}_{\hat{A}}$ whose cosupport is the point 0. If we let $\mathcal{I}_0$ denote the ideal sheaf of the point, the problem is to show that $\mathscr{J} = \mathcal{I}_0$. Here we shall make use of the universal property of the Poincaré bundle on $A \times \hat{A}$, in particular, the fact that it is trivial on $A \times \{0\}$, but not on any bigger subscheme.

Our main technical tool will be Grothendieck duality, applied to the second projection $p_2 \colon A \times \hat{A} \to \hat{A}$. Recall from Theorem 23.2 that

$$\mathrm{Hom}\big(\mathbf{R}p_{2*}P, G\big) \simeq \mathrm{Hom}\big(P, p_2^! G\big).$$

Because the canonical bundles of A and $\hat{A}$ are trivial, we get

$$p_2^! G \simeq \omega_{A \times \hat{A}/A}[g] \otimes \mathbf{L}p_2^* G \simeq \mathbf{L}p_2^* G[g].$$

Since $\mathbf{R}p_{2*}P \simeq R^g p_{2*} P[-g]$, we can put the isomorphism from Grothendieck duality into the more convenient form

$$\mathrm{Hom}\big(R^g p_{2*} P, \mathscr{G}\big) \simeq \mathrm{Hom}\big(P, p_2^* \mathscr{G}\big),$$

where $\mathscr{G}$ can be an arbitrary coherent sheaf on $\hat{A}$.

Lemma 24.5. *In the above notation, we have $\mathscr{J} = \mathcal{I}_0$.*

Proof. Let $Z \subseteq \hat{A}$ denote the closed subscheme defined by the ideal sheaf $\mathscr{J}$; recall that $R^g p_{2*} P \simeq \mathscr{O}_Z$. Because Grothendieck duality is functorial, we obtain a commutative diagram

$$\begin{tikzcd} \operatorname{Hom}\bigl(\mathscr{O}_Z, \mathscr{O}_Z\bigr) \arrow[r, equal] \arrow[d] & \operatorname{Hom}\bigl(P, \mathscr{O}_{A\times Z}\bigr) \arrow[d] \\ \operatorname{Hom}\bigl(\mathscr{O}_Z, \mathscr{O}_0\bigr) \arrow[r, equal] & \operatorname{Hom}\bigl(P, \mathscr{O}_{A\times\{0\}}\bigr). \end{tikzcd}$$

The identity morphism $\mathscr{O}_Z \to \mathscr{O}_Z$ corresponds to a morphism $P \to \mathscr{O}_{A\times Z}$; by adjunction, we obtain a morphism $\varphi\colon P\big|_{A\times Z} \to \mathscr{O}_{A\times Z}$. Similarly, the quotient morphism $\mathscr{O}_Z \to \mathscr{O}_0$ corresponds to a nontrivial morphism $\bar{\varphi}\colon P\big|_{A\times\{0\}} \to \mathscr{O}_{A\times\{0\}}$; the commutativity of the diagram means that $\bar{\varphi}$ is nothing but the reduction of φ modulo the ideal sheaf of $A \times \{0\}$. Now the restriction of P to $A \times \{0\}$ is trivial, and so $\bar{\varphi}$ must be an isomorphism. By Nakayama's lemma, φ itself is also an isomorphism. In other words, the restriction of P to the subscheme $A\times Z$ is trivial; by the universal property of P, the subscheme in question has to be contained in $A \times \{0\}$, which implies that Z is reduced, and hence that $\mathscr{J} = \mathcal{I}_0$. □

This concludes the proof that $\mathbf{R}p_{2*}P = \mathscr{O}_0[-g]$.

Mukai's theorem. Now our goal is to show that $\mathbf{R}\Phi_P\colon \mathrm{D}^b_{coh}(\mathscr{O}_A) \to \mathrm{D}^b_{coh}(\mathscr{O}_{\hat{A}})$ is an equivalence of categories; we shall do this by constructing an explicit inverse.

$$\begin{tikzcd} A \times \hat{A} \arrow[r, "p_2"] \arrow[d, "p_1"] & \hat{A} \\ A, & \end{tikzcd}$$

It is not hard to write down a functor going in the opposite direction – we only have to interchange the role of A and $\hat{A}$. We then obtain a second exact functor

$$\mathbf{R}\Psi_P\colon \mathrm{D}^b_{coh}(\mathscr{O}_{\hat{A}}) \to \mathrm{D}^b_{coh}(\mathscr{O}_A), \quad G \mapsto \mathbf{R}p_{1*}\bigl(p_2^* G \otimes P\bigr).$$

There is, however, no reason why $\mathbf{R}\Psi_P$ should be the inverse of $\mathbf{R}\Phi_P$. To see what is happening, let us revisit the two examples from above. We found that

$$\mathbf{R}\Phi_P(\mathscr{O}_a) \simeq \hat{P}_a$$

for any point $a \in A$, as well as $\mathbf{R}\Phi_P(\mathscr{O}_A) \simeq \mathscr{O}_0[-g]$. The latter can easily be generalized to arbitrary elements of $\operatorname{Pic}^0(A)$: the result is that

$$\mathbf{R}\Phi_P(P_\alpha) \simeq \mathscr{O}_{-\alpha}[-g]$$

for any $\alpha \in \hat{A}$. Indeed, the support of the transform must be the point $-\alpha$ (corresponding to the line bundle P_α^{-1}) because $P_\alpha \otimes P_\beta \simeq P_{\alpha+\beta}$ only has nontrivial cohomology when $\alpha + \beta = 0$. Starting from a point $a \in A$, we then have

$$(\mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P)(\mathscr{O}_a) \simeq \mathbf{R}\Psi_P(\hat{P}_a) \simeq \mathscr{O}_{-a}[-g].$$

The composition is not the identity; but it is not off by much, either, because both the shift $[-g]$ and the inverse $\iota\colon A \to A$, $a \mapsto -a$, are invertible operations. This calculation suggests the following general result.

Theorem 24.6 (Mukai)**.** *One has natural isomorphisms of functors*

$$\mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P \simeq \iota^*[-g] \quad \text{and} \quad \mathbf{R}\Phi_P \circ \mathbf{R}\Psi_P \simeq \iota^*[-g].$$

In particular, both $\mathbf{R}\Phi_P$ and $\mathbf{R}\Psi_P$ are equivalences of categories.

Note. Having a "natural isomorphism of functors" means that one should have a collection of isomorphisms $(\mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P)(F) \simeq \iota^* F[-g]$ that are functorial in F; we shall see during the proof that these isomorphisms come from the projection formula, flat base change, etc.

Proof. The general idea is to compute the composition with the help of the projection formula and base change, using a few special properties of the Poincaré bundle along the way. We shall use the following big diagram:

$$
\begin{tikzcd}
A \times \hat{A} & A \times A \times \hat{A} \arrow[l, "m\times\mathrm{id}"'] \arrow[r, "p_{23}"] \arrow[d, "p_{13}"] \arrow[rr, bend left, "p_2"] \arrow[dd, bend right, "p_1"'] & A \times \hat{A} \arrow[r, "p_1"] \arrow[d, "p_2"] & A \\
& A \times \hat{A} \arrow[r, "p_2"] \arrow[d, "p_1"] & \hat{A} & \\
& A & &
\end{tikzcd}
\tag{24.7}
$$

Fix an object $F \in \mathrm{D}^b_{coh}(\mathscr{O}_A)$. Our goal is to calculate the composition

$$F' = (\mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P)(F) = \mathbf{R}p_{1*}\Big(P \otimes p_2^*\big(\mathbf{R}p_{2*}(P \otimes p_1^* F)\big)\Big).$$

Flat base change, applied to the square in (24.7), gives

$$p_2^*\big(\mathbf{R}p_{2*}(P \otimes p_1^* F)\big) \simeq \mathbf{R}p_{23*}\big(p_{13}^*(P \otimes p_1^* F)\big) \simeq \mathbf{R}p_{23*}\big(p_{13}^* P \otimes p_1^* F\big).$$

The projection formula for p_{23} lets us put everything on $A \times A \times \hat{A}$:

$$P \otimes \mathbf{R}p_{23*}\big(p_{13}^* P \otimes p_1^* F\big) \simeq \mathbf{R}p_{23*}\big(p_{23}^* P \otimes p_{13}^* P \otimes p_1^* F\big).$$

Now we use the identity $(m \times \mathrm{id})^* P \simeq p_{13}^* P \otimes p_{23}^* P$ from the proof of Lemma 16.2; here $m \colon A \times A \to A$ is the addition morphism. It gives us

$$F' \simeq \mathbf{R}p_{1*}\mathbf{R}p_{23*}\big((m \times \mathrm{id})^* P \otimes p_1^* F\big) \simeq \mathbf{R}p_{2*}\big((m \times \mathrm{id})^* P \otimes p_1^* F\big).$$

To simplify this further, consider the diagram

$$
\begin{tikzcd}
A \times \hat{A} \arrow[r, "p_1"] & A & \\
A \times A \times \hat{A} \arrow[u, "m\times\mathrm{id}"] \arrow[r, "p_{12}"] \arrow[dr, bend right, "p_1"'] & A \times A \arrow[u, "m"'] \arrow[r, "p_2"] \arrow[d, "p_1"] & A \\
& A &
\end{tikzcd}
$$

By decomposing both projections in the manner indicated above, we obtain

$$F' \simeq \mathbf{R}p_{2*}\mathbf{R}p_{12*}\Big((m \times \mathrm{id})^* P \otimes p_{12}^*\big(p_1^* F\big)\Big) \simeq \mathbf{R}p_{2*}\Big(p_1^* F \otimes \mathbf{R}p_{12*}(m \times \mathrm{id})^* P\Big);$$

in the second step, we used the projection formula for p_{12}. Another application of flat base change, and the important formula in Proposition 24.3, yield

$$\mathbf{R}p_{12*}(m \times \mathrm{id})^* P \simeq m^* \mathbf{R}p_{1*} P \simeq m^* \mathscr{O}_0[-g].$$

If we define a closed embedding $i \colon A \to A \times A$ by $i(a) = (-a, a)$, then another simple base change calculation shows that $m^* \mathscr{O}_0 \simeq i_* \mathscr{O}_A$. Putting everything together,

$$F' \simeq \mathbf{R}p_{2*}\big(p_1^* F \otimes i_* \mathscr{O}_A[-g]\big).$$

Now consider the third (and last) diagram below:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[rr, bend left, "p_2"] \arrow[dr, bend right, "\iota"'] & A \times A \arrow[r, "p_2"] \arrow[d, "p_1"] & A \\
& A &
\end{tikzcd}$$

A final application of the projection formula (for the closed embedding i) gives

$$F' \simeq \mathbf{R}p_{2*}\mathbf{R}i_*\big(\mathbf{L}i^*p_1^*F \otimes \mathscr{O}_A[-g]\big) \simeq \mathbf{R}p_{2*}\iota^*F[-g] \simeq \iota^*F[-g].$$

This is the result we were after. Observe that all isomorphisms that we used during the calculation either affected only the Poincaré bundle, or came from the projection formula or the base change theorem; in particular, they are functorial in F. The same formula holds for $\mathbf{R}\Phi_P \circ \mathbf{R}\Psi_P$: just swap the roles of A and $\hat{A}$. $\square$

An important consequence of Mukai's theorem is that we can recover an object $F \in \mathrm{D}^b_{coh}(\mathscr{O}_A)$, up to canonical isomorphism, from its Fourier-Mukai transform $\mathbf{R}\Phi_P(F)$. When we pass from F to $\mathbf{R}\Phi_P(F)$, we therefore lose no information.

Ample line bundles. During the proof of Hacon's theorem, we will need to know how ample line bundles behave under the Fourier-Mukai transform.

Example 24.8. Let L be an ample line bundle on A; we would like to know $\mathbf{R}\Phi_P(L)$. As before, we can use the base change theorem to describe this object. By the Kodaira vanishing theorem,

$$H^i(A, L \otimes P_\alpha) \simeq H^i(A, \omega_A \otimes L \otimes P_\alpha) = 0$$

for $i > 0$, because $L \otimes P_\alpha$ is ample. It follows that

$$R^i\Phi_P(L) = R^ip_{2*}\big(P \otimes p_1^*L\big) = 0$$

for $i > 0$, which means that

$$\mathbf{R}\Phi_P(L) = p_{2*}\big(P \otimes p_1^*L\big)$$

is a sheaf. The dimension of $H^0(A, L \otimes P_\alpha)$ is constant by the Riemann-Roch theorem, and so this sheaf is locally free of rank $h^0(A, L)$.

Mukai observed that one can describe the vector bundle $\mathscr{E}_L = R^0\Phi_P(L)$ explicitly – not on $\hat{A}$ itself, but after passing to a finite étale cover.

Proposition 24.9. *We have* $\varphi_L^*\mathbf{R}\Phi_P(L) \simeq H^0(A, L) \otimes L^{-1}$.

Proof. Recall that $\varphi_L \colon A \to \hat{A}$ is the morphism of abelian varieties given by $\varphi_L(a) = (t_a^*L) \otimes L^{-1}$; it is surjective and of degree $(h^0(A, L))^2$. The calculation is similar to the other one; here is the diagram of morphisms:

$$\begin{tikzcd}
A \times A \arrow[r, "p_2"] \arrow[d, "\mathrm{id} \times \varphi_L"] \arrow[dd, bend right=60, "p_1"'] & A \arrow[d, "\varphi_L"] \\
A \times \hat{A} \arrow[r, "p_2"] \arrow[d, "p_1"] & \hat{A} \\
A &
\end{tikzcd}$$

We define $\mathscr{E}_L = \mathbf{R}\Phi_P(L)$. To begin with, we use flat base change to get

$$\varphi_L^*\mathscr{E}_L \simeq \varphi_L^*\mathbf{R}p_{2*}\big(P \otimes p_1^*L\big) \simeq \mathbf{R}p_{2*}\big((\mathrm{id} \times \varphi_L)^*P \otimes p_1^*L\big).$$

The Poincaré bundle has the property that $(\mathrm{id} \times \varphi_L)^* P \simeq m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$. If we substitute this into the previous line, and also use the projection formula,

$$\varphi_L^* \mathscr{E}_L \simeq \mathbf{R}p_{2*}\big(m^* L \otimes p_2^* L^{-1}\big) \simeq L^{-1} \otimes \mathbf{R}p_{2*} m^* L.$$

It remains to show that $\mathbf{R}p_{2*} m^* L$ is isomorphic to the trivial bundle $H^0(A, L) \otimes \mathscr{O}_A$. To do this, we use a clever factorization of m:

$$\begin{tikzcd}
A \times A \arrow[r, "f"] \arrow[rr, bend left, "p_2"] \arrow[dr, bend right, "m"'] & A \times A \arrow[r, "p_2"] \arrow[d, "p_1"] & A \\
& A &
\end{tikzcd}$$

Here $f(a, b) = (a + b, b)$ is an isomorphism. This yields

$$\mathbf{R}p_{2*} m^* L \simeq \mathbf{R}p_{2*} \mathbf{R}f_* \big(f^* p_1^* L\big) \simeq \mathbf{R}p_{2*}\big(p_1^* L \otimes \mathbf{R}f_* \mathscr{O}_{A \times A}\big) \simeq \mathbf{R}p_{2*}(p_1^* L).$$

Another simple application of flat base change gives us that

$$\mathbf{R}p_{2*}(p_1^* L) \simeq p^* \mathbf{R}p_* L \simeq H^0(A, L) \otimes \mathscr{O}_A,$$

where $p \colon A \to \operatorname{Spec} \mathbb{C}$ is the morphism to a point. $\square$

Exercises.

Exercise 24.1. Show that $\iota^* \mathbf{R}\Psi_P[g]$ is a left adjoint to $\mathbf{R}\Phi_P$; in other words, that

$$\operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_{\hat{A}})}\big(G, \mathbf{R}\Phi_P(F)\big) \simeq \operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_A)}\big(\iota^* \mathbf{R}\Psi_P(G)[g], F\big),$$

with the isomorphism being functorial in F and G.

Exercise 24.2. Show that $\mathbf{R}\Phi_P$ interchanges translations by points with tensor products by line bundles: for $a \in A$ and $\alpha \in \hat{A}$, one has

$$\mathbf{R}\Phi_P \circ t_a^* \simeq (\hat{P}_{-a} \otimes -) \circ \mathbf{R}\Phi_P \quad \text{and} \quad t_\alpha^* \circ \mathbf{R}\Phi_P \simeq \mathbf{R}\Phi_P \circ (P_\alpha \otimes -).$$

Exercise 24.3. Prove the identity

$$(\mathrm{id} \times \varphi_L)^* P \simeq m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

that was used during the proof. (*Hint:* Use the see-saw theorem.)

Lecture 25

The generic vanishing theorem. Having completed our review of derived categories and the Fourier-Mukai transform, we are now ready for understanding the algebraic proof of the generic vanishing theorem. Let me first remind you of the statement again – the version below is equivalent to the original generic vanishing theorem (in Theorem 6.6) because of Serre duality.

Theorem. *Let X be a smooth projective variety. Then*

$$\operatorname{codim}\bigl\{\, L \in \operatorname{Pic}^0(X) \bigm| H^i(X, \omega_X \otimes L) \neq 0 \,\bigr\} \geq i - \bigl(\dim X - \dim \operatorname{alb}(X)\bigr)$$

for every integer $i \geq 0$.

Note that $\dim X - \dim \operatorname{alb}(X)$ is the dimension of the general fiber of the Albanese mapping; we shall see in a moment where this comes from. To simplify the notation, we let $A = \operatorname{Alb}(X)$ denote the Albanese variety of X, and $f \colon X \to A$ the Albanese mapping (for some choice of base point). We also put $\hat{A} = \operatorname{Pic}^0(A)$ and let P be the normalized Poincaré bundle on the product $A \times \hat{A}$; as before, the line bundle corresponding to $\alpha \in \hat{A}$ will be denoted by P_α. Because $\operatorname{Pic}^0(X)$ is isomorphic to $\hat{A}$, the generic vanishing theorem is equivalent to

$$\operatorname{codim} S^i(X, \omega_X) = \operatorname{codim}\bigl\{\, \alpha \in \hat{A} \bigm| H^i(X, \omega_X \otimes f^* P_\alpha) \neq 0 \,\bigr\} \geq i - k, \tag{25.1}$$

where $k = \dim X - \dim f(X)$.

To prove (25.1), we shall reduce it to a statement about certain coherent sheaves on the abelian variety A. Given a point $\alpha \in \hat{A}$, we can use the isomorphism $\mathbf{R}\Gamma(X, -) \simeq \mathbf{R}\Gamma(A, -) \circ \mathbf{R}f_*$ and the projection formula to get

$$H^i\bigl(X, \omega_X \otimes f^* P_\alpha\bigr) \simeq R^i\Gamma\bigl(A, \mathbf{R}f_*\omega_X \otimes P_\alpha\bigr). \tag{25.2}$$

According to Kollár's results about higher direct images of dualizing sheaves (in Theorem 23.6), one has

$$\mathbf{R}f_*\omega_X \simeq \bigoplus_{j=0}^{k} R^j f_*\omega_X[-j]$$

in the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_A)$; recall that the summation stops at $j = k$ because the sheaves $R^j f_*\omega_X$, viewed as coherent sheaves on their support $f(X)$, are torsion-free. If we substitute this isomorphism into (25.2), we obtain

$$H^i\bigl(X, \omega_X \otimes f^* P_\alpha\bigr) \simeq \bigoplus_{j=0}^{k} H^{i-j}\bigl(A, R^j f_*\omega_X \otimes P_\alpha\bigr).$$

Obviously, the left-hand side is nonzero if and only if one of the summands on the right-hand side is nonzero; on the level of cohomology support loci, this means that

$$S^i(X, \omega_X) = \bigcup_{j=0}^{k} S^{i-j}\bigl(A, R^j f_*\omega_X\bigr), \tag{25.3}$$

as subsets of $\hat{A}$. This reduces the proof of (25.1) to the more uniform statement

$$\operatorname{codim} S^i\bigl(A, R^j f_*\omega_X\bigr) \geq i \quad \text{for every } i \geq 0. \tag{25.4}$$

Hacon's key insight is that this collection of inequalities is equivalent to a vanishing theorem. Here is the general result.

Theorem 25.5 (Hacon)**.** *Let $\mathscr{F}$ be a coherent sheaf on a complex abelian variety. The following four conditions are equivalent to each other:*

(a) One has $\operatorname{codim}\bigl\{\, \alpha \in \hat{A} \bigm| H^i(A, \mathscr{F} \otimes P_\alpha) \neq 0 \,\bigr\} \geq i$ *for all $i \in \mathbb{Z}$.*

(b) *The Fourier-Mukai transform* $\mathbf{R}\Phi_P(\mathscr{F})$ *satisfies*

$$\operatorname{codim} \operatorname{Supp} R^i\Phi_P(\mathscr{F}) \geq i \quad \textit{for all } i \in \mathbb{Z}.$$

(c) *For every finite étale morphism* $\varphi \colon B \to A$ *of abelian varieties, and every ample line bundle* L *on* B*, one has*

$$H^i(B, L \otimes \varphi^*\mathscr{F}) = 0 \quad \textit{for } i > 0.$$

(d) *There is a coherent sheaf* $\mathscr{G}$ *with the property that* $\mathbf{R}\Phi_P(\mathscr{F}) \simeq \mathbf{R}\mathcal{H}om(\mathscr{G}, \mathscr{O}_{\hat{A}})$.

Before we get to the proof of Hacon's theorem, let us first see why it implies the generic vanishing theorem. From Theorem 4.6, we already know that the coherent sheaves $R^j f_*\omega_X$ satisfy the vanishing

$$H^i\big(A, L \otimes R^j f_*\omega_X\big) = 0 \quad \text{for every ample } L \text{ and every } i > 0.$$

It is not hard to strengthen this to the condition in (c).

Lemma 25.6. *Let* $\varphi \colon B \to A$ *be a finite morphism of abelian varieties. Then*

$$H^i\big(B, L \otimes \varphi^* R^j f_*\omega_X\big) = 0$$

for every ample line bundle L *on* B *and every* $i > 0$.

Proof. If we let $Y = B \times_A X$ be the fiber product, we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & X \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow{\varphi} & A \end{array}$$

in which ψ is also finite étale. By flat base change,

$$\varphi^* R^j f_*\omega_X \simeq R^j g_*\psi^*\omega_X \simeq R^j g_*\omega_Y,$$

and so the assertion follows from Theorem 23.6, applied to the morphism g. $\square$

Combining this lemma with (25.3) and Theorem 25.5, we conclude that

$$\operatorname{codim} S^i(X, \omega_X) \geq i - k,$$

which proves the generic vanishing theorem.

Proof of Hacon's theorem. We now give the proof of Theorem 25.5. Throughout, $\mathscr{F}$ denotes a fixed coherent sheaf on the abelian variety A. We first prove that (a) and (b) are equivalent to each other; this is actually a direct consequence of the base change theorem. Recall that

$$\mathbf{R}\Phi_P(\mathscr{F}) = \mathbf{R}p_{2*}\big(p_1^*\mathscr{F} \otimes P\big);$$

the restriction of $p_1^*\mathscr{F} \otimes P$ to the fiber $p_2^{-1}(\alpha) = A \times \{\alpha\}$ is therefore isomorphic to $\mathscr{F} \otimes P_\alpha$, and so the groups

$$H^i(A, \mathscr{F} \otimes P_\alpha)$$

are precisely the fiberwise cohomology groups. By the base change theorem,

$$\bigcup_{i \geq n} \operatorname{Supp} R^i\Phi_P(\mathscr{F}) = \bigcup_{i \geq n} \big\{\, \alpha \in \hat{A} \;\big|\; H^i(A, \mathscr{F} \otimes P_\alpha) \neq 0 \,\big\}.$$

From this, one can deduce quite easily (by descending induction on $i \geq 0$) that (a) and (b) are equivalent.

Next, let us see why (b) implies (c). To simplify the notation, we shall only do the case $\varphi = \mathrm{id}$; the general case is similar. Let $G = \mathbf{R}\Phi_P(\mathscr{F})$ denote the Fourier-Mukai transform of $\mathscr{F}$. According to Mukai's Theorem 24.6,

$$\mathscr{F} \simeq \iota^*\mathbf{R}\Psi_P(G)[g],$$

and so we can recover $\mathscr{F}$, up to canonical isomorphism, from its Fourier-Mukai transform. Now let L be any ample line bundle on A; then

$$H^i(A, \mathscr{F} \otimes L) \simeq \operatorname{Hom}\bigl(\mathscr{O}_A, \mathscr{F} \otimes L[i]\bigr) \simeq \operatorname{Hom}\Bigl(\mathscr{O}_A, L \otimes \iota^* \mathbf{R}\Psi_P(G)[g+i]\Bigr),$$

where Hom means the morphisms in the derived category $\mathrm{D}^b_{coh}(\mathscr{O}_A)$. After replacing the original ample line bundle L by $\iota^* L$, which is of course equally ample, we therefore need to prove that

$$H^i(A, \mathscr{F} \otimes \iota^* L) \simeq \operatorname{Hom}\bigl(\mathscr{O}_A, L \otimes \mathbf{R}\Psi_P(G)[g+i]\bigr) = 0$$

for every $i > 0$. Using adjointness and the projection formula,

$$\begin{aligned} \operatorname{Hom}\bigl(\mathscr{O}_A, L \otimes \mathbf{R}\Psi_P(G)[g+i]\bigr) &\simeq \operatorname{Hom}\bigl(\mathscr{O}_{A \times \hat{A}}, P \otimes p_1^* L \otimes p_2^* G[g+i]\bigr) \\ &\simeq \operatorname{Hom}\bigl(\mathscr{O}_{\hat{A}}, \mathbf{R}\Phi_P(L) \otimes G[g+i]\bigr). \end{aligned}$$

We know from Proposition 24.9 that $\mathscr{E}_L = \mathbf{R}\Phi_P(L)$ is a locally free sheaf of rank $\dim H^0(A, L)$; this gives

$$H^i(A, \mathscr{F} \otimes \iota^* L) \simeq R^{g+i}\Gamma\bigl(\hat{A}, \mathscr{E}_L \otimes \mathbf{R}\Phi_P(\mathscr{F})\bigr).$$

We can now get the desired vanishing from dimension considerations. As usual, we make use of the hypercohomology spectral sequence

$$E_2^{p,q} = H^p\bigl(\hat{A}, \mathscr{E}_L \otimes R^q\Phi_P(\mathscr{F})\bigr) \Longrightarrow H^{p+q-g}(A, \mathscr{F} \otimes \iota^* L).$$

By assumption, the dimension of $\operatorname{Supp} R^q\Phi_P(\mathscr{F})$ is at most $g - q$; for that reason, $E_2^{p,q} = 0$ once $p > g - q$. We conclude that $H^i(A, \mathscr{F} \otimes \iota^* L) = 0$ for $i > 0$.

Continuing with the proof, we have to show that (c) implies (d). This is a long (but purely formal) calculation with derived functors. We first note that

$$\mathbf{R}\mathcal{H}om\Bigl(\mathbf{R}\mathcal{H}om(\mathscr{G}, \mathscr{O}_{\hat{A}}), \mathscr{O}_{\hat{A}}\Bigr) \simeq \mathscr{G},$$

because the dual complex $\mathbf{R}\mathcal{H}om(\mathscr{G}, \mathscr{O}_{\hat{A}})$ is computed by taking a locally free resolution of $\mathbf{R}\Phi_P(\mathscr{F})$ and applying $\mathcal{H}om(-, \mathscr{O}_{\hat{A}})$ to each term. It is therefore sufficient to prove that $\mathbf{R}\mathcal{H}om\bigl(\mathbf{R}\Phi_P(\mathscr{F}), \mathscr{O}_{\hat{A}}\bigr)$ is isomorphic to a sheaf – in other words, that the cohomology sheaves

$$R^i\mathcal{H}om\bigl(\mathbf{R}\Phi_P(\mathscr{F}), \mathscr{O}_{\hat{A}}\bigr)$$

are zero for $i \neq 0$. Since $\hat{A}$ is projective, we can decide whether or not this is the case with the help of a sufficiently ample line bundle.

Lemma 25.7. *Let X be a smooth projective variety, and $G \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ an object of the derived category. Then one has $\mathcal{H}^i(G) = 0$ if and only if $R^i\Gamma(X, G \otimes L) = 0$ for every sufficiently ample line bundle L.*

Proof. This is an easy consequence of Serre's theorems about ample line bundles. Consider the hypercohomology spectral sequence

$$E_2^{p,q} = H^p\bigl(X, \mathcal{H}^q(G) \otimes L\bigr) \Longrightarrow R^{p+q}\Gamma(X, G \otimes L).$$

If L is sufficiently ample, $E_2^{p,q} = 0$ for $p > 0$ by Serre's vanishing theorem, whence

$$H^0\bigl(X, \mathcal{H}^q(G) \otimes L\bigr) \simeq R^q\Gamma(X, G \otimes L).$$

Because the coherent sheaf $\mathcal{H}^q(G) \otimes L$ is generated by its global sections when L is sufficiently ample, the vanishing of $R^q\Gamma(X, G \otimes L)$ is therefore equivalent to that of $\mathcal{H}^q(G)$. $\square$

Returning to the proof that (c) implies (d), we therefore need to show that the complex of vector spaces

$$\mathbf{R}\Gamma\Big(\hat{A}, \mathbf{R}\mathcal{H}om\big(\mathbf{R}\Phi_P(\mathscr{F}), \mathscr{O}_{\hat{A}}\big) \otimes L\Big) \simeq \mathbf{R}\operatorname{Hom}\big(\mathbf{R}\Phi_P(\mathscr{F}), L\big)$$

has cohomology only in degree 0. Using the isomorphism between Ext-groups and morphisms in the derived category, we get

$$R^i \operatorname{Hom}\big(\mathbf{R}\Phi_P(\mathscr{F}), L\big) \simeq \operatorname{Hom}_{\mathrm{D}^b_{coh}(\mathscr{O}_{\hat{A}})}\big(\mathbf{R}\Phi_P(\mathscr{F}), L[i]\big);$$

given (c), this will turn out to be zero for every ample line bundle L and every $i \neq 0$. (From now on, all morphisms will be taken in the derived category, and so we shall leave out the subscript on Hom.) The proof is a calculation with the properties of various functors. To begin with, Grothendieck duality gives us

$$\operatorname{Hom}\big(\mathbf{R}\Phi_P(\mathscr{F}), L[i]\big) \simeq \operatorname{Hom}\big(P \otimes p_1^*\mathscr{F}, p_2^! L[i]\big) \simeq \operatorname{Hom}\big(P \otimes p_1^*\mathscr{F}, p_2^* L[g+i]\big),$$

where $g = \dim A$. We can rewrite this in the form

$$\operatorname{Hom}\big(p_1^*\mathscr{F}, P^{-1} \otimes p_2^* L[g+i]\big) \simeq \operatorname{Hom}\big(\mathscr{F}, \mathbf{R}p_{1*}(P^{-1} \otimes p_2^* L)[g+i]\big).$$

Now $P^{-1} \simeq (\mathrm{id} \times \iota)^* P$, and a simple base change argument shows that

$$\mathbf{R}p_{1*}(P^{-1} \otimes p_2^* L) \simeq \mathbf{R}p_{1*}(P \otimes p_2^* \iota^* L) \simeq \mathbf{R}\Psi_P\big(\iota^* L\big).$$

Combining the three preceding isomorphisms, we get

$$\operatorname{Hom}\big(\mathbf{R}\Phi_P(\mathscr{F}), L[i]\big) \simeq \operatorname{Hom}\big(\mathscr{F}, \mathbf{R}\Psi_P(\iota^* L)[g+i]\big).$$

Because $\iota^* L$ is still ample, we can replace $\iota^* L$ by L without affecting the statement we are trying to prove; in other words, it will be enough for us to show that

$$\operatorname{Hom}\big(\mathscr{F}, \mathbf{R}\Psi_P(L)[g+i]\big) = 0$$

for every ample line bundle L on $\hat{A}$ and every integer $i \neq 0$.

Now it is time to make use of the vanishing in (c). Consider the finite étale morphism $\varphi_L \colon \hat{A} \to A$ determined by the ample line bundle L. Recall from Proposition 24.9 that $\varphi_L^* \mathbf{R}\Psi_P(L) \simeq H^0(\hat{A}, L) \otimes L^{-1}$. Because φ_L is finite étale, the structure sheaf $\mathscr{O}_A$ is a direct summand of $\varphi_{L*}\mathscr{O}_{\hat{A}}$; together with the projection formula, this says that

$$\operatorname{Hom}\big(\mathscr{F}, \mathbf{R}\Psi_P(L)[g+i]\big)$$

is a direct summand of

$$\operatorname{Hom}\big(\varphi_L^*\mathscr{F}, \varphi_L^* \mathbf{R}\Psi_P(L)[g+i]\big) \simeq \operatorname{Hom}\big(\varphi_L^*\mathscr{F}, L^{-1}[g+i]\big) \otimes H^0(\hat{A}, L).$$

We are thus reduced to proving the vanishing of $\operatorname{Hom}\big(\varphi_L^*\mathscr{F}, L^{-1}[g+i]\big)$ for $i \neq 0$. By Serre duality, this is equivalent to the vanishing of

$$\operatorname{Hom}\big(L^{-1}[g+i], \varphi_L^*\mathscr{F} \otimes \omega_A[g]\big) \simeq \operatorname{Hom}\big(\mathscr{O}_{\hat{A}}, L \otimes \varphi_L^*\mathscr{F}[-i]\big) \simeq H^{-i}(\hat{A}, L \otimes \varphi_L^*\mathscr{F}).$$

For $i > 0$, this is obvious because $\mathscr{F}$ is a sheaf; for $i < 0$, it follows from (c). This concludes the proof that (c) implies (d).

It remains to show that (d) implies (b); this implication is a general fact about coherent sheaves on nonsingular varieties. By assumption, the cohomology sheaves of the Fourier-Mukai transform satisfy

$$R^i\Phi_P(\mathscr{F}) \simeq \mathcal{E}xt^i(\mathscr{G}, \mathscr{O}_{\hat{A}}).$$

Now $\hat{A}$ is a nonsingular scheme, and so the local ring at every (possibly non-closed) point is a regular. Because $\mathscr{G}$ is coherent, we can apply the following result from commutative algebra to conclude that the support of $\mathcal{E}xt^i(\mathscr{G}, \mathscr{O}_{\hat{A}})$ has codimension at least i.

Lemma 25.8. *Let $(A, \mathfrak{m})$ be a regular local ring. If M is a finitely generated A-module, then* $\operatorname{codim} \operatorname{Supp} \operatorname{Ext}^i_A(M, A) \geq i$ *for every $i \in \mathbb{Z}$.*

Proof. The support of $\operatorname{Ext}^i_A(M, A)$ consists of all prime ideals $P \subseteq A$ with the property that

$$\operatorname{Ext}^i_A(M, A) \otimes_A A_P \simeq \operatorname{Ext}^i_{A_P}(M_P, A_P) \neq 0;$$

and the codimension of the closed subscheme determined by P is equal to $\dim A_P$. After replacing A by A_P (which is still regular) and M by M_P, it is therefore enough to show that $\operatorname{Ext}^i_A(M, A) \neq 0$ implies that $\dim A \geq i$; we shall prove the equivalent statement that

$$\operatorname{Ext}^i_A(M, A) = 0 \quad \text{for } i > \dim A.$$

This follows from the fact that M has a free resolution of length at most $\dim A$, which is a consequence of the Auslander-Buchsbaum formula $\operatorname{pd}_A M + \operatorname{depth}_A M = \dim A$. Here is a direct proof that $\operatorname{pd}_A M \leq \dim A$. Take a minimal free resolution

$$\dots \to F_2 \to F_1 \to F_0 \to M \to 0$$

of the A-module A; recall that this means that all differentials in the complex $F_\bullet$ have entries in the maximal ideal. With $k = A/\mathfrak{m}$, one has

$$\operatorname{rk} F_i = \dim_k \bigl(F_i \otimes_A k\bigr) = \dim_k \operatorname{Tor}_i^A(M, k),$$

and so it suffices to show that $\operatorname{Tor}_i^A(M, k) = 0$ for $i > \dim A$; the trick is that Tor can also be computed from a free resolution of k. Because A is regular, the maximal ideal $\mathfrak{m}$ is generated by a regular sequence of length $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$, and the Koszul complex gives a free resolution of k of the same length. From this resolution, it is obvious that $\operatorname{Tor}^i_A(M, k) = 0$ for $i > \dim A$; as explained above, it follows that $F_i = 0$, and hence also that $\operatorname{Ext}^i_A(M, A) = 0$, as soon as $i > \dim A$. $\square$

Lecture 26

A conjecture by Green-Lazarsfeld. One advantage of Hacon's approach is that it gives certain additional results that are hard to get by classical means. For instance, Green and Lazarsfeld had made the following conjecture at the end of their second paper: when X has maximal Albanese dimension, the higher direct image sheaves of the universal line bundle on $X \times \operatorname{Pic}^0(X)$ should vanish in all degrees below $\dim X$. Hacon showed that this is the case; in fact, he got the following stronger result.

Theorem 26.1. *Let X be a smooth complex projective variety, and let P_X denote a universal line bundle on $X \times \operatorname{Pic}^0(X)$. Then one has*

$$R^i p_{2*} P_X = 0 \quad \text{for } i < \dim \operatorname{alb}(X),$$

where $\operatorname{alb}\colon X \to \operatorname{Alb}(X)$ *is the Albanese mapping of X.*

This can be proved in a similar manner as Theorem 25.5.

Exercise 26.1. Prove Theorem 26.1 by showing that

$$R^i\Gamma\big(\hat{A}, L \otimes \mathbf{R}p_{2*}P_X\big) = 0$$

for every $i < \dim f(X)$ and every ample line bundle L on $\hat{A}$.

In fact, one can be a bit more precise. If we denote the Albanese mapping of X by the letter $f\colon X \to A$, then we showed last time that

$$\mathbf{R}\Phi_P(R^j f_* \omega_X) \simeq \mathbf{R}\mathcal{H}om(\mathscr{G}_j, \mathscr{O}_{\hat{A}})$$

for certain coherent sheaves $\mathscr{G}_j$ on the dual abelian variety $\hat{A}$. If we denote as usual by $\iota\colon \hat{A} \to \hat{A}$ the inverse morphism, then what is true is that

$$R^{\dim X - j} p_{2*} P_X \simeq \iota^* \mathscr{G}_j, \tag{26.2}$$

and so Theorem 26.1 follows from the fact that, because of Kollár's theorem, $\mathscr{G}_j = 0$ outside the range $0 \le j \le \dim X - \dim f(X)$. This means that the conjecture of Green and Lazarsfeld is actually *equivalent* to the generic vanishing theorem – but this only becomes clear if one uses the formalism of derived categories.

Proof of Theorem 26.1. Let us prove the more precise result in (26.2). If we use the same base point for normalizing the universal line bundle P_X and the Albanese mapping $f\colon X \to A$, we have $P_X \simeq (f \times \mathrm{id})^* P$, where P is the Poincaré bundle on $A \times \hat{A}$. To keep the proof short, we shall use the following local version of Grothendieck duality: if $f\colon X \to Y$ is a projective morphism between two smooth varieties, then one has

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_Y}\big(\mathbf{R}f_*F, G\big) \simeq \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathscr{O}_X}\big(F, \omega_{X/Y}[\dim X - \dim Y] \otimes \mathbf{L}f^*G\big),$$

and the isomorphism is functorial in $F \in \mathrm{D}^b_{coh}(\mathscr{O}_X)$ and $G \in \mathrm{D}^b_{coh}(\mathscr{O}_Y)$.

$$\begin{tikzcd}
X \arrow[d, "f"] & X \times \hat{A} \arrow[l, "p_1"'] \arrow[d, "f \times \mathrm{id}"] \arrow[dr, bend left, "p_2"] & \\
A & A \times \hat{A} \arrow[l, "p_1"'] \arrow[r, "p_2"] & \hat{A}
\end{tikzcd}$$

If we apply the local version of Grothendieck duality to the second projection $p_2\colon X \times \hat{A} \to \hat{A}$, whose relative dimension is $n = \dim X$, we get

$$\mathbf{R}\mathcal{H}om\big(\mathbf{R}p_{2*}P_X, \mathscr{O}_{\hat{A}}\big) \simeq \mathbf{R}p_{2*}\mathbf{R}\mathcal{H}om\big(P_X, p_1^*\omega_X[n]\big) \simeq \mathbf{R}p_{2*}\Big(P_X^{-1} \otimes p_1^*\omega_X[n]\Big).$$

For the remainder of the calculation, please refer to the commutative diagram above. We first use the projection formula to rewrite the right-hand side as

$$\begin{aligned}\mathbf{R}p_{2*}\Big(P_X^{-1}\otimes p_1^*\omega_X[n]\Big) &\simeq \mathbf{R}p_{2*}\mathbf{R}(f\times \mathrm{id})_*\Big((f\times\mathrm{id})^*P^{-1}\otimes p_1^*\omega_X[n]\Big)\\ &\simeq \mathbf{R}p_{2*}\Big(P^{-1}\otimes \mathbf{R}(f\times\mathrm{id})_*p_1^*\omega_X[n]\Big)\\ &\simeq \mathbf{R}p_{2*}\Big(P^{-1}\otimes p_1^*\mathbf{R}f_*\omega_X[n]\Big),\end{aligned}$$

where the last isomorphism involves flat base change. Now Kollár's theorem gives

$$\mathbf{R}p_{2*}\Big(P^{-1}\otimes p_1^*\mathbf{R}f_*\omega_X[n]\Big)\simeq\bigoplus_{j=0}^{k}\mathbf{R}p_{2*}\Big(P^{-1}\otimes R^jf_*\omega_X\Big)[n-j],$$

where $k=\dim X-\dim f(X)$ is again the dimension of the general fiber of the Albanese mapping. Since $P^{-1}\simeq(\mathrm{id}\times\iota)^*P$, it is not hard to see that

$$\mathbf{R}p_{2*}\Big(P^{-1}\otimes R^jf_*\omega_X\Big)\simeq\iota^*\mathbf{R}\Phi_P(R^jf_*\omega_X)\simeq\iota^*\mathbf{R}\mathcal{H}om(\mathscr{G}_j,\mathscr{O}_{\hat{A}}).$$

Putting everything together, we have shown that

$$\mathbf{R}\mathcal{H}om\big(\mathbf{R}p_{2*}P_X,\mathscr{O}_{\hat{A}}\big)\simeq\bigoplus_{j=0}^{k}\iota^*\mathbf{R}\mathcal{H}om(\mathscr{G}_j,\mathscr{O}_{\hat{A}})[n-j].$$

After dualizing again, this is clearly equivalent to

$$\mathbf{R}p_{2*}P_X\simeq\bigoplus_{j=0}^{k}\iota^*\mathscr{G}_j[j-n],$$

and so we get the asserted isomorphisms by passing to cohomology. □

GV-sheaves. The four equivalent conditions in Theorem 25.5 describe a certain class of coherent sheaves on an abelian variety; following Pareschi and Popa, we shall refer to them as *GV-sheaves.*

Definition 26.3. A coherent sheaf $\mathscr{F}$ on a complex abelian variety A is said to be a *GV-sheaf* if its cohomology support loci

$$S^i(A,\mathscr{F})=\big\{\,\alpha\in\hat{A}\;\big|\;H^i(A,\mathscr{F}\otimes P_\alpha)\neq 0\,\big\}$$

satisfy the inequalities $\operatorname{codim} S^i(A,\mathscr{F})\geq i$ for every $i\geq 0$.

We will see below that most of the results connected with the generic vanishing theorem – with the exception of structural results such as Theorem 11.1 – are true for arbitrary GV-sheaves; this makes them very useful in applications. Let me remind you that, according to Theorem 25.5 from last time, each of the following three conditions is equivalent to $\mathscr{F}$ being a GV-sheaf:

(a) The Fourier-Mukai transform $\mathbf{R}\Phi_P(\mathscr{F})$ satisfies

$$\operatorname{codim}\operatorname{Supp}R^i\Phi_P(\mathscr{F})\geq i\quad\text{for all } i\geq 0.$$

(b) For every finite étale morphism $\varphi\colon B\to A$ of abelian varieties, and every ample line bundle L on B, one has

$$H^i(B,L\otimes\varphi^*\mathscr{F})=0\quad\text{for } i>0.$$

(c) There is a coherent sheaf $\mathscr{G}$ with the property that $\mathbf{R}\Phi_P(\mathscr{F})\simeq\mathbf{R}\mathcal{H}om(\mathscr{G},\mathscr{O}_{\hat{A}})$.

In practice, to show that something is a GV-sheaf, one typically uses the second condition; on the other hand, the third condition is what is responsible for many of the interesting properties of GV-sheaves. The reason is that

$$R^i\Phi_P(\mathscr{F}) \simeq \mathscr{E}xt^i(\mathscr{G}, \mathscr{O}_{\hat{A}}),$$

and because both sides are computed by taking an (injective respectively locally free) resolution of a single coherent sheaf, this leads to many nontrivial relations.

Here are some examples of GV-sheaves. The first one already appeared in Hacon's proof of the generic vanishing theorem.

Example 26.4. If $f\colon X \to A$ is a morphism from a smooth projective variety X to an abelian variety A, then the sheaves $R^i f_*\omega_X$ are GV-sheaves. As we saw earlier (in Lemma 25.6), this follows more or less immediately from Kollár's theorem.

The following generalization will be useful for us later.

Example 26.5. Continuing with the previous example, suppose that $L \in \operatorname{Pic}(X)$ is a line bundle with $L^d \simeq \mathscr{O}_X$. Then the sheaves $R^i f_*(\omega_X \otimes L)$ are GV-sheaves. To prove this, we use the fact that L determines a finite étale covering $p\colon Y \to X$ of degree d, and that

$$p_*\mathscr{O}_Y \simeq \mathscr{O}_X \oplus L^{-1} \oplus \cdots \oplus L^{-(d-1)}.$$

This was the content of Proposition 4.2. Because Y is étale over X, we get

$$p_*\omega_Y \simeq p_*p^*\omega_X \simeq \omega_X \otimes p_*\mathscr{O}_Y;$$

in particular, $\omega_X \otimes L$ is a direct summand of $p_*\omega_Y$. This means that $R^i f_*(\omega_X \otimes L)$ is a direct summand of $R^i(f \circ p)_*\omega_Y$, and therefore itself a GV-sheaf.

Another very elementary class of examples are ample line bundles.

Example 26.6. Any ample line bundle L on A is a GV-sheaf. This is clear because we saw earlier on that $\mathbf{R}\Phi_P(L)$ is a locally free sheaf.

Properties of GV-sheaves. We shall now take a more careful look at the properties of GV-sheaves. The first property is suggested by a result of Green and Lazarsfeld about varieties of maximal Albanese dimension: as we showed in Proposition 8.15, the cohomology support loci of X satisfy

$$\operatorname{Pic}^0(X) \supseteq S^n(X) \supseteq \cdots \supseteq S^1(X) \supseteq S^0(X) = \{\mathscr{O}_X\},$$

provided that $\dim \operatorname{alb}(X) = n = \dim X$. The exact same result is true for arbitrary GV-sheaves; the difference in indexing comes from the fact that, by Serre duality, $S^i(X, \omega_X) = -S^{n-i}(X)$.

Proposition 26.7. *Let $\mathscr{F}$ be a GV-sheaf on an abelian variety A. Then*

$$\hat{A} \supseteq S^0(A, \mathscr{F}) \supseteq S^1(A, \mathscr{F}) \supseteq \cdots \supseteq S^g(A, \mathscr{F}).$$

Proof. The assertion is that $H^i(\hat{A}, \mathscr{F} \otimes P_\alpha) = 0$ implies $H^{i+1}(\hat{A}, \mathscr{F} \otimes P_\alpha) = 0$. This turns out to be a formal consequence of the fact that $\mathbf{R}\Phi_P(\mathscr{F}) \simeq \mathbf{R}\mathscr{H}om(\mathscr{G}, \mathscr{O}_{\hat{A}})$ for a coherent sheaf $\mathscr{G}$ on $\hat{A}$. By the base change theorem,

$$H^i(A, \mathscr{F} \otimes P_\alpha) \simeq R^i\Gamma\bigl(\hat{A}, \mathbf{R}\Phi_P(\mathscr{F}) \overset{\mathbf{L}}{\otimes} \mathscr{O}_\alpha\bigr) \simeq R^i\Gamma\bigl(\hat{A}, \mathbf{R}\mathscr{H}om(\mathscr{G}, \mathscr{O}_{\hat{A}}) \overset{\mathbf{L}}{\otimes} \mathscr{O}_\alpha\bigr).$$

This may be rewritten in the form

$$R^i\Gamma\bigl(\hat{A}, \mathbf{R}\mathscr{H}om(\mathscr{G}, \mathscr{O}_\alpha)\bigr) \simeq H^0\bigl(\hat{A}, \mathscr{E}xt^i(\mathscr{G}, \mathscr{O}_\alpha)\bigr),$$

because the support of $\mathscr{E}xt^i(\mathscr{G}, \mathscr{O}_\alpha)$ is a point. This reduces the problem to to showing that $\mathscr{E}xt^i(\mathscr{G}, \mathscr{O}_\alpha) = 0$ implies $\mathscr{E}xt^{i+1}(\mathscr{G}, \mathscr{O}_\alpha) = 0$. After localizing at the point α, this follows from a general result about local rings; see the exercise below for details. $\square$

Exercise 26.2. Let $(A, \mathfrak{m})$ be a local ring with residue field $k = A/\mathfrak{m}$. Show that

$$\operatorname{Ext}^i_A(M,k) = 0 \quad \text{implies} \quad \operatorname{Ext}^{i+1}_A(M,k) = 0$$

for every finitely generated A-module M. (*Hint:* Use a minimal free resolution.)

By definition, we also have $\operatorname{codim} S^i(A, \mathscr{F}) \geq i$ for all $i \geq 0$. One surprising property of GV-sheaves is that this inequality cannot always be strict.

Lemma 26.8. *If $\mathscr{F}$ is a nonzero GV-sheaf on A, then one has* $\operatorname{codim} S^i(A, \mathscr{F}) = i$ *for at least one value of $0 \leq i \leq g$.*

Proof. If $\operatorname{codim} S^i(A, \mathscr{F}) > i$ for every $i \geq 0$, then by base change, one also has $\operatorname{codim} \operatorname{Supp} R^i\Phi_P(\mathscr{F}) > i$ for every $i \geq 0$. By the same reasoning as in the proof of Theorem 25.5, this collection of inequalities implies that

$$H^0(A, \mathscr{F} \otimes \iota^* L) = 0$$

for every ample line bundle L on A. Let us briefly recall the argument. If $G = \mathbf{R}\Phi_P(\mathscr{F})$ denotes the Fourier-Mukai transform of $\mathscr{F}$, then according to Mukai's Theorem 24.6,

$$\mathscr{F} \simeq \iota^* \mathbf{R}\Psi_P(G)[g].$$

Based on this isomorphism, a calculation with base change and the projection formula shows that

$$H^0(A, \mathscr{F} \otimes \iota^* L) \simeq R^g\Gamma\big(\hat{A}, \mathscr{E}_L \otimes \mathbf{R}\Phi_P(\mathscr{F})\big),$$

where $\mathscr{E}_L$ is the locally free sheaf that we get by taking the Fourier-Mukai transform of the ample line bundle L. By assumption, the dimension of $\operatorname{Supp} R^q\Phi_P(\mathscr{F})$ is strictly less than $g - q$; in the hypercohomology spectral sequence

$$E_2^{p,q} = H^p\big(\hat{A}, \mathscr{E}_L \otimes R^q\Phi_P(\mathscr{F})\big) \Longrightarrow R^{p+q}\Gamma\big(\hat{A}, \mathscr{E}_L \otimes \mathbf{R}\Phi_P(\mathscr{F})\big),$$

we therefore have $E_2^{p,q} = 0$ for $p + q \geq g$ (for dimension reasons). We conclude that $H^0(A, \mathscr{F} \otimes \iota^* L) = 0$; taking L sufficiently ample, this forces $\mathscr{F} = 0$. $\square$

For most GV-sheaves, one has $S^0(A, \mathscr{F}) = \hat{A}$, and so equality happens for $i = 0$.

Example 26.9. A typical example is the canonical bundle ω_X of a smooth subvariety $i\colon X \hookrightarrow A$ that is not fibered in subtori. In that case, ω_X is known to be ample, and so $H^0(X, \omega_X \otimes i^* P_\alpha) \neq 0$ for every $\alpha \in \hat{A}$. Thus $S^0(A, i_*\omega_X) = \hat{A}$.

Recall from the discussion after Theorem 26.1 that the conjecture of Green and Lazarsfeld about the vanishing of certain cohomology sheaves of $\mathbf{R}p_{2*}P_X$ turned out to be a consequence of the generic vanishing theorem. Something similar happens for arbitrary GV-sheaves: if it happens that $S^0(A, \mathscr{F}) \neq \hat{A}$, then the Fourier-Mukai transform $\mathbf{R}\Phi_P(\mathscr{F})$ is forced to be concentrated in certain degrees.

Proposition 26.10. *One has $R^i\Phi_P(\mathscr{F}) = 0$ for every $i <$* $\operatorname{codim} S^0(A, \mathscr{F})$.

Proof. Recall that $\mathbf{R}\Phi_P(\mathscr{F}) \simeq \mathbf{R}\mathcal{H}om(\mathscr{G}, \mathscr{O}_{\hat{A}})$ for a coherent sheaf $\mathscr{G}$ on $\hat{A}$. Now

$$\begin{aligned} \operatorname{Supp} \mathscr{G} &= \operatorname{Supp} \mathbf{R}\Phi_P(\mathscr{F}) \\ &= \bigcup_{i \geq 0} \operatorname{Supp} R^i\Phi_P(\mathscr{F}) = \bigcup_{i \geq 0} S^i(A, \mathscr{F}) = S^0(A, \mathscr{F}) \end{aligned} \tag{26.11}$$

by the base change theorem and the fact that $S^i(A, \mathscr{F}) \subseteq S^0(A, \mathscr{F})$ for all $i \geq 0$. Therefore the assertion is that

$$R^i\Phi_P(\mathscr{F}) \simeq \mathscr{E}xt^i(\mathscr{G}, \mathscr{O}_{\hat{A}}) = 0$$

for every $i < \operatorname{codim} \operatorname{Supp} \mathscr{G}$. Since $\hat{A}$ is nonsingular, this follows from the following general result in commutative algebra: Let $(A, \mathfrak{m})$ be a regular (or just Cohen-Macaulay) local ring, and let M be a finitely generated A-module. Then

$$\min\left\{ i \geq 0 \;\middle|\; \operatorname{Ext}^i_A(M, A) \neq 0 \right\} = \operatorname{codim} \operatorname{Supp} M = \dim A / \operatorname{Ann}(M).$$

The quantity on the left is sometimes called the *grade* of M. The proof is basically by induction on the codimension of $\operatorname{Supp} M$, starting from the fact that M a torsion module exactly when $\operatorname{Hom}(M, A) = 0$. You can find the argument in §16 and §17 of Matsumura's book. $\square$

Exercise 26.3. Give a geometric proof for the fact that

$$\min\left\{ i \geq 0 \;\middle|\; \mathscr{E}xt^i(\mathscr{G}, \mathscr{O}_X) \neq 0 \right\} \geq \operatorname{codim} \operatorname{Supp} \mathscr{G},$$

where $\mathscr{G}$ is a coherent sheaf on a smooth projective variety X.

The result above also has a "local" variant that is very useful in practice. The idea is that, instead of imposing a condition on all of $S^0(A, \mathscr{F})$, we only consider one irreducible component.

Proposition 26.12. *Suppose that $Z \subseteq S^0(A, \mathscr{F})$ is an irreducible component of codimension k. Then Z is actually an irreducible component of $S^k(A, \mathscr{F})$; in particular, we must have* $\dim \operatorname{Supp} \mathscr{F} \geq k$.

Proof. Since we have already convinced ourselves that

$$\operatorname{Supp} \mathscr{G} = \operatorname{Supp} \mathbf{R}\Phi_P(\mathscr{F}) = S^0(A, \mathscr{F}),$$

we know that Z is also an irreducible component of $\operatorname{Supp} \mathscr{G}$. By applying the same argument as before to the local ring at the generic point of Z, we deduce that the sheaves $R^i\Phi_P(\mathscr{F}) \simeq \mathscr{E}xt^i(\mathscr{G}, \mathscr{O}_{\hat{A}})$ for $i < k$ have to be zero in a Zariski-open neighborhood of the generic point of Z. Now it follows from (26.11) that Z is an irreducible component of $\operatorname{Supp} R^k\Phi_P(\mathscr{F})$, and therefore also of $S^k(A, \mathscr{F})$. Because

$$S^k(A, \mathscr{F}) = \left\{ \alpha \in \hat{A} \;\middle|\; H^k(A, \mathscr{F} \otimes P_\alpha) \neq 0 \right\},$$

this can only happen if $\dim \operatorname{Supp} \mathscr{F} \geq k$. $\square$

This result can be put to use in several ways. If we happen to know that $S^0(A, \mathscr{F})$ contains an isolated point, then the same point must also lie in $S^g(A, \mathscr{F})$, where $g = \dim A$; for dimension reasons, this means that $\mathscr{F}$ has to be supported on all of A. In applications, $\mathscr{F}$ is typically of the form $R^i f_* \omega_X$, and the support of $\mathscr{F}$ is therefore equal to the image $f(X)$; in the situation above, we could then conclude for instance that f has to be surjective.

LECTURE 27

Applications of GV-sheaves. From now on, we are going to study several applications of the theory of GV-sheaves. Let me remind you briefly of the basic properties of GV-sheaves that we proved last time. If $\mathscr{F}$ is a GV-sheaf on an abelian variety A, then:

(1) One has $\operatorname{codim} S^i(A, \mathscr{F}) \geq i$ for all $i \geq 0$ (by definition).
(2) Moreover, $\operatorname{codim} S^i(A, \mathscr{F}) = i$ for some $0 \leq i \leq \dim A$, unless $\mathscr{F} = 0$.
(3) One has $S^0(A, \mathscr{F}) \supseteq S^1(A, \mathscr{F}) \supseteq \cdots \supseteq S^{\dim A}(A, \mathscr{F})$.
(4) If $Z \subseteq S^0(A, \mathscr{F})$ is a component of codimension k, then $Z \subseteq S^k(A, \mathscr{F})$; in particular, $\dim \operatorname{Supp} \mathscr{F} \geq k$.
(5) Finally, $R^i \Phi_P(\mathscr{F}) = 0$ for every $i < \operatorname{codim} S^0(A, \mathscr{F})$.

Perhaps the most interesting application is the birational characterization of abelian varieties by Jungkai Chen and Christopher Hacon. As a warm-up exercise, let us first see how GV-sheaves can be used to give algebraic proofs for some of the results of Green and Lazarsfeld (that we discussed during the first half of the course).

For the time being, we let X be a smooth projective variety of dimension n, and $\operatorname{alb}\colon X \to \operatorname{Alb}(X)$ its Albanese mapping (for some choice of base point, irrelevant for what follows). Recall that every irreducible component of

$$S^i(X) = \left\{ L \in \operatorname{Pic}^0(X) \mid H^i(X, L) \neq 0 \right\}$$

is a translate of a subtorus (Theorem 11.1) by a point of finite order (Theorem 15.2); in this case, a subtorus is automatically an abelian variety because $\operatorname{Pic}^0(X)$ is projective. By Serre duality, $H^i(X, L)$ is dual to $H^{n-i}(X, \omega_X \otimes L^{-1})$, and so every irreducible component of

$$S^i(X, \omega_X) = \left\{ L \in \operatorname{Pic}^0(X) \mid H^i(X, \omega_X \otimes L) \neq 0 \right\} = -S^{n-i}(X)$$

is also a translate of an abelian variety by a point of finite order.

Our first application of GV-sheaves is to give a different proof for Theorem 12.4, which was itself a generalization of Beauville's Theorem 10.2.

Theorem 27.1. *Let $Z \subseteq S^i(X, \omega_X)$ be an irreducible component. Then there exists a normal projective variety Y with $\dim Y \leq n - i$, and a surjective morphism $g\colon X \to Y$ with connected fibers, such that Z is contained in a translate of $g^* \operatorname{Pic}^0(Y)$. Any resolution of singularities of Y has maximal Albanese dimension.*

Proof. Recall from Theorem 11.1 that Z is a translate of an abelian subvariety of $\operatorname{Pic}^0(X)$ by a torsion point $L \in \operatorname{Pic}^0(X)$. If we let A denote the dual abelian variety, we obtain a morphism

$$f\colon X \to A$$

by composing $\operatorname{Alb}(X) \to A$ with the Albanese morphism of X. We then have

$$H^i\big(X, \omega_X \otimes L \otimes f^* P_\alpha\big) \neq 0$$

for every $\alpha \in \hat{A}$. As explained last time, Kollár's theorem still applies to the sheaf $\omega_X \otimes L$ because L has finite order; consequently, we get

$$H^i\big(X, \omega_X \otimes L \otimes f^* P_\alpha\big) \simeq \bigoplus_{j=0}^{k} H^{i-j}\big(A, R^j f_*(\omega_X \otimes L) \otimes P_\alpha\big)$$

where now $k = \dim X - \dim f(X) = n - \dim f(X)$. Because the sheaves $R^j f_*(\omega_X \otimes L)$ on the right-hand side are GV-sheaves, the left-hand side can only be nonzero for every $\alpha \in \hat{A}$ if we have $i - j = 0$ for some $0 \leq j \leq k$; this implies that

$$i \leq k = n - \dim f(X)$$

or equivalently $\dim f(X) \leq n - i$. We now define $g\colon X \to Y$ by taking the Stein factorization of $f\colon X \to f(X)$; in the resulting diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend left, "f"] & Y \arrow[r, "h"] & A, \end{tikzcd}$$

h is a finite morphism, and g is a surjective morphism with connected fibers. By construction, $\dim Y = \dim f(X) \leq n - i$, and every resolution of singularities of Y has maximal Albanese dimension; it is also clear from the diagram that the image of $f^*\colon \operatorname{Pic}^0(A) \to \operatorname{Pic}^0(X)$ is contained in that of $g^*\colon \operatorname{Pic}^0(Y) \to \operatorname{Pic}^0(X)$. □

The key point in the proof was to show that $\dim f(X) \leq n - i$; notice how this kind of numerical result follows very easily by combining the structure theorem for cohomology support loci with results about GV-sheaves. By almost the same method, we can reprove the result of Ein and Lazarsfeld about the Albanese image of varieties of maximal Albanese dimension with $\chi(X, \omega_X) = 0$.

Theorem 27.2. *Let X be a smooth projective variety of maximal Albanese dimension. If $\chi(X, \omega_X) = 0$, then the Albanese image* $\operatorname{alb}(X)$ *is fibered by abelian varieties.*

Proof. Since X has maximal Albanese dimension, the Albanese mapping $\operatorname{alb}\colon X \to \operatorname{Alb}(X)$ is generically finite over its image, and so $R^i \operatorname{alb}_* \omega_X = 0$ for $i > 0$ by Kollár's theorem. In particular, we have

$$S^i(X, \omega_X) = S^i\bigl(\operatorname{Alb}(X), \operatorname{alb}_* \omega_X\bigr)$$

for every $0 \leq i \leq n$. Because $\operatorname{alb}_* \omega_X$ is a GV-sheaf, $S^i(X, \omega_X)$ is a proper subvariety of $\operatorname{Pic}^0(X)$ for $i > 0$; the condition $\chi(X, \omega_X) = 0$ says exactly that $S^0(X, \omega_X)$ is also a proper subvariety of $\operatorname{Pic}^0(X)$. We can now apply Proposition 26.12: for some $i \geq 1$, the locus $S^i(X, \omega_X) = S^i\bigl(\operatorname{Alb}(X), \operatorname{alb}_* \omega_X\bigr)$ must have an irreducible component of codimension i. As before, such a component gives rise to a morphism

$$\begin{tikzcd} X \arrow[r, "\operatorname{alb}"] \arrow[dr, bend right, "f"'] & \operatorname{Alb}(X) \arrow[d, "p"] \\ & A \end{tikzcd}$$

to an abelian variety A such that $\dim f(X) \leq n - i$. Because $\dim A = g - i$, the fibers of p are finite unions of abelian varieties of dimension i; on the other hand, every fiber of $p\colon \operatorname{alb}(X) \to f(X)$ has dimension at least

$$\dim \operatorname{alb}(X) - \dim f(X) = \dim X - \dim f(X) \geq i.$$

This can only happen if $\operatorname{alb}(X)$ is a union of connected components of fibers of p, and therefore a union of i-dimensional abelian varieties □

The birational characterization of abelian varieties. We now come to the most surprising application of GV-sheaves, namely the birational characterization of abelian varieties by Chen and Hacon. Before stating the theorem, let me briefly describe the background.

A fundamental problem in algebraic geometry is to characterize certain classes of varieties (up to birational equivalence) by their numerical invariants. In the case of surfaces, the Enriques-Kodaira classification shows that a minimal smooth projective surface S is abelian if and only if $\kappa(S) = 0$ and $q(S) = 2$. The first general result in higher dimensions is due to Kawamata, who showed that an n-dimensional smooth projective variety X is birational to an abelian variety exactly when $\kappa(X) = 0$ and $\dim H^1(X, \mathscr{O}_X) = n$. In essence, Kawamata proved that if $\kappa(X) = 0$, then

$$\mathrm{alb}\colon X \to \mathrm{Alb}(X)$$

is surjective with connected fibers; this is the first part of Ueno's Conjecture 12.7. If one knows in addition that $\dim \mathrm{Alb}(X) = n$, then alb must be birational, and so X is birational to an abelian variety.

Of course, this result is not "effective", because in order to be sure that $\kappa(X) = 0$, one has to know that all the plurigenera $P_m(X) = \dim H^0(X, \omega_X^{\otimes m})$ are bounded (and actually equal to 1 for every sufficiently large and divisible m). The first effective result is due to Kollár, who showed that $\dim H^1(X, \mathscr{O}_X) = n$ and $P_m(X) = 1$ for some $m \geq 3$ are sufficient to conclude that X is birational to an abelian variety. Kollár also conjectured the following optimal statement, which was subsequently proved by Chen and Hacon.

Theorem 27.3 (Chen-Hacon)**.** *Let X be a smooth projective variety with $P_1(X) = P_2(X) = 1$ and $\dim H^1(X, \mathscr{O}_X) = \dim X$. Then X is birational to an abelian variety.*

The arguments of Chen and Hacon were greatly simplified by Giuseppe Pareschi (in his nice survey article *Basic results on irregular varieties via Fourier-Mukai methods*); the resulting proof is another very pretty application of the theory of GV-sheaves.

Recall that, by the work of Ein and Lazarsfeld, $P_1(X) = P_2(X) = 1$ guarantees that the Albanese mapping of X is surjective. As a preparation for understanding Pareschi's proof, let us review their argument. The first step is to observe that the condition on plurigenera gives information about $S^0(X, \omega_X)$.

Proposition 27.4. *If $P_1(X) = P_2(X) = 1$, then the origin must be an isolated point of $S^0(X, \omega_X)$.*

Proof. Since $P_1(X) \neq 0$, we have $H^0(X, \omega_X) \neq 0$, and so $\mathscr{O}_X \in S^0(X, \omega_X)$. Suppose that it is not an isolated point. Then by Theorem 11.1, $S^0(X, \omega_X)$ contains an abelian variety Z of positive dimension. In particular, Z is a subgroup, and so if $L \in Z$, then also $L^{-1} \in Z$. This means that the image of the multiplication map

$$H^0(X, \omega_X \otimes L) \otimes H^0(X, \omega_X \otimes L^{-1}) \to H^0(X, \omega_X^{\otimes 2})$$

is nonzero for every $L \in Z$. Now $\omega_X^{\otimes 2}$ only has one global section because $P_2(X) = 1$; let D be the corresponding effective divisor on X. By the above, the divisor of any global section of $\omega_X \otimes L$ has to be contained in D; but because D has only finitely many irreducible components, we can find two distinct points $L_1, L_2 \in Z$, and nontrivial sections $s_1 \in H^0(X, \omega_X \otimes L_1)$ and $s_2 \in H^0(X, \omega_X \otimes L_2)$, such that $\operatorname{div} s_1 = \operatorname{div} s_2$. But then $\omega_X \otimes L_1 \simeq \omega_X \otimes L_2$, which contradicts the fact that L_1 and L_2 are distinct points. $\square$

The second step is the following geometric result; this time around, we can give a shorter proof based on Proposition 26.12.

Proposition 27.5. *If the origin is an isolated point of $S^0(X, \omega_X)$, then the Albanese mapping* $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ *is surjective.*

Proof. We always have $S^0(X, \omega_X) = S^0\bigl(\mathrm{Alb}(X), \mathrm{alb}_* \omega_X\bigr)$. Now $\mathrm{alb}_* \omega_X$ is a GV-sheaf on $\mathrm{Alb}(X)$; according to Proposition 26.12, the origin is therefore automatically a point of $S^g\bigl(\mathrm{Alb}(X), \mathrm{alb}_* \omega_X\bigr)$, where $g = \dim \mathrm{Alb}(X)$. In particular, we must have $\dim \mathrm{alb}(X) \geq g$, which means that alb is surjective. $\square$

If we assume in addition that $\dim \mathrm{Alb}(X) = \dim X$, then we can conclude from $P_1(X) = P_2(X) = 1$ that the Albanese mapping of X is generically finite and surjective; in particular, X has maximal Albanese dimension. This is almost what we want: to prove Theorem 27.3, we just need to show that the degree of alb is equal to 1.

Pareschi's criterion for birationality. Pareschi observed that the results about GV-sheaves lead to the following criterion for the Albanese mapping to be birational. (This leads to an interesting point about the set $S^0(X, \omega_X)$: the "local" property that $\mathscr{O}_X$ is an isolated point ensures that alb is generically finite; the "global" assumption that all points are isolated ensures that alb is birational.)

Proposition 27.6. *Let X be a smooth projective variety of maximal Albanese dimension such that* $\dim S^0(X, \omega_X) = 0$*. Then* $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ *is birational.*

Proof. To simplify the notation, we shall write $f\colon X \to A$ for the Albanese mapping of X. As explained above, f is surjective and generically finite; in particular, $g = \dim A = \dim X = n$. We have to prove that $\deg f = 1$; more precisely, we shall show that the sheaf $f_* \omega_X$ is isomorphic to $\mathscr{O}_A$.

To begin with, $f_* \omega_X$ is a GV-sheaf on A. Kollár's theorem shows that

$$S^i(X, \omega_X) = S^i(A, f_* \omega_X)$$

for every $i \geq 0$. By assumption, every point of $S^0(A, f_* \omega_X)$ is an isolated point, and therefore actually contained in $S^g(A, f_* \omega_X)$ by virtue of Proposition 26.12. But $S^g(A, f_* \omega_X) = S^n(X, \omega_X)$ consists of just the origin; we deduce that $S^i(A, f_* \omega_X) = \{0\}$ for every $0 \leq i \leq g$. Looking back at Proposition 26.10, it follows that the Fourier-Mukai transform $\mathbf{R}\Phi_P(f_* \omega_X)$ is supported at the origin in $\hat{A}$, and that $R^i \Phi_P(f_* \omega_X) = 0$ for $i < n$. A simple calculation gives

$$R^n \Phi_P(f_* \omega_X) \simeq R^n p_{2*}\big(P_X \otimes p_1^* \omega_X\big),$$

where $P_X = (f \times \mathrm{id})^* P$ is the universal line bundle on $X \times \mathrm{Pic}^0(X)$. By the same argument as in the proof of Lemma 24.5, one can show that this equals $\mathscr{O}_0$. Putting everything together, we find that

$$\mathbf{R}\Phi_P(f_* \omega_X) \simeq R^n \Phi_P(f_* \omega_X)[-n] \simeq \mathscr{O}_0[-n] \simeq \mathbf{R}\Phi_P(\mathscr{O}_A).$$

But we know from Theorem 24.6 that the Fourier-Mukai transform is an equivalence of categories; the conclusion is that $f_* \omega_X \simeq \mathscr{O}_A$, and hence that f is birational. ☐

Exercises.

Exercise 27.1. Describe all line bundles on an abelian variety that are GV-sheaves.

Exercise 27.2. Prove the following partial generalization of Proposition 24.3: Let X be a smooth projective variety of dimension n, and let P_X denote a universal line bundle on $X \times \mathrm{Pic}^0(X)$. Then one has $R^n p_{2*}\big(P_X \otimes p_1^* \omega_X\big) \simeq \mathscr{O}_0$. (*Hint:* Imitate the proof of Lemma 24.5.)

LECTURE 28

The proof of the Chen-Hacon theorem. In the first half of today's class, we shall finish the proof of Theorem 27.3. The statement was that if X is a smooth projective variety of dimension n, and if $P_1(X) = P_2(X) = 1$ and $\dim \mathrm{Alb}(X) = n$, then X must be birational to an abelian variety; more precisely, the Albanese mapping $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ must be birational. We have already seen how the two assumptions imply that the Albanese mapping is generically finite. Pareschi's criterion in Proposition 27.6 therefore reduces the problem to proving that $\dim S^0(X, \omega_X) = 0$. The following result describes what happens if $\dim S^0(X, \omega_X) \neq 0$.

Proposition 28.1. *Let X be a smooth projective variety of maximal Albanese dimension. If $\dim S^0(X, \omega_X) \neq 0$, then the intersection*

$$S^0(X, \omega_X) \cap \iota^* S^0(X, \omega_X)$$

also has positive dimension.

Proof. Let $Z \subseteq S^0(X, \omega_X)$ be an irreducible component of positive dimension, and let k denote its codimension; by construction, $k < g = \dim \mathrm{Alb}(X)$. We know from Theorem 15.2 that Z is a translate of an abelian variety by a torsion point $L \in \mathrm{Pic}^0(X)$. If Z contains the origin, then we are clearly done; for the remainder of the argument, we may therefore assume that Z does not contain the origin. By the same construction as in the proof of Theorem 27.1, we obtain a morphism

$$\begin{tikzcd} X \arrow[r, "\mathrm{alb}"] \arrow[dr, bend right, "f"'] & \mathrm{Alb}(X) \arrow[d, "p"] \\ & A \end{tikzcd}$$

to the dual abelian variety A, such that $\dim A = g - k$ and $\dim f(X) = n - k$. An important point is that we know the dimension of $f(X)$; this is where we need the assumption that X is of maximal Albanese dimension. The remainder of the proof divides itself into four steps.

Step 1. We show that $R^k f_*(\omega_X \otimes L) \neq 0$. Recall that Z has codimension k and is an irreducible component of

$$S^0(X, \omega_X) = S^0\big(\mathrm{Alb}(X), \mathrm{alb}_* \omega_X\big);$$

because $\mathrm{alb}_* \omega_X$ is a GV-sheaf, Z is automatically contained in

$$S^k(X, \omega_X) = S^k\big(\mathrm{Alb}(X), \mathrm{alb}_* \omega_X\big).$$

An application of Kollár's theorem shows that

$$H^k(X, \omega_X \otimes L \otimes f^* P_\alpha) \simeq \bigoplus_{j=0}^{k} H^{k-j}\big(A, P_\alpha \otimes R^j f_*(\omega_X \otimes L)\big).$$

By construction, the left-hand side is nonzero for every $\alpha \in \hat{A}$; in terms of cohomology support loci, this says that

$$\hat{A} = \bigcup_{j=0}^{k} S^{k-j}\big(A, R^j f_*(\omega_X \otimes L)\big).$$

But the sheaves $R^j f_*(\omega_X \otimes L)$ are GV-sheaves on A, and so the cohomology support loci with $k - j \geq 1$ are proper subvarieties of $\hat{A}$; this is only possible if

$$\hat{A} = S^0\big(A, R^k f_*(\omega_X \otimes L)\big).$$

In particular, the sheaf $R^k f_*(\omega_X \otimes L)$ has to be nonzero; recall from Kollár's theorem that it is torsion-free when viewed as a coherent sheaf on $f(X)$.

Step 2. We show that $R^k f_*(\omega_X \otimes L^{-1}) \neq 0$. The idea is to exploit the fact that the general fiber of $f\colon X \to f(X)$ is k-dimensional. Since the fibers may not be connected, consider the Stein factorization of f:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend left, "f"] & Y \arrow[r, "p"] & A \end{tikzcd}$$

Let F denote a general fiber of g; then F is a smooth projective variety of dimension k. We know that $R^k g_*(\omega_X \otimes L)$ is a nontrivial torsion-free sheaf on $g(X)$; by base change, it follows that

$$H^k\bigl(F, \omega_F \otimes L\big|_F\bigr) \simeq H^k\bigl(F, (\omega_X \otimes L)\big|_F\bigr) \neq 0.$$

Since $\dim F = k$, we can use Serre duality to conclude that

$$L \in \ker\bigl(\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(F)\bigr).$$

But the kernel is a group, and so it also contains L^{-1}; by running the same argument backwards, we get that $R^k f_*(\omega_X \otimes L^{-1}) \neq 0$, too.

Step 3. We produce a nontrivial subset of $\iota^* S^0(X, \omega_X)$. Observe that, because of the isomorphism

$$H^k(X, \omega_X \otimes L^{-1} \otimes f^* P_\alpha) \simeq \bigoplus_{j=0}^{k} H^{k-j}\bigl(A, P_\alpha \otimes R^j f_*(\omega_X \otimes L^{-1})\bigr),$$

the entire subset

$$L^{-1} \otimes f^* S^0\bigl(A, R^k f_*(\omega_X \otimes L^{-1})\bigr) \subseteq \iota^* Z$$

is contained in $S^k(X, \omega_X)$. But X has maximal Albanese dimension, and so Proposition 26.7 shows that it is also contained in $S^0(X, \omega_X)$. Since $\iota^* Z \subseteq \iota^* S^0(X, \omega_X)$, this reduces the whole problem to proving that

$$\dim S^0\bigl(A, R^k f_*(\omega_X \otimes L^{-1})\bigr) \geq 1.$$

Step 4. We prove that $S^0\bigl(A, R^k f_*(\omega_X \otimes L^{-1})\bigr)$ does not contain any isolated points. First of all, we observe that $R^k f_*(\omega_X \otimes L^{-1})$ is a nonzero GV-sheaf on A, and so

$$S^0\bigl(A, R^k f_*(\omega_X \otimes L^{-1})\bigr) \neq 0$$

by Lemma 26.8 and Proposition 26.7. Suppose that $\alpha \in S^0\bigl(A, R^k f_*(\omega_X \otimes L^{-1})\bigr)$ was an isolated point. Since $\dim A = g - k$, we could then apply Proposition 26.12 and conclude that

$$\alpha \in S^{g-k}\bigl(A, R^k f_*(\omega_X \otimes L^{-1})\bigr).$$

Because $\dim f(X) = n - k \geq g - k$, it would follow that $f(X) = A$ and $n = g$, and hence that

$$0 \neq H^{n-k}\bigl(A, P_\alpha \otimes R^k f_*(\omega_X \otimes L^{-1})\bigr) \subseteq H^n\bigl(X, \omega_X \otimes L^{-1} \otimes f^* P_\alpha\bigr);$$

the inclusion again comes from Kollár's theorem. But then $L^{-1} \otimes f^* P_\alpha$ would be the trivial line bundle, which would mean that Z contains the origin. Since we are assuming that this is not the case, $S^0\bigl(A, R^k f_*(\omega_X \otimes L^{-1})\bigr)$ cannot contain any isolated points. This finishes the proof. $\square$

Now suppose that $P_1(X) = P_2(X) = 1$. Then by the same argument is in the proof of Proposition 27.4, the intersection $S^0(X, \omega_X) \cap \iota^* S^0(X, \omega_X)$ cannot be of positive dimension. According to Proposition 28.1, $\dim S^0(X, \omega_X) = 0$; Pareschi's birationality criterion in Proposition 27.6 therefore applies, and so X is birational to its Albanese variety $\operatorname{Alb}(X)$.

Varieties of Kodaira dimension zero. Kawamata proved that if X is a smooth projective variety of Kodaira dimension zero, then $\mathrm{alb}\colon X \to \mathrm{Alb}(X)$ is surjective and has connected fibers. Last time, we reviewed the proof that $P_1(X) = P_2(X) = 1$ implies the surjectivity of the Albanese mapping. It turns out that this condition is also enough to ensure that the fibers are connected; this optimal result is due to Zhi Jiang.

Theorem 28.2. *Let X be a smooth projective variety with $P_1(X) = P_2(X) = 1$. Then the fibers of the Albanese mapping are connected.*

Proof. We already know that the Albanese mapping of X is surjective. Let us consider the Stein factorization

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{alb}"'] & Y \arrow[r, "p"] & \mathrm{Alb}(X); \end{tikzcd}$$

the assertion is that the finite morphism p is an isomorphism. Note that Y may be singular – all we know is that it is normal. After resolving singularities, we can arrange that Y is smooth and that p is generically finite; this does not change the fact that $P_1(X) = P_2(X) = 1$. Since $\dim Y = \dim \mathrm{Alb}(X)$, it is clear that Y has maximal Albanese dimension; moreover, the above factorization implies that $\mathrm{Pic}^0(X)$ is a factor of $\mathrm{Pic}^0(Y)$.

To prove the theorem, we need to show that Y is birational to an abelian variety; the universal property of the Albanese morphism will then imply that p is birational. As Y has maximal Albanese dimension, this is equivalent to the condition that $P_1(Y) = P_2(Y) = 1$. We are going to argue that $P_m(Y) \leq P_m(X)$; because of the assumptions on X, this will clearly do the job. The problem is that there is no direct relationship between sections of ω_Y and sections of ω_X; in fact, $n = \dim X$ is typically greater than $g = \dim Y$. We shall overcome this problem by proving that $\omega_X \otimes f^*\omega_Y^{-1}$ is effective. Unfortunately, I only know how to prove this with the help of Hodge theory – it would be nice to have a proof that only uses properties of GV-sheaves.

Let us first see how far we can get with the help of GV-sheaves. The starting point is Proposition 27.4, which says that the origin is isolated in

$$S^0(X, \omega_X) = S^0\big(\mathrm{Alb}(X), \mathrm{alb}_* \omega_X\big).$$

If we apply Proposition 26.12 to the GV-sheaf $\mathrm{alb}_* \omega_X$, we see that the origin must be contained in $S^g\big(\mathrm{Alb}(X), \mathrm{alb}_* \omega_X\big)$; together with Kollár's theorem, this says that $H^g(X, \omega_X) \neq 0$. Now Hodge theory and Serre duality show that

$$H^0(X, \Omega_X^{n-g}) \simeq \overline{H^{n-g}(X, \mathscr{O}_X)} \neq 0.$$

We would like to say that the (essentially unique) section of ω_X is the wedge product of a holomorphic $(n-g)$-form with the pullback of a section of ω_Y. For that, we need additional information about the space $H^0(X, \Omega_X^{n-g})$, and so we have to leave the theory of GV-sheaves and turn to Hodge theory.

We know that the origin is an isolated point in $S^0(X, \omega_X)$, and therefore also in the intersection $S^0(Y, f_*\omega_X) \cap \mathrm{Pic}^0(X)$; recall that $\mathrm{Pic}^0(X)$ is a factor of $\mathrm{Pic}^0(Y)$. Because of Corollary 12.3, this means that the derivative complex

$$0 \longrightarrow H^0(Y, f_*\omega_X) \xrightarrow{v\cup} H^1(Y, f_*\omega_X) \xrightarrow{v\cup} \cdots \xrightarrow{v\cup} H^g(Y, f_*\omega_X) \longrightarrow 0$$

is exact for every nonzero $v \in H^1(X, \mathscr{O}_X)$. It follows that the complex of locally free sheaves

$$0 \to H^0(Y, f_*\omega_X) \otimes \mathscr{O}_{\mathbb{P}}(-g) \to H^1(Y, f_*\omega_X) \otimes \mathscr{O}_{\mathbb{P}}(-g+1) \to \cdots$$
$$\cdots \to H^g(f_*\omega_X) \otimes \mathscr{O}_{\mathbb{P}} \to 0$$

on $\mathbb{P} = \mathbb{P}^{g-1}$ is exact; we have already used this trick once before, during the proof of Lemma 13.4. Recall from the discussion there that the differential is given by the formula $\sum_j v_j \otimes t_j$, where $v_1, \dots, v_g$ is a basis for $H^1(X, \mathscr{O}_X)$, and $t_1, \dots, t_g$ are the corresponding homogeneous coordinates on $\mathbb{P}$. By analyzing the hypercohomology spectral sequence, we find that

$$v_1 \cup \dots \cup v_g \colon H^0(Y, f_*\omega_X) \to H^g(Y, f_*\omega_X)$$

is an isomorphism. As usual, Kollár's theorem shows that $H^g(Y, f_*\omega_X)$ is a direct summand in $H^g(X, \omega_X)$. We have a commutative diagram

$$\begin{tikzcd}
H^0(Y, f_*\omega_X) \arrow[r, "v_1 \cup \dots \cup v_g"] \arrow[d, equal] & H^g(Y, f_*\omega_X) \arrow[d, hook] \\
H^0(X, \omega_X) \arrow[r, "v_1 \cup \dots \cup v_g"] & H^g(X, \omega_X)
\end{tikzcd}$$

and so cup product with $v_1 \cup \dots \cup v_g$ embeds $H^0(X, \omega_X)$ as a direct summand into $H^g(X, \omega_X)$. If we apply Serre duality again, it follows that

$$v_1 \cup \dots \cup v_g \colon H^{n-g}(X, \mathscr{O}_X) \to H^n(X, \mathscr{O}_X)$$

is surjective. Now let $\omega_i = \overline{v_i} \in H^0(X, \Omega_X^1)$; after conjugating, we find that

$$\omega_1 \wedge \dots \wedge \omega_g \colon H^0(X, \Omega_X^{n-g}) \to H^0(X, \omega_X)$$

is also surjective. If we consider $\omega_1 \wedge \dots \wedge \omega_g$ as a holomorphic g-form on $\mathrm{Alb}(X)$, then $p^*(\omega_1 \wedge \dots \wedge \omega_g)$ is a section of ω_Y; it follows that the (essentially unique) section of ω_X can be written as the wedge product of a section of $f^*\omega_Y$ and of a holomorphic $(n-g)$-form. This clearly means that $\omega_X \otimes f^*\omega_Y^{-1}$ is effective.

We conclude that $P_m(Y) \leq P_m(X)$ for all $m \geq 1$; in particular, we have $P_1(Y) = P_2(Y) = 1$, and so Y is birational to an abelian variety by Theorem 27.3. This finishes the proof that alb has connected fibers. $\square$

Notice how the argument with GV-sheaves and the argument with derivative complexes both lead to the conclusion that $H^g(X, \omega_X) \neq 0$; the main advantage of the second method is that it produces an explicit embedding of $H^0(X, \omega_X)$ into $H^g(X, \omega_X)$.

Exercises.

Exercise 28.1. Let X be a smooth projective variety of maximal Albanese dimension. Show that $S^0(X, \omega_X)$ is a finite union of subsets of the form

$$\left\{ L^k \;\middle|\; \gcd(k, d) = 1 \right\} \otimes A,$$

where $L \in \mathrm{Pic}^0(X)$ is a point of order d, and $A \subseteq \mathrm{Pic}^0(X)$ is an abelian subvariety.

Exercise 28.2. Let $\mathscr{F}$ be a GV-sheaf on an abelian variety A, and suppose that the origin is an isolated point of $S^0(A, \mathscr{F})$. As we know, this implies that the origin belongs to $S^g(A, \mathscr{F})$, and hence that $H^g(A, \mathscr{F}) \neq 0$. Is there a more direct relationship between $H^0(A, \mathscr{F})$ and $H^g(A, \mathscr{F})$, similar to what appeared during the proof of Theorem 28.2?

Lecture 29

Inequalities among Hodge numbers. The theory of Green and Lazarsfeld can also be used to prove numerical statements about irregular varieties. The general idea is that certain geometric assumptions about a smooth projective variety X lead to inequalities among certain numerical invariants such as the dimension of X, the Hodge numbers $h^{0,q} = \dim H^q(X, \mathscr{O}_X)$, or the holomorphic Euler characteristic $\chi(X, \omega_X)$. We have already seen one example of this phenomenon in Corollary 8.13: if X has maximal Albanese dimension, then $\chi(X, \omega_X) \geq 0$.

In fact, the prototypical example is an old theorem by Castelnuovo and de Franchis about surfaces. Let S be a smooth projective surface, and denote by $p_g(S)$ its geometric genus, and by $q(S)$ its irregularity. The following result is known as the *Castelnuovo-de Franchis inequality.*

Theorem 29.1. *If $p_g(S) \leq 2q(S) - 4$, then there exists a fibration (= a surjective morphism with connected fibers) from S to a curve of genus ≥ 2.*

Proof. The proof is not all that difficult. Consider the Plücker embedding

$$G\big(2, H^0(S, \Omega_S^1)\big) \to G\Big(1, \bigwedge^2 H^0(S, \Omega_S^1)\Big)$$

of the Grassmannian of 2-planes in $H^0(S, \Omega_S^1)$. The image is a closed subvariety of dimension $2q(S) - 4$, and because we are assuming that $p_g(S) \leq 2q(S) - 4$, it has to intersect the hyperplane corresponding to the kernel of

$$\bigwedge^2 H^0(S, \Omega_S^1) \to H^0(S, \Omega_S^2).$$

This means that there are two linearly independent holomorphic one-forms $\alpha, \beta \in H^0(S, \Omega_S^1)$ whose wedge product $\alpha \wedge \beta \in H^0(S, \Omega_S^2)$ is zero. During the proof of Beauville's Theorem 10.2, we showed how to construct from α and β a surjective morphism with connected fibers from S to a curve of genus ≥ 2. (In fact, we proved a slightly more general result in Proposition 10.12. The idea was that $\beta = f \cdot \alpha$ for some meromorphic function f on S; we then got the fibration by taking the Stein factorization of the resulting morphism to $\mathbb{P}^1$.) □

Equivalently, we can say that if S does not admit any fibration over a curve of genus ≥ 2, then the inequality $p_g(S) \geq 2q(S) - 3$ must hold.

What about a higher-dimensional version? Since $\chi(S, \omega_S) = p_g(S) - q(S) + 1$, we can rewrite the Castelnuovo-de Franchis inequality in the form

$$\chi(S, \omega_S) \geq q(S) - \dim S.$$

This suggests a statement in arbitrary dimension; it was proved by Giuseppe Pareschi and Mihnea Popa. But before we can state this and related inequalities, we have to decide what the correct generalization of an irregular fibration (= a fibration over a curve of genus ≥ 2) should be. One possibility would be fibrations over varieties of general type; from the point of view of Green-Lazarsfeld theory, a more natural class is varieties of maximal Albanese dimension.

Definition 29.2. An *irregular fibration* on a smooth projective variety X is a morphism $f \colon X \to Y$ with connected fibers to a normal projective variety Y, such that $1 \leq \dim Y \leq \dim X - 1$, and such that (any smooth model of) Y has maximal Albanese dimension.

Let me now explain a nice result from a recent paper by Rob Lazarsfeld and Mihnea Popa. They showed that when X admits no irregular fibrations, its Hodge numbers $h^{0,q} = \dim H^q(X, \mathscr{O}_X)$ are related to each other by many nontrivial inequalities. The basic idea is to relate the absence of irregular fibrations to the cohomology support loci $S^i(X)$. Here is how this works.

Suppose that X admits no irregular fibrations. Then clearly X itself has to be of maximal Albanese dimension – otherwise, $\mathrm{alb}\colon X \to \mathrm{alb}(X)$ would be an irregular fibration. But more is true. Recall that when we proved the structure theorem for cohomology support loci, we showed (in Theorem 12.4) that any irreducible component of $S^i(X)$ of positive dimension gives rise to a morphism

$$f\colon X \to Y$$

with connected fibers, such that $\dim Y \leq i$, and such that any smooth model of Y has maximal Albanese dimension. Since $S^0(X) = \{\mathscr{O}_X\}$, the non-existence of irregular fibrations on X therefore implies that

$$\dim S^i(X) = 0 \quad \text{for every } i < n = \dim X.$$

Putting this together with Proposition 8.15, we find that $\mathscr{O}_X$ is an isolated point of $S^i(X)$ for every $i < n$.

In order to derive numerical consequences from this fact, Lazarsfeld and Popa use the derivative complex. Recall that the infinitesimal properties of the loci $S^i(X)$ near the point $\mathscr{O}_X$ are governed by the complex

$$H^0(X, \mathscr{O}_X) \to H^1(X, \mathscr{O}_X) \to \cdots \to H^n(X, \mathscr{O}_X);$$

the differentials are given by cup product with $v \in H^1(X, \mathscr{O}_X)$, which we think of as a tangent vector to $\mathrm{Pic}^0(X)$. Recall from Corollary 12.3 that $\mathscr{O}_X$ is an isolated point of $S^i(X)$ if and only if the derivative complex is exact in degree i for every nonzero $v \in H^1(X, \mathscr{O}_X)$. This criterion, in turn, was based on the fact that the complex $\mathbf{R}p_{2*}P_X$ on $\mathrm{Pic}^0(X)$ is, in a sufficiently small neighborhood of the point $\mathscr{O}_X$, quasi-isomorphic to a linear complex.

Now let $\mathbb{P} = \mathbb{P}^{g-1}$ denote the projective space of lines in $H^1(X, \mathscr{O}_X)$. As in the previous lecture, we consider the complex of vector bundles

$$H^0(X, \mathscr{O}_X) \otimes \mathscr{O}_{\mathbb{P}}(-n) \to H^1(X, \mathscr{O}_X) \otimes \mathscr{O}_{\mathbb{P}}(-n+1) \to \cdots \to H^n(X, \mathscr{O}_X) \otimes \mathscr{O}_{\mathbb{P}};$$

the differentials are given by the formula

$$\sum_{j=1}^{g} v_j \otimes t_j,$$

where $v_1, \dots, v_g$ are a basis of $H^1(X, \mathscr{O}_X)$, and $t_1, \dots, t_g$ the corresponding homogeneous coordinates on $\mathbb{P}$. If X does not admit irregular fibrations, this complex of vector bundles is exact except possibly at the right end; this suggests defining a coherent sheaf $\mathscr{F}_X$ as the cokernel of the right-most differential.

Lemma 29.3. *If X does not admit irregular fibrations, then $\mathscr{F}_X$ is locally free and*

$$0 \to H^0(X, \mathscr{O}_X) \otimes \mathscr{O}_{\mathbb{P}}(-n) \to \cdots \to H^n(X, \mathscr{O}_X) \otimes \mathscr{O}_{\mathbb{P}} \to \mathscr{F}_X \to 0 \tag{29.4}$$

is exact. In particular, $\mathscr{F}_X$ is a globally generated vector bundle of rank $\chi(X, \omega_X)$.

Proof. We have already seen that the complex in (29.4) resolves $\mathscr{F}_X$. Because the derivative complex is exact for every nonzero $v \in H^1(X, \mathscr{O}_X)$, the restriction of (29.4) to any point of $\mathbb{P}$ is also exact; this implies that $\mathscr{F}_X$ is locally free. Now

$$\mathrm{rk}\, \mathscr{F}_X = \sum_{i=0}^{n} (-1)^i \dim H^{n-i}(X, \mathscr{O}_X) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, \omega_X) = \chi(X, \omega_X)$$

because the complex of vector bundles is exact. □

From the fact that $\mathscr{F}_X$ is globally generated, we can obtain various inequalities among the Hodge numbers $h^{0,q} = \dim H^q(X, \mathscr{O}_X)$. The idea is that the Chern classes $c_i = c_i(\mathscr{F}_X)$ are represented by effective cycles, and therefore nonnegative.

Note. More generally, this holds for any Schur polynomial in the c_i. The first few Schur polynomials are

$$c_1,\ c_2,\ c_1^2 - c_2,\ c_3,\ c_1c_2 - c_3,\ c_1^3 - 2c_1c_2 + c_3.$$

For the precise definition, and a proof that the Schur polynomials in the Chern classes of an ample/globally generated vector bundle are positive/nonnegative, see Section 8.3 of Lazarsfeld's book *Positivity in algebraic geometry.*

To obtain formulas for the Chern classes, consider the Chern polynomial

$$c(\mathscr{F}_X, t) = 1 + c_1 t + c_2 t^2 + \cdots + c_r t^r,$$

where $r = \operatorname{rk} \mathscr{F}_X = \chi(X, \omega_X)$. The Chern polynomial is multiplicative in short exact sequences, and so the resolution in the lemma yields

$$c(\mathscr{F}_X, t) \equiv \prod_{j=0}^{n} (1 - jt)^{(-1)^j h^{0,n-j}} \mod t^g$$

because the Chern polynomial of $\mathscr{O}_{\mathbb{P}}(-j)$ is equal to $1 - jt$. The formula is an identity between formal power series; since we are on $\mathbb{P}^{g-1}$, this relation is only meaningful for the coefficients at $1, t, \dots, t^{g-1}$.

Extracting information from this formula is a bit tedious; we shall therefore consider only one example. By looking at the linear terms, one finds that

$$c_1(\mathscr{F}_X) = \sum_{j=0}^{n} (-1)^{j-1} j \cdot h^{0,n-j} \geq 0;$$

in a slightly rearranged form, this becomes

$$h^{0,n-1} \geq 2h^{0,n-2} - 3h^{0,n-3} + 4h^{0,n-4} - 5h^{0,n-5} + \cdots + (-1)^{n-1} n.$$

The higher Chern classes of $\mathscr{F}_X$ (and Schur polynomials in them) lead to many additional polynomial inequalities of this type.

A higher-dimensional Castelnuovo-de Franchis inequality. Another very pretty result, due to Pareschi and Popa, is the following higher-dimensional generalization of the Castelnuovo-de Franchis inequality.

Theorem 29.5. *Let X be a smooth complex projective variety. Then one has*

$$\chi(X, \omega_X) \geq \dim \operatorname{Pic}^0(X) - \dim X,$$

provided that X does not admit irregular fibrations.

Proof. As usual, we let $n = \dim X$ and $g = \dim \operatorname{Pic}^0(X)$. Since X has maximal Albanese dimension, we have $g - n \geq 0$; by Corollary 8.13, also $\chi(X, \omega_X) \geq 0$.

We begin by dealing with the case $\chi(X, \omega_X) = 0$. In that case, $S^n(X)$ is a proper subset of $\operatorname{Pic}^0(X)$, and therefore equal to $S^{n-1}(X)$. According to the discussion above, the origin is an isolated point of $S^n(X)$, and we conclude from Proposition 12.10 that the Albanese mapping of X is surjective. Consequently, $n = g$, and so the inequality is satisfied in this case.

We may therefore assume for the remainder of the argument that $\chi(X, \omega_X) \geq 1$. We shall deduce the inequality from the exactness of the complex in Lemma 29.3. Our main tool will be a splitting criterion due to Ein, which says that if $\mathscr{E}$ is a vector bundle of rank r on $\mathbb{P}^d$, and if one knows that

$$H^i\big(\mathbb{P}^d, \mathscr{E}(j)\big) = 0 \quad \text{for } 1 \leq i \leq r-1 \text{ and every } j,$$

then $\mathscr{E}$ splits into a direct sum of line bundles. (The converse is obviously true.) You can find a short proof of this result, based on vanishing theorems and Castelnuovo-Mumford regularity, in Ein's paper *An analogue of Max Noether's theorem.*

We can take the exact sequence in (29.4), tensor it by $\mathscr{O}_{\mathbb{P}}(j)$, and then take cohomology; the most convenient way to organize the resulting information is by using the hypercohomology spectral sequence. Except for the column with $\mathscr{F}_X$, all entries on the E_1-page are of the form

$$E_1^{p,q}(j) = H^{n-p}(X, \mathscr{O}_X) \otimes H^q\big(\mathbb{P}^{g-1}, \mathscr{O}_{\mathbb{P}}(-p+j)\big)$$

with $-n \leq p \leq 0$ and $0 \leq q \leq g-1$. Here is a picture of the E_1-page:

$$\begin{array}{ccccc}
E_1^{-n,g-1}(j) & E_1^{-n+1,g-1}(j) & \cdots & E_1^{0,g-1}(j) & H^{g-1}\big(\mathbb{P}^{g-1}, \mathscr{F}_X(j)\big) \\
0 & 0 & \cdots & 0 & H^{g-2}\big(\mathbb{P}^{g-1}, \mathscr{F}_X(j)\big) \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & 0 & H^{1}\big(\mathbb{P}^{g-1}, \mathscr{F}_X(j)\big) \\
E_1^{-n,0}(j) & E_1^{-n+1,0}(j) & \cdots & E_1^{0,0}(j) & H^{0}\big(\mathbb{P}^{g-1}, \mathscr{F}_X(j)\big)
\end{array}$$

A little bit of diagram chasing shows that we have

$$H^i\big(\mathbb{P}^{g-1}, \mathscr{F}_X(j)\big) = 0 \quad \text{for } 1 \leq i \leq g-n-2 \text{ and every } j.$$

Now let us suppose that the asserted inequality for the holomorphic Euler characteristic was violated. Then

$$0 \leq \operatorname{rk} \mathscr{F}_X - 1 = \chi(X, \omega_X) - 1 \leq g - n - 2,$$

and so we have the vanishing necessary to apply Ein's theorem and to conclude that $\mathscr{F}_X$ splits into a sum of line bundles; say

$$\mathscr{F}_X \simeq \mathscr{O}_{\mathbb{P}}(a_1) \oplus \mathscr{O}_{\mathbb{P}}(a_2) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}}(a_r).$$

Because $\mathscr{F}_X$ is globally generated, it is clear that $a_1, \dots, a_r \geq 0$.

Now we observe that the spectral sequence degenerates at E_2; in fact, because $g - n - 2 \geq 0$, there is no room for any nonzero differentials other than d_1. As the spectral sequence is converging to zero, this means that the complex

$$0 \to E_1^{-n,0}(j) \to E_1^{-n+1,0}(j) \to \cdots \to E_1^{0,0}(j) \to H^0\big(\mathbb{P}^{g-1}, \mathscr{F}_X(j)\big) \to 0 \tag{29.6}$$

in the bottom row of the E_1-page is exact. From this, we can easily obtain a contradiction. Indeed, if at least one $a_k \geq 1$, then we get a contradiction by choosing $j = -1$ (because $E_1^{0,0}(-1) = 0$, but $r \geq 1$). The only possibility is that $\mathscr{F}_X$ is a trivial bundle of rank r; by taking $j = 0$, we see that $H^n(X, \mathscr{O}_X) \otimes \mathscr{O}_{\mathbb{P}} \to \mathscr{F}_X$ must be an isomorphism. If we now choose $j \geq 1$ minimal with the property that $H^{n-j}(X, \mathscr{O}_X) \neq 0$, we obtain a contradiction to the exactness of (29.6). $\square$

Lecture 30

In the next two lectures, I would like to discuss a more recent result that appeared after I wrote the first version of these notes. Let $f\colon X \to A$ be a morphism from a smooth projective variety to an abelian variety. We already know that the sheaves $R^j f_* \omega_X$ are GV-sheaves; the following theorem gives a much better result.

Theorem 30.1. *Let $f\colon X \to A$ be a morphism from a smooth projective variety to an abelian variety. Then each $R^j f_* \omega_X$ decomposes into a finite direct sum of sheaves of the form*

$$L \otimes q^* \mathscr{F},$$

where $q\colon A \to B$ is a surjective morphism of abelian varieties with connected fibers, $L \in \operatorname{Pic}^0(A)$ is a line bundle of finite order, and $\mathscr{F}$ is an M-regular coherent sheaf on B.

M-regular sheaves are a special class of GV-sheaves; roughly speaking, if GV-sheaves are "nef", then M-regular sheaves are "ample". The decomposition in the theorem is called the Chen-Jiang decomposition. The history is as follows. Jungkai Chen and Zhi Jiang originally proved the result for the Albanese mapping on varieties of maximal Albanese dimension (by a geometric argument). Pareschi, Popa, and I then generalized it arbitrary morphisms to abelian varieties, but our proof used the theory of Hodge modules. Later on, my former student Mads Villadsen found a more elementary argument that combines the methods of Chen and Jiang with some results about variations of Hodge structure. I will present an outline of the proof (in the case of maximal Albanese dimension) next time; today, we will concentrate on M-regular sheaves and their properties.

M-regular sheaves. M-regularity was introduced by Pareschi and Popa in a series of papers, as a generalization of Castelnuovo-Mumford regularity on projective space. (The letter M apparently stands for Mukai.)

Definition 30.2. Let $\mathscr{F}$ be a coherent sheaf on an abelian variety A. We say that $\mathscr{F}$ is *M-regular* if $\operatorname{codim} S^i(A, \mathscr{F}) \geq i + 1$ for every $i \geq 1$.

An M-regular sheaf is obviously a GV-sheaf, but being M-regular is a much stronger condition: in fact, the small change in the numbers turns out to have rather drastic consequences.

Example 30.3. An ample line bundle L is M-regular, because $S^i(A, L) = \emptyset$ for $i \geq 1$. On the other hand, the trivial line bundle $\mathscr{O}_A$ is not M-regular, because $S^g(A, \mathscr{O}_A)$ is nonempty for $g = \dim A$.

The condition of being M-regular has the following nice interpretation in terms of the Fourier-Mukai transform.

Proposition 30.4. *Let $\mathscr{F}$ be a GV-sheaf on an abelian variety A, so that $\mathbf{R}\Phi_P(\mathscr{F}) \cong \mathbf{R}\mathcal{H}om(\mathscr{G}, \mathscr{O}_{\hat{A}})$ for a coherent sheaf $\mathscr{G}$ on the dual abelian variety $\hat{A} = \operatorname{Pic}^0(A)$. Then $\mathscr{F}$ is M-regular if and only if $\mathscr{G}$ is torsion-free.*

For example, the Fourier-Mukai transform of an ample line bundle L is

$$\mathbf{R}\Phi_P(L) = (p_2)_*(p_1^* L \otimes P),$$

and we saw earlier that this is the dual of an ample vector bundle. On the other hand, the Fourier-Mukai transform of $\mathscr{O}_A$ is the dual of a skyscraper sheaf, and therefore torsion.

Proof. One consequence of the base change theorem (in [?prop:base-change-Si]) was that
$$\bigcup_{i\geq n} \operatorname{Supp} R^i\Phi_P(\mathscr{F}) = \bigcup_{i\geq n} S^i(A,\mathscr{F}).$$
Consequently, $\mathscr{F}$ is M-regular if and only if
$$\operatorname{codim} \operatorname{Supp} \mathcal{E}xt^i(\mathscr{G}, \mathscr{O}_{\hat{A}}) = \operatorname{codim} \operatorname{Supp} R^i\Phi_P(\mathscr{F}) \geq i+1$$
for every $i \geq 1$. Now we only have to explain why this condition on the Ext-sheaves is equivalent to $\mathscr{G}$ being torsion-free. This is a local problem, and so we may assume that $(R,\mathfrak{m})$ is a regular local ring – in fact, a local integral domain would be enough – and that M is a finitely-generated R-module. The claim is that M is torsion-free if and only if $\operatorname{codim} \operatorname{Ext}^i(M,R) \geq i+1$ for all $i \geq 1$.

Suppose first that M is torsion-free. For any nonzero $f \in R$, the sequence
$$0 \to M \xrightarrow{f} M \to M/fM \to 0$$
is exact. By looking at the long exact sequence for Ext, we deduce the exactness of
$$\operatorname{Ext}^i(M,R) \xrightarrow{f} \operatorname{Ext}^i(M,R) \to \operatorname{Ext}^{i+1}(M/fM,R)$$
for every $i \geq 1$. Since $\operatorname{codim} \operatorname{Ext}^i(M,R) \geq i$, we can choose a nonzero element $f \in R$ that annihilates $\operatorname{Ext}^i(M,R)$; but then $\operatorname{Ext}^i(M,R)$ injects into $\operatorname{Ext}^{i+1}(M/fM,R)$, and since $\operatorname{codim} \operatorname{Ext}^{i+1}(M/fM,R) \geq i+1$, this gives us what we want.

Now suppose that $\operatorname{codim} \operatorname{Ext}^i(M,R) \geq i+1$ for $i \geq 1$. We need to prove that the multiplication map $f\colon M \to M$ is injective for nonzero $f \in R$. Let K be the kernel, so that we have a short exact sequence
$$0 \to K \to M \to fM \to 0.$$
Since K is torsion, we have $\operatorname{Hom}(K,R) = 0$. From the long exact sequence, we again get the exactness of
$$\operatorname{Ext}^i(M,R) \to \operatorname{Ext}^i(K,R) \to \operatorname{Ext}^{i+1}(fM,R),$$
and so $\operatorname{codim} \operatorname{Ext}^i(K,R) \geq i+1$ for all $i \in \mathbb{Z}$ (including $i=0$). Fro the same reason as in Lemma 26.8, this implies that $K = 0$. □

One very nice property of M-regular sheaves is that they are "essentially" globally generated, in the following sense.

Proposition 30.5. *Let $\mathscr{F}$ be an M-regular coherent sheaf on A. Then there is an isogeny $\varphi\colon A \to A$ such that the pullback sheaf $\varphi^*\mathscr{F}$ is globally generated.*

This generalizes a classical result for ample line bundles on abelian varieties. Namely, an ample line bundle L on A always has global sections (because $h^0(A,L) = \chi(A,L) \neq 0$), but the example of the Theta divisor shows that it can have just one global section, and therefore fail to be globally generated. But it is known that L^2 is always globally generated, and that L^3 is always very ample.

Proof. Let us again write $\mathbf{R}\Phi_P(\mathscr{F}) \cong \mathbf{R}\mathcal{H}om(\mathscr{G}, \mathscr{O}_{\hat{A}})$, with $\mathscr{G}$ torsion-free on $\hat{A}$. The nice thing about this result is that the proof shows very clearly why it is useful to have $\mathscr{G}$ be torsion-free.

Since it is difficult to produce global sections of $\mathscr{F}$ itself, we instead look at global sections of the twists $\mathscr{F} \otimes P_\alpha$. For any $\alpha \in \hat{A}$, consider the evaluation map
$$H^0(A, \mathscr{F} \otimes P_\alpha) \otimes P_\alpha^{-1} \to \mathscr{F}.$$
By Nakayama's lemma, this is surjective on a neigborhood of a point $a \in A$ if and only if the map on fibers
$$H^0(A, \mathscr{F} \otimes P_\alpha) \otimes P_\alpha^{-1}\big|_a \to \mathscr{F}\big|_a$$

is surjective. A short calculation with the Fourier-Mukai transform and duality shows that this linear mapping is dual to the restriction mapping

$$H^0(\hat{A}, \mathscr{G} \otimes \hat{P}_a^{-1}) \to (\mathscr{G} \otimes \hat{P}_a^{-1})\big|_\alpha.$$

Now $\mathscr{G}$ is torsion-free on $\hat{A}$, and points of finite order are obviously dense in $\hat{A}$, and so any global section of $\mathscr{G} \otimes \hat{P}^{-1}$ is uniquely determined by its values at sufficiently many points of finite order. If we write $\hat{A}[m]$ for the set of $\alpha \in Ah$ with $m\alpha = 0$, this means that

$$H^0(\hat{A}, \mathscr{G} \otimes \hat{P}_a^{-1}) \to \bigoplus_{\alpha \in \hat{A}[m]} (\mathscr{G} \otimes \hat{P}_a^{-1})\big|_\alpha$$

is injective for $m \gg 0$. Dually, it follows that

$$\bigoplus_{\alpha \in \hat{A}[m]} H^0(A, \mathscr{F} \otimes P_\alpha) \otimes P_\alpha^{-1} \to \mathscr{F}$$

is surjective in a neighborhood of the given point $a \in A$. Since A is compact, finitely many such neighborhoods cover A; consequently,

$$\bigoplus_{\alpha \in \hat{A}[m]} H^0(A, \mathscr{F} \otimes P_\alpha) \otimes P_\alpha^{-1} \to \mathscr{F}$$

must be surjective for $m \gg 0$. If we now pull back by the isogeny $\varphi \colon A \to A$, $\varphi(a) = m \cdot a$, all the line bundle P_α become trivial, and therefore $\varphi^* \mathscr{F}$ becomes globally generated. □

The Chen-Jiang decomposition. Let us now go back to Theorem 30.1 and try to understand what it is saying. Consider one of the summands $L \otimes q^* \mathscr{F}$, where $q \colon A \to B$ is a morphism of abelian varieties (surjective with connected fibers), and $\mathscr{F}$ is M-regular on B. Since the Fourier-Mukai transform of $\mathscr{F}$ is the dual of a torsion-free sheaf, we have $S^0(B, \mathscr{F}) = \hat{B}$. Let us compute the cohomology support loci of $L \otimes q^* \mathscr{F}$. Because the fibers of $q \colon A \to B$ are abelian varieties,

$$H^i(A, L \otimes q^* \mathscr{F} \otimes P_\alpha)$$

can only be nonzero if $L \otimes P_\alpha$ is trivial on the fibers, and therefore of the form $q^* P_\beta$ for some $\beta \in \hat{B}$. By the projection formula, we get

$$H^i(A, L \otimes q^* \mathscr{F} \otimes P_\alpha) \cong \bigoplus_{j=0}^{k} H^{i-j}(B, \mathscr{F} \otimes P_\beta \otimes R^j q_* \mathscr{O}_A)$$

where $k = \dim A - \dim B$. Now $R^j q_* \mathscr{O}_A$ is a trivial bundle of rank $\binom{k}{j}$, and so the right-hand side is nonzero for every $\beta \in \operatorname{Pic}^0(B)$, as long as $0 \le i \le k$. This gives

$$S^0(A, L \otimes q^* \mathscr{F}) = \cdots = S^k(A, L \otimes q^* \mathscr{F}) = L^{-1} \otimes q^* \operatorname{Pic}^0(B) \subseteq \operatorname{Pic}^0(A).$$

In particular, $S^k(A, L \otimes q^* \mathscr{F})$ has codimension exactly k.

Now suppose that, as in Theorem 30.1, we have a GV-sheaf $\mathscr{F}$ (such as $R^j f_* \omega_X$) with a Chen-Jiang decomposition

$$\mathscr{F} \cong \bigoplus_{i=1}^{n} L_i \otimes q_i^* \mathscr{F}_i$$

with $q_i \colon A \to B_i$ and $\mathscr{F}_i$ M-regular on B_i. Then $S^0(A, \mathscr{F})$ is the union of the translated subtori $L_i^{-1} \otimes q_i^* \operatorname{Pic}^0(B_i)$; more precisely, the i-th summand is responsible for a component of codimension k_i in $S^{k_i}(A, \mathscr{F})$, where $k_i = \dim A - \dim B_i$.

There is also a very useful interpretation using the Fourier-Mukai transform. We have $\mathbf{R}\Phi_P(\mathscr{F}) \cong \mathbf{R}\mathcal{H}om(\mathscr{G}, \mathscr{O}_{\hat{A}})$ for a coherent sheaf $\mathscr{G}$ on $\hat{A}$, and $\mathbf{R}\Phi_P(\mathscr{F}_i) \cong$

$\mathbf{R}\mathcal{H}om(\mathscr{G}_i, \mathscr{O}_{\hat{B}})$ for torsion-free coherent sheaves $\mathscr{G}_i$ on $\hat{B}_i$. Then a calculation (that we will do next time) shows that

$$\mathscr{G} \cong \bigoplus_{i=1}^{n} (t_{L_i})_* (\hat{q}_i)_* \mathscr{G}_i,$$

where $\hat{q}_i \colon \hat{B}_i \to A$ is the closed embedding dual to $q_i \colon A \to B_i$, and t_{L_i} means translation by the point $L_i \in \hat{A}$. The i-th summand is the pushforward of the torsion-free sheaf $\mathscr{G}_i$ by the closed embedding $t_{L_i} \circ \hat{q}_i \colon \hat{B}_i \to A$; this kind of sheaf is sometimes called a "pure sheaf" in the literature. So we see that $\mathscr{G}$ is very special: it is a direct sum of pure sheaves, and so each component of its support is "explained" by one of the summands in the decomposition.

Lecture 31

Today, we are going to look at the proof of Theorem 30.1. We will prove the result only in the special case where X has maximal Albanese dimension and $f\colon X \to A$ is the Albanese mapping; this is the situation originally considered by Chen and Jiang. In that case, the theorem is claiming that

$$f_*\omega_X \cong \bigoplus_{i=1}^n L_i \otimes q_i^* \mathscr{F}_i,$$

where $q_i\colon A \to B_i$ is surjective with connected fibers and $\mathscr{F}_i$ is an M-regular coherent sheaf on the abelian variety B_i.

Functoriality of the Fourier-Mukai transform. But first, I want to prove a result that we used near the end of the previous lecture. It describes how the Fourier-Mukai transform behaves under morphisms between abelian varieties. Let $p\colon A \to B$ be a morphism between two abelian varieties (hence, in particular, a group homomorphism), and let $\hat{p}\colon \hat{B} \to \hat{A}$ denote the induced morphism from $\hat{B} = \operatorname{Pic}^0(B)$ to $\hat{A} = \operatorname{Pic}^0(A)$. Denote by P_A and P_B the normalized Poincaré bundles on $A \times \hat{A}$ and $B \times \hat{B}$.

Lemma 31.1. *There are natural isomorphisms of functors*

$$\mathbf{R}\Phi_{P_B} \circ \mathbf{R}p_* \cong \mathbf{L}\hat{p}^* \circ \mathbf{R}\Phi_{P_A} \quad \textit{and} \quad \mathbf{R}\Phi_{P_A} \circ \mathbf{L}p^*[d] \cong \mathbf{R}\hat{p}_* \circ \iota_{\hat{B}}^* \circ \mathbf{R}\Phi_{P_B} \circ \iota_B^*$$

where $d = \dim A - \dim B$ is the relative dimension.

Proof. Let us first recall how the morphism $\hat{p}\colon \hat{B} \to \hat{A}$ is constructed. By the universal property of $\operatorname{Pic}^0(A)$, such morphisms are in one-to-one correspondence with line bundles on the product $A \times \hat{B}$ (whose restriction to each copy of $\hat{B}$ must be of the same type as the trivial line bundle). The pullback $(p \times \mathrm{id})^* P_B$ along the morphism $p \times \mathrm{id}\colon A \times \hat{B} \to B \times \hat{B}$ is such a line bundle; consequently, there is a unique morphism $\hat{p}\colon \hat{B} \to \hat{A}$ for which

$$(p \times \mathrm{id})^* P_B \cong (\mathrm{id} \times \hat{p})^* P_A. \tag{31.2}$$

Now let $K \in \mathrm{D}^b_{coh}(\mathscr{O}_A)$ be an object in the derived category on A. Then

$$\mathbf{R}\Phi_{P_B}\big(\mathbf{R}p_* K\big) = \mathbf{R}(p_2)_*\big(p_1^* \mathbf{R}p_* K \otimes P_B\big),$$

and by using flat base change, we get $p_1^*\mathbf{R}p_*K \cong \mathbf{R}(p \times \mathrm{id})_* p_1^* K$. Together with the projection formula and the identity in (31.2), this lets us rewrite the expression for the Fourier-Mukai transform of $\mathbf{R}p_*K$ in the form

$$\begin{aligned}\mathbf{R}\Phi_{P_B}\big(\mathbf{R}p_* K\big) &\cong \mathbf{R}(p_2)_* \mathbf{R}(p \times \mathrm{id})_*\big(p_1^* K \otimes (p \times \mathrm{id})^* P_B\big) \\ &\cong \mathbf{R}(p_2)_*\big(p_1^* K \otimes (\mathrm{id} \times \hat{p})^* P_A\big) \cong \mathbf{R}(p_2)_* \mathbf{L}(\mathrm{id} \times \hat{p})^*\big(p_1^* K \otimes P_A\big).\end{aligned}$$

Refer to the following commutative diagram for the morphisms:

$$\begin{tikzcd}
 & \hat{A} & \hat{B} \arrow[l, "\hat{p}"'] & & \\
 A \times \hat{A} \arrow[ur, phantom, ""] & & & & \\
\end{tikzcd}$$

$$\begin{array}{ccccccc}
\hat{A} & \xleftarrow{\ \hat{p}\ } & \hat{B} & & & & \\
\uparrow{\scriptstyle p_2} & & \uparrow{\scriptstyle p_2} & & & & \\
A \times \hat{A} & \xleftarrow{\mathrm{id} \times \hat{p}} & A \times \hat{B} & \xrightarrow{p \times \mathrm{id}} & B \times \hat{B} & \xrightarrow{p_2} & \hat{B} \\
 & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1} & & \\
 & & A & \xrightarrow{\quad p \quad} & B & &
\end{array}$$

After using flat base change again, this becomes

$$\mathbf{R}\Phi_{P_B}\big(\mathbf{R}p_* K\big) \cong \mathbf{L}\hat{p}^* \mathbf{R}(p_2)_*\big(p_1^* K \otimes P_A\big) \cong \mathbf{L}\hat{p}^*\big(\mathbf{R}\Phi_{P_A}(K)\big).$$

This proves the first identity. The second one follows by applying Mukai's formula for the inverse of the Fourier-Mukai transform (in Theorem 24.6). Indeed, if we denote by $\iota_A\colon A \to A$ and $\iota_B\colon B \to B$ the inverse morphisms in the group law on the two abelian varieties, we have

$$\iota_B^* \circ \mathbf{R}p_*[-\dim B] \cong \mathbf{R}\Psi_{P_B} \circ \mathbf{R}\Phi_{P_B} \circ \mathbf{R}p_* \cong \mathbf{R}\Psi_{P_B} \circ \mathbf{L}\hat{p}^* \circ \mathbf{R}\Phi_{P_A}$$

and therefore

$$\iota_B^* \circ \mathbf{R}p_* \circ \mathbf{R}\Psi_{P_A}[-\dim B] \cong \mathbf{R}\Psi_{P_B} \circ \mathbf{L}\hat{p}^* \circ \iota_{\hat{A}}^*[-\dim A].$$

After swapping the two morphisms $p\colon A \to B$ and $\hat{p}\colon \hat{B} \to \hat{A}$, this becomes

$$\iota_{\hat{A}}^* \circ \mathbf{R}\hat{p}_* \circ \mathbf{R}\Phi_{P_B}[-\dim A] \cong \mathbf{R}\Phi_{P_A} \circ \mathbf{L}p^* \circ \iota_B^*[-\dim B].$$

and if we rearrange the terms a bit more, we finally get

$$\mathbf{R}\Phi_{P_A} \circ \mathbf{L}p^*[\dim A - \dim B] \cong \mathbf{R}\hat{p}_* \circ \iota_{\hat{B}}^* \circ \mathbf{R}\Phi_{P_B} \circ \iota_B^*. \qquad \square$$

Proof of the Chen-Jiang decomposition theorem. Let us now try to go through the proof of Theorem 30.1. As in the original paper, we assume that X is a smooth projective variety of maximal Albanese dimension, and that $f\colon X \to A$ is generically finite over its image. We set $n = \dim X$ and $g = \dim A$, so that $\dim f(X) = n$ as well. Our goal is to construct a decomposition

$$f_*\omega_X \cong \bigoplus_i L_i \otimes q_i^* \mathscr{F}_i,$$

where $q_i\colon A \to B_i$ are surjective morphisms of abelian varieties with connected fibers, $L_i \in \mathrm{Pic}^0(A)$ have finite order, and $\mathscr{F}_i$ is M-regular on the abelian variety B_i. We know from last time that each summand contributes a component of codimension k to some $S^k(A, f_*\omega_X)$, where $k = \dim A - \dim B_i$.

The general idea is as follows. If we have $\operatorname{codim} S^i(A, f_*\omega_X) \geq i+1$ for every $i \geq 1$, then $f_*\omega_X$ is itself M-regular, and so there is nothing to prove. The obstruction to this is the finite set of subvarieties of $\mathrm{Pic}^0(A)$ that show up as components of codimension exactly i in the set $S^i(A, f_*\omega_X)$, for some $i \geq 1$. Each subvariety in this set is a translate of a subtorus by a point of finite order. For each of these subvarieties, we are going to construct a summand of $f_*\omega_X$ of the desired kind that accounts for that subvariety.

The construction of the summands works better if all the subvarieties in question pass through the origin in $\mathrm{Pic}^0(A)$. The following lemma allows us to reduce to that case.

Lemma 31.3. *There is an isogeny $\varphi\colon A' \to A$ such that on the fiber product*

$$\begin{array}{ccc} X' & \xrightarrow{\varphi'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ A' & \xrightarrow{\varphi} & A, \end{array}$$

every irreducible component of every $S^i(A', f'_*\omega_{X'})$ contains the origin.

Proof. Each irreducible component of $S^i(A, f_*\omega_X)$ is a translate of a subtorus by a point of finite order, and there are finitely many such components. Let $m \geq 1$ be the least common multiples of the orders of the finitely many points that we get in this way, and let $\varphi\colon A \to A$ be the isogeny $\varphi(a) = m \cdot a$. Let $f'\colon X' \to A$ be the base change. Then $\omega_{X'}$ is isomorphic to the pullback of ω_X, and by flat base change, we get $f'_*\omega_{X'} \cong \varphi^* f_*\omega_X$. For any $\alpha \in \mathrm{Pic}^0(A)$, we have

$$H^i\big(A, f'_*\omega_{X'} \otimes P_\alpha\big) \cong H^i\big(A, f_*\omega_X \otimes \varphi_* P_\alpha\big).$$

Since $\varphi\colon A \to A$ is a finite covering space whose group of deck transformation is the abelian group $(\mathbb{Z}/m\mathbb{Z})^{\oplus 2g}$, it is easy to see that

$$\varphi_* P_\alpha \cong \bigoplus_{m\beta=\alpha} P_\beta.$$

After substituting this into the identity from above, it follows that

$$S^i(A, f'_*\omega_{X'}) = \left\{ \alpha \in \operatorname{Pic}^0(A) \mid \alpha = m\beta \text{ for some } \beta \in S^i(A, f_*\omega_X) \right\},$$

and by our choice of m, every irreducible component of this set contains the origin. □

The lemma reduces the proof of Theorem 30.1 to the case where all irreducible components of the loci $S^i(A, f_*\omega_X)$ pass through the origin. Indeed, if we know the result in that case, then $f'_*\omega_{X'}$ has a Chen-Jiang decomposition. From that, it is not hard to deduce that $\varphi_* f'_*\omega_{X'}$ also has a Chen-Jiang decomposition. But

$$\varphi_* f'_*\omega_{X'} \cong \varphi_*\varphi^* f_*\omega_X \cong f_*\omega_X \otimes \varphi_*\mathscr{O}_A \cong f_*\omega_X \otimes \bigoplus_{m\alpha=0} P_\alpha$$

contains $f_*\omega_X$ as a direct summand, and so we can apply the following lemma.

Lemma 31.4. *If a coherent sheaf $\mathscr{F}$ on an abelian variety has a Chen-Jiang decomposition, then any direct summand of $\mathscr{F}$ also has a Chen-Jiang decomposition.*

Now let us fix one irreducible component of $S^k(A, f_*\omega_X)$ of codimension exactly $k \geq 1$. It is the image of the closed embedding $\hat{q}\colon \hat{B} \to \hat{A}$, where $q\colon A \to B$ is a surjective morphism of abelian varieties with connected fibers, and $\dim B = \dim A - k$. From this data, we need to construct a summand of $f_*\omega_X$. The construction has two parts. The first is a computation that we have already done several times. Because we are starting from an irreducible component of $S^k(A, f_*\omega_X)$, we have

$$H^k(A, f_*\omega_X \otimes q^* P_\beta) \neq 0$$

for every $\beta \in \hat{B}$. The morphism $f\colon X \to A$ is generically finite over its image, and therefore $\mathbf{R}f_*\omega_X \cong f_*\omega_X$. Applying Kollár's theorem, we get

$$\mathbf{R}q_*(f_*\omega_X) \cong \mathbf{R}q_*(\mathbf{R}f_*\omega_X) \cong \mathbf{R}(q\circ f)_*\omega_X \cong \bigoplus_{j=0}^{k} R^j q_*(f_*\omega_X)[-j].$$

Together with the projection formula, this gives

$$H^k(A, f_*\omega_X \otimes q^* P_\beta) \cong \bigoplus_{j=0}^{k} H^{k-j}\bigl(B, R^j q_*(f_*\omega_X) \otimes P_\beta\bigr)$$

Each sheaf $R^j q_*(f_*\omega_X) \cong R^j (q\circ f)_*\omega_X$ is a GV-sheaf on B, and so only the term with H^0 can be nonzero for every $\beta \in \hat{B}$. Consequently, the projection

$$H^k(A, f_*\omega_X \otimes q^* P_\beta) \to H^0\bigl(B, R^k q_*(f_*\omega_X) \otimes P_\beta\bigr)$$

is an isomorphism for general $\beta \in \hat{B}$ (and always surjective).

The second part of the construction is more geometric. We aim to find a morphism $g\colon Y \to B$ from a smooth projective variety Y of dimension $\dim Y = n-k$ that is generically finite over its image, such that $R^k q_*(f_*\omega_X) \cong g_*\omega_Y$. Let us start by considering the Stein factorization of $q\circ f\colon X \to B$:

(31.5)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & A \arrow[d, "q"] \\ Y \arrow[r, "g"] & B \end{tikzcd}$$

Here $p\colon X \to Y$ has connected fibers and $g\colon Y \to B$ is finite, but Y is of course only a normal projective variety in general. Let $\mu\colon \tilde{Y} \to Y$ be a resolution of singularities, and consider the fiber product $\tilde{X} = \tilde{Y} \times_B A$, as in the following commutative diagram:

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\nu} & X & \xrightarrow{f} & A \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ \tilde{Y} & \xrightarrow{\mu} & Y & \xrightarrow{g} & B \end{array}$$

The morphism $p_1\colon \tilde{X} \to \tilde{Y}$ is smooth of relative dimension k, and all the fibers are isomorphic to the k-dimensional abelian variety $\ker q$. Since $\mu\colon \tilde{Y} \to Y$ is birational, it is not hard to see that $\nu\colon \tilde{X} \to X$ is also birational. Since the original variety X is smooth, this gives us

$$(f \circ \nu)_* \omega_{\tilde{X}} = f_*(\nu_* \omega_{\tilde{X}}) \cong f_* \omega_X.$$

After replacing X by $\tilde{X}$, we may therefore assume without loss of generality that the morphism $p\colon X \to Y$ in (31.5) is smooth of relative dimension k, with fibers isomorphic to a fixed k-dimensional abelian variety. We obtain $\omega_X \cong p^*\omega_Y \otimes \omega_{X/Y}$, and so the projection formula gives

$$R^k q_*(f_*\omega_X) \cong R^k (q \circ f)_* \omega_X \cong g_*\big(R^k p_* \omega_X\big) \cong g_*\big(\omega_Y \otimes R^k p_* \omega_{X/Y}\big) \cong g_* \omega_Y,$$

since $R^k p_* \omega_{X/Y} \cong \mathscr{O}_Y$. Substituting this into the morphism that we constructed in the first step, we find that

$$H^k(A, f_*\omega_X \otimes q^* P_\beta) \to H^0\big(B, g_*\omega_Y \otimes P_\beta\big) \neq 0$$

is an isomorphism for general $\beta \in \hat{B}$; in particular, $H^0(B, g_*\omega_Y \otimes P_\beta) \neq 0$ for all $\beta \in \hat{B}$.

It remains to find the desired summand of $f_*\omega_X$. Recall that $\dim B = \dim A - k \leq \dim A - 1$. By induction on the dimension of the abelian variety, the sheaf $g_*\omega_Y$ therefore has a Chen-Jiang decomposition on B. As $H^0\big(B, g_*\omega_Y \otimes P_\beta\big) \neq 0$ for every $\beta \in \hat{B}$, this decomposition must contain an M-regular summand. In other words, there is an M-regular coherent sheaf $\mathscr{F}$ on B such that $\mathscr{F}$ is a direct summand of $g_*\omega_Y$ and such that

$$H^k(A, f_*\omega_X \otimes q^* P_\beta) \to H^0\big(B, g_*\omega_Y \otimes P_\beta\big) \to H^0(B, \mathscr{F} \otimes P_\beta)$$

is an isomorphism for general $\beta \in \hat{B}$.

Now all we need is two morphisms

$$q^*\mathscr{F} \to f_*\omega_X \quad \text{and} \quad f_*\omega_X \to q^*\mathscr{F}$$

whose composition is the identity. For that, it suffices to construct morphisms

$$q^*(g_*\omega_Y) \to f_*\omega_X \quad \text{and} \quad f_*\omega_X \to q^*(g_*\omega_Y)$$

with the same property. The first one is easy: we have

$$q_*(f_*\omega_X) \cong g_*(p_*\omega_X) \cong g_*\big(\omega_Y \otimes p_*\omega_{X/Y}\big),$$

and because all fibers of $p\colon X \to Y$ are isomorphic to a fixed k-dimesional abelian variety, we have $p_*\omega_{X/Y} \cong \mathscr{O}_Y$, and therefore $q_*(f_*\omega_X) \cong g_*\omega_Y$. Because q^* is the left adjoint of q_*, we obtain the desired morphism

$$q^*(g_*\omega_Y) \to f_*\omega_X.$$

For the other morphism, we use duality. We already know that $R^k q_*(f_*\omega_X) \cong g_*\omega_Y$, and together with Kollár's theorem, this gives us a morphism

$$\mathbf{R}q_*(f_*\omega_X) \to R^k q_*(f_*\omega_X)[-k] \cong g_*\omega_Y[-k].$$

The right adjoint of the functor $\mathbf{R}q_*$ is the exceptional pullback functor $q^!$, which is this case equals $q^* \otimes \omega_{X/Y}[k]$. We therefore have a morphism

$$f_*\omega_X \to q^!\big(g_*\omega_Y[-k]\big) \cong q^*(g_*\omega_Y) \otimes \omega_{X/Y} \cong q^*(g_*\omega_Y).$$

Then one checks that the composition of the two morphisms is the identity; consequently, $q^*(g_*\omega_Y)$, and therefore $q^*\mathscr{F}$, is isomorphic to a direct summand of $f_*\omega_X$.

The summand $q^*\mathscr{F}$ accounts for the entire component $\hat{q}(\hat{B})$ of $S^k(A, f_*\omega_X)$, because we know that

$$H^k(A, f_*\omega_X \otimes q^*P_\beta) \to H^0(B, \mathscr{F} \otimes P_\beta)$$

is an isomorphism for general $\beta \in \hat{B}$. The rest of the proof then proceeds as follows. First, one argues that all the different summands are compatible, and hence that there is a decomposition

$$f_*\omega_X \cong \bigoplus_{i=0}^{m} q_i^*\mathscr{F}_i,$$

where $B_0 = A$ and $\dim B_i < \dim A$ for $i = 1, \dots, m$. This comes down to the fact that the Fourier-Mukai transform of $q_i^*\mathscr{F}_i$ is the dual of a torsion-free sheaf supported on $\hat{q}_i(\hat{B}_i)$, and that there are no nontrivial morphism from a torsion sheaf to a torsion-free sheaf. Since the summands $q_i^*\mathscr{F}_i$ with $i = 1, \dots, m$ account for all the components of codimension k in $S^k(A, f_*\omega_X)$, we must have $\operatorname{codim} S^k(A, \mathscr{F}_0) \geq k+1$ for $k \geq 1$, which means that $\mathscr{F}_0$ is M-regular on A.

An application. One interesting application of the Chen-Jiang decomposition in Theorem 30.1 is the following global generation result.

Corollary 31.6. *Let $f\colon X \to A$ be a morphism from a smooth projective variety to an abelian variety. Then there is an isogeny $\varphi\colon A \to A$ such that $\varphi^* R^j f_*\omega_X$ is globally generated for every $j \in \mathbb{N}$.*

Proof. This follows from Proposition 30.5. □

In particular, this means that if $f\colon X \to A$ is generically finite over its image, then the sheaf $f_*\omega_X$ is essentially globally generated. This result is very useful for studying the birational geometry of varieties of maximal Albanese dimension.

Lecture 32

There are several other interesting applications of the theory we have developed in this course. At this point, it should be possible for you to read those papers!

Numerical characterization of theta divisors. In the spirit of Theorem 27.3, there is also a numerical characterization of theta divisors in principally polarized abelian varieties, due to Hacon. The precise result is that a smooth projective variety X with $\chi(X, \omega_X) = 1$ and $\dim X < \dim H^1(X, \mathscr{O}_X)$ is birational to a theta divisor if and only if $\operatorname{codim} S^i(X, \omega_X) > i + 1$ for all $0 < i < \dim X$. A simplified proof, based on the theory of GV-sheaves, is explained in the last section of Pareschi's survey.

Holomorphic Euler characteristic. As I mentioned during Lecture 13, an open problem is to classify smooth projective varieties of general type that have maximal Albanese dimension and satisfy $\chi(X, \omega_X) = 0$. The first such example (in dimension three) was found by Ein and Lazarsfeld; other examples were constructed by Chen and Hacon. In `arXiv:1105.3418`, Chen, Debarre, and Jiang propose a conjecture about the structure of such varieties.

Regularity on abelian varieties. In a series of articles, Pareschi and Popa have developed the theory of M-regularity on abelian varieties (similar to Castelnuovo-Mumford regularity on projective space); a good survey is `arXiv:0802.1021`. It is closely related to the theory of GV-sheaves: by definition, a coherent sheaf $\mathscr{F}$ on an abelian variety A is M-regular if

$$\operatorname{codim} S^i(A, \mathscr{F}) \geq i + 1 \quad \text{for all } i \geq 1.$$

An equivalent condition is that $\mathbf{R}\Phi_P(\mathscr{F}) \simeq \mathbf{R}\mathcal{H}om(\mathscr{G}, \mathscr{O}_{\hat{A}})$, where the coherent sheaf $\mathscr{G}$ is *torsion-free*. The theory of M-regularity has many applications to the study of linear series on curves and abelian varieties.

Pluricanonical maps. Chen and Hacon were the first to study the pluricanonical maps ϕ_m on varieties of maximal Albanese dimension. In `arXiv:1111.6279`, Zhi Jiang, Martì Lahoz, and Sofia Tirabassi subsequently proved the optimal result, by showing that ϕ_4 induces the Iitaka fibration, and that ϕ_3 is birational when X is of general type. An interesting result along the way is that, for varieties of maximal Albanese dimension, one can read off the Kodaira dimension $\kappa(X)$ from $S^0(X, \omega_X)$.

Abundance conjecture. Recall from Theorem 15.2 that all components of

$$\left\{ L \in \operatorname{Pic}^0(X) \;\middle|\; \dim H^0(X, \omega_X \otimes L) \geq m \right\}$$

are translates of abelian varieties by points of finite order. Using some clever arguments with branched coverings, one can extend this result to the sets

$$\left\{ L \in \operatorname{Pic}^0(X) \;\middle|\; \dim H^0(X, \omega_X^k \otimes L) \geq m \right\}$$

with $k \geq 1$ (see `arXiv:0912.3012` for the case $m = 1$). In `arXiv:1002.2682`, Kawamata used this to prove the simplest case of the abundance conjecture: if the numerical Kodaira dimension $\nu(X)$ is equal to zero, then also $\kappa(X) = 0$.

Positive characteristic. There are two recent papers that explore generic vanishing in positive characteristic. In `arXiv:1212.5105`, Hacon and Kovács show that the generic vanishing theorem does not remain true in characteristic p. The issue seems to be the failure of Kollár's theorem: there are birational morphisms $f \colon X \to Y$ with X nonsingular but $R^i f_* \omega_X \neq 0$ for $i > 0$. In `arXiv:1310.2996`, Hacon and Zsolt Patakfalvi prove a weaker result, which is still good enough for certain applications (such as Kawamata's theorem in characteristic p).

Generic vanishing for Hodge modules. One can get many additional examples of GV-sheaves by looking at Hodge modules on abelian varieties. Hodge modules are basically pairs $(\mathcal{M}, F_\bullet \mathcal{M})$ with special properties, where $\mathcal{M}$ is a regular holonomic $\mathscr{D}_A$-module, and $F_\bullet \mathcal{M}$ a filtration by coherent subsheaves. In `arXiv:1112.3058`, Popa and I showed that the coherent sheaves

$$\operatorname{gr}_k^F \mathcal{M} = F_k \mathcal{M} / F_{k-1} \mathcal{M}$$

are always GV-sheaves; we also proved that their cohomology support loci

$$S_m^i(A, \operatorname{gr}_k^F \mathcal{M})$$

are finite unions of translates of abelian varieties by points of finite order, provided that the Hodge module is "of geometric origin". A special case of this are the results about $R^i f_* \omega_X$ that we discussed in class. There are several new applications: the generic vanishing theorem for holomorphic forms in Theorem 9.8; and the proof that, on a smooth projective variety of general type, every holomorphic one-form has nonempty zero locus.