Class 1. Overview

Introduction. The subject of this course is *complex manifolds*. Recall that a smooth manifold is a space in which some neighborhood of every point is homeomorphic to an open subset of $\mathbb{R}^n$, such that the transitions between those open sets are given by smooth functions. Similarly, a complex manifold is a space in which some neighborhood of every point is homeomorphic to an open subset of $\mathbb{C}^n$, such that the transitions between those open sets are given by holomorphic functions.

Here is a brief overview of what we are going to do this semester. The first few classes will be taken up with studying holomorphic functions in several variables; in some ways, they are similar to the familiar theory of functions in one complex variable, but there are also many interesting differences. Afterwards, we will use that basic theory to define complex manifolds.

The study of complex manifolds has two different subfields:

(1) Function theory: concerned with properties of holomorphic functions on open subsets $D \subseteq \mathbb{C}^n$.
(2) Geometry: concerned with global properties of (for instance, compact) complex manifolds.

In this course, we will be more interested in global results; we will develop the local theory only as needed.

Two special classes of complex manifolds will appear very prominently in this course. The first is *Kähler manifolds*; these are (usually, compact) complex manifolds that are defined by a differential-geometric condition. Their study involves a fair amount of differential geometry, which will be introduced at the right moment. The most important example of a Kähler manifold is complex projective space $\mathbb{P}^n$ (and any submanifold). This space is also very important in algebraic geometry, and we will see many connections with that field as we go along. (Note that no results from algebraic geometry will be assumed, but if you already know something, this course will show you a different and more analytic point of view towards complex algebraic geometry.) Three of the main results that we will prove about compact Kähler manifolds are:

(1) The Hodge theorem. It says that the cohomology groups $H^*(X, \mathbb{C})$ of a compact Kähler manifold have a special structure, with many useful consequences for their geometry and topology.
(2) The Kodaira embedding theorem. It gives necessary and sufficient conditions for being able to embed X into projective space.
(3) Chow's theorem. It says that a complex submanifold of projective space is actually an algebraic variety.

The second class is *Stein manifolds*; here the main example is $\mathbb{C}^n$ (and its submanifolds). Since the 1950s, the main tool for studying Stein manifolds has been the theory of coherent sheaves. Sheaves provide a formalism for passing from local results (about holomorphic functions on small open subsets of $\mathbb{C}^n$, say) to global results, and we will carefully define and study coherent sheaves. Time permitting, we will prove the following two results:

(1) The embedding theorem. It says that a Stein manifold can always be embedded into $\mathbb{C}^n$ for sufficiently large n.

(2) The finiteness theorem. It says that the cohomology groups of a coherent sheaf on a compact complex manifold are finite-dimensional vector spaces; the proof uses the theory of Stein manifolds.

Along the way, we will introduce many useful techniques, and prove many other interesting theorems.

Holomorphic functions. Our first task is to generalize the notion of holomorphic function from one to several complex variables. There are many equivalent ways of saying that a function $f(z)$ in one complex variable is holomorphic (e.g., the derivative $f'(z)$ exists; f can be locally expanded into a convergent power series; f satisfies the Cauchy-Riemann equations; etc.). Perhaps the most natural definition in several variables is the following:

Definition 1.1. Let D be an open subset of $\mathbb{C}^n$, and let $f\colon D \to \mathbb{C}$ be a complex-valued function on D. Then f is *holomorphic* in D if each point $a \in D$ has an open neighborhood U, such that the function f can be expanded into a power series

$$f(z) = \sum_{k_1,\dots,k_n=0}^{\infty} c_{k_1,\dots,k_n}(z_1-a_1)^{k_1}\cdots(z_n-a_n)^{k_n} \tag{1.2}$$

which converges for all $z \in U$. We denote the set of all holomorphic functions on D by the symbol $\mathscr{O}(D)$.

More generally, we say that a mapping $f\colon D \to E$ between open sets $D \subseteq \mathbb{C}^n$ and $E \subseteq \mathbb{C}^m$ is *holomorphic* if its m coordinate functions $f_1,\dots,f_m\colon D \to \mathbb{C}$ are holomorphic functions on D.

It is often convenient to use multi-index notation with formulas in several variables: for $k = (k_1,\dots,k_n) \in \mathbb{Z}^n$ and $z \in \mathbb{C}^n$, we let $z^k = z_1^{k_1}\cdots z_n^{k_n}$; we can then write the formula in (1.2) more compactly as

$$f(z) = \sum_{k\in\mathbb{N}^n} c_k(z-a)^k.$$

The familiar convergence results from one complex variable carry over to this setting (with the same proofs). For example, if the series (1.2) converges at a point $b \in \mathbb{C}^n$, then it converges absolutely and uniformly on the open polydisk

$$\Delta(a;r) = \left\{\, z \in \mathbb{C}^n \;\middle|\; |z_j - a_j| < r_j \,\right\},$$

where $r_j = |b_j - a_j|$ for $j = 1,\dots,n$. In particular, a holomorphic function f is automatically continuous, being the uniform limit of continuous functions. A second consequence is that the series (1.2) can be rearranged arbitrarily; for instance, we may give certain values $b_1,\dots,b_{j-1},b_{j+1},\dots,b_n$ to the coordinates $z_1,\dots,z_{j-1},z_{j+1},\dots,z_n$, and then (1.2) can be rearranged into a convergent power series in $z_j - a_j$ alone. This means that a holomorphic function $f \in \mathscr{O}(D)$ is holomorphic in each variable separately, in the sense that $f(b_1,\dots,b_{j-1},z,b_{j+1},\dots,b_n)$ is a holomorphic function of z, provided only that $(b_1,\dots,b_{j-1},z,b_{j+1},\dots,b_n) \in D$.

Those observations have a partial converse, known as Osgood's lemma; it is often useful for proving that some function is holomorphic.

Lemma 1.3. *Let f be a complex-valued function on an open subset $D \subseteq \mathbb{C}^n$. If f is continuous and holomorphic in each variable separately, then it is holomorphic on D.*

Proof. Let $a \in D$ be an arbitrary point, and choose a closed polydisk
$$\overline{\Delta}(a;r) = \left\{ z \in \mathbb{C}^n \;\middle|\; |z_j - a_j| \leq r_j \right\}$$
contained in D. On an open neighborhood of $\Delta(a;r)$, the function f is holomorphic in each variable separately. We may therefore apply Cauchy's integral formula for functions of one complex variable repeatedly, until we arrive at the formula
$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1 - a_1| = r_1} \cdots \int_{|\zeta_n - a_n| = r_n} f(\zeta_1, \dots, \zeta_n) \frac{d\zeta_n}{\zeta_n - z_n} \cdots \frac{d\zeta_1}{\zeta_1 - z_1},$$
valid for any $z \in \Delta(a;r)$. For fixed z, the integrand is a continuous function on the compact set
$$S(a,r) = \left\{ \zeta \in \mathbb{C}^n \;\middle|\; |\zeta_j - a_j| = r_j \right\},$$
and so Fubini's theorem allows us to replace the iterated integral above by
$$f(z) = \frac{1}{(2\pi i)^n} \int_{S(a,r)} \frac{f(\zeta_1, \dots, \zeta_n) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}. \tag{1.4}$$
Now for any point $z \in \Delta(a;r)$, the power series
$$\frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} = \sum_{k_1, \dots, k_n = 0}^{\infty} \frac{(z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n}}{(\zeta_1 - a_1)^{k_1 + 1} \cdots (\zeta_n - a_n)^{k_{n+1}}}$$
converges absolutely and uniformly on S. We may therefore substitute this series expansion into (1.4); after interchanging summation and integration, and reordering the series, it follows that $f(z)$ has a convergent series expansion as in (1.2) on $\Delta(a;r)$, where
$$c_{k_1, \dots, k_n} = \frac{1}{(2\pi i)^n} \int_{S(a,r)} \frac{f(\zeta_1, \dots, \zeta_n) d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - a_1)^{k_1 + 1} \cdots (\zeta_n - a_n)^{k_{n+1}}}$$
This concludes the proof. ∎

In fact, Lemma 1.3 remains true without the assumption that f is continuous; this is the content of Hartog's theorem, which we do not prove here.

The formula in (1.4) generalizes the Cauchy integral formula to holomorphic functions of several complex variables. But, different from the one-variable case, the integral in (1.4) is not taken over the entire boundary of the polydisk $\Delta(a;r)$, but only over the n-dimensional subset $S(a,r)$.

Cauchy-Riemann equations. In one complex variable, holomorphic functions are characterized by the Cauchy-Riemann equations: a continuously differentiable function $f = u + iv$ in the variable $z = x + iy$ is holomorphic iff $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. With the help of the two operators
$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$
these equations can be written more compactly as $\partial f/\partial \bar{z} = 0$. Osgood's lemma shows that this characterization holds in several variables as well: a continuously differentiable function $f \colon D \to \mathbb{C}$ is holomorphic iff it satisfies
$$\frac{\partial f}{\partial \bar{z}_1} = \cdots = \frac{\partial f}{\partial \bar{z}_n} = 0. \tag{1.5}$$
Indeed, such a function f is continuous and holomorphic in each variable separately, and therefore holomorphic by Lemma 1.3.

The operators $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ are very useful in studying holomorphic functions. It is easy to see that

$$\frac{\partial z_j}{\partial \bar{z}_k} = \frac{\partial \bar{z}_j}{\partial z_k} = 0 \quad \text{while} \quad \frac{\partial z_j}{\partial z_k} = \frac{\partial \bar{z}_j}{\partial \bar{z}_k} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to express the coefficients in the power series (1.2) in terms of f: termwise differentiation proves the formula

$$c_{k_1,\dots,k_n} = \frac{1}{(k_1!)\cdots(k_n!)} \cdot \frac{\partial^{k_1+\dots+k_n} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(a). \tag{1.6}$$

Class 2. Local theory

As another application of the differential operators $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$, let us show that the composition of holomorphic mappings is holomorphic. It clearly suffices to show that if $f\colon D \to E$ is a holomorphic mapping between open subsets $D \subseteq \mathbb{C}^n$ and $E \subseteq \mathbb{C}^m$, and $g \in \mathscr{O}(E)$, then $g \circ f \in \mathscr{O}(D)$. Let $z = (z_1, \dots, z_n)$ denote the coordinates on D, and $w = (w_1, \dots, w_m)$ those on E; then $w_j = f_j(z_1, \dots, z_n)$. By the chain rule, we have

$$\frac{\partial (g \circ f)}{\partial \bar{z}_k} = \sum_j \left(\frac{\partial g}{\partial w_j} \frac{\partial f_j}{\partial \bar{z}_k} + \frac{\partial g}{\partial \bar{w}_j} \frac{\partial \bar{f}_j}{\partial \bar{z}_k} \right) = 0,$$

because $\partial f_j/\partial \bar{z}_k = 0$ and $\partial g/\partial \bar{w}_j = 0$.

Actually, the property of preserving holomorphic functions completely characterizes holomorphic mappings.

Lemma 2.1. *A mapping $f\colon D \to E$ between open subsets $D \subseteq \mathbb{C}^n$ and $E \subseteq \mathbb{C}^m$ is holomorphic iff $g \circ f \in \mathscr{O}(D)$ for every holomorphic function $g \in \mathscr{O}(E)$.*

Proof. One direction has already been proved; the other is trivial, since $f_j = w_j \circ f$, where w_j are the coordinate functions on E. □

Basic properties. Before undertaking a more careful study of holomorphic functions, we prove a few basic results that are familiar from the function theory of one complex variable. The first is the identity theorem.

Theorem 2.2. *Let D be a connected open subset of $\mathbb{C}^n$. If f and g are holomorphic functions on D, and if $f(z) = g(z)$ for all points z in a nonempty open subset $U \subseteq D$, then $f(z) = g(z)$ for all $z \in D$.*

Proof. By looking at $f - g$, we are reduced to considering the case where $g = 0$. Since f is continuous, the set of points $z \in D$ where $f(z) = 0$ is relatively closed in D; let E be its interior. By assumption, E is nonempty; to prove that $E = D$, it suffices to show that E is relatively closed in D, because D is connected. To that end, let $a \in D$ be any point in the closure of E, and choose a polydisk $\Delta(a; r) \subseteq D$. Since $a \in \bar{E}$, there is a point $b \in E \cap \Delta(a; r/2)$, and then $a \in \Delta(b; r/2) \subseteq D$. Now f can be expanded into a power series

$$f(z) = \sum_{k \in \mathbb{N}^n} c_k (z - b)^k$$

that converges on $\Delta(b; r/2)$; on the other hand, f is identically zero in a neighborhood of the point b, and so we have $c_k = 0$ for all $k \in \mathbb{N}^k$ by (1.6). It follows that $\Delta(b; r/2) \subseteq E$, and hence that $a \in E$, proving that E is relatively closed in D. □

The second is the following generalization of the maximum principle.

Theorem 2.3. *Let D be a connected open subset of $\mathbb{C}^n$, and $f \in \mathscr{O}(D)$. If there is a point $a \in D$ with $|f(a)| \geq |f(z)|$ for all $z \in D$, then f is constant.*

Proof. Choose a polydisk $\Delta(a;r) \subseteq D$. For any choice of $b \in \Delta(a;r)$, the one-variable function $g(t) = f\big(a+t(b-a)\big)$ is holomorphic on a neighborhood of the unit disk in $\mathbb{C}$, and $|g(0)| \geq |g(t)|$. By the maximum principle, g has to be constant, and so $f(b) = g(1) = g(0) = f(a)$. Thus f is constant on $\Delta(a;r)$; since D is connected, we conclude from Theorem 2.2 that $f(z) = f(a)$ for all $z \in D$. □

Germs of holomorphic functions. In one complex variable, the local behavior of holomorphic functions is very simple. Consider a holomorphic function $f(z)$ that is defined on some open neighborhood of $0 \in \mathbb{C}$. Then we can uniquely write $f(z) = u(z) \cdot z^k$, where $k \in \mathbb{N}$ and $u(z)$ is a unit, meaning that $u(0) \neq 0$, or equivalently that $1/u(z)$ is holomorphic near the origin. In several variables, the situation is much more complicated.

Fix an integer $n \geq 0$. We begin the local study of holomorphic functions in n variables by recalling the notion of a germ. Consider holomorphic functions $f \in \mathscr{O}(U)$ that are defined in some neighborhood U of the origin in $\mathbb{C}^n$. We say that $f \in \mathscr{O}(U)$ and $g \in \mathscr{O}(V)$ are equivalent if there is an open set $W \subseteq U \cap V$, containing the origin, such that $f|_W = g|_W$. The equivalence class of $f \in \mathscr{O}(U)$ is called the *germ* of f at $0 \in \mathbb{C}^n$. We denote the set of all germs of holomorphic functions by $\mathscr{O}_n$. Obviously, germs of holomorphic functions can be added and multiplied, and so $\mathscr{O}_n$ is a (commutative) ring. We have $\mathbb{C} \subseteq \mathscr{O}_n$ through the germs of constant functions.

The ring $\mathscr{O}_n$ can be described more formally as the direct limit

$$\mathscr{O}_n = \lim_{U \ni 0} \mathscr{O}(U),$$

where U ranges over all open neighborhoods of $0 \in \mathbb{C}^n$, ordered by inclusion. For $V \subseteq U$, we have the restriction map $\mathscr{O}(U) \to \mathscr{O}(V)$, and the limit is taken with respect to this family of maps.

Either way, we think of $f \in \mathscr{O}_n$ as saying that f is a holomorphic function on some (unspecified) neighborhood of the origin in $\mathbb{C}^n$. Note that the value $f(0) \in \mathbb{C}$ is well-defined for germs, but the same is not true at other points of $\mathbb{C}^n$. By definition, a function $f \in C(U)$ is holomorphic at $0 \in \mathbb{C}^n$ if it can be expanded into a convergent power series $\sum c_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n}$. It follows immediately that

$$\mathscr{O}_n \simeq \mathbb{C}\{z_1, \dots, z_n\}$$

is isomorphic to the ring of convergent power series in the variables $z_1, \dots, z_n$.

Example 2.4. For $n = 0$, we have $\mathscr{O}_0 \simeq \mathbb{C}$. For $n = 1$, we have $\mathscr{O}_1 \simeq \mathbb{C}\{z\}$. The simple local form of holomorphic functions in one variable corresponds to the simple algebraic structure of the ring $\mathbb{C}\{z\}$: it is a discrete valuation ring, meaning that all of its ideals are of the form (z^k) for $k \in \mathbb{N}$.

As in the example, the philosophy behind the local study of holomorphic functions is to relate local properties of holomorphic functions to algebraic properties of the ring $\mathscr{O}_n$. This is the purpose of the next few lectures.

The first observation is that $\mathscr{O}_n$ is a semi-local[1] ring. Recall that a ring A is called *semi-local* if it has a unique maximal ideal $\mathfrak{m}$, and every element $a \in A$ is either a unit (meaning that it has an inverse a^{-1} in A), or belongs to $\mathfrak{m}$. The unique maximal ideal is

$$\mathfrak{m}_n = \left\{ f \in \mathscr{O}_n \;\middle|\; f(0) = 0 \right\};$$

indeed, if $f \in \mathscr{O}_n$ satisfies $f(0) \neq 0$, then f^{-1} is holomorphic in a neighborhood of the origin, and therefore $f^{-1} \in \mathscr{O}_n$. Note that the residue field $\mathscr{O}_n/\mathfrak{m}_n$ is isomorphic to $\mathbb{C}$. We digress to point out that integer n (the dimension of $\mathbb{C}^n$) can be recovered from the ring $\mathscr{O}_n$, because of the following lemma.

Lemma 2.5. *The quotient $\mathfrak{m}_n/\mathfrak{m}_n^2$ is a complex vector space of dimension n.*

Proof. Since $\mathscr{O}_n \simeq \mathbb{C}\{z_1, \dots, z_n\}$, the maximal ideal is generated by $z_1, \dots, z_n$, and their images give a basis for the quotient $\mathfrak{m}_n/\mathfrak{m}_n^2$. □

Here is another basic property of the ring $\mathscr{O}_n$.

Proposition 2.6. *The ring $\mathscr{O}_n$ is a domain.*

Proof. We have to show that there are no nontrivial zero-divisors in $\mathscr{O}_n$. So suppose that we have $f, g \in \mathscr{O}_n$ with $fg = 0$ and $g \neq 0$. Let $\Delta(0; r)$ be a polydisk on which both f and g are holomorphic functions. Since $g \neq 0$, there is some point $a \in \Delta(0; r)$ with $g(a) \neq 0$; then g is nonzero, and therefore f is identically zero, in some neighborhood of a. By Theorem 2.2, it follows that $f(z) = 0$ for every $z \in \Delta(0; r)$; in particular, $f = 0$ in $\mathscr{O}_n$. □

Weierstraß polynomials. To get at the deeper properties of the ring $\mathscr{O}_n$, we have to study the local structure of holomorphic functions more carefully. We will proceed by induction on $n \geq 0$, by using the inclusions of rings

$$\mathscr{O}_{n-1} \subseteq \mathscr{O}_{n-1}[z_n] \subseteq \mathscr{O}_n. \tag{2.7}$$

Elements of the intermediate ring are polynomials of the form $z_n^k + a_1 z_n^{k-1} + \dots + a_k$ with coefficients $a_1, \dots, a_k \in \mathscr{O}_{n-1}$; they are obviously holomorphic germs. The first inclusion in (2.7) is a simple algebraic extension; we will see that the second one is of a more analytic nature.

Throughout this section, we write the coordinates on $\mathbb{C}^n$ in the form $z = (w, z_n)$, so that $w = (z_1, \dots, z_{n-1})$. To understand the second inclusion in (2.7), we make the following definition.

Definition 2.8. An element $h = z_n^d + a_1 z_n^{d-1} + \dots + a_d \in \mathscr{O}_{n-1}[z_n]$ with $d \geq 1$ is called a *Weierstraß polynomial* if $a_1, \dots, a_d \in \mathfrak{m}_{n-1}$.

In analogy with the one-variable case, we will show that essentially every $f \in \mathscr{O}_n$ can be written in the form $f = uh$ with h a Weierstraß polynomial and u a unit. This statement has to be qualified, however, because we have $h(0, z_n) = z_n^d$, which means that if $f = uh$, then the restriction of f to the line $w = 0$ cannot be identically zero.

Definition 2.9. Let U be an open neighborhood of $0 \in \mathbb{C}^n$, and $f \in \mathscr{O}(U)$. We say that f is *regular* (in z_n) if the holomorphic function $f(0, z_n)$ is not identically equal to zero.

[1] $\mathscr{O}_n$ is even a local ring, since it is also Noetherian (meaning that every ideal is finitely generated); however, it will takes us some time to prove this.

If f is regular, we can write $f(0, z_n) = u(z_n) z_n^d$, where d is the order of vanishing of $f(0, z_n)$ at the origin, and $u(0) \neq 0$. We may summarize this by saying that f is *regular of order* d. The notion of regularity also makes sense for elements of $\mathscr{O}_n$, since it only depends on the behavior of f in arbitrarily small neighborhoods of the origin.

Class 3. The Weierstrass theorems

We continue to write the coordinates on $\mathbb{C}^n$ in the form $z = (w, z_n)$.

Recall that a function $f \in \mathscr{O}_n$ is said to be regular in z_n if $f(0, z_n)$ is not identically equal to zero. Of course, not every holomorphic function is regular (for instance, z_j for $j < n$ is not), but if $f \neq 0$, then we can always make it regular by changing the coordinate system.

Lemma 3.1. *Given finitely many nonzero elements of $\mathscr{O}_n$, there is a linear change of coordinates that makes all of them regular in the variable z_n.*

Proof. By taking the product of the finitely many germs, we reduce to the case of a single $f \in \mathscr{O}_n$. Since $f \neq 0$, there is some vector $a \in \mathbb{C}^n$ such that the holomorphic function $f(t \cdot a)$ is not identically zero for $t \in \mathbb{C}$ sufficiently close to 0. After making a change of basis in the vector space $\mathbb{C}^n$, we can assume that $a = (0, \dots, 0, 1)$; but then $f(0, z_n)$ is not identically zero, proving that f is regular in z_n. $\square$

The following fundamental result is known as the Weierstraß preparation theorem; it is the key to understanding the second inclusion in (2.7).

Theorem 3.2. *If $f \in \mathscr{O}_n$ is regular of order d in the variable z_n, then there exists a unique Weierstraß polynomial $h \in \mathscr{O}_{n-1}[z_n]$ of degree d such that $f = uh$ for some unit $u \in \mathscr{O}_n$.*

The idea of the proof is quite simple: Fix $w \in \mathbb{C}^n$ sufficiently close to 0, and consider $f_w(z_n) = f(w, z_n)$ as a holomorphic function of z_n. Since $f_0(z_n)$ vanishes to order d when z_n, each $f_w(z_n)$ will have exactly d zeros (counted with multiplicities) close to the origin; call them $\zeta_1(w), \dots, \zeta_d(w)$. Now if $f = uh$ for a unit u and a monic polynomial h, then we should have $h_w(z_n) = \big(z_n - \zeta_1(w)\big) \cdots \big(z_n - \zeta_d(w)\big)$. The main point is to show that, after expanding this into a polynomial, the coefficients are holomorphic functions of w. Here is the rigorous proof.

Proof. The germ $f \in \mathscr{O}_n$ can be represented by a holomorphic function on some neighborhood of $0 \in \mathbb{C}^n$. We begin by constructing the required Weierstraß polynomial h. Since f is regular in the variable z_n, we have $f(0, z_n) \neq 0$ for sufficiently small $z_n \neq 0$. We can therefore find $r > 0$ and $\delta > 0$ with the property that $|f(0, z_n)| \geq \delta$ for $|z_n| = r$; because f is continuous, we can then choose $\varepsilon > 0$ such that $|f(w, z_n)| \geq \delta/2$ as long as $|z_n| = r$ and $|w| \leq \varepsilon$.

For any fixed $w \in \mathbb{C}^{n-1}$ with $|w| \leq \varepsilon$, consider the integral

$$N(w) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{(\partial f / \partial z_n)(w, \zeta)}{f(w, \zeta)} d\zeta;$$

by the residue theorem, it counts the zeros of the holomorphic function $f(w, \zeta)$ inside the disk $|\zeta| < r$ (with multiplicities). We clearly have $N(0) = d$, and so by continuity, $N(w) = d$ whenever $|w| \leq \varepsilon$. We can therefore define $\zeta_1(w), \dots, \zeta_d(w)$ to be those zeros (in any order). We also set

$$h(w, z_n) = \prod_{j=1}^{d} \bigl(z_n - \zeta_j(w)\bigr) = z_n^d - \sigma_1(w) z_n^{d-1} + \dots + (-1)^d \sigma_d(w),$$

where $\sigma_1(w), \dots, \sigma_d(w)$ are the elementary symmetric polynomials in the roots $\zeta_j(w)$.

Of course, each $\zeta_j(w)$ by itself is probably not holomorphic (or even continuous) in w. But by invoking the residue theorem one more time, we see that

$$\zeta_1(w)^k + \dots + \zeta_d(w)^k = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{\zeta^k \cdot (\partial f/\partial z_n)(w, \zeta)}{f(w, \zeta)} d\zeta,$$

which is holomorphic in w (this can be seen by differentiating under the integral sign). By Newton's formulas, the elementary symmetric polynomials $\sigma_j(w)$ are therefore holomorphic functions in w as well; it follows that h is holomorphic for $|w| < \varepsilon$ and $|z_n| < r$. The regularity of f implies that $\sigma_j(0) = 0$ for all j, and therefore h is a Weierstraß polynomial of degree d.

For $|w| < \varepsilon$ and $|z_n| < r$, we consider the quotient

$$u(w, z_n) = \frac{f(w, z_n)}{h(w, z_n)},$$

which is a holomorphic function outside the zero set of h. For fixed w, the singularities of the function $u(w, z_n)$ inside the disk $|z_n| < r$ are removable by construction, and so $u(w, z_n)$ is holomorphic in z_n. But by the Cauchy integral formula, we then have

$$u(w, z_n) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{u(w, \zeta)}{\zeta - z_n} d\zeta,$$

and so u is actually a holomorphic function of (w, z_n). To conclude that u is a unit, note that $u(0, z_n) = f(0, z_n)/h(0, z_n) = f(0, z_n)/z_n^d$, whose value at 0 is nonzero by assumption. We now have the desired representation $f = uh$ where h is a Weierstraß polynomial and u a unit.

The uniqueness of the Weierstraß polynomial for given f is clear: indeed, since u is a unit, $h(w, z_n)$ necessarily has the same zeros as $f(w, z_n)$ for every $w \in \mathbb{C}^{n-1}$ near the origin, and so its coefficients have to be given by the $\sigma_j(w)$, which are uniquely determined by f. $\square$

The preparation theorem allows us to deduce one important property of the ring $\mathscr{O}_n$, namely that it has unique factorization. Recall that in a domain A, an element $a \in A$ is called *irreducible* if in any factorization $a = bc$, either b or c has to be a unit. Moreover, A is called a *unique factorization domain* (UFD) if every nonzero element $a \in A$ can be uniquely factored into a product of irreducible elements, each unique up to units.

Theorem 3.3. *The ring $\mathscr{O}_n$ is a unique factorization domain.*

Proof. We argue by induction on $n \geq 0$; the case $n = 0$ is trivial since $\mathscr{O}_0 \simeq \mathbb{C}$ is a field. We may suppose that $\mathscr{O}_{n-1}$ is a UFD; by Gauß' lemma, the polynomial ring $\mathscr{O}_{n-1}[z_n]$ is then also a UFD. Let $f \in \mathscr{O}_n$ be any nonzero element; without loss of generality, we may assume that it is regular in z_n. According to Theorem 3.2, we have $f = uh$ for a unique Weierstraß polynomial $h \in \mathscr{O}_{n-1}[z_n]$.

Now suppose that we have a factorization $f = f_1 f_2$ in $\mathscr{O}_n$. Then each f_j is necessarily regular in z_n, and can therefore be written as $f_j = u_j h_j$ with h_j a Weierstraß polynomial and u_j a unit. Then

$$uh = f = (u_1 u_2) \cdot (h_1 h_2),$$

and the uniqueness part of Theorem 3.2 shows that $h = h_1 h_2$. Existence and uniqueness of a factorization for f are thus reduced to the corresponding problems for h in the ring $\mathscr{O}_{n-1}[z_n]$; but $\mathscr{O}_{n-1}[z_n]$ is already known to be a UFD. □

The division theorem. The next result is the so-called Weierstraß division theorem; it shows that one can do long division with Weierstraß polynomials, in the same way as in the ring $\mathbb{C}[z]$. We continue to write the coordinates on $\mathbb{C}^n$ in the form $z = (w, z_n)$, in order to do to induction on $n \geq 0$.

Theorem 3.4. *Let $h \in \mathscr{O}_{n-1}[z_n]$ be a Weierstraß polynomial of degree d. Then any $f \in \mathscr{O}_n$ can be uniquely written in the form $f = qh + r$, where $q \in \mathscr{O}_n$, and $r \in \mathscr{O}_{n-1}[z_n]$ is a polynomial of degree $< d$. Moreover, if $f \in \mathscr{O}_{n-1}[z_n]$, then also $q \in \mathscr{O}_{n-1}[z_n]$.*

Proof. As in the proof of Theorem 3.2, we can choose $\rho, \varepsilon > 0$ sufficiently small, to insure that for each fixed $w \in \mathbb{C}^{n-1}$ with $|w| < \varepsilon$, the polynomial $h(w, z_n)$ has exactly d zeros in the disk $|z_n| < \rho$. For $|z_n| < \rho$ and $|w| < \varepsilon$, we may then define

$$q(w, z_n) = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(w, \zeta)}{h(w, \zeta)} \frac{d\zeta}{\zeta - z_n}.$$

As usual, differentiation under the integral sign shows that q is holomorphic; hence $r = f - qh$ is holomorphic as well. The function r can also be written as an integral,

$$\begin{aligned} r(w, z_n) &= f(w, z_n) - q(w, z_n) h(w, z_n) \\ &= \frac{1}{2\pi i} \int_{|\zeta| = \rho} \left(f(w, \zeta) - h(w, z_n) \frac{f(w, \zeta)}{h(w, \zeta)} \right) \frac{d\zeta}{\zeta - z_n} \\ &= \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(w, \zeta)}{h(w, \zeta)} \cdot p(z, \zeta, z_n) d\zeta, \end{aligned}$$

where we have introduced the new function

$$p(w, \zeta, z_n) = \frac{h(w, \zeta) - h(w, z_n)}{\zeta - z_n}.$$

Now $h \in \mathscr{O}_{n-1}[z_n]$ is a monic polynomial of degree d, and so $\zeta - z_n$ divides the numerator; therefore $p \in \mathscr{O}_{n-1}[z_n]$ is monic of degree $d - 1$. Writing

$$p(w, \zeta, z_n) = a_0(w, \zeta) z_n^{d-1} + a_1(w, \zeta) z_n^{d-2} + \cdots + a_{d-1}(w, \zeta),$$

we then have $r(w, z_n) = b_0(w) z_n^{d-1} + b_1(w) z_n^{d-2} + \cdots + b_{d-1}(w)$, where the coefficients are given by the integrals

$$b_j(w) = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(w, \zeta)}{h(w, \zeta)} \cdot a_j(w, \zeta) d\zeta.$$

This proves that $r \in \mathscr{O}_{n-1}[z_n]$ is a polynomial of degree $< d$, and completes the main part of the proof.

To prove the uniqueness of q and r, it suffices to consider the case $f = 0$. Suppose then that we have $0 = qh + r$, where $r \in \mathscr{O}_{n-1}[z_n]$ has degree $< d$. For fixed w with $|w| < \varepsilon$, the function $r(w, z_n) = -q(w, z_n) h(w, z_n)$ has at least d zeros in the disk

$|z_n| < \varepsilon$; but since it is a polynomial in z_n of degree $< d$, this can only happen if $r = 0$, and hence $q = 0$.

Finally, suppose that $f \in \mathscr{O}_{n-1}[z_n]$. Because h is monic, we can apply the division algorithm for polynomials to obtain $f = q'h + r'$ with $q', r' \in \mathscr{O}_{n-1}[z_n]$. By uniqueness, $q' = q$ and $r' = r$, and so q is a polynomial in that case. □

Class 4. Analytic sets

We now come to another property of the ring $\mathscr{O}_n$ that is of great importance in the local theory. Recall that a (commutative) ring A is called *Noetherian* if every ideal of A can be generated by finitely many elements. An equivalent definition is that any increasing chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ has to stabilize (to see why, note that the union of all I_k is generated by finitely many elements, which will already be contained in one of the I_k). Also, A is said to be a *local* ring if it is semi-local and Noetherian.

Theorem 4.1. *The ring $\mathscr{O}_n$ is Noetherian, and therefore a local ring.*

Proof. Again, we argue by induction on $n \geq 0$, the case $n = 0$ being trivial. We may assume that $\mathscr{O}_{n-1}$ is already known to be Noetherian. Let $I \subseteq \mathscr{O}_n$ be a nontrivial ideal, and choose a nonzero element $h \in I$. After a change of coordinates, we may assume that h is regular in z_n; by Theorem 3.2, we can then multiply h by a unit and assume from the outset that h is a Weierstraß polynomial.

For any $f \in I$, Theorem 3.4 shows that $f = qh + r$, where $r \in \mathscr{O}_{n-1}[z_n]$. Set $J = I \cap \mathscr{O}_{n-1}[z_n]$; then we have $r \in J$, and so $I = J + (h)$. According to Hilbert's basis theorem, the polynomial ring $\mathscr{O}_{n-1}[z_n]$ is Noetherian; consequently, the ideal J can be generated by finitely many elements; it follows that I is also finitely generated, concluding the proof. □

Analytic sets. Our next topic—and one reason for having proved all those theorems about the structure of the ring $\mathscr{O}_n$—is the study of so-called analytic sets, that is, sets defined by holomorphic equations.

Definition 4.2. Let $D \subseteq \mathbb{C}^n$ be an open set. A subset $Z \subseteq D$ is said to be *analytic* if every point $p \in D$ has an open neighborhood U, such that $Z \cap U$ is the common zero set of a collection of holomorphic functions on U.

Note that we are not assuming that $Z \cap U$ is defined by finitely many equations; but we will soon prove that finitely many equations are enough.

Since holomorphic functions are continuous, an analytic set is automatically closed in D; but we would like to know more about its structure. The problem is trivial for $n = 1$: the zero set of a holomorphic function (or any collection of them) is a set of isolated points. In several variables, the situation is again more complicated.

Example 4.3. The zero set $Z(f)$ of a single holomorphic function $f \in \mathscr{O}(D)$ is called a *complex hypersurface*. In one of the exercises, we have seen that $Z(f)$ has Lebesgue measure zero.

We begin our study of analytic sets by considering their local structure; without loss of generality, we may suppose that $0 \in Z$, and restrict our attention to small neighborhoods of the origin. To begin with, note that Z determines an ideal $I(Z)$ in the ring $\mathscr{O}_n$, namely $I(Z) = \{\, f \in \mathscr{O}_n \mid f \text{ vanishes on } Z \,\}$. Since $I(Z)$ contains the holomorphic functions defining Z, it is clear that Z is the common zero locus of the elements of $I(Z)$. Moreover, it is easy to see that if $Z_1 \subseteq Z_2$, then $I(Z_2) \subseteq I(Z_1)$.

The next observation is that, in some neighborhood of 0, the set Z can actually be defined by *finitely* many holomorphic functions. Indeed, on a suitable neighborhood U of the origin, $Z \cap U$ is the common zero locus of its ideal $I(Z)$; but since $\mathscr{O}_n$ is Noetherian, $I(Z)$ is generated by finitely many elements $f_1, \dots, f_r$, say. After shrinking U, we then have $Z \cap U = Z(f_1) \cap \dots \cap Z(f_r)$ defined by the vanishing of finitely many holomorphic equations.

We say that an analytic set Z is *reducible* if it can be written as a union of two analytic sets in a nontrivial way; if this is not possible, then Z is called *irreducible.* At least locally, irreducibility is related to the following algebraic condition on the ideal $I(Z)$.

Lemma 4.4. *An analytic set Z is irreducible in some neighborhood of $0 \in \mathbb{C}^n$ iff $I(Z)$ is a prime ideal in the ring $\mathscr{O}_n$.*

Proof. Recall that an ideal I in a ring A is called *prime* if, whenever $a \cdot b \in I$, either $a \in I$ or $b \in I$. One direction is obvious: if we have $fg \in I(Z)$, then $Z \subseteq Z(f) \cap Z(g)$; since Z is irreducible, either $Z \subseteq Z(f)$ or $Z \subseteq Z(g)$, which implies that either $f \in I(Z)$ or $g \in I(Z)$. For the converse, suppose that we have a nontrivial decomposition $Z = Z_1 \cup Z_2$. Since Z_1 is the common zero locus of $I(Z_1)$, we can find a holomorphic function $f_1 \in I(Z_1)$ that does not vanish everywhere on Z_2; similarly, we get $f_2 \in I(Z_2)$ that does not vanish everywhere on Z_1. Then the product $f_1 f_2$ belongs to $I(Z)$, while neither of the factors does, contradicting the fact that $I(Z)$ is a prime ideal. □

A useful property of analytic sets is that they can be locally decomposed into irreducible components; this type of result may be familiar to you from algebraic geometry.

Proposition 4.5. *Let Z be an analytic set in $D \subseteq \mathbb{C}^n$, with $0 \in Z$. Then in some neighborhood of the origin, there is a decomposition $Z = Z_1 \cup \dots \cup Z_r$ into irreducible analytic sets Z_j. If we require that there are no inclusions among the Z_j, then the decomposition is unique up to reordering.*

Proof. Suppose that Z could not be written as a finite union of irreducible analytic sets. Then Z has to be reducible, and so $Z = Z_1 \cup Z_2$ in some neighborhood of 0. At least one of the two factors is again reducible, say $Z_1 = Z_{1,1} \cup Z_{1,2}$. Continuing in this manner, we obtain a strictly decreasing chain of analytic subsets

$$Z \supset Z_1 \supset Z_{1,1} \supset \cdots ,$$

and correspondingly, a strictly increasing chain of ideals

$$I(Z) \subset I(Z_1) \subset I(Z_{1,1}) \subset \cdots .$$

But $\mathscr{O}_n$ is Noetherian, and hence such a chain cannot exist. We conclude that $Z = Z_1 \cup \dots \cup Z_r$, where the Z_j are irreducible in a neighborhood of 0, and where we may clearly assume that there are no inclusions $Z_j \subseteq Z_k$ for $j \neq k$.

To prove the uniqueness, let $Z = Z_1' \cup \dots \cup Z_s'$ is another decomposition without redundant terms. Then

$$Z_j' = (Z_j' \cap Z_1) \cup \dots \cup (Z_j' \cap Z_r),$$

and so by irreducibility, $Z'_j \subseteq Z_k$ for some k. Conversely, we have $Z_k \subseteq Z'_l$ for some l, and since the decompositions are irredundant, it follows that $j = l$ and $Z'_j = Z_k$. It is then easy to show by induction that $r = s$ and $Z'_j = Z_{\sigma(j)}$ for some permutation σ of $\{1, \dots, r\}$. □

Implicit mapping theorem. To say more about the structure of analytic sets, we need a version of the implicit function theorem (familiar from multi-variable calculus). It gives a sufficient condition (in terms of partial derivatives of the defining equations) for being able to parametrize the points of an analytic set by an open set in $\mathbb{C}^k$.

We note that if $Z \subseteq D$ is defined by holomorphic equations $f_1, \dots, f_m$, we can equivalently say that $Z = f^{-1}(0)$, where $f \colon D \to \mathbb{C}^m$ is the holomorphic mapping with coordinate functions f_j. We take this more convenient point of view in this section. As usual, we denote the coordinates on $\mathbb{C}^n$ by $z_1, \dots, z_n$. If $f \colon D \to \mathbb{C}^m$ is holomorphic, we let

$$J(f) = \frac{\partial(f_1, \dots, f_m)}{\partial(z_1, \dots, z_n)}$$

be the matrix of its partial derivatives; in other words, $J(f)_{j,k} = \partial f_j / \partial z_k$ for $1 \leq j \leq m$ and $1 \leq k \leq n$.

In order to state the theorem, we also introduce the following notation: Let $m \leq n$, and write the coordinates on $\mathbb{C}^n$ in the form $z = (z', z'')$ with $z' = (z_1, \dots, z_m)$ and $z'' = (z_{m+1}, \dots, z_n)$. Similarly, we let $r = (r', r'')$, so that $\Delta(0; r) = \Delta(0; r') \times \Delta(0; r'') \subseteq \mathbb{C}^m \times \mathbb{C}^{n-m}$. For a holomorphic mapping $f \colon D \to \mathbb{C}^n$, we then have

$$J(f) = \bigl(J'(f), J''(f)\bigr),$$

where $J'(f) = \partial f / \partial z'$ is an $m \times m$-matrix, and $J''(f) = \partial f / \partial z''$ is an $m \times (n-m)$-matrix.

Theorem 4.6. *Let f be a holomorphic mapping from an open neighborhood of $0 \in \mathbb{C}^n$ into $\mathbb{C}^m$ for some $m \leq n$, and suppose that $f(0) = 0$. If the matrix $J'(f)$ is nonsingular at the point 0, then for some polydisk $\Delta(0; r)$, there exists a holomorphic mapping $\phi \colon \Delta(0; r'') \to \Delta(0; r')$ with $\phi(0) = 0$, such that*

$$f(z) = 0 \text{ for some point } z \in \Delta(0; r) \text{ precisely when } z' = \phi(z'').$$

Proof. The proof is by induction on the dimension m. First consider the case $m = 1$, where we have a single holomorphic function $f \in \mathscr{O}_n$ with $f(0) = 0$ and $\partial f / \partial z_1 \neq 0$. This means that f is regular in z_1 of order 1; by Theorem 3.2, we can therefore write

$$f(z) = u(z) \cdot \bigl(z_1 - a(z_2, \dots, z_n)\bigr),$$

where $u \in \mathscr{O}_n$ is a unit, and $a \in \mathfrak{m}_{n-1}$. Consequently, $u(0) \neq 0$ and $a(0) = 0$; on a suitable polydisk around 0, we therefore obtain the assertion with $\phi = a$.

Now consider some dimension $m > 1$, assuming that the theorem has been proved in dimension $m - 1$. After a linear change of coordinates in $\mathbb{C}^m$, we may further assume that $J'(f) = \mathrm{id}_m$ at the point $z = 0$. Then $\partial f_1 / \partial z_1(0) = 1$, and it follows from the case $m = 1$ that there is a polydisk $\Delta(0; r)$ and a holomorphic function $\phi_1 \colon \Delta(0; r_2, \dots, r_n) \to \Delta(0; r_1)$ with $\phi_1(0) = 0$, such that $f_1(z) = 0$ precisely when $z_1 = \phi_1(z_2, \dots, z_n)$.

Define a holomorphic mapping $g \colon \Delta(0; r_2, \dots, r_n) \to \mathbb{C}^{m-1}$ by setting

$$g_j(z_2, \dots, z_n) = f_j\bigl(\phi_1(z_2, \dots, z_n), z_2, \dots, z_n\bigr)$$

for $2 \leq j \leq m$. Then clearly $g(0) = 0$, and $\partial(g_2, \dots, g_m)/\partial(z_2, \dots, z_m) = \mathrm{id}_{m-1}$ at the point $z = 0$. It follows from the induction hypothesis that, after further shrinking the polydisk $\Delta(0; r)$ if necessary, there is a holomorphic mapping

$$\psi \colon \Delta(0; r'') \to \Delta(0; r_2, \dots, r_m)$$

with $\psi(0) = 0$, such that $g(z_2, \dots, z_n) = 0$ exactly when $(z_2, \dots, z_m) = \psi(z'')$.

Now evidently $f(z) = 0$ at some point $z \in \Delta(0; r)$ iff $z_1 = \phi_1(z_2, \dots, z_n)$ and $g(z_2, \dots, z_n) = 0$. Hence it is clear that the mapping

$$\phi(z) = \Big(\phi_1\big(\psi(z''), z''\big), \psi(z'')\Big)$$

has all the desired properties. □

Class 5. Complex manifolds

The implicit mapping theorem basically means the following: if $J'(f)$ has maximal rank, then the points of the analytic set $Z = f^{-1}(0)$ can be parametrized by an open subset of $\mathbb{C}^{n-m}$; in other words, Z looks like $\mathbb{C}^{n-m}$ in some neighborhood of the origin. This is one of the basic examples of a complex manifold.

A smooth manifold is a space that locally looks like an open set in $\mathbb{R}^n$; similarly, a complex manifold should be locally like an open set in $\mathbb{C}^n$. To see that something more is needed, take the example of $\mathbb{C}^n$. It is at the same time a topological space, a smooth manifold (isomorphic to $\mathbb{R}^{2n}$), and presumably a complex manifold; what distinguishes between these different structures is the class of functions that one is interested in. In other words, $\mathbb{C}^n$ becomes a smooth manifold by having the notion of smooth function; and a complex manifold by having the notion of holomorphic function.

Geometric spaces. We now introduce a convenient framework that includes smooth manifolds, complex manifolds, and many other kinds of spaces. Let X be a topological space; we shall always assume that X is Hausdorff and has a countable basis. For every open subset $U \subseteq X$, let $C(U)$ denote the ring of complex-valued continuous functions on U; the ring operations are defined pointwise.

Definition 5.1. A *geometric structure* $\mathscr{O}$ on the topological space X is a collection of subrings $\mathscr{O}(U) \subseteq C(U)$, where U runs over the open sets in X, subject to the following three conditions:

(1) The constant functions are in $\mathscr{O}(U)$.
(2) If $f \in \mathscr{O}(U)$ and $V \subseteq U$, then $f|_V \in \mathscr{O}(V)$.
(3) If $f_i \in \mathscr{O}(U_i)$ is a collection of functions satisfying $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is a unique $f \in \mathscr{O}(U)$ such that $f_i = f|_{U_i}$, where $U = \bigcup_{i \in I} U_i$.

The pair $(X, \mathscr{O})$ is called a *geometric space*; functions in $\mathscr{O}(U)$ will sometimes be called *distinguished*.

The second and third condition together mean that being distinguished is a local property; the typical example is differentiability (existence of a limit) or holomorphicity (power series expansion). In the language of sheaves, which will be introduced later in the course, we may summarize them by saying that $\mathscr{O}$ is a subsheaf of the sheaf of continuous functions on X.

Example 5.2. Let D be an open set in $\mathbb{C}^n$, and for every open subset $U \subseteq D$, let $\mathscr{O}(U) \subseteq C(U)$ be the subring of holomorphic functions on U. Since Definition 1.1 is clearly local, the pair $(D, \mathscr{O})$ is a geometric space.

Example 5.3. Let X be an open set in $\mathbb{R}^n$, and for every open subset $U \subseteq X$, let $\mathscr{A}(U) \subseteq C(U)$ be the subring of smooth (meaning, infinitely differentiable) functions on U. Then $(X, \mathscr{A})$ is again a geometric space.

Definition 5.4. A *morphism* $f\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of geometric spaces is a continuous map $f\colon X \to Y$, with the following additional property: whenever $U \subseteq Y$ is open, and $g \in \mathscr{O}_Y(U)$, the composition $g \circ f$ belongs to $\mathscr{O}_X\bigl(f^{-1}(U)\bigr)$.

Example 5.5. Let $D \subseteq \mathbb{C}^n$ and $E \subseteq \mathbb{C}^m$ be open subsets. Then a morphism of geometric spaces $f\colon (D, \mathscr{O}) \to (E, \mathscr{O})$ is the same as a holomorphic mapping $f\colon D \to E$. This is because a continuous map $f\colon D \to E$ is holomorphic iff it preserves holomorphic functions (by Lemma 2.1).

For a morphism $f\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$, we typically write

$$f^*\colon \mathscr{O}_Y(U) \to \mathscr{O}_X\bigl(f^{-1}(U)\bigr)$$

for the induced ring homomorphisms. We say that f is an *isomorphism* if it has an inverse that is also a morphism; this means that $f\colon X \to Y$ should be a homeomorphism, and that each map $f^*\colon \mathscr{O}_Y(U) \to \mathscr{O}_X\bigl(f^{-1}(U)\bigr)$ should be an isomorphism of rings.

Example 5.6. If $(X, \mathscr{O})$ is a geometric space, then any open subset $U \subseteq X$ inherits a geometric structure $\mathscr{O}|_U$, by setting $\bigl(\mathscr{O}|_U\bigr)(V) = \mathscr{O}(V)$ for $V \subseteq U$ open. With this definition, the natural inclusion map $(U, \mathscr{O}|_U) \to (X, \mathscr{O})$ becomes a morphism.

Complex manifolds. We now define a complex manifold as a geometric space that is locally isomorphic to an open subset of $\mathbb{C}^n$ (with the geometric structure given by Example 5.2).

Definition 5.7. A *complex manifold* is a geometric space $(X, \mathscr{O}_X)$ in which every point has an open neighborhood $U \subseteq X$, such that $(U, \mathscr{O}_X|_U) \simeq (D, \mathscr{O})$ for some open subset $D \subseteq \mathbb{C}^n$ and some $n \in \mathbb{N}$.

The integer n is called the *dimension* of the complex manifold X at the point x, and denoted by $\dim_x X$. In fact, it is uniquely determined by the rings $\mathscr{O}_X(U)$, as U ranges over sufficiently small open neighborhoods of x. Namely, define the *local ring* of X at the point x to be

$$\mathscr{O}_{X,x} = \lim_{U \ni x} \mathscr{O}_X(U);$$

as in the case of $\mathscr{O}_n$, its elements are germs of holomorphic functions in a neighborhood of $x \in X$. A moment's thought shows that we have $\mathscr{O}_{X,x} \simeq \mathscr{O}_n$, and therefore $\mathscr{O}_{X,x}$ is a local ring by Theorem 4.1. The integer n can now be recovered from $\mathscr{O}_{X,x}$ by Lemma 2.5, since $n = \dim_{\mathbb{C}} \mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the ideal of functions vanishing at the point x. In particular, the dimension is preserved under isomorphisms of complex manifolds, and is therefore a well-defined notion.

It follows that the function $x \mapsto \dim_x X$ is locally constant; if X is connected, the dimension is the same at each point, and the common value is called the *dimension* of the complex manifold X, denoted by $\dim X$. In general, the various connected components of X need not be of the same dimension, however.

A morphism of complex manifolds is also called a *holomorphic mapping*; an isomorphism is said to be a *biholomorphic mapping* or a *biholomorphism*. Example 5.5 shows that this agrees with our previous definitions for open subsets of $\mathbb{C}^n$.

Charts and atlases. Note that smooth manifolds can be defined in a similar way: as those geometric spaces that are locally isomorphic to open subsets of $\mathbb{R}^n$ (as in Example 5.3). More commonly, though, smooth manifolds are described by atlases: a collection of charts (or local models) is given, together with transition functions that describe how to pass from one chart to another. Since it is also convenient, let us show how to do the same for complex manifolds.

In the alternative definition, let X be a topological space (again, Hausdorff and with a countable basis). An *atlas* is a covering of X by open subsets $U_i \subseteq X$, indexed by $i \in I$, together with a set of homeomorphisms $\phi_i \colon U_i \to D_i$, where D_i is an open subset of some $\mathbb{C}^n$; the requirement is that the *transition functions*

$$g_{i,j} = \phi_i \circ \phi_j^{-1} \colon \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j),$$

which are homeomorphisms, should actually be biholomorphic mappings. Each $\phi_i \colon U_i \to D_i$ is then called a *coordinate chart* for X, and X is considered to be described by the atlas.

Proposition 5.8. *The alternative definition of complex manifolds is equivalent to Definition 5.7.*

Proof. One direction is straightforward: If we are given a complex manifold $(X, \mathscr{O}_X)$ in the sense of Definition 5.7, we can certainly find for each $x \in X$ an open neighborhood U_x, together with an isomorphism of geometric spaces $\phi_x \colon (U_x, \mathscr{O}_X|_{U_x}) \to (D_x, \mathscr{O})$, for $D_x \subseteq \mathbb{C}^n$ open. Then $g_{x,y}$ is an isomorphism between $\phi_x(D_x \cap D_y)$ and $\phi_y(D_x \cap D_y)$ as geometric spaces, and therefore a biholomorphic map.

For the converse, we assume that the topological space X is given, together with an atlas of coordinate charts $\phi_i \colon U_i \to D_i$. To show that X is a complex manifold, we first have to define a geometric structure: for $U \subseteq X$ open, set

$$\mathscr{O}_X(U) = \big\{ f \in C(U) \bigm| (f|_{U \cap U_i}) \circ \phi_i^{-1} \text{ holomorphic on } \phi_i(U \cap U_i) \text{ for all } i \in I \big\}.$$

The definition makes sense because the transition functions $g_{i,j}$ are biholomorphic. It is easy to see that $\mathscr{O}_X$ satisfies all three conditions in Definition 5.1, and so $(X, \mathscr{O}_X)$ is a geometric space. It is also a complex manifold, because every point has an open neighborhood (namely one of the U_i) that is isomorphic to an open subset of $\mathbb{C}^n$. □

The following class of examples should be familiar from last semester.

Example 5.9. Any Riemann surface is a one-dimensional complex manifold; this follows from Proposition 5.8. In fact, Riemann surfaces are precisely the (connected) one-dimensional complex manifolds.

Projective space. Projective space $\mathbb{P}^n$ is the most important example of a compact complex manifold, and so we spend some time defining it carefully. Basically, $\mathbb{P}^n$ is the set of lines in $\mathbb{C}^{n+1}$ passing through the origin. Each such line is spanned by a nonzero vector $(a_0, a_1, \dots, a_n) \in \mathbb{C}^{n+1}$, and two vectors a, b span the same line iff $a = \lambda b$ for some $\lambda \in \mathbb{C}^*$. We can therefore define

$$\mathbb{P}^n = \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\mathbb{C}^*,$$

and make it into a topological space with quotient topology. Consequently, a subset $U \subseteq \mathbb{P}^n$ is open iff its preimage $q^{-1}(U)$ under the quotient map $q\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is open. It is not hard to see that $\mathbb{P}^n$ is Hausdorff and compact, and that q is an open mapping.

The equivalence class of a vector $a \in \mathbb{C}^{n+1} - \{0\}$ is denoted by $[a]$; thus points of $\mathbb{P}^n$ can be described through their *homogeneous coordinates* $[a_0, a_1, \dots, a_n]$.

We would like to make $\mathbb{P}^n$ into a complex manifold, in such a way that the quotient map q is holomorphic. This means that if f is holomorphic on $U \subseteq \mathbb{P}^n$, then $g = f \circ q$ should be holomorphic on $q^{-1}(U)$, and invariant under scaling the coordinates. We therefore define

$$\begin{aligned}\mathscr{O}_{\mathbb{P}^n}(U) = \Big\{ f \in C(U) \;\Big|\; & g = f \circ q \text{ is holomorphic on } q^{-1}(U), \text{ and} \\ & g(\lambda a) = g(a) \text{ for } a \in \mathbb{C}^{n+1} \setminus \{0\} \text{ and } \lambda \in \mathbb{C}^* \Big\}.\end{aligned}$$

This definition is clearly local, and satisfies the conditions in Definition 5.1.

Class 6. Examples of complex manifolds

In the previous lecture, we defined projective space as the quotient

$$\mathbb{P}^n = \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\mathbb{C}^*;$$

with the quotient topology, it is a compact (Hausdorff) space. We also introduced the following geometric structure on it:

$$\begin{aligned}\mathscr{O}_{\mathbb{P}^n}(U) = \Big\{ f \in C(U) \;\Big|\; & g = f \circ q \text{ is holomorphic on } q^{-1}(U), \text{ and} \\ & g(\lambda a) = g(a) \text{ for } a \in \mathbb{C}^{n+1} \setminus \{0\} \text{ and } \lambda \in \mathbb{C}^* \Big\}\end{aligned}$$

It remains to show that the geometric space $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n})$ is actually a complex manifold. For this, we note that $\mathbb{P}^n$ is covered by the open subsets

$$U_i = \big\{ [a] \in \mathbb{P}^n \;\big|\; a_i \neq 0 \big\}.$$

To simplify the notation, we consider only the case $i = 0$. The map

$$\phi_0\colon U_0 \to \mathbb{C}^n, \quad [a] \mapsto \big(a_1/a_0, \dots, a_n/a_0\big)$$

is a homeomorphism; its inverse is given by sending $z \in \mathbb{C}^n$ to the point with homogeneous coordinates $[1, z_1, \dots, z_n]$.

$$\begin{tikzcd} q^{-1}(U_0) \arrow[r, "q"] \arrow[d] & U_0 \arrow[r, "\phi_0"] \arrow[d] & \mathbb{C}^n \\ \mathbb{C}^{n+1} \setminus \{0\} \arrow[r, "q"] & \mathbb{P}^n \end{tikzcd}$$

We claim that ϕ_0 is an isomorphism between the geometric spaces $(U_0, \mathscr{O}_{\mathbb{P}^n}|_{U_0})$ and $(\mathbb{C}^n, \mathscr{O})$. Since it is a homeomorphism, we only need to show that ϕ_0 induces an isomorphism between $\mathscr{O}(D)$ and $\mathscr{O}_{\mathbb{P}^n}\big(\phi_0^{-1}(D)\big)$, for any open set $D \subseteq \mathbb{C}^n$. This amounts to the following statement: a function $f \in C(D)$ is holomorphic iff $g = f \circ \phi_0 \circ q$ is holomorphic on $(\phi_0 \circ q)^{-1}(D)$. But that is almost obvious: on the one hand, we have

$$f(z_1, \dots, z_n) = g(1, z_1, \dots, z_n),$$

and so f is holomorphic if g is; on the other hand, on the open set where $a_0 \neq 0$, we have

$$g(a_0, a_1, \dots, a_n) = f\big(a_1/a_0, \dots, a_n/a_n\big),$$

and so g is holomorphic if f is. Similarly, one proves that each U_i is isomorphic to $\mathbb{C}^n$ as a geometric space; since $U_0, U_1, \dots, U_n$ together cover $\mathbb{P}^n$, it follows that $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n})$ is a complex manifold in the sense of Definition 5.7.

Quotients. Another basic way to construct complex manifolds is by dividing a given manifold by a group of automorphisms; a familiar example is the construction of elliptic curves as quotients of $\mathbb{C}$ by lattices.

First, a few definitions. An *automorphism* of a complex manifold X is a biholomorphic self-mapping from X onto itself. The *automorphism group* $\operatorname{Aut}(X)$ is the group of all automorphisms. A subgroup $\Gamma \subseteq \operatorname{Aut}(X)$ is said to be *properly discontinuous* if for any two compact subsets $K_1, K_2 \subseteq X$, the intersection $\gamma(K_1) \cap K_2$ is nonempty for only finitely many $\gamma \in \Gamma$. Finally, Γ is said to be *without fixed points* if $\gamma(x) = x$ for some $x \in X$ implies that $\gamma = \mathrm{id}$.

Example 6.1. Any lattice $\Lambda \subseteq \mathbb{C}$ acts on $\mathbb{C}$ by translation; the action is clearly properly discontinuous and without fixed points.

Define X/Γ as the set of equivalence classes for the action of Γ on X; that is to say, two points $x, y \in X$ are equivalent if $y = \gamma(x)$ for some $\gamma \in \Gamma$. We endow X/Γ with the quotient topology, making the quotient map $q\colon X \to X/\Gamma$ continuous. Note that q is also an open mapping: if $U \subseteq X$ is open, then

$$q^{-1}\big(q(U)\big) = \bigcup_{\gamma \in \Gamma} \gamma(U)$$

is clearly open, proving that $q(U)$ is an open subset of the quotient.

Proposition 6.2. *Let X be a complex manifold, and let $\Gamma \subseteq \operatorname{Aut}(X)$ be a properly discontinuous group of automorphisms of X without fixed points. Then the quotient X/Γ is naturally a complex manifold, and the quotient map $q\colon X \to X/\Gamma$ is holomorphic and locally a biholomorphism.*

Note that in order for q to be holomorphic and locally biholomorphic, the geometric structure on the quotient has to be given by

$$\mathscr{O}_{X/\Gamma}(U) = \big\{\, f \in \mathscr{O}_X\big(q^{-1}(U)\big) \;\big|\; f \circ \gamma = f \text{ for every } \gamma \in \Gamma \,\big\}.$$

Example 6.3. Let $\Lambda \subseteq \mathbb{C}^n$ be a lattice, that is, a discrete subgroup isomorphic to $\mathbb{Z}^{2n}$. Then Λ acts on $\mathbb{C}^n$ by translations, and the action is again properly discontinuous and without fixed points. Proposition 6.2 shows that the quotient is a complex manifold. As in the case of elliptic curves, one can easily show that $\mathbb{C}^n/\Lambda$ is compact; indeed, if $\lambda_1, \dots, \lambda_{2n}$ are a basis for Λ, then the map

$$[0,1]^{2n} \to \mathbb{C}^n/\Lambda, \quad (x_1, \dots, x_{2n}) \mapsto x_1\lambda_1 + \dots + x_{2n}\lambda_{2n} + \Lambda$$

is surjective. $\mathbb{C}^n/\Lambda$ is called a *complex torus* of dimension n.

Blowing up a point. Let M be a complex manifold and $p \in M$ a point with $\dim_p M = n$. The blow-up construction produces another complex manifold $\operatorname{Bl}_p M$, in which the point p is replaced by a copy of $\mathbb{P}^{n-1}$ that parametrizes all possible directions from p into M.

We first consider the case of the origin in $\mathbb{C}^n$. Each point $z \in \mathbb{C}^n$ determines a unique line through the origin, and hence a point in $\mathbb{P}^{n-1}$, except when $z = 0$. Thus if we define

$$\operatorname{Bl}_0 \mathbb{C}^n = \big\{\, (z, L) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \;\big|\; z \text{ lies on the line } L \subseteq \mathbb{C}^n \,\big\},$$

then the projection map $\pi\colon \mathrm{Bl}_0\,\mathbb{C}^n \to \mathbb{C}^n$ is bijective for $z \neq 0$, but contains an extra copy of $\mathbb{P}^{n-1}$ over the point $z = 0$. We call $\mathrm{Bl}_0\,\mathbb{C}^n$ the *blow-up* of $\mathbb{C}^n$ at the origin, and $\pi^{-1}(0)$ the *exceptional set*.

Lemma 6.4. $\mathrm{Bl}_0\,\mathbb{C}^n$ *is a complex manifold of dimension n, and the projection map $\pi\colon \mathrm{Bl}_0\,\mathbb{C}^n \to \mathbb{C}^n$ is holomorphic. Moreover, the exceptional set is a submanifold of dimension $n-1$.*

Proof. On $\mathbb{P}^{n-1}$, we use homogeneous coordinates $[a_1, \dots, a_n]$; then the condition defining the blow-up is that the vectors $(z_1, \dots, z_n)$ and $(a_1, \dots, a_n)$ should be linearly dependent. This translates into the equations $z_i a_j = a_i z_j$ for $1 \leq i, j \leq n$.

Let $q\colon \mathrm{Bl}_0\,\mathbb{C}^n \to \mathbb{P}^{n-1}$ be the other projection. On $\mathbb{P}^{n-1}$, we have natural coordinate charts U_i defined by the condition $a_i \neq 0$, and

$$U_i = \left\{ [a] \in \mathbb{P}^{n-1} \;\middle|\; a_i \neq 0 \right\} \simeq \mathbb{C}^{n-1}, \quad [a] \mapsto \left(\frac{a_1}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i} \right).$$

Consequently, the blow-up is covered by the n open sets $V_i = q^{-1}(U_i)$, and from the equations relating the two vectors z and a, we find that

$$V_i = \left\{ \big(z, [a]\big) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \;\middle|\; a_i \neq 0 \text{ and } z_j = z_i a_j / a_i \text{ for } j \neq i \right\} \simeq \mathbb{C}^n.$$

Explicitly, the isomorphism is given by the formula

$$f_i \colon V_i \to \mathbb{C}^n, \quad \big(z, [a]\big) \mapsto \left(\frac{a_1}{a_i}, \dots, \frac{a_{i-1}}{a_i}, z_i, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i} \right),$$

and so the inverse mapping takes $b \in \mathbb{C}^n$ to the point with coordinates $\big(z, [a]\big)$, where $a = (b_1, \dots, b_{i-1}, 1, b_{i+1}, \dots, b_n)$, and $z = b_i a$. In this way, we obtain n coordinate charts whose union covers the blow-up.

It is a simple matter to compute the transition functions. For $i \neq j$, the composition $g_{i,j} = f_i \circ f_j^{-1}$ takes the form $g_{i,j}(b_1, \dots, b_n) = (c_1, \dots, c_n)$, where

$$c_k = \begin{cases} b_k / b_i & \text{if } k \neq i, j, \\ b_i b_j & \text{if } k = i, \\ 1/b_i & \text{if } k = j. \end{cases}$$

We observe that $f_j(V_i \cap V_j)$ is the set of points $b \in \mathbb{C}^n$ with $b_i \neq 0$, which means that each $g_{i,j}$ is a holomorphic mapping. Consequently, the n coordinate charts determine a holomorphic atlas, and we can conclude from Proposition 5.8 that $\mathrm{Bl}_0\,\mathbb{C}^n$ is an n-dimensional complex manifold.

To prove that the mapping π is holomorphic, note that $\pi \circ f_i^{-1}$ is given in coordinates by the formula

$$\pi\big(f_i^{-1}(b_1, \dots, b_n)\big) = (b_i b_1, \dots, b_i b_{i-1}, b_i, b_i b_{i+1}, \dots, b_i b_n)$$

which is clearly holomorphic on $\mathbb{C}^n$. We see from this description that the intersection $\pi^{-1}(0) \cap U_i$ is mapped, under f_i, to the hyperplane $b_i = 0$. This means that $\pi^{-1}(0)$ is a complex submanifold of dimension $n-1$ (the precise definition of a submanifold will be given later). $\square$

Class 7. Further examples of complex manifolds

Recall that we constructed the blow-up $\pi\colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$ at the origin. Similarly, for any open subset $D \subseteq \mathbb{C}^n$ containing the origin, we define $\operatorname{Bl}_0 D$ as $\pi^{-1}(D)$, where $\pi\colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$ is as above.

We are now in a position to construct the blow-up $\operatorname{Bl}_p M$ of a point on an arbitrary complex manifold. Choose a coordinate chart $f\colon U \to D$ centered at the point p, and let $\tilde{D} = \operatorname{Bl}_0 D$ be the blow-up of D at the origin. Also let $M^* = M - \{p\}$, and $U^* = U \cap M^*$; then U^* is isomorphic to the complement of the exceptional set in $\tilde{D}$, and we can glue M^* and $\tilde{D}$ together along this common open subset. More precisely, define $\operatorname{Bl}_p M$ as the quotient of the disjoint union $M^* \sqcup \tilde{D}$ by the equivalence relation that identifies $q \in M^*$ and $x \in \tilde{D}$ whenever $q \in U^*$ and $f(q) = \pi(x)$. Since f is biholomorphic, and π is biholomorphic outside the origin, it is easy to see that transition functions between coordinate charts on M^* and on $\tilde{D}$ are biholomorphic. Thus $\operatorname{Bl}_p M$ is a complex manifold, and the projection map $\operatorname{Bl}_p M \to M$ is holomorphic.

It remains to show that the construction is independent of the choice of coordinate chart. In order to deal with this technical point, we first prove the following property of $\operatorname{Bl}_0 \mathbb{C}^n$. (The same result is then of course true for the blow-up of a point on any complex manifold.)

Lemma 7.1. *Let $f\colon M \to \mathbb{C}^n$ be a holomorphic mapping from a connected complex manifold. Suppose that $f(M) \neq \{0\}$, and that at every point $p \in M$ with $f(p) = 0$, the ideal generated by $f_1, \dots, f_n$ in the local ring $\mathscr{O}_{M,p}$ is principal. Then there is a unique holomorphic mapping $\tilde{f}\colon M \to \operatorname{Bl}_0 \mathbb{C}^n$ such that $f = \pi \circ \tilde{f}$.*

Proof. Since π is an isomorphism over $\mathbb{C}^n - \{0\}$, the uniqueness of $\tilde{f}$ follows easily from the identity theorem. Because of the uniqueness statement, the existence of $\tilde{f}$ becomes a local problem; we may therefore assume that we are dealing with a holomorphic map $f\colon D \to \mathbb{C}^n$, where D is an open neighborhood of $0 \in \mathbb{C}^m$, and $f(0) = 0$. By assumption, the ideal $(f_1, \dots, f_n) \subseteq \mathscr{O}_m$ is generated by a single element $g \in \mathscr{O}_m$; after possibly shrinking D, we may furthermore assume that $g = a_1 f_1 + \dots + a_n f_n$ and $f_j = b_j g$ for suitable holomorphic functions $a_j, b_j \in \mathscr{O}(D)$. We then have $a_1 b_1 + \dots + a_n b_n = 1$, and so at each point of D, at least one of the functions $b_1, \dots, b_n$ is nonzero. Since, in addition, $[f_1(z), \dots, f_n(z)] = [b_1(z), \dots, b_n(z)] \in \mathbb{P}^{n-1}$, we can now define

$$\tilde{f}\colon D \to \operatorname{Bl}_0 \mathbb{C}^n, \quad \tilde{f}(z) = \big(f_1(z), \dots, f_n(z), [b_1(z), \dots, b_n(z)]\big),$$

which clearly has the required properties. □

Now suppose we have a second coordinate chart centered at $p \in M$; without loss of generality, we may assume that it is of the form $\phi \circ f$, where $\phi\colon D \to E$ is biholomorphic and satisfies $\phi(0) = 0$. To prove that $\operatorname{Bl}_p M$ is independent of the choice of chart, we have to show that ϕ induces an isomorphism $\tilde{\phi}\colon \operatorname{Bl}_0 D \to \operatorname{Bl}_0 E$. By Lemma 7.1, it suffices to show that m coordinate functions of $\phi \circ \pi\colon \operatorname{Bl}_0 D \to E$ generate a principal ideal in the local ring at each point of $\operatorname{Bl}_0 D$. We may consider this question in one of the coordinate charts $f_i\colon V_i \to \mathbb{C}^n$ introduced during the proof of Lemma 6.4. Thus let $\psi = \phi \circ \pi \circ f_i^{-1}\colon \mathbb{C}^n \to E$; we then have

$$\psi(w) = \phi(w_i w_1, \dots, w_i w_{i-1}, w_i, w_i w_{i+1}, \dots, w_i w_n) \tag{7.2}$$

for any $w \in \mathbb{C}^n$.

Now we fix a point $b \in \mathbb{C}^n$, and let $I \subseteq \mathscr{O}_{\mathbb{C}^n,b}$ be the ideal generated by the functions $\psi_1(w), \dots, \psi_n(w)$ in the local ring at b. If $b_i \neq 0$, then since ϕ is bijective, at least one of the values $\psi_j(b)$ has to be nonzero, and so I is the unit ideal. We may therefore assume that $b_i = 0$; we shall argue that $I = (w_i)$. Because $\phi(0) = 0$, we can clearly write

$$\bigl(\phi_1(z), \dots, \phi_n(z)\bigr) = (z_1, \dots, z_n) \cdot A(z)$$

for a certain $n \times n$-matrix of holomorphic functions; upon substituting (7.2), we find that every function $\psi_j(w)$ is a multiple of w_i, and therefore $I \subseteq (w_i)$. On the other hand, $A(0) = J(\phi)|_{z=0}$ is invertible, and so $A(z)^{-1}$ is holomorphic on a suitable polydisk $\Delta(0; r) \subseteq D$. If we again substitute (7.2) into the resulting identity

$$(z_1, \dots, z_n) = \bigl(\phi_1(z), \dots, \phi_n(z)\bigr) \cdot A(z)^{-1},$$

we see that w_i can itself be expressed as a linear combination of $\psi_1(w), \dots, \psi_n(w)$ in a neighborhood of the point b. (More precisely, we need $|w_i| < r_i$ and $|w_i||b_j| < r_j$ for $j \neq i$.) This proves that $I = (w_i)$ in $\mathscr{O}_{\mathbb{C}^n,b}$, and completes the proof.

Vector bundles. Another useful class of complex manifolds is given by holomorphic vector bundles. Since we will be using vector bundles frequently during the course, we begin by reviewing some general theory. Let $\mathbb{K}$ be one of $\mathbb{R}$ or $\mathbb{C}$. Recall that if M is a topological space, then a $\mathbb{K}$-vector bundle on M is a mapping $\pi \colon E \to M$ of topological spaces, such that all fibers $E_p = \pi^{-1}(p)$ have the structure of $\mathbb{K}$-vector spaces in a compatible way. Informally, we think of a vector bundle as a continuously varying family of vector spaces E_p; here is the precise definition.

Definition 7.3. A $\mathbb{K}$-*vector bundle* of rank k on a topological space M is a continuous mapping $\pi \colon E \to M$, such that the following two conditions are satisfied:

(1) For each point $p \in M$, the *fiber* $E_p = \pi^{-1}(p)$ is a $\mathbb{K}$-vector space of dimension k.
(2) For every $p \in M$, there is an open neighborhood U and a homeomorphism

$$\phi \colon \pi^{-1}(U) \to U \times \mathbb{K}^k$$

mapping E_p into $\{p\} \times \mathbb{K}^k$, such that the composition $E_p \to \{p\} \times \mathbb{K}^k \to \mathbb{K}^k$ is an isomorphism of $\mathbb{K}$-vector spaces.

The pair (U, ϕ) is called a *local trivialization* of the vector bundle; also, E is called the *total space* and M the *base space*.

Any two local trivializations (U_α, ϕ_α) and (U_β, ϕ_β) can be compared over $U_\alpha \cap U_\beta$. Because of the second condition in the definition, the composition

$$\phi_\alpha \circ \phi_\beta^{-1} \colon (U_\alpha \cap U_\beta) \times \mathbb{K}^k \to (U_\alpha \cap U_\beta) \times \mathbb{K}^k$$

is necessarily of the form $(\mathrm{id}, g_{\alpha,\beta})$ for a continuous mapping

$$g_{\alpha,\beta} \colon U_{\alpha,\beta} \to \mathrm{GL}_k(\mathbb{K}).$$

These so-called *transition functions* satisfy the following compatibility conditions:

$$\begin{aligned} g_{\alpha,\beta} \cdot g_{\beta,\gamma} \cdot g_{\gamma,\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma; \\ g_{\alpha,\alpha} &= \mathrm{id} \quad \text{on } U_\alpha. \end{aligned} \tag{7.4}$$

When M is a smooth manifold, we say that E is a *smooth vector bundle* if the transition functions $g_{\alpha,\beta}$ are smooth maps. (Note that the group $\mathrm{GL}_k(\mathbb{K})$ has a natural manifold structure, being an open subset of the space of all $k \times k$-matrices over $\mathbb{K}$.) In that case, it is easy to see that E is itself a smooth manifold: indeed, each product $U_\alpha \times \mathbb{K}^k$ is a smooth manifold, and the transition functions $(\mathrm{id}, g_{\alpha,\beta})$ between them are diffeomorphisms. Clearly, the map $\pi\colon E \to M$ and the local trivializations ϕ_α are then smooth maps.

Similarly, when M is a complex manifold, we say that a $\mathbb{C}$-vector bundle E is *holomorphic* if the transition functions $g_{\alpha,\beta}$ are holomorphic maps. (This uses the fact that $\mathrm{GL}_k(\mathbb{C})$ is naturally a complex manifold.) In that case, it follows from Proposition 5.8 that E is itself a complex manifold , and that the map $\pi\colon E \to M$ as well as the local trivializations ϕ_α become holomorphic mappings.

It is possible to describe a vector bundle entirely through its transition functions, because the following result shows that the $g_{\alpha,\beta}$ uniquely determine the bundle.

Proposition 7.5. *Let M be a topological space, covered by open subsets U_α, and let $g_{\alpha,\beta}\colon \mathrm{GL}_k(\mathbb{K})$ be a collection of continuous mappings satisfying the conditions in* (7.4)*. Then the $g_{\alpha,\beta}$ are the transition functions for a (essentially unique) vector bundle E of rank k on M. If M is a smooth (resp., complex) manifold and the $g_{\alpha,\beta}$ are smooth (resp., holomorphic) maps, then E is a smooth (resp., holomorphic) vector bundle.*

Proof. We first define E as a topological space. On the disjoint union

$$\bigsqcup_\alpha U_\alpha \times \mathbb{K}^k,$$

there is a natural equivalence relation: two points $(p, v) \in U_\alpha \times \mathbb{K}^k$ and $(q, w) \in U_\beta \times \mathbb{K}^k$ are equivalent if $p = q$ and $v = g_{\alpha,\beta}(p) \cdot w$. This does define an equivalence relation because of the conditions in (7.4), and so we can let E be the quotient space. The obvious projection map $\pi\colon E \to M$ is then continuous, and it is easy to verify that E is a vector bundle of rank k with transition functions given by $g_{\alpha,\beta}$. The remaining assertion follows from the comments made above. ☐

Definition 7.6. A *section* of a vector bundle $\pi\colon E \to M$ over an open set $U \subseteq M$ is a continuous map $s\colon U \to E$ with the property that $\pi \circ s = \mathrm{id}_U$. We denote the set of all sections of E over U by the symbol $\Gamma(U, E)$.

When E is a smooth (resp., holomorphic) vector bundle, we usually require sections to be smooth (resp., holomorphic). It is a simple matter to describe sections in terms of transition functions: Suppose we are given a section $s\colon M \to E$. For each local trivialization $\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^k$, the composition $\phi_\alpha \circ s$ is necessarily of the form (id, s_α) for a continuous mapping $s_\alpha\colon U_\alpha \to \mathbb{K}^k$, and one checks that

$$g_{\alpha,\beta} \cdot s_\beta = s_\alpha \quad \text{on } U_\alpha \cap U_\beta. \tag{7.7}$$

Conversely, every collection of mappings s_α that satisfies these identities describes a section of E. Since (7.7) is clearly $\mathbb{K}$-linear, it follows that the set $\Gamma(U, E)$ is actually a $\mathbb{K}$-vector space.

Tangent spaces and tangent bundles. On a manifold, the most natural example of a vector bundle is the tangent bundle. Before discussing complex manifolds, we first review the basic properties of the tangent bundle on a smooth manifold.

Let M be a smooth manifold; to simplify the discussion, we assume that M is connected and let $n = \dim M$. Given any point $p \in M$, there is an isomorphism $f\colon U \to D$ between a neighborhood of p and an open subset $D \subseteq \mathbb{R}^n$; we may clearly assume that $f(p) = 0$. By composing the coordinate functions $x_1, \dots, x_n$ on

$\mathbb{R}^n$ with f, we obtain n smooth functions on U; they form a *local coordinate system* around the point $p \in M$. Despite the minor ambiguity, we continue to denote the coordinate functions by $x_1, \dots, x_n \in \mathscr{A}_M(U)$. Note that we have $x_j(p) = 0$ for every j.

On $\mathbb{R}^n$, we have n vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$ that act as derivations on the ring of smooth functions on D. By composing with f, we can view them as smooth vector fields on $U \subseteq M$; the action on $\mathscr{A}_M(U)$ is now given by the rule

$$\frac{\partial}{\partial x_j}\psi = \frac{\partial(\psi \circ f^{-1})}{x_j} \circ f$$

for any smooth function $\psi \colon U \to \mathbb{R}$. The values of those vector fields at the point p give a basis for the *real tangent space*

$$T_{\mathbb{R},p}M = \mathbb{R}\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}.$$

The tangent bundle $T_{\mathbb{R}}M$ is the smooth vector bundle with fibers $T_{\mathbb{R},p}M$; its sections are smooth vector fields. To obtain transition functions for $T_{\mathbb{R}}M$, let us see how vector fields transform between coordinate charts. To simplify the notation, let $f \colon U \to D$ and $g \colon U \to E$ be two charts with the same domain; we denote the coordinates on D by $x_1, \dots, x_n$, and the coordinates on E by $y_1, \dots, y_n$. As usual, we let $h = f \circ g^{-1} \colon E \to D$ be the diffeomorphism that compares the two charts.

Now say

$$\sum_{j=1}^n a_j(x)\frac{\partial}{\partial x_j} \quad \text{and} \quad \sum_{k=1}^n b_k(y)\frac{\partial}{\partial y_k}$$

are smooth vector fields on D and E, respectively, that represent the same vector field on U. Let $\psi \colon D \to \mathbb{R}$ be a smooth function; then since $\psi(x) = \psi\bigl(h(y)\bigr)$, we compute with the help of the chain rule that

$$\frac{\partial}{\partial y_k}\psi = \frac{\partial(\psi \circ h)}{\partial y_k} \circ h^{-1} = \sum_{j=1}^n \left(\frac{\partial h_j}{\partial y_k} \circ h^{-1}\right) \cdot \frac{\partial \psi}{\partial x_j}.$$

This means that, as vector fields on D,

$$\frac{\partial}{\partial y_k} = \sum_{j=1}^n \frac{\partial h_j}{\partial y_k}\bigl(h^{-1}(x)\bigr)\frac{\partial}{\partial x_j},$$

and so it follows that the coefficients in the two coordinate systems are related by the identity

$$a_j(x) = \sum_{k=1}^n \frac{\partial h_j}{\partial y_k}\bigl(h^{-1}(x)\bigr) \cdot b_k\bigl(h^{-1}(x)\bigr).$$

If we compose with $f \colon U \to D$ and note that $h^{-1} = g \circ f^{-1}$, we find that

$$a_j \circ f = \sum_{k=1}^n \left(\frac{\partial h_j}{\partial y_k} \circ g\right) \cdot \bigl(b_k \circ g\bigr)$$

Now if $a \colon U \to \mathbb{R}^n$ and $b \colon U \to \mathbb{R}^n$ represent the same smooth section of the tangent bundle, then we can read off the transition functions by comparing the formula we have just derived with (7.7). This leads to the following conclusion.

Definition 7.8. Let M be a (connected) smooth manifold of dimension n. Cover M by coordinate charts $f_\alpha \colon U_\alpha \to D_\alpha$, where $D_\alpha \subseteq \mathbb{R}^n$ is an open subset with coordinates $x_\alpha = (x_{\alpha,1}, \dots, x_{\alpha,n})$, and as usual set $h_{\alpha,\beta} = f_\alpha \circ f_\beta^{-1}$. Then the *real tangent bundle* $T_\mathbb{R} M$ is the smooth vector bundle of rank n defined by the collection of transition functions

$$g_{\alpha,\beta} = J_\mathbb{R}(h_{\alpha,\beta}) \circ h_\beta^{-1} \colon U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{R}),$$

where $J_\mathbb{R}(h_{\alpha,\beta}) = \partial h_{\alpha,\beta}/\partial x_\beta$ is the matrix of partial derivates of $h_{\alpha,\beta}$.

Class 8. Complex submanifolds

Now let M be a complex manifold, and let $p \in M$ be any point. Again, there is an isomorphism $f \colon U \to D$ between a neighborhood of p and an open subset $D \subseteq \mathbb{C}^n$, satisfying $f(p) = 0$; it defines a local holomorphic coordinate system $z_1, \dots, z_n \in \mathscr{O}_M(U)$ centered at the point p.

We can write $z_j = x_j + i y_j$, where both x_j and y_j are smooth real-valued functions on U. Then $(x_1, \dots, x_n, y_1, \dots, y_n)$ gives an isomorphism between U and an open subset of $\mathbb{R}^{2n}$; this illustrates the obvious fact that M is also a smooth manifold of real dimension $2n$. Consequently, the *real tangent space* at the point p is now

$$T_{\mathbb{R},p} M = \mathbb{R}\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\}.$$

Another useful notion is the *complexified tangent space*

$$\begin{aligned} T_{\mathbb{C},p} M &= \mathbb{C}\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\} \\ &= \mathbb{C}\left\{ \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right\}, \end{aligned}$$

where the alternative basis in the second line is again given by

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

Finally, the two subspaces

$$T'_p M = \mathbb{C}\left\{ \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\} \quad \text{and} \quad T''_p M = \mathbb{C}\left\{ \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right\}$$

of the complexified tangent space are called the *holomorphic* and *antiholomorphic tangent spaces*, respectively.

The holomorphic and antiholomorphic tangent spaces give a direct sum decomposition

$$T_{\mathbb{C},p} M = T'_p M \oplus T''_p M.$$

Evidently, $\partial/\partial \bar{z}_j$ is the complex conjugate of $\partial/\partial z_j$, and so complex conjugation interchanges $T'_p M$ and $T''_p M$. Therefore the map

$$T_{\mathbb{R},p} M \hookrightarrow T_{\mathbb{C},p} M \twoheadrightarrow T'_p M$$

is an isomorphism of $\mathbb{R}$-vector spaces; it maps $\partial/\partial x_j$ to $\partial/\partial z_j$ and $\partial/\partial y_j$ to $i \cdot \partial/\partial z_j$. The relationship between the different tangent spaces is one of the useful features of calculus on complex manifolds.

Example 8.1. The holomorphic tangent spaces $T'_p M$ are the fibers of a holomorphic vector bundle $T'M$, the *holomorphic tangent bundle* of M.

To describe a set of transition functions for the tanget bundle, we continue to assume that $\dim M = n$, and cover M by coordinate charts $f_\alpha \colon U_\alpha \to D_\alpha$, with $D_\alpha \subseteq \mathbb{C}^n$ open. Let

$$h_{\alpha,\beta} = f_\alpha \circ f_\beta^{-1} \colon f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$$

give the transitions between the charts. Then the differential $J(h_{\alpha,\beta})$ can be viewed as a holomorphic mapping from $f_\beta(U_\alpha \cap U_\beta)$ into $\mathrm{GL}_n(\mathbb{C})$; by analogy with the smooth case, we expect the transition functions for $T'M$ to be given by the formula

$$g_{\alpha,\beta} = J(h_{\alpha,\beta}) \circ f_\beta,$$

where $J(h_{\alpha,\beta}) = \partial h_{\alpha,\beta}/\partial z_\beta$ is now the matrix of all holomorphic partial derivatives. Let us verify that the compatibility conditions in (7.4) hold. By the chain rule,

$$\begin{aligned} g_{\alpha,\beta} \cdot g_{\beta,\gamma} &= \bigl(J(h_{\alpha,\beta}) \circ f_\beta\bigr) \cdot \bigl(J(h_{\beta,\gamma}) \circ f_\gamma\bigr) = \Bigl(\bigl(J(h_{\alpha,\beta}) \circ h_{\beta,\gamma}\bigr) \cdot J(h_{\beta,\gamma})\Bigr) \circ f_\gamma \\ &= J(h_{\alpha,\beta} \circ h_{\beta,\gamma}) \circ f_\gamma = J(h_{\alpha,\gamma}) \circ f_\gamma = g_{\alpha,\gamma}, \end{aligned}$$

and so the $g_{\alpha,\beta}$ are the transition functions for a holomorphic vector bundle $\pi \colon T'M \to M$ of rank n. The same calculation as in the smooth case shows that sections of $T'M$ are holomorphic vector fields.

Complex submanifolds. Let $(X, \mathscr{O}_X)$ be a geometric space, and $Z \subseteq X$ any subset. There is a natural way to make Z into a geometric space: First, we give Z the induced topology. We call a continuous function $f \colon V \to \mathbb{C}$ on an open subset $V \subseteq Z$ *distinguished* if every point $a \in Z$ admits an open neighborhood U_a in X, such that there exists $f_a \in \mathscr{O}_X(U_a)$ with the property that $f(z) = f_a(z)$ for every $z \in V \cap U_a$. One can easily check that this defines a geometric structure on Z, which we denote by $\mathscr{O}_X|_Z$.

Now suppose that X is a complex manifold. We are interested in finding conditions under which $(Z, \mathscr{O}_X|_Z)$ is also a complex manifold. The following example illustrates the situation.

Example 8.2. Consider $\mathbb{C}^k$ as a subset of $\mathbb{C}^n$ (for $n \geq k$), by means of the embedding $(z_1, \dots, z_k) \mapsto (z_1, \dots, z_k, 0, \dots, 0)$. If f is a holomorphic function on an open subset $V \subseteq \mathbb{C}^k$, then f is distinguished in the above sense, since it obviously extends to a holomorphic function on $V \times \mathbb{C}^{n-k}$. Thus we have $\mathscr{O}_{\mathbb{C}^n}|_{\mathbb{C}^k} = \mathscr{O}_{\mathbb{C}^k}$.

The example motivates the following definition.

Definition 8.3. A subset Z of a complex manifold $(X, \mathscr{O}_X)$ is called *smooth* if, for every point $a \in Z$, there exists a chart $\phi \colon U \to D \subseteq \mathbb{C}^n$ such that $\phi(U \cap Z)$ is the intersection of D with a linear subspace of $\mathbb{C}^n$. In that case, we say that $(Z, \mathscr{O}_X|_Z)$ is a *complex submanifold* of X.

Calling Z a complex submanifold is justified, because Z is obviously itself a complex manifold. Indeed, if $\phi \colon U \to D$ is a local chart for X as in the definition, then the restriction of ϕ to $U \cap Z$ provides a local chart for Z.

The submanifold theorem. Examples of submanifolds are given by level sets of holomorphic mappings whose differential has constant rank. A similar result, sometimes called the submanifold theorem, should be familiar from the theory of smooth manifolds.

Let $f\colon M \to N$ be a holomorphic mapping between two complex manifolds M and N; recall that this means that f is continuous, and $g \circ f \in \mathscr{O}_M(f^{-1}(U))$ for every holomorphic function $g \in \mathscr{O}_N(U)$ and every open subset $U \subseteq N$.

Fix a point $p \in M$, and let $z_1, \dots, z_n$ be holomorphic coordinates centered at p; also let $w_1, \dots, w_m$ be holomorphic coordinates centered at $q = f(p)$. We can express f in those coordinates as $w_k = f_k(z_1, \dots, z_n)$, with $f_1, \dots, f_m$ holomorphic functions in a neighborhood of $0 \in \mathbb{C}^n$. In particular, each f_j is then a smooth function, and so f is also a smooth mapping. It therefore induces a linear map

$$f_*\colon T_{\mathbb{R},p}M \to T_{\mathbb{R},q}N$$

between real tangent spaces, and therefore also a map between the complexified tangent spaces. By the complex version of the chain rule, we have

$$f_* \frac{\partial}{\partial z_j} = \sum_{k=1}^{m} \left(\frac{\partial f_k}{\partial z_j} \frac{\partial}{\partial w_k} + i \cdot \frac{\partial \bar{f}_k}{\partial z_j} \frac{\partial}{\partial \bar{w}_k} \right) = \sum_{k=1}^{m} \frac{\partial f_k}{\partial z_j} \frac{\partial}{\partial w_k} \tag{8.4}$$

because each f_j is holomorphic; therefore f_* maps T'_pM into T'_qN. (In fact, one can show that a smooth map $f\colon M \to N$ is holomorphic iff f_* preserves holomorphic tangent spaces.)

We digress to explain the relationship between $J_{\mathbb{R}}(f)$ and $J(f)$. In the complexified tangent space $T_{\mathbb{C},p}M$, we may use the basis given by $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$, and for $T_{\mathbb{C},q}N$ the basis given by $\partial/\partial w_k$ and $\partial/\partial \bar{w}_k$. According to (8.4), the map f_* is then represented by the $2m \times 2n$-matrix

$$J_{\mathbb{C}}(f) = \begin{pmatrix} J(f) & 0 \\ 0 & \overline{J(f)} \end{pmatrix},$$

where $J(f) = \partial f/\partial z$ is the $m \times n$-matrix with entries $\partial f_k/\partial z_j$. This simple calculation shows that if M and N have the same dimension (i.e., $m = n$), then

$$\det J_{\mathbb{R}}(f) = |\det J(f)|^2. \tag{8.5}$$

This relationship makes it possible to deduce the holomorphic submanifold theorem from its usual version on smooth manifolds (which is a fairly difficult result).

Since we already have the implicit mapping theorem (in Theorem 4.6, whose proof used the Weierstraß theorems), we can give a direct proof.

Theorem 8.6. *Let $f\colon M \to N$ be a holomorphic mapping between complex manifolds, and suppose that the differential $f_*\colon T_pM \to T_{f(p)}N$ has constant rank r at every point $p \in M$. Then for every $q \in N$, the level set $f^{-1}(q)$ is either empty, or a complex submanifold of M. Moreover, if $f(p) = q$, then we have*

$$\dim_p f^{-1}(q) = \dim_p M - r.$$

Proof. We shall suppose that we have a point $p \in M$ with $f(p) = q$. By choosing local coordinates centered at p and q respectively, we reduce to the case where $D \subseteq \mathbb{C}^n$ is an open neighborhood of the origin, and $f\colon D \to \mathbb{C}^m$ is a holomorphic mapping with $f(0) = 0$ and $J(f)$ has rank r throughout D. Moreover, after making linear changes of coordinates and shrinking D, we may clearly assume that the submatrix $\partial(f_1, \dots, f_r)/\partial(z_1, \dots, z_r)$ is nonsingular. The theorem will be proved if we show that, in some neighborhood of the origin, $f^{-1}(0)$ is a complex submanifold of D of dimension $n-r$.

Introduce a holomorphic mapping $g\colon D \to \mathbb{C}^n$ by setting

$$g_j(z) = \begin{cases} f_j(z) & \text{for } j = 1, \dots, r, \\ z_j & \text{for } j = r+1, \dots, n. \end{cases}$$

Clearly $J(g)$ is nonsingular for $z = 0$, and so by the inverse mapping theorem (from the exercises), g is a biholomorphism between suitable open neighborhoods of $0 \in \mathbb{C}^n$; this means that g can be used to define a new coordinate system. After making that change of coordinates (which amounts to replacing f by $f \circ g^{-1}$), we can therefore assume that f has the form

$$f(z) = \bigl(z_1, \dots, z_r, f_{r+1}(z), \dots, f_m(z)\bigr).$$

The remaining functions $f_{r+1}(z), \dots, f_m(z)$ can only depend on $z_1, \dots, z_r$; indeed, since $\operatorname{rk} J(f) = r$, we necessarily have $\partial f_j / \partial z_k = 0$ for all $j, k > r$. But this implies that the level set $f^{-1}(0)$ is the intersection of D with the linear subspace $z_1 = \cdots = z_r = 0$, and therefore a complex submanifold of dimension $n-r$. □

Example 8.7. Let $f\colon D \to \mathbb{C}$ be a holomorphic function on an open subset of $\mathbb{C}^n$. Then the level sets $f^{-1}(a)$ are complex submanifolds of D, provided that at each point $z \in D$, at least one of the partial derivatives $\partial f / \partial z_j$ is nonzero. Submanifolds defined by a single holomorphic function are called *hypersurfaces*.

Analytic sets. Definition 4.2 can easily be extended to subsets of arbitrary complex manifolds: a subset $Z \subseteq M$ is said to be *analytic* if it is locally defined by the vanishing of a (finite) collection of holomorphic functions. Proposition 4.5 shows that, at least locally, analytic sets can always be decomposed into finitely many irreducible components.

Sometimes, an analytic subset $Z \subseteq M$ is actually a complex submanifold: if $z_1, \dots, z_n$ are local coordinates centered at a point $p \in Z$, and Z can be defined in a neighborhood U of the point by holomorphic functions $f_1, \dots, f_m$ with the property that $\partial(f_1, \dots, f_m)/\partial(z_1, \dots, z_n)$ has constant rank r, then $Z \cap U$ is a complex submanifold of U of dimension $n-r$. This is exactly the content of Theorem 8.6.

We call a point $p \in Z$ a *smooth point* if Z is a submanifold of M in some neighborhood of p; otherwise, p is said to be *singular*. The set of all singular points of Z is denoted by Z^s, and is called the *singular locus* of Z.

Example 8.8. Let $f(z, w) = z^2 + w^3 \in \mathscr{O}(\mathbb{C}^2)$. The partial derivatives are $\partial f / \partial z = 2z$ and $\partial f / \partial w = 3w^2$, and both vanish together exactly at the point $(0, 0)$. Thus $Z(f)$ is a submanifold of $\mathbb{C}^2$ at every point except the origin, and $Z^s = \{(0, 0)\}$.

Every analytic set Z is a submanifold at most of its points, because of the following lemma (whose proof is contained in the exercises).

Lemma 8.9. *Let $Z \subseteq M$ be an analytic set in a complex manifold M. Then the singular locus Z^s is contained in an analytic subset strictly smaller than Z.*

Note that points where several irreducible components of Z meet are necessarily singular points. In fact, a much stronger statement is true: Z^s is itself an analytic subset of Z. But the proof of this fact requires more theory, and will have to wait until later in the semester.

Class 9. Differential forms

Differential forms. We now turn to calculus on complex manifolds; just as for smooth manifolds, differential forms are a highly useful tool for this purpose. We briefly recall the definition. Let M be a smooth manifold, with real tangent bundle $T_{\mathbb{R}}M$. A differential k-form ω is a section of the smooth vector bundle $\bigwedge^k T_{\mathbb{R}}^* M$; in other words, it associates to k smooth vector fields $\xi_1, \dots, \xi_k$ a smooth function $\omega(\xi_1, \dots, \xi_k)$, and is multilinear and alternating in its arguments. We denote the space of all differential k-forms on M by $A^k(M)$.

Let $U \subseteq \mathbb{R}^n$ be an open subset, with coordinates $x_1, \dots, x_n$. We then have the basic one-forms $dx_1, \dots, dx_n$, defined by

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Any one-form can then be written as $\varphi_1 dx_1 + \dots + \varphi_n dx_n$, for smooth functions $\varphi_j \in A(U)$. Similarly, every $\omega \in A^k(U)$ can be expressed as

$$\omega = \sum_{i_1 < \dots < i_k} \varphi_{i_1, \dots, i_k}(x_1, \dots, x_n) \cdot dx_{i_1} \wedge \dots \wedge dx_{i_k}; \tag{9.1}$$

the coefficient $\varphi_{i_1, \dots, i_k}$ is the smooth function $\omega\big(\partial/\partial x_{i_1}, \dots, \partial/\partial x_{i_k}\big)$. We often use multi-index notation, and write (9.1) more compactly as $\omega = \sum_{|I|=k} \varphi_I dx_I$.

Returning to the case of a smooth manifold M, any $\omega \in A^k(M)$ is locally given by an expression as in (9.1), where $x_1, \dots, x_n$ are now local coordinates. With some patience and the chain rule, one can work out how to transform such expressions from one coordinate system to another, and thereby determine the transition functions for the vector bundle $\bigwedge^k T_{\mathbb{R}}^* M$. For example, for n-forms, they are

$$\det J_{\mathbb{R}}(h_{\alpha,\beta}) \circ f_\beta \tag{9.2}$$

where $f_\alpha \colon U_\alpha \to D_\alpha \subseteq \mathbb{R}^n$ are local charts, and $h_{\alpha,\beta} = f_\alpha \circ f_\beta^{-1}$ as usual.

Given a k-form ω, we can define its *exterior derivative* $d\omega$; it is a $(k+1)$-form, which is given in local coordinates by the rule

$$d\omega = \sum_{j=1}^n \sum_I \frac{\partial \varphi_I}{\partial x_j} \cdot dx_j \wedge dx_I.$$

For instance, if f is a smooth function, then $df = \sum \partial f / \partial x_i \cdot dx_i$. Exterior differentiation gives a map $d \colon A^k(M) \to A^{k+1}(M)$, which satisfies the Leibniz rule $d(f\omega) = df \wedge \omega + f d\omega$. One can easily check that $d \circ d = 0$; this means that

$$A^0(M) \xrightarrow{d} A^1(M) \xrightarrow{d} A^2(M) \longrightarrow \cdots \longrightarrow A^{n-1}(M) \xrightarrow{d} A^n(M)$$

is a complex of vector spaces. According to the de Rham theorem, this complex computes the singular cohomology of the manifold M: for every $k = 0, \dots, n$,

$$H^k(M, \mathbb{R}) \simeq \frac{\ker d \colon A^k(M) \to A^{k+1}(M)}{\operatorname{coker} d \colon A^{k-1}(M) \to A^k(M)}.$$

In other words, the space of closed differential forms ($d\omega = 0$) modulo the space of exact differential forms ($\omega = d\psi$) is exactly the singular cohomology of M.

Finally, recall that differential forms can be pulled back along smooth mappings $f \colon M \to N$. For $\omega \in A^k(M)$, the pullback $f^*\omega \in A^k(M)$ is a differential k-form on M; the operation is easily described in local coordinates. Let $x_1, \dots, x_n$ be coordinates centered at $p \in M$, and $y_1, \dots, y_m$ coordinates centered at $f(p) \in N$, and write the components of f as $y_i = f_i(x_1, \dots, x_n)$. Then we have

$$f^* dy_i = df_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j,$$

and from this, we can derive a (somewhat complicated) formula for the pullback of any differential form.

Types. Having briefly reviewed the smooth case, suppose now that M is a complex manifold, and let $z_1, \dots, z_n$ be local coordinates. If we set $z_j = x_j + iy_j$, then as noted previously, the smooth functions $x_1, \dots, x_n, y_1, \dots, y_n$ give a local coordinate system for M as a smooth manifold, and we can consequently talk about differential forms on M, and define the spaces $A^k(M)$ as above. But observe that we have

$$dz_j = dx_j + i dy_j \quad \text{and} \quad d\bar{z}_j = dx_j - i dy_j;$$

we can therefore use $dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$ instead of $dx_1, \dots, dx_n, dy_1, \dots, dy_n$, and write any k-form locally as

$$\omega = \sum_{|I|+|J|=k} \varphi_{I,J} \cdot dz_I \wedge d\bar{z}_J,$$

where each $\varphi_{I,J}$ is a again a smooth function on U.

Definition 9.3. We say that $\omega \in A^k(M)$ is *of type* (p, q) if it can locally be written in the form

$$\omega = \sum_{|I|=p} \sum_{|J|=q} \varphi_{I,J} \cdot dz_I \wedge d\bar{z}_J.$$

The space of all such (p, q)-forms is denoted by $A^{p,q}(M)$.

Using the chain rule, it is easy to check that this definition is independent of the choice of local coordinate system. We can also decompose the exterior derivative by type as $d = \partial + \bar{\partial}$, where $\partial \colon A^{p,q}(M) \to A^{p+1,q}(M)$ and $\bar{\partial} \colon A^{p,q}(M) \to A^{p,q+1}(M)$; in local coordinates, we have

$$\bar{\partial}\left(\sum \varphi_{I,J} dz_I \wedge d\bar{z}_J\right) = \sum \frac{\partial \varphi_{I,J}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J.$$

As before, we get a complex since $\bar{\partial} \circ \bar{\partial} = 0$; we define the *Dolbeault cohomology* of the complex manifold M as

$$H^{p,q}(M) = \frac{\ker \bar{\partial} \colon A^{p,q}(M) \to A^{p,q+1}(M)}{\operatorname{coker} \bar{\partial} \colon A^{p,q-1}(M) \to A^{p,q}(M)}. \tag{9.4}$$

Note that $H^{p,0}(M)$ is the space of holomorphic p-forms on M, that is, the space of $\omega \in A^p(M)$ that can locally be written as $\sum_{|I|=p} f_I dz_I$ with f_I holomorphic.

The $\bar{\partial}$-Poincaré lemma. The key step in proving de Rham's theorem is to show that closed forms are always locally exact. The same result is true for Dolbeault cohomology, and is the content of the so-called $\bar{\partial}$-Poincaré lemma.

Lemma. *Let $D \subseteq \mathbb{C}^n$ be an open subset, and $\omega \in A^{p,q+1}(D)$ be a $\bar{\partial}$-closed form with $q \geq 0$. Then for any relatively compact open set U with $\bar{U} \subseteq D$, there is a (p, q)-form $\psi \in A^{p,q}(U)$ such that $\omega = \bar{\partial}\psi$ on U.*

Class 10. The Poincaré lemma

As a warm-up, let us prove the $\bar{\partial}$-Poincaré lemma in one complex variable.

Lemma 10.1. *Let $g\colon \mathbb{C} \to \mathbb{C}$ be a smooth function with compact support. Then the (singular) integral*

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(w)}{w-z} dw \wedge d\bar{w} \tag{10.2}$$

converges for every $z \in \mathbb{C}$, and defines a smooth, compactly supported function with $\partial f/\partial \bar{z} = g$.

Proof. Recall the more precise form of Cauchy's formula: Let $D = \Delta(z;R)$ and $D_\varepsilon = \Delta(z;\varepsilon)$. If f is smooth in a neighborhood of the closed disk $\overline{D}$, then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i}\int_D \frac{\partial f}{\partial \bar{w}}(w)\frac{dw \wedge d\bar{w}}{w-z}. \tag{10.3}$$

This is proved by letting $\alpha = (2\pi i)^{-1} f(w) dw/(w-z)$, and applying Stokes' theorem

$$\int_{D \setminus D_\varepsilon} d\alpha = \int_{\partial D} \alpha - \int_{\partial D_\varepsilon} \alpha.$$

to obtain the identity

$$-\frac{1}{2\pi i}\int_{D\setminus D_\varepsilon} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{f(w)}{w-z} dw.$$

After setting $w = re^{i\theta} + z$ and computing that $dw \wedge d\bar{w} = 2ir \cdot d\theta \wedge dr$, this becomes

$$-\frac{1}{\pi}\int_{D\setminus D_\varepsilon} \frac{\partial f}{\partial \bar{w}}(z + re^{i\theta}) \cdot e^{-i\theta} d\theta \wedge dr = \int_0^{2\pi} f(z + Re^{i\theta})\frac{d\theta}{2\pi} - \int_0^{2\pi} f(z + \varepsilon e^{i\theta})\frac{d\theta}{2\pi}$$

and converges to the asserted formula as $\varepsilon \to 0$, because the integrands are smooth functions.

We now prove the lemma. Changing to polar coordinates by again setting $w = re^{i\theta} + z$, the integral in (10.2) becomes

$$f(z) = \frac{1}{\pi}\int_{\mathbb{C}} g(z + re^{i\theta}) \cdot e^{-i\theta} d\theta \wedge dr.$$

Since g has compact support, it is clear from this expression that f is well-defined and smooth on $\mathbb{C}$. Interchanging the order of differentiation and integration, and undoing the change of coordinates, we then have

$$\frac{\partial f}{\partial \bar{z}}(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g}{\partial \bar{w}}(z + re^{i\theta}) \cdot e^{-i\theta} d\theta \wedge dr = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial g}{\partial \bar{w}}(w)\frac{dw \wedge d\bar{w}}{w-z}.$$

Now the support of g is contained in $D = \Delta(z;R)$ for sufficiently large R, and so we get the result by applying (10.3), noting that the integral over ∂D is zero. $\square$

We can now prove the higher-dimensional version of the $\bar{\partial}$-Poincaré lemma from last time.

Lemma 10.4. *Let $D \subseteq \mathbb{C}^n$ be an open subset, and $\omega \in A^{p,q+1}(D)$ be a $\bar{\partial}$-closed form with $q \geq 0$. Then for any relatively compact open set U with $\bar{U} \subseteq D$, there is a (p,q)-form $\psi \in A^{p,q}(U)$ such that $\omega = \bar{\partial}\psi$ on U.*

Proof. The proof of the lemma works by induction; the k-th step is to show that the statement is true when ω does not depend on $d\bar{z}_{k+1}, \dots, d\bar{z}_n$. This is clearly trivial when $k = 0$, and gives us the desired result when $k = n$). Suppose that the statement has been proved for $k-1$, and that ω does not involve $d\bar{z}_{k+1}, \dots, d\bar{z}_n$. Write ω in the form $\alpha \wedge d\bar{z}_k + \beta$, where $\alpha \in A^{p,q}(\mathbb{C}^n)$ and $\beta \in A^{p,q+1}(\mathbb{C}^n)$ do not depend on $d\bar{z}_k, \dots, d\bar{z}_n$. As usual, let $\alpha = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J$; then $\bar{\partial}\omega = 0$ implies that $\partial \alpha_{I,J} / \partial \bar{z}_j = 0$ for every $j > k$.

Now choose a smooth function ρ with compact support inside D that is identically equal to 1 on an open neighborhood V of $\bar{U}$. By the above,

$$\varphi_{I,J}(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \alpha_{I,J}(z_1, \dots, z_{k-1}, w, z_{k+1}, \dots, z_n) \frac{\rho(w)}{w - z_k} dw \wedge d\bar{w}$$

is a smooth function on D; it satisfies $\partial \varphi_{I,J} / \partial \bar{z}_j = 0$ for $j > k$, and $\partial \varphi_{I,J} / \partial \bar{z}_k = \alpha_{I,J}$ at every point of V. If we now let $\varphi = \sum \varphi_{I,J} dz_I \wedge d\bar{z}_J$, then $\omega - \bar{\partial}\varphi$ is independent of $\bar{z}_k, \dots, \bar{z}_n$ on V. By induction, we can find $\psi' \in A^{p,q}(U)$ such that $\omega - \bar{\partial}\varphi = \bar{\partial}\psi'$, and then $\psi = \varphi + \psi'$ does the job. $\square$

By writing any (generalized) polydisk as an increasing union of relatively compact polydisks, one can then deduce the following proposition.

Proposition 10.5. *Let $D = \left\{ z \in \mathbb{C}^n \;\middle|\; |z_j| < r_j \right\}$, where we allow the possibility that some or all $r_j = \infty$. Then $H^{p,q}(D) = 0$ for $q \geq 1$.*

Class 11. Integration and Riemannian manifolds

Integration. Differential forms are connected with integration on manifolds, as follows. Suppose that M is an oriented manifold, meaning that we have a consistent choice of orientation on each tangent space $T_{\mathbb{R},p}M$. It then makes sense to talk about the orientation of a system of local coordinates: $x_1, \dots, x_n$ is positively oriented if the vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$ form a positive basis in each $T_{\mathbb{R},p}M$. (A necessary and sufficient condition for being orientable is that the transition functions $h_{\alpha,\beta}$ between local charts are orientation preserving, in the sense that $\det J_{\mathbb{R}}(h_{\alpha,\beta}) > 0$.)

Let $\omega \in A^n(M)$ be a smooth n-form with compact support. We can cover the support of ω by finitely many coordinate charts U_α, and choose a partition of unity $1 = \sum \rho_\alpha$ subordinate to the covering. In positively oriented local coordinates $x_{\alpha,1}, \dots, x_{\alpha,n}$, we have

$$(\rho_\alpha \omega)|_{U_\alpha} = \varphi_\alpha dx_{\alpha,1} \wedge \cdots \wedge dx_{\alpha,n},$$

where φ_α are smooth functions with compact support in $D_\alpha \subseteq \mathbb{R}^n$. We then define the integral of ω over M by the formula

$$\int_M \omega = \sum_\alpha \int_{D_\alpha} \varphi_\alpha d\mu, \tag{11.1}$$

where μ is Lebesgue measure on $\mathbb{R}^n$. Note that this definition makes sense: by (9.2), we have

$$dx_{\alpha,1} \wedge \cdots \wedge dx_{\alpha,n} = \left(\det J_{\mathbb{R}}(h_{\alpha,\beta}) \circ h_{\alpha,\beta}^{-1} \right) \cdot dx_{\beta,1} \wedge \cdots \wedge dx_{\beta,n},$$

and since M is orientable, there is no problem with the choice of sign. It follows from the usual change of variables formula for integrals that the definition does not depend on the choice of coordinates.

As in calculus, Stokes' theorem is valid: if $\psi \in A^{n-1}(M)$ has compact support, then $\int_M d\psi = 0$. This proves the familiar fact that, on a compact orientable n-dimensional manifold, $H^n(X, \mathbb{R}) \simeq \mathbb{R}$, where the isomorphism is given by integration over M.

An important fact in complex geometry is that any complex manifold M is automatically orientable. Indeed, the transition functions $h_{\alpha,\beta}$ between coordinate charts are now biholomorphic, and we have seen in (8.5) that $\det J_{\mathbb{R}}(h_{\alpha,\beta}) = |J(h_{\alpha,\beta})|^2 > 0$. We take the natural orientation to be the one given in local coordinates $z_j = x_j + iy_j$ by the ordering

$$x_1, y_1, x_2, y_2, \dots, x_n, y_n.$$

We can therefore integrate any compactly supported form $\omega \in A^{n,n}(M)$, and the integral $\int_M \omega$ is a complex number. Noting that $dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy$, we compute that

$$(dx_1 \wedge dy_1) \wedge \dots \wedge (dx_n \wedge dy_n) = \frac{i^n}{2^n}(dz_1 \wedge d\bar{z}_1) \wedge \dots \wedge (dz_n \wedge d\bar{z}_n);$$

this takes the place of Lebesgue measure in the definition of the integral above.

Riemannian manifolds. Let M be a smooth manifold of dimension n. Recall that a Riemannian metric on M is a collection of positive definite symmetric bilinear forms $g_p \colon T_{\mathbb{R},p}M \otimes T_{\mathbb{R},p}M \to \mathbb{R}$ that vary smoothly with $p \in M$. In other words, for any pair of smooth vector fields $\xi, \eta \in \Gamma(U, T_{\mathbb{R}}M)$ on an open subset $U \subseteq M$, the real-valued function $g(\xi, \eta)$ is required to be smooth on U. In local coordinates $x_1, \dots, x_n$, we define the smooth functions

$$g_{i,j}(x) = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right),$$

and then the $n \times n$-matrix $G(x)$ with those entries is symmetric and positive definite at each point of U.

Example 11.2. On $\mathbb{R}^n$, we have the Euclidean metric for which $g_{i,j} = 1$ if $i = j$, and 0 otherwise. Since the n-sphere $\mathbb{S}^n$ is contained in $\mathbb{R}^{n+1}$, it inherits a Riemannian metric (by noting that $T_{\mathbb{R},p}\mathbb{S}^n \subseteq T_{\mathbb{R},p}\mathbb{R}^{n+1}$ at each point). It is a good exercise to compute the coefficients $g_{i,j}$ for $\mathbb{S}^2$, in the two coordinate charts given by stereographic projection.

On an oriented manifold, the Riemannian metric also determines a differential form in $A^n(M)$, called the volume form. Let us first consider the case of a real vector space V of dimension n. Recall that the vector space $\bigwedge^n V$ is one-dimensional, and that an orientation of V consists in choosing one of the two connected components of $\bigwedge^n V \setminus \{0\}$ and calling it the positive one. (We may then say that a basis $v_1, \dots, v_n$ is positive if $v_1 \wedge \dots \wedge v_n$ lies in that component.) Now suppose that V is endowed with an inner product $g \colon V \otimes V \to \mathbb{R}$. It induces inner products on each of the spaces $\bigwedge^k V$, with the property that

$$g\big(v_1 \wedge \dots \wedge v_k, w_1 \wedge \dots \wedge w_k\big) = \det\big(g(v_j, w_k)\big)_{1 \le j,k \le n}.$$

This allows us to choose a distinguished generator for $\bigwedge^n V$, namely the unique positive element φ with the property that $g(\varphi, \varphi) = 1$; it is usually called the *fundamental element*. To describe it directly, let $e_1, \dots, e_n$ be a positively oriented orthonormal basis for V; then $\varphi = e_1 \wedge \dots \wedge e_n$.

Through the isomorphism $V \to V^*$ given by $v \mapsto g(v, -)$, the dual space $V^* = \operatorname{Hom}(V, \mathbb{R})$ also inherits an orientation and an inner product, and we have a fundamental element in $\bigwedge^n V^*$. In fact, it is not hard to see that the latter is given by the formula $g(\varphi, -)$.

Let M be an oriented Riemannian manifold of dimension n. Then the *volume form* is the unique smooth form $vol(g) \in A^n(M)$ whose value at any point $p \in M$ is the fundamental element in $\bigwedge^n T^*_{\mathbb{R},p}M$. If $x_1, \dots, x_n$ are local coordinates on an open subset $U \subseteq M$, such that $\partial/\partial x_1, \dots, \partial/\partial x_n$ is a positive basis at each point, then we have

$$vol(g)|_U = \sqrt{\det G(x)} \cdot dx_1 \wedge \cdots \wedge dx_n. \tag{11.3}$$

We define the *volume* of M to be $\operatorname{vol}(M) = \int_M vol(g)$; note that this integral may be infinite if M is noncompact.

Example 11.4. If we let M be the sphere of radius r in $\mathbb{R}^3$, with the induced Riemannian metric, then $\operatorname{vol}(M) = 4\pi r^2$.

Class 12. Hermitian manifolds

Linear algebra. We begin by looking at some linear algebra on a complex vector space V. To begin with, V is also a real vector space (of twice the dimension); when considering V as a real vector space, we use the symbol J to denote multiplication by i for clarity. $J \in \operatorname{End}_{\mathbb{R}}(V)$ satisfies $J \circ J = -\operatorname{id}$, and contains the information about the original complex structure on V.

A *Hermitian form* on V is a map $h \colon V \times V \to \mathbb{C}$ which is $\mathbb{C}$-linear in its first argument, and such that $h(v_2, v_1) = \overline{h(v_1, v_2)}$ for all $v_1, v_2 \in V$. It follows that h is $\mathbb{C}$-antilinear in its second argument. We say that h is *positive definite* if $h(v, v) > 0$ for every nonzero $v \in V$; note that $h(v, v) \in \mathbb{R}$.

It is not hard to verify that if h is positive definite, then its real part

$$g(v_1, v_2) = \operatorname{Re} h(v_1, v_2) = \frac{1}{2}\big(h(v_1, v_2) + h(v_2, v_1)\big)$$

defines an inner product on the underlying real vector space; h is uniquely determined by g, as a brief calculation shows that $h(v_1, v_2) = g(v_1, v_2) + ig(v_1, Jv_2)$. (In fact, this formula defines a Hermitian form iff g is compatible with J, in the sense that $g(Jv_1, Jv_2) = g(v_1, v_2)$ for all $v_1, v_2 \in V$.)

Consider next the imaginary part of h, or

$$\omega(v_1, v_2) = -\operatorname{Im} h(v_1, v_2) = \frac{i}{2}\big(h(v_1, v_2) - h(v_2, v_1)\big).$$

It follows from the properties of h that ω is a real bilinear form that is alternating, meaning that $\omega(v_2, v_1) = -\omega(v_1, v_2)$. One easily sees that $\omega(v_1, Jv_2) = g(v_1, v_2)$; consequently, an alternating real-valued form ω comes from a Hermitian form iff $\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$ for all $v_1, v_2 \in V$; moreover, ω uniquely determines h.

Hermitian manifolds. We now generalize this to complex manifolds. Recall that if $p \in M$ is a point on a complex manifold, then the composition $T_{\mathbb{R},p}M \hookrightarrow T_{\mathbb{C},p}M \twoheadrightarrow T'_pM$ is an isomorphism of real vector spaces. We use this isomorphism to identify the underlying real vector space of T'_pM with $T_{\mathbb{R},p}M$; we continue to denote by J the endomorphism of $T_{\mathbb{R},p}M$ induced from multiplication by i.

Definition 12.1. A *Hermitian metric* h on a complex manifold M is a collection of positive definite Hermitian forms h_p on the holomorphic tangent spaces $T_{\mathbb{C},p}M$, whose real parts $g_p = \operatorname{Re} h_p$ induce a Riemannian metric on the underlying smooth manifold.

On a Hermitian manifold (M, h), we thus have a Riemannian metric $g = \operatorname{Re} h$ and a real-valued differential 2-form $\omega = -\operatorname{Im} h$.

We make the definition more concrete by writing down formulas in local holomorphic coordinates $z_1, \dots, z_n$. First off, we let H be the $n \times n$-matrix with entries the smooth functions

$$h_{j,k} = h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right);$$

at each point, H is Hermitian-symmetric and positive definite.

To find the Riemannian metric, let $z_j = x_j + iy_j$, and recall that, under our identification of $T_{\mathbb{R},p}M$ with T'_pM, the vector field $\partial/\partial x_j$ corresponds to $\partial/\partial z_j$, and $\partial/\partial y_j = J\partial/\partial x_j$ to $i \cdot \partial/\partial z_j$. Thus we have for instance that

$$g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = \operatorname{Re} h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = \operatorname{Re} h_{j,k},$$

while

$$g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\right) = g\left(\frac{\partial}{\partial x_j}, J\frac{\partial}{\partial x_k}\right) = \operatorname{Re} h\left(\frac{\partial}{\partial z_j}, i\frac{\partial}{\partial z_k}\right) = \operatorname{Im} h_{j,k}.$$

In the basis $\partial/\partial x_1, \dots, \partial/\partial x_n, \partial/\partial y_1, \dots, \partial/\partial y_n$, the Riemannian metric g is therefore given by the $2n \times 2n$-matrix

$$G = \begin{pmatrix} \operatorname{Re} H & \operatorname{Im} H \\ -\operatorname{Im} H & \operatorname{Re} H \end{pmatrix},$$

Note that G is a symmetric matrix, as expected: H being Hermitian symmetric, it follows that $\operatorname{Re} H$ is symmetric, while $-\operatorname{Im} H = H^T$.

Finally, consider the 2-form ω; we compute that

$$\omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = -\operatorname{Im} h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = -\operatorname{Im} h_{j,k},$$

while

$$\omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\right) = -\operatorname{Im} h\left(\frac{\partial}{\partial z_j}, i\frac{\partial}{\partial z_k}\right) = \operatorname{Re} h_{j,k}.$$

To make sense of these formulas, let us view ω as a complex-valued 2-form by extending it bilinearly to the complexified tangent spaces $T_{\mathbb{C},p}M$; here we have to be careful to distinguish multiplication by i and the effect of the operator J. We would now like express ω in terms of $dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$. We compute that

$$\begin{aligned} 4\omega\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right) &= \omega\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right) \\ &= \omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) - i\omega\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k}\right) + i\omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\right) + \omega\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}\right), \end{aligned}$$

which, by the above formulas, equals $-\operatorname{Im} h_{j,k} + i\operatorname{Re} h_{j,k} + i\operatorname{Re} h_{j,k} - \operatorname{Im} h_{j,k} = 2ih_{j,k}$. Similarly, one proves that $\omega(\partial/\partial z_j, \partial/\partial z_k) = \omega(\partial/\partial \bar{z}_j, \partial/\partial \bar{z}_k) = 0$, and so

$$\omega = \frac{i}{2}\sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\bar{z}_k. \tag{12.2}$$

It follows that ω is of type $(1,1)$; this justifies calling it the *associated* $(1,1)$*-form* of the metric h.

Example 12.3. Consider $\mathbb{C}^n$ with the metric in which the $\partial/\partial z_j$ form a unitary basis; in the notation from above, $H = \mathrm{id}_n$. Then g is the standard Euclidean metric on $\mathbb{R}^{2n}$, and

$$\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

This is one of the reasons for defining $\omega = -\operatorname{Im} h$.

Example 12.4. Let $N \subseteq M$ be a submanifold. For every $p \in N$, we have $T'_p N \subseteq T'_p M$, and so a Hermitian metric h_M on M naturally induces one on N. If we denote the latter by h_N, then a brief computation in local coordinates shows that $\omega_N = i^* \omega_M$, where $i \colon N \to M$ is the inclusion map.

The Fubini-Study metric. We now come to an important example: on $\mathbb{P}^n$, there is a natural Hermitian metric called the Fubini-Study metric. It will be easiest to describe the metric through its associated $(1,1)$-form ω_{FS}. Recall that $\mathbb{P}^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by $\mathbb{C}^*$, and that the quotient map

$$q \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$$

is holomorphic. Then ω_{FS} is the unique $(1,1)$-form on $\mathbb{P}^n$ whose pullback via the map q to $\mathbb{C}^{n+1} \setminus \{0\}$ is given by the formula

$$q^* \omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log\bigl(|z_0|^2 + |z_1|^2 + \dots + |z_n|^2\bigr). \tag{12.5}$$

One readily derives formulas in local coordinates: for example, in the chart $U_0 \subseteq \mathbb{P}^n$ with coordinates $[1, z_1, \dots, z_n]$, we have

$$\begin{aligned} \omega_{FS}|_{U_0} &= \frac{i}{2\pi} \partial \bar{\partial} \log\bigl(1 + |z|^2\bigr) \\ &= \frac{i}{2\pi} \left(\frac{1}{1+|z|^2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j - \frac{1}{(1+|z|^2)^2} \sum_{j,k=1}^{n} \bar{z}_j z_k dz_j \wedge d\bar{z}_k \right), \end{aligned}$$

where we have set $|z|^2 = |z_1|^2 + \dots + |z_n|^2$. One can read off the coefficients of the associated metric using (12.2), and this shows that we have really defined a metric on $\mathbb{P}^n$.

We note two useful properties of the Fubini-Study metric. The first is its invariance under unitary automorphisms of $\mathbb{P}^n$. Suppose that $A \in U(n+1)$ is a unitary matrix; it defines an automorphism f_A of $\mathbb{P}^n$ by the formula $[z] \mapsto [Az]$. Since $|Az| = |z|$ for every $z \in \mathbb{C}^{n+1}$, we clearly have $f_A^* \omega_{FS} = \omega_{FS}$. The second is that ω_{FS} is both d-closed and $\bar{\partial}$-closed, and therefore defines cohomology classes in the de Rham cohomology group $H^2(\mathbb{P}^n, \mathbb{R})$ and in the Dolbeault cohomology group $H^{1,1}(\mathbb{P}^n)$. Both cohomology groups are one-dimensional, and the class of ω_{FS} is the natural generator. The reason for the normalizing factor $1/2\pi$ in the definition of the Fubini-Study metric can be found in one of the exercises: on $\mathbb{P}^1$, we have $\int_{\mathbb{P}^1} \omega_{FS} = 1$.

The Wirtinger theorem. Let (M, h) be a complex manifold with a Hermitian metric. Locally, there always exist unitary frames for the metric h, that is, smooth sections $\zeta_1, \dots, \zeta_n$ of $T'M$ whose values give a unitary basis for the holomorphic tangent space T'_pM at each point. For such a frame, we have

$$h(\zeta_j, \zeta_k) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

One way to construct such a unitary frame is to start from an arbitrary frame (for instance, the coordinate vector fields $\partial/\partial z_1, \dots, \partial/\partial z_n$), and then apply the Gram-Schmidt process. If we let $\theta_1, \dots, \theta_n$ be a dual basis of smooth $(1,0)$-forms, in the sense that $\theta_j(\zeta_k) = 1$ if $j = k$, and 0 otherwise, then we have

$$\omega = \frac{i}{2} \sum_{j=1}^{n} \theta_j \wedge \overline{\theta_j}.$$

From this, we compute that

$$\omega^{\wedge n} = \omega \wedge \cdots \wedge \omega = n! \cdot \frac{i^n}{2^n} (\theta_1 \wedge \overline{\theta_1}) \wedge \cdots \wedge (\theta_n \wedge \overline{\theta_n}) = n! \cdot vol(g),$$

and so the volume form on the underlying oriented Riemannian manifold is given by Wirtinger's formula

$$vol(g) = \frac{1}{n!} \omega^{\wedge n}.$$

If we suppose in addition that M is compact, then we can conclude that

$$\operatorname{vol}(M) = \int_M vol(g) = \frac{1}{n!} \int_M \omega^{\wedge n}.$$

Since the volume of M is necessarily nonzero, it follows from Stokes' theorem that $\omega^{\wedge n}$ cannot be exact, and therefore that ω itself can never be an exact form.

Let $N \subseteq M$ be a complex submanifold, with the induced Hermitian metric. We then have $\omega_N = i^* \omega$, and if we set $m = \dim N$, then

$$\operatorname{vol}(N) = \frac{1}{m!} \int_N i^* \omega^{\wedge m}.$$

In particular, the volume of any submanifold is given by the integral of a globally defined differential form on M, which is very special to complex manifolds.

Example 12.6. The flat metric on $\mathbb{C}^n$ from Example 12.3 induces a Hermitian metric h_M on every complex torus $M = \mathbb{C}^n/\Lambda$. To compute the volume of M, we choose a fundamental domain $D \subseteq \mathbb{C}^n$ for the lattice; then the interior of D maps isomorphically to its image in M, and so

$$\operatorname{vol}(M) = \int_M vol(g_M) = \int_D vol(g) = \int_D d\mu$$

is the usual Lebesgue measure of D.

Class 13. Sheaves and cohomology

Introduction. Sheaves are a useful tool for relating local to global data. We begin with a nice example, taken from "Principles of Algebraic Geometry" by Griffiths and Harris, that shows this passage from local to global.

Let M be a Riemann surface, not necessarily compact. Recall that a meromorphic function on M is a mapping $f \colon M \to \mathbb{C} \cup \{\infty\}$ that can locally be written as a quotient of two holomorphic functions, with denominator not identically zero. (Equivalently, a meromorphic function is a holomorphic mapping from M to the Riemann sphere $\mathbb{P}^1$, not identically equal to ∞.) In a neighborhood of any point $p \in M$, we can choose a holomorphic coordinate z centered at p, and write f in the form $\sum_{j \geq -N} a_j z^j$. The polar part of f is the sum $\pi_p(f) = \sum_{j<0} a_j z^j$; clearly f is holomorphic at p iff the polar part is zero.

A classical problem, named after Mittag-Leffler, is whether one can find a meromorphic function with prescribed polar parts at a discrete set of points $p_1, p_2, \dots$. One can approach this question from two different points of view.

For the first, let U_i be a small open neighborhood of p_i not containing any of the other points, and let π_i be the desired polar part at p_i. Also let $U_0 = M \setminus \{p_1, p_2, \dots\}$, and set $\pi_0 = 0$. On the intersection $U_i \cap U_j$, the difference $g_{i,j} = \pi_i - \pi_j$ is a holomorphic function. Now if we can find a meromorphic function f with those polar parts, then $f - \pi_i$ is holomorphic on U_i, and so $g_{i,j} = (f - \pi_j) - (f - \pi_i)$ is actually the difference of two holomorphic functions. Conversely, if there are holomorphic functions $f_i \in \mathscr{O}_M(U_i)$ such that $g_{i,j} = f_j - f_i$, then the individual functions $f_i + \pi_i$ agree on overlaps, and therefore define a global meromorphic function with the correct polar parts.

Note that $g_{i,j} + g_{j,k} = g_{i,k}$ on $U_i \cap U_j \cap U_k$. If we let $\mathbf{U}$ denote the given open cover, and $\mathbf{g}$ the collection of holomorphic functions $g_{i,j} \in \mathscr{O}_M(U_i \cap U_j)$, then we can summarize our observations as follows: Whether or not the Mittag-Leffler problem has a solution is measured by the class of $\mathbf{g}$ in the vector space

$$H^1(\mathbf{U}, \mathscr{O}_M) = Z^1(\mathbf{U}, \mathscr{O}_M)/B^1(\mathbf{U}, \mathscr{O}_M);$$

here $Z^1(\mathbf{U}, \mathscr{O}_M) = \left\{ \mathbf{g} \;\middle|\; g_{i,j} + g_{j,k} = g_{i,k} \text{ on } U_i \cap U_j \cap U_k \right\}$ is the space of so-called 1-cocycles, and $B^1(\mathbf{U}, \mathscr{O}_M) = \left\{ \mathbf{g} \;\middle|\; g_{i,j} = f_j - f_i \text{ for suitable } f_i \in \mathscr{O}_M(U_i) \right\}$ the space of 1-coboundaries. The quotient is the first *Čech cohomology group* for the sheaf $\mathscr{O}_M$ and the given open cover.

The second point of view is more analytic in nature. With the same notation as above, let ρ_i be a smooth function with compact support inside U_i, and identically equal to 1 in a neighorhood of the point p_i. Then

$$\omega = \sum_{i=0}^{\infty} \bar{\partial}(\rho_i \pi_i) = \sum_{i=0}^{\infty} \pi_i \cdot \bar{\partial} \rho_i$$

is a smooth $(0,1)$-form on M, identically equal to zero in a neighborhood of each point p_i. Suppose now that $\omega = \bar{\partial}\phi$ for some smooth function ϕ on M. Then ϕ is holomorphic in a neighborhood of each p_i, and the difference $f = \sum_i \rho_i \pi_i - \phi$ is holomorphic on U_0, and clearly has the correct polar part π_i at each point p_i. Since the converse is easily shown to be true as well, we arrive at the following conclusion: Whether or not the Mittag-Leffler problem has a solution is measured by the first Dolbeault cohomology group

$$H^{0,1}(M) = \frac{\{\, \omega \in A^{0,1}(M) \mid \bar{\partial}\omega = 0 \,\}}{\{\, \bar{\partial}\phi \mid \phi \in A^{0,0}(M) \,\}}.$$

Since we already know that $H^{0,1}(\mathbb{C}) = 0$ (by Proposition 10.5), we deduce the well-known fact that the Mittag-Leffler problem can always be solved on $\mathbb{C}$.

To summarize: Since the problem can always be solved locally, the only issue is the existence of a global solution. In either approach, the obstruction to finding a global solution lies in a certain cohomology group. In fact, as we will later see, $H^1(\mathbf{U}, \mathscr{O}_M) \simeq H^{0,1}(M)$.

Sheaves. We now introduce the useful concept of sheaves.

Definition 13.1. Let X be a topological space. A *sheaf* (of abelian groups) on X assigns to every open set $U \subseteq X$ a group $\mathscr{F}(U)$, called the *sections* of the sheaf, and to every inclusion $V \subseteq U$ a *restriction homomorphism* $\rho_V^U \colon \mathscr{F}(U) \to \mathscr{F}(V)$, subject to the following two conditions:

(1) If $W \subseteq V \subseteq U$ are open sets, then $\rho_W^V \circ \rho_V^U = \rho_W^U$. One can therefore write $s|_V$ in place of $\rho_V^U(s)$ without loss of information.
(2) If $s_i \in \mathscr{F}(U_i)$ is a collection of sections satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is a unique $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$, where $U = \bigcup_{i \in I} U_i$.

In practice, a sheaf often has additional structure: for instance, we say that $\mathscr{F}$ is a sheaf of rings if every $\mathscr{F}(U)$ is a (commutative) ring, and if the restriction maps are ring homomorphisms. Similarly, there are sheaves of vector spaces, etc. For clarity, we sometimes denote the set of sections of $\mathscr{F}$ by the symbol $\Gamma(U, \mathscr{F})$ instead of the $\mathscr{F}(U)$ in the definition.

Example 13.2. A geometric structure $\mathscr{O}$ on a topological space X is a sheaf of rings: each $\mathscr{O}(U)$ is a subring of the ring of continuous functions on U, and the conditions in the two definitions are more or less identical.

Example 13.3. Let $\pi \colon E \to X$ be a vector bundle on X. Then assigning to every open set $U \subseteq X$ the space of continuous sections of the vector bundle over U defines a sheaf of vector spaces on X. When E is smooth (or holomorphic), we usually consider smooth (or holomorphic) sections instead.

On a complex manifold M, there are by and large three interesting classes of sheaves. The first are the so-called *locally constant sheaves*; for example, assigning to every open set U the set of locally constant maps from U into $\mathbb{Z}$ defines a sheaf $\mathbb{Z}_M$; one similarly defines $\mathbb{R}_M$ and $\mathbb{C}_M$. Such sheaves contain information about M as a topological space: for instance, $\Gamma(M, \mathbb{C}_M)$ is a $\mathbb{C}$-vector space whose dimension equals the number of connected components of M (since a locally constant functions has to be constant on each connected component).

The second class of sheaves are sections of smooth vector bundles, as in Example 13.3 above. The most important examples are the sheaf of sections of the tangent bundle, which assigns to every open set $U \subseteq M$ the space of smooth vector fields on U, and sheaves of differential forms. We let $\mathscr{A}^k$ be the sheaf that assigns to an open set U the space of smooth k-forms on U (these are sections of the vector bundle $\bigwedge^k T_{\mathbb{R}}^* M$). Likewise, the sections of the sheaf $\mathscr{A}^{p,q}$ are the (p,q)-forms on

U. Such sheaves contain information about M as a smooth manifold, and are very useful for doing calculus.

The third class are those sheaves that are connected to the complex structure on M. Examples are the structure sheaf $\mathscr{O}_M$, whose sections are the holomorphic functions; the sheaves Ω_M^p, where $\Omega_M^p(U)$ is the space of holomorphic forms of type $(p,0)$ on U; the sheaf of units $\mathscr{O}_M^*$, defined by letting $\mathscr{O}_M^*(U)$ be the set of nowhere vanishing holomorphic functions on U; and, more generally, the sheaf of holomorphic sections of any holomorphic vector bundle on M.

Stalks and operations. Let $\mathscr{F}$ be a sheaf on a topological space X, and let $x \in X$ be a point. The *stalk* of the sheaf is the direct limit

$$\mathscr{F}_x = \lim_{U \ni x} \mathscr{F}(U),$$

taken over all open neighborhoods of the point. The stalk is again an abelian group; it is a ring (or vector space) if $\mathscr{F}$ is a sheaf of rings (or vector spaces). We think of elements of the stalk as germs of sections at x.

Example 13.4. On a complex manifold M, the local ring $\mathscr{O}_{M,p}$ is the stalk of the sheaf $\mathscr{O}_M$ at the point p.

A *morphism* of sheaves $f \colon \mathscr{F} \to \mathscr{G}$ is a collection of group homomorphisms $f_U \colon \mathscr{F}(U) \to \mathscr{G}(U)$, compatible with restriction maps in the sense that $\rho_V^U \circ f_U = f_V \circ \rho_V^U$ for every inclusion $V \subseteq U$. If each f_U is the inclusion of a subgroup, we say that $\mathscr{F}$ is a *subsheaf* of $\mathscr{G}$.

The *kernel* of a morphism of sheaves is the subsheaf of $\mathscr{F}$ defined by setting

$$\Gamma(U, \ker f) = \left\{\, s \in \mathscr{F}(U) \;\middle|\; f_U(s) = 0 \,\right\};$$

it is not hard to verify that $\ker f$ is indeed a sheaf. A morphism of sheaves also has an *image* $\operatorname{im} f$, which is a subsheaf of $\mathscr{G}$; but the definition is more complicated since the groups $\operatorname{im} f_U$ do not form a sheaf. To ensure that the second condition in Definition 13.1 is satisfied, we are forced instead to set

$$\Gamma(U, \operatorname{im} f) = \left\{\, s \in \mathscr{G}(U) \;\middle|\; s|_{U_i} \in \operatorname{im} f_{U_i} \text{ for some open cover } U = \bigcup_{i \in I} U_i \,\right\}.$$

We say that f is *injective* if $\ker f = 0$, and that f is *surjective* if $\operatorname{im} f = \mathscr{G}$. Finally, we say that a sequence of sheaves

$$\mathscr{F}^0 \xrightarrow{f^0} \mathscr{F}^1 \xrightarrow{f^1} \mathscr{F}^2 \longrightarrow \cdots \longrightarrow \mathscr{F}^{k-1} \xrightarrow{f^{k-1}} \mathscr{F}^k$$

is a *complex* if $f_{i+1} \circ f_i = 0$ for every i, and that it is *exact* if $\ker f_{i+1} = \operatorname{im} f_i$ at all places.

Example 13.5. If $\mathscr{F}$ is a subsheaf of $\mathscr{G}$, one can also define a quotient sheaf $\mathscr{G}/\mathscr{F}$, in such a way that there is an exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{G}/\mathscr{F} \to 0$. It is a good exercise to work out the correct definition.

The following example illustrates these notions; it is one of the most important exact sequences of sheaves on a complex manifold M.

Example 13.6. On a complex manifold M, the so-called *exponential sequence*

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow \mathscr{O}_M \xrightarrow{\exp} \mathscr{O}_M^* \longrightarrow 0$$

is an exact sequence of sheaves. (The group operation on $\mathscr{O}_M^*(U)$ is multiplication.) The first map is given by the inclusion $\mathbb{Z}_M(U) \subseteq \mathscr{O}_M(U)$, using that locally constant functions are holomorphic. The second map takes a holomorphic function $f \in \mathscr{O}_M(U)$ to the nowhere vanishing holomorphic function $\exp_U(f) = e^{2\pi i f}$. It is easy to see that the sequence is exact at $\mathbb{Z}_M$ and at $\mathscr{O}_M$; in fact, if $e^{2\pi i f} = 1$ for some holomorphic function f, then f is integer-valued, and hence locally constant.

Exactness at $\mathscr{O}_M^*$ means the surjectivity of exp; according to the definition above, this amounts to saying that a nowhere vanishing holomorphic function g can *locally* be written in the form $e^{2\pi i f}$. After choosing local coordinates, we can reduce to the case $g \in \mathscr{O}(D)$, where $D \subseteq \mathbb{C}^n$ is a small polydisk. After choosing a suitable branch of the logarithm, we can then take $f = \log g$ on D.

Note that the individual maps $\exp_U \colon \mathscr{O}_M(U) \to \mathscr{O}_M^*(U)$ need not be surjective; with $M = \mathbb{C}$ and $U = \mathbb{C} \setminus \{0\}$, for example, the holomorphic function z cannot be written in the form $e^{2\pi i f}$ with f holomorphic on U.

The example shows that a morphism $f \colon \mathscr{F} \to \mathscr{G}$ can be surjective, even though the individual maps $f_U \colon \mathscr{F}(U) \to \mathscr{G}(U)$ are not.

We note that a morphism $f \colon \mathscr{F} \to \mathscr{G}$ always induces homomorphisms $f_x \colon \mathscr{F}_x \to \mathscr{G}_x$ between stalks. The following proposition shows that injectivity, surjectivity, and so forth, can be verified at the level of stalks; this means that they are local properties.

Proposition 13.7. *Let $f \colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. Then f is surjective (resp., injective) iff for every point $x \in X$, the induced map on stalks $f_x \colon \mathscr{F}_x \to \mathscr{G}_x$ is injective (resp., surjective). Likewise, a sequence of sheaves*

$$\mathscr{F}^0 \xrightarrow{f^0} \mathscr{F}^1 \xrightarrow{f^1} \mathscr{F}^2 \longrightarrow \cdots \longrightarrow \mathscr{F}^{k-1} \xrightarrow{f^{k-1}} \mathscr{F}^k$$

is exact iff the induced sequence of abelian groups

$$\mathscr{F}_x^0 \xrightarrow{f_x^0} \mathscr{F}_x^1 \xrightarrow{f_x^1} \mathscr{F}_x^2 \longrightarrow \cdots \longrightarrow \mathscr{F}_x^{k-1} \xrightarrow{f_x^{k-1}} \mathscr{F}_x^k$$

is exact for every point $x \in X$.

Sheaf cohomology. The following lemma is easy to prove from the definitions.

Lemma 13.8. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves on a topological space X, then $0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X)$ is an exact sequence of abelian groups.*

In general, the map $\mathscr{F}(X) \to \mathscr{F}''(X)$ need not be surjective; we have already seen an example of this above. But in practice, one often needs to know whether or not a given section of $\mathscr{F}''$ can be lifted to a section of $\mathscr{F}$. Sheaf cohomology solves this problem by giving us a long exact sequence of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathscr{F}') \arrow[r] & H^0(X,\mathscr{F}) \arrow[r] & H^0(X,\mathscr{F}'') \arrow[dll, rounded corners] \\
& H^1(X,\mathscr{F}') \arrow[r] & H^1(X,\mathscr{F}) \arrow[r] & H^1(X,\mathscr{F}'') \arrow[dll, rounded corners] \\
& H^2(X,\mathscr{F}') \arrow[r] & H^2(X,\mathscr{F}) \arrow[r] & H^2(X,\mathscr{F}'') \arrow[r] & \cdots
\end{tikzcd}$$

Here $H^0(X, \mathscr{F}) = \mathscr{F}(X)$, and so the higher cohomology groups $H^i(X, \mathscr{F})$ extend the sequence in Lemma 13.8. This means that a section in $\mathscr{F}''(X)$ can be lifted to a section in $\mathscr{F}(X)$ iff its image in the first cohomology group $H^1(X, \mathscr{F}')$ is zero.

To define the cohomology groups of a sheaf, we introduce the following notion: A sheaf $\mathscr{F}$ on a topological space is called *flabby* if the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective for every open set $U \subseteq X$. With flabby sheaves, taking global sections preserves exactness.

Lemma 13.9. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves on a topological space X, and if $\mathscr{F}'$ is flabby, then $0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X) \to 0$ is an exact sequence of abelian groups.*

Proof. Let $\alpha\colon \mathscr{F}' \to \mathscr{F}$ and $\beta\colon \mathscr{F} \to \mathscr{F}''$ denote the maps. By virtue of Lemma 13.8, it suffices to show that $\beta_X : \mathscr{F}(X) \to \mathscr{F}''(X)$ is surjective. This is most easily done by using Zorn's lemma. Fix a global section $s'' \in \mathscr{F}''(X)$, and consider the set of all pairs (U, s), where $U \subseteq X$ is open and $s \in \mathscr{F}(U)$ satisfies $\beta_U(s) = s''|_U$. It is clear that this set is nonempty, because β is surjective on stalks.

We put a partial order on our set of pairs by declaring that $(U_1, s_1) \leq (U_2, s_2)$ if $U_1 \subseteq U_2$ and $s_2|_{U_1} = s_1$. Since $\mathscr{F}$ is a sheaf, every chain $\{(U_i, s_i)\}_{i \in I}$ has an upper bound (U, s): take $U = \bigcup_{i \in I} U_i$ and let $s \in \mathscr{F}(U)$ be the unique section with $s|_{U_i} = s_i$ for all $i \in I$. By Zorn's lemma, there is a maximal element (U_{max}, s_{max}). To complete the proof, we need to show that $U_{max} = X$.

To that end, let $x \in X$ be any point. Then $\beta_x\colon \mathscr{F}_x \to \mathscr{F}''_x$ is onto, and so we can find a pair (U, s) with $x \in U$. On $V = U \cap U_{max}$, we now have two sections lifting s'', and so by Lemma 13.8, there is a unique section $s' \in \mathscr{F}'(U \cap U_{max})$ with $\alpha_V(s') = (s_{max} - s)|_V$. But now $\mathscr{F}'$ is flabby, and so we can find $t' \in \mathscr{F}'(U)$ with $t'|_V = s'$; then $s_{max} \in \mathscr{F}(U_{max})$ and $s + \alpha_U(t') \in \mathscr{F}(U)$ agree on V, and therefore define a section in $\mathscr{F}(U \cup U_{max})$ that still maps to s''. By maximality, we have $U \cup U_{max} = U_{max}$, and therefore $x \in U_{max}$. This proves that $U_{max} = X$, and shows that $s_{max} \in \mathscr{F}(X)$ satisfies $\beta_X(s_{max}) = s''$. $\square$

Next, we show that any sheaf has a canonical resolution by flabby sheaves. Given any sheaf $\mathscr{F}$, let $T(\mathscr{F}) = \coprod_{x \in X} \mathscr{F}_x$ be the disjoint union of its stalks; we can then define the *sheaf of discontinuous sections* $\operatorname{ds}\mathscr{F}$ by setting

$$\Gamma(U, \operatorname{ds}\mathscr{F}) = \left\{ s\colon U \to T(\mathscr{F}) \;\middle|\; s(x) \in \mathscr{F}_x \text{ for all } x \in X \right\}.$$

It is obvious from the definition that $\operatorname{ds}\mathscr{F}$ is a flabby sheaf; moreover, we have an injective map of sheaves $\varepsilon\colon \mathscr{F} \to \operatorname{ds}\mathscr{F}$, taking a section $s \in \mathscr{F}(U)$ to the map $x \mapsto s_x$. This construction gives us an exact sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varepsilon} \mathscr{F}^0 \xrightarrow{d^0} \mathscr{F}^1 \xrightarrow{d^1} \mathscr{F}^2 \xrightarrow{d^2} \cdots, \tag{13.10}$$

in which the $\mathscr{F}^i$ are flabby sheaves, as follows: Define $\mathscr{F}^0 = \operatorname{ds}\mathscr{F}$, and let $\varepsilon\colon \mathscr{F} \to \mathscr{F}^0$ be the map from above. Next, form the quotient sheaf $\mathscr{G}^0 = \mathscr{F}^0 / \operatorname{im}\varepsilon$, let $\mathscr{F}^1 = \operatorname{ds}\mathscr{G}^0$, and let $d^1\colon \mathscr{F}^0 \to \mathscr{F}^1$ be the composition of the two natural maps. Continuing in this way, we obtain a commutative diagram of the type

$$\begin{tikzcd}
\mathscr{F} \arrow[dr, hook, "\varepsilon"] & & & & \mathscr{G}^1 \arrow[dr, hook] & & & & \mathscr{G}^3 \arrow[dr, hook] & \\
& \mathscr{F}^0 \arrow[rr, "d^0"] \arrow[dr] & & \mathscr{F}^1 \arrow[rr, "d^1"] \arrow[ur] & & \mathscr{F}^2 \arrow[rr, "d^2"] \arrow[dr] & & \mathscr{F}^3 \arrow[rr] \arrow[ur] & & \cdots \\
& & \mathscr{G}^0 \arrow[ur, hook] & & & & \mathscr{G}^2 \arrow[ur, hook] & & &
\end{tikzcd} \tag{13.11}$$

continuing to the right; at each stage, $\mathscr{F}^k = \operatorname{ds}\mathscr{G}^{k-1}$, and $\mathscr{G}^k$ is the quotient of $\mathscr{F}^k$ by its subsheaf $\mathscr{G}^{k-1}$. Since the diagonal sequences are all exact, it is not hard to prove (by looking at stalks) that (13.10) is itself exact. We refer to it as the *Godement resolution* of the sheaf $\mathscr{F}$.

Definition 13.12. For a sheaf $\mathscr{F}$ on a topological space X, we define $H^i(X, \mathscr{F})$ to be the i-th cohomology group of the complex of abelian groups

$$0 \longrightarrow \mathscr{F}^0(X) \to \mathscr{F}^1(X) \to \mathscr{F}^2(X) \to \cdots .$$

It follows from Lemma 13.8 that the sequence $0 \to \mathscr{F}(X) \to \mathscr{F}^0(X) \to \mathscr{F}^1(X)$ is exact, and therefore that $H^0(X, \mathscr{F}) \simeq \mathscr{F}(X)$. Note also that when $\mathscr{F}$ is a sheaf of vector spaces, each $H^i(X, \mathscr{F})$ is again a vector space. As promised, we always have a long exact sequence in cohomology.

Proposition 13.13. *A short exact sequence* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *of sheaves gives rise to a long exact sequence of cohomology groups.*

Proof. A morphism $f\colon \mathscr{F} \to \mathscr{G}$ induces maps on stalks, and hence a morphism $\mathrm{ds}\,\mathscr{F} \to \mathrm{ds}\,\mathscr{G}$ between the sheaves of discontinuous sections. Using this fact, one can easily show that the Godement resolutions for the three sheaves fit into a commutative diagram

$$\begin{tikzcd}
\mathscr{F}'^0 \arrow[r] \arrow[d, hook] & \mathscr{F}'^1 \arrow[r] \arrow[d, hook] & \mathscr{F}'^2 \arrow[r] \arrow[d, hook] & \cdots \\
\mathscr{F}^0 \arrow[r] \arrow[d, two heads] & \mathscr{F}^1 \arrow[r] \arrow[d, two heads] & \mathscr{F}^2 \arrow[r] \arrow[d, two heads] & \cdots \\
\mathscr{F}''^0 \arrow[r] & \mathscr{F}''^1 \arrow[r] & \mathscr{F}''^2 \arrow[r] & \cdots
\end{tikzcd}$$

with exact columns. Since each $\mathscr{F}'^k$ is flabby, it follows from Lemma 13.9 that, even after taking global sections, the columns in

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}'^0(X) \arrow[r] \arrow[d, hook] & \mathscr{F}'^1(X) \arrow[r] \arrow[d, hook] & \mathscr{F}'^2(X) \arrow[r] \arrow[d, hook] & \cdots \\
0 \arrow[r] & \mathscr{F}^0(X) \arrow[r] \arrow[d, two heads] & \mathscr{F}^1(X) \arrow[r] \arrow[d, two heads] & \mathscr{F}^2(X) \arrow[r] \arrow[d, two heads] & \cdots \\
0 \arrow[r] & \mathscr{F}''^0(X) \arrow[r] & \mathscr{F}''^1(X) \arrow[r] & \mathscr{F}''^2(X) \arrow[r] & \cdots
\end{tikzcd}$$

are short exact sequences of abelian groups. The long exact sequence of cohomology groups is then obtained by applying a form of the Snake Lemma, which is a basic result in homological algebra. □

To conclude our discussion of flabby sheaves, we would like to show that the higher cohomology groups of flabby sheaves are zero. We begin with a small lemma.

Lemma 13.14. *If* $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence with* $\mathscr{F}'$ *and* $\mathscr{F}$ *flabby, then* $\mathscr{F}''$ *is also flabby.*

Proof. For any open subset $U \subseteq X$, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{F}(X) \arrow[r, two heads] \arrow[d] & \mathscr{F}''(X) \arrow[d] \\
\mathscr{F}(U) \arrow[r, two heads] & \mathscr{F}''(U).
\end{tikzcd}$$

The surjectivity of the two horizontal maps is due to Lemma 13.9, and that of the vertical restriction map comes from the flabbiness of $\mathscr{F}$. We conclude that $\mathscr{F}''(X) \to \mathscr{F}''(U)$ is also surjective, proving that $\mathscr{F}''$ is flabby. □

We can now prove that flabby sheaves have trivial cohomology.

Proposition 13.15. *If $\mathscr{F}$ is a flabby sheaf, then $H^i(X, \mathscr{F}) = 0$ for $i > 0$.*

Proof. According to the preceding lemma, the quotient of a flabby sheaf by a flabby subsheaf is again flabby. This fact implies that in (13.11), all the the sheaves $\mathscr{G}^j$ are also flabby sheaves. Consequently, the entire diagram remains exact after taking global sections, which shows that $0 \to \mathscr{F}(X) \to \mathscr{F}^0(X) \to \mathscr{F}^1(X) \to \cdots$ is an exact sequence of abelian groups. But this means that $H^i(X, \mathscr{F}) = 0$ for $i > 0$. □

Example 13.16. Let us return to the exponential sequence on a complex manifold M. From Proposition 13.13, we obtain a long exact sequence

$$0 \longrightarrow H^0(M, \mathbb{Z}_M) \longrightarrow H^0(M, \mathscr{O}_M) \longrightarrow H^0(M, \mathscr{O}_M^*) \longrightarrow H^1(M, \mathbb{Z}_M) \longrightarrow \cdots$$

One can show that the cohomology groups $H^i(M, \mathbb{Z}_M)$ are (naturally) isomorphic to the singular cohomology groups $H^i(M, \mathbb{Z})$ defined in algebraic topology. Thus whether or not the map $\mathscr{O}_M(M) \to \mathscr{O}_M^*(M)$ is surjective depends on the group $H^1(M, \mathbb{Z})$; for instance, $H^1(\mathbb{C}^*, \mathbb{Z}) \simeq \mathbb{Z}$, and this explains the failure of surjectivity. On the other hand, if M is simply connected, then $H^1(M, \mathbb{Z}) = 0$, and therefore $\mathscr{O}_M(M) \to \mathscr{O}_M^*(M)$ is surjective.

Čech cohomology. In addition to the general framework introduced above, there are many other cohomology theories; one that is often convenient for calculations is Čech cohomology. We shall limit our discussion to a special case that will be useful later.

Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups. Fix an open cover $\mathbf{U}$ of X. The group of *p-cochains* for the cover is the product

$$C^p(\mathbf{U}, \mathscr{F}) = \prod_{U_0, \dots, U_p \in \mathbf{U}} \mathscr{F}(U_0 \cap U_1 \cap \cdots \cap U_p);$$

we denote a typical element by $\mathbf{g}$, with components $g_{U_0,\dots,U_p} \in \mathscr{F}(U_0 \cap \cdots \cap U_p)$. The restriction maps for the sheaf $\mathscr{F}$ allow us to define a *differential* $\delta^p \colon C^p(\mathbf{U}, \mathscr{F}) \to C^{p+1}(\mathbf{U}, \mathscr{F})$ by setting $\delta^p(\mathbf{g}) = \mathbf{h}$, where

$$h_{U_0,\dots,U_{p+1}} = \sum_{k=0}^{p+1} (-1)^k g_{U_0,\dots,U_{k-1},U_{k+1},\dots,U_{p+1}} |_{U_0 \cap U_1 \cap \cdots \cap U_{p+1}}.$$

Then a somewhat tedious computation shows that $\delta^{p+1} \circ \delta^p = 0$, and thus

$$0 \longrightarrow C^0(\mathbf{U}, \mathscr{F}) \xrightarrow{\delta^0} C^1(\mathbf{U}, \mathscr{F}) \xrightarrow{\delta^1} C^2(\mathbf{U}, \mathscr{F}) \xrightarrow{\delta^2} \cdots \tag{13.17}$$

is a complex of abelian groups. We define the *Čech cohomology group* $H^i(\mathbf{U}, \mathscr{F})$ to be the i-th cohomology group of the complex.

Example 13.18. From the sheaf axioms, one readily proves that $H^0(\mathbf{U}, \mathscr{F}) \simeq \mathscr{F}(X)$.

Example 13.19. Let $L \to M$ be a holomorphic line bundle on a complex manifold M. The transition functions $g_{\alpha,\beta} \in \mathscr{O}_M^*(U_\alpha \cap U_\beta)$ satisfy the relations $g_{\alpha,\beta} \cdot g_{\beta,\gamma} = g_{\alpha,\gamma}$. In other words, we have a cohomology class in $H^1(\mathbf{U}, \mathscr{O}_M^*)$. If this class is trivial, we have $g_{\alpha,\beta} = s_\beta / s_\alpha$ for $s_\alpha \in \mathscr{O}_M^*(U_\alpha)$, which means that the s_α^{-1} form a nowhere vanishing section of the line bundle. Thus we can think of $H^1(\mathbf{U}, \mathscr{O}_M^*)$ as the obstruction to the existence of such a section.

One can define Čech cohomology groups more generally, but unless the topological space X is nice, they lack the good properties of Godement's theory (for instance, there is not in general a long exact cohomology sequence). This drawback notwithstanding, Čech cohomology can frequently be used to compute the groups $H^i(X, \mathscr{F})$. The following result, known as Cartan's lemma, is the main result in this direction.

Theorem 13.20. *Suppose that the cover* $\mathbf{U}$ *is* acyclic *for the sheaf* $\mathscr{F}$*, in the sense that* $H^i(U_1 \cap \dots \cap U_p, \mathscr{F}) = 0$ *for every* $U_1, \dots, U_p \in \mathbf{U}$ *and every* $i > 0$*. Then there are natural isomorphisms*

$$H^i(\mathbf{U}, \mathscr{F}) \simeq H^i(X, \mathscr{F})$$

between the Čech cohomology and the usual cohomology of $\mathscr{F}$*.*

The proof is not that difficult, but we leave it out since it requires a knowledge of spectral sequences.

Example 13.21. Let $\mathbf{U} = \{U_0, U_1\}$ be the standard open cover of $\mathbb{P}^1$. A good excercise in the use of Čech cohomology is to prove that $H^0(\mathbf{U}, \mathscr{O}) = \mathbb{C}$, while $H^j(\mathbf{U}, \mathscr{O}) = 0$ for $j \geq 1$. Next time, we will see that this cover is acyclic, and therefore $H^j(\mathbb{P}^1, \mathscr{O}) = 0$ for $j \geq 1$.

Class 14. Dolbeault cohomology

On a complex manifold M, there is another way to compute the cohomology groups of the sheaves $\mathscr{O}_M$ and Ω_M^p (and, more generally, of the sheaf of sections of any holomorphic vector bundle), by relating them to Dolbeault cohomology. Recall that we had defined the Dolbeault cohomology groups

$$H^{p,q}(M) = \frac{\ker \bar{\partial} \colon A^{p,q}(M) \to A^{p,q+1}(M)}{\operatorname{coker} \bar{\partial} \colon A^{p,q-1}(M) \to A^{p,q}(M)},$$

where $A^{p,q}(M)$ denotes the space of smooth (p,q)-forms on M. Clearly, each $H^{p,q}(M)$ is a complex vector space, and can also be viewed as the q-th cohomology group of the complex

$$0 \longrightarrow A^{p,0}(M) \xrightarrow{\bar{\partial}} A^{p,1}(M) \xrightarrow{\bar{\partial}} A^{p,2}(M) \longrightarrow \cdots \longrightarrow A^{p,n}(M) \longrightarrow 0.$$

The purpose of today's class is to prove the following result, usually referred to as Dolbeault's theorem.

Theorem 14.1. *On a complex manifold* M*, we have natural isomorphisms*

$$H^q\big(M, \Omega_M^p\big) \simeq H^{p,q}(M)$$

for every $p, q \in \mathbb{N}$*.*

The proof is based on the $\bar{\partial}$-Poincaré lemma (Lemma 10.4) and some general sheaf theory. We fix an integer $p \geq 0$, and consider the complex of sheaves

$$0 \longrightarrow \Omega_M^p \longrightarrow \mathscr{A}^{p,0} \xrightarrow{\bar{\partial}} \mathscr{A}^{p,1} \xrightarrow{\bar{\partial}} \mathscr{A}^{p,2} \longrightarrow \cdots \longrightarrow \mathscr{A}^{p,n} \longrightarrow 0. \tag{14.2}$$

It is a complex because $\bar{\partial} \circ \bar{\partial} = 0$; the first observation is that it is actually exact.

Lemma 14.3. *The complex of sheaves in* (14.2) *is exact.*

Proof. It suffices to prove the exactness at the level of stalks; after fixing a point of M and choosing local coordinates, we may assume without loss of generality that M is an open subset of $\mathbb{C}^n$. Now let $\omega \in A^{p,q}(U)$ be defined on some open neighborhood of the point in question, and suppose that $\bar{\partial}\omega = 0$. If $q = 0$, this means that ω is holomorphic, and therefore $\omega \in \Omega_M^p(U)$, proving that the complex is exact at $\mathscr{A}^{p,0}$. If, on the other hand, $q > 0$, then Lemma 10.4 shows that there is a possibly smaller open neighborhood $V \subseteq U$ such that $\omega = \bar{\partial}\psi$ for some $\psi \in A^{p,q-1}(V)$, and so we have exactness on stalks. $\square$

We will show in a moment that the higher cohomology groups for each of the sheaves $\mathscr{A}^{p,q}$ vanish. Assuming this for the time being, let us complete the proof of Theorem 14.1

Proof. Probably the most convenient way to get the conclusion is by using a spectral sequence; but since it is not difficult either, will shall give a more basic proof. We begin by breaking up (14.2) into several short exact sequences:

(14.4)
$$\begin{tikzcd}
\Omega_M^p \arrow[dr, hook] & & & & \mathscr{Q}^2 \arrow[dr, hook] & & & & \mathscr{Q}^4 \arrow[dr, hook] & \\
& \mathscr{A}^{p,0} \arrow[rr, "\bar{\partial}"] \arrow[dr] & & \mathscr{A}^{p,1} \arrow[rr, "\bar{\partial}"] \arrow[ur] & & \mathscr{A}^{p,2} \arrow[rr, "\bar{\partial}"] \arrow[dr] & & \mathscr{A}^{p,3} \arrow[rr] \arrow[ur] & & \cdots \\
& & \mathscr{Q}^1 \arrow[ur, hook] & & & & \mathscr{Q}^3 \arrow[ur, hook] & & &
\end{tikzcd}$$

Here $\mathscr{Q}^k = \ker\bigl(\bar{\partial}\colon \mathscr{A}^{p,k} \to \mathscr{A}^{p,k+1}\bigr) = \operatorname{im}\bigl(\bar{\partial}\colon \mathscr{A}^{p,k-1} \to \mathscr{A}^{p,k}\bigr)$, using that the original complex is exact.

Now recall that we have $H^0\bigl(M, \mathscr{A}^{p,q}\bigr) = A^{p,q}(M)$. Since $\mathscr{Q}^{q+1}$ is a subsheaf of $\mathscr{A}^{p,q+1}$, the sequence $0 \to \mathscr{Q}^q \to \mathscr{A}^{p,q} \to \mathscr{A}^{p,q+1}$ is exact. After passage to cohomology, we find that

$$\ker\bigl(\bar{\partial}\colon A^{p,q}(M) \to A^{p,q+1}(M)\bigr) \simeq H^0\bigl(M, \mathscr{Q}^q\bigr).$$

Also, $0 \to \mathscr{Q}^{q-1} \to \mathscr{A}^{p,q-1} \to \mathscr{Q}^q \to 0$ is exact, and as part of the corresponding long exact sequence, we have

$$A^{p,q-1}(M) \to H^0\bigl(M, \mathscr{Q}^q\bigr) \longrightarrow H^1\bigl(M, \mathscr{Q}^{q-1}\bigr) \to H^1\bigl(M, \mathscr{A}^{p,q}\bigr).$$

The fourth term vanishes, and we conclude that $H^{p,q}(M) \simeq H^1\bigl(M, \mathscr{Q}^{q-1}\bigr)$. Continuing in this manner, we then obtain a string of isomorphisms

$$H^{p,q}(M) \simeq H^1\bigl(M, \mathscr{Q}^{q-1}\bigr) \simeq H^2\bigl(M, \mathscr{Q}^{q-2}\bigr) \simeq \cdots \simeq H^{q-1}\bigl(M, \mathscr{Q}^1\bigr) \simeq H^q\bigl(M, \Omega_M^p\bigr),$$

which is the desired result. $\square$

Applications. As an application of Dolbeault's theorem, we will now solve a classical problem about the geometry of $\mathbb{C}^n$. Let $X \subseteq \mathbb{C}^n$ be a *hypersurface*; this means that X is an analytic subset, locally defined by the vanishing of a single holomorphic function. We would like to show that, actually, $X = Z(f)$ for a global $f \in \mathscr{O}(\mathbb{C}^n)$.

This in another instance of a local-to-global problem, and we should expect the answer to come from cohomology. By assumption, X can locally be defined by a one holomorphic equation, and so we may cover $\mathbb{C}^n$ by open sets U_j, with the property that $X \cap U_j = Z(f_j)$ for certain $f_j \in \mathscr{O}(U_j)$; if an open set U_j does not meet X, we simply take $f_j = 1$. More precisely, we shall assume that each U_j is a polybox, that is, an open set of the form

$$\left\{ z \in \mathbb{C}^n \;\middle|\; |x_j - a_j| < r_j \text{ and } |y_j - b_j| < s_j \right\}.$$

Since the intersection of two open intervals is again an open interval, it is clear that every finite intersection of open sets in the cover $\mathbf{U}$ is again a polybox, and in particular contractible. Moreover, if we take the defining equation f_j not divisible by the square of any nonunit, then it is unique up to multiplication by units.

Next, we observe that if $D \subseteq \mathbb{C}^n$ is an arbitrary polybox, then $H^q(D, \Omega_D^p) = 0$ for $q > 0$; indeed, this group is isomorphic to $H^{p,q}(D)$, which vanishes for polyboxes by a result analogous to Proposition 10.5. In particular, the cover $\mathbf{U}$ is acyclic for the sheaf $\mathscr{O}$, and we have

$$H^q(\mathbf{U}, \mathscr{O}) \simeq H^q(\mathbb{C}^n, \mathscr{O}) \simeq H^{0,q}(\mathbb{C}^n) \simeq 0$$

by Cartan's lemma (Theorem 13.20) and Proposition 10.5.

Returning to the problem at hand, consider the intersection $U_j \cap U_k$. There, we have $f_j = g_{j,k} \cdot f_k$ for a nowhere vanishing holomorphic function $g_{j,k} \in \mathscr{O}^*(U_j \cap U_k)$. Now $U_j \cap U_k$ is contractible, and so $H^1(U_j \cap U_k, \mathbb{Z}) = 0$. From the exponential sequence

$$0 \longrightarrow \mathbb{Z}_{\mathbb{C}^n} \longrightarrow \mathscr{O}_{\mathbb{C}^n} \longrightarrow \mathscr{O}^*_{\mathbb{C}^n} \longrightarrow 0,$$

it follows that $g_{j,k} = e^{2\pi i h_{j,k}}$ for holomorphic functions $h_{j,k}$ on $U_j \cap U_k$. Observe that we have $g_{j,k} g_{k,l} = g_{j,l}$, and that $a_{j,k,l} = h_{j,l} - h_{j,k} - h_{k,l}$ is therefore an integer. These integers define a class in the Čech cohomology group

$$H^2(\mathbf{U}, \mathbb{Z}_{\mathbb{C}^n}) \simeq H^2(\mathbb{C}^n, \mathbb{Z}_{\mathbb{C}^n}) \simeq H^2(\mathbb{C}^n, \mathbb{Z}) \simeq 0.$$

The first isomorphism is because of Cartan's lemma (Theorem 13.20), since every intersection of open sets in the cover is contractible; the second and third isomorphisms are facts from algebraic topologyy. We thus have $a_{j,k,l} = b_{k,l} - b_{j,l} + b_{j,k}$ for integers $b_{j,k}$. Replacing $h_{j,k}$ by $h_{j,k} + b_{j,k}$, we may thus assume from the start that $h_{j,k} + h_{k,l} = h_{j,l}$ on $U_j \cap U_k \cap U_l$. This means that $\mathbf{h}$ defines an element of the Čech cohomology group $H^1(\mathbf{U}, \mathscr{O})$.

But as observed above, we have $H^1(\mathbf{U}, \mathscr{O}) \simeq 0$; this means that $h_{j,k} = h_k - h_j$ for holomorphic functions $h_j \in \mathscr{O}(U_j)$. This essentially completes the proof: By construction, $f_j = e^{2\pi i(h_k - h_j)} f_k$, and so $f_j e^{2\pi i h_j} = f_k e^{2\pi i h_k}$ on $U_j \cap U_k$. Since $\mathscr{O}$ is a sheaf, there is a holomorphic function $f \in \mathscr{O}(\mathbb{C}^n)$ with $f|_{U_j} = f_j e^{2\pi i h_j}$; clearly, we have $Z(f) = X$, proving that the hypersurface X is indeed defined by a single holomorphic equation.

Note. We proved the vanishing of the Dolbeault cohomology groups by purely analytic means in Proposition 10.5. To deduce from it the vanishing of Čech cohomology, we first go from Dolbeault cohomology to sheaf cohomology (Dolbeault's theorem), and then from sheaf cohomology to Čech cohomology (Cartan's lemma).

Fine and soft sheaves. We now have to explain why the higher cohomology groups of $\mathscr{A}^{p,q}$ vanish. This is due to the fact that sections of this sheaf are smooth forms, and that we have partitions of unity.

A few basic definitions first. An open covering $X = \bigcup_{i \in I} U_i$ of a topological space is *locally finite* if every point is contained in at most finitely many U_i. A topological space is called *paracompact* if every open cover can be refined to a locally finite open cover. It is not hard to see that a locally compact Hausdorff space with a countable basis is paracompact; in particular, every complex manifold is paracompact.

Definition 14.5. A sheaf $\mathscr{F}$ on a paracompact space X is *fine* if for every locally finite open cover $X = \bigcup_{i \in I} U_i$, there are sheaf homomorphisms $\eta_i \colon \mathscr{F} \to \mathscr{F}$, with the following two properties:

(1) There are open sets $V_i \supseteq X \setminus U_i$, such that $\eta_i \colon \mathscr{F}_x \to \mathscr{F}_x$ is the zero map for every $x \in V_i$.
(2) As morphisms of sheaves, $\sum_{i \in I} \eta_i = \mathrm{id}_{\mathscr{F}}$.

The first condition is saying that the support of $\eta_i(s)$ lies inside U_i; the second condition means that $s = \sum_{i \in I} \eta_i(s)$, which makes sense since the sum is locally finite. Note that if $s \in \mathscr{F}(U_i)$, then $\rho_i(s)$ may be considered as an element of $\mathscr{F}(X)$: by assumption, $\rho_i(s)$ is zero near the boundary of U_i, and can therefore be extended by zero using the sheaf axioms.

Example 14.6. On a complex manifold M, each $\mathscr{A}^{p,q}$ is a fine sheaf. Indeed, given any locally finite open covering $M = \bigcup_{i \in I} U_i$, we can find a partition of unity $1 = \sum_{i \in I} \rho_i$ subordinate to that cover; this means that each ρ_i is a smooth function with values in $[0, 1]$, and zero on an open neighborhood $V_i \supseteq M \setminus U_i$. We can now define $\eta_i \colon \mathscr{A}^{p,q} \to \mathscr{A}^{p,q}$ as multiplication by ρ_i; then both conditions in the definition are clearly satisfied.

Example 14.7. One can also show that the sheaf of discontinuous sections $\mathrm{ds}\,\mathscr{F}$ is always a fine sheaf.

We would like to show that fine sheaves have vanishing higher cohomology. But unfortunately, being fine does not propagate very well along the Godement resolution of a sheaf; this leads us to introduce a weaker property that does behave well in exact sequences of sheaves. We first observe that, just as in the case of geometric spaces, a sheaf $\mathscr{F}$ can be restricted to any closed subset $Z \subseteq X$; at each point $x \in Z$, the stalk of the restriction $\mathscr{F}|_Z$ is equal to $\mathscr{F}_x$. The precise definition is as follows: for $U \subseteq Z$, we let $\Gamma\big(U, \mathscr{F}|_Z\big)$ be the set of maps $s \colon U \to T(\mathscr{F})$ with $s(x) \in \mathscr{F}_x$ for every $x \in Z$, such that s is locally the restriction of a section of $\mathscr{F}$. (Here $T(\mathscr{F})$ is the disjoint union of all the stalks of $\mathscr{F}$.) We sometimes write $\mathscr{F}(Z)$ in place of the more correct $\Gamma(Z, \mathscr{F}|_Z)$.

Definition 14.8. A sheaf $\mathscr{F}$ on a paracompact topological space is called *soft* if, for every closed subset $Z \subseteq X$, the restriction map $\Gamma(X, \mathscr{F}) \to \Gamma(Z, \mathscr{F}|_Z)$ is surjective.

It is clear that the sheaf of discontinuous sections $\mathrm{ds}\,\mathscr{F}$ is soft for every sheaf $\mathscr{F}$. Let us now see why fine sheaves are soft. Fix an arbitrary section $t \in \Gamma(Z, \mathscr{F}|_Z)$; we need to show that it can be extended to a section of $\mathscr{F}$ on all of X. By definition, there certainly exist local extensions, and so we can find open sets $U_i \subseteq X$ whose union covers Z, and sections $s_i \in \Gamma(U_i, \mathscr{F})$ with $s_i(x) = t(x)$ for every $x \in Z$. We will assume that $U_0 = X \setminus Z$ is one of the open sets, with $s_0 = 0$. Since X is paracompact, we can assume after suitable refinement that the open cover of X by the U_i is locally finite; as $\mathscr{F}$ is fine, we can then find morphisms $\rho_i \colon \mathscr{F} \to \mathscr{F}$ as in Definition 14.5. After extending by zero, we may again consider $\rho_i(s_i) \in \mathscr{F}(X)$.

Now let

$$s = \sum_{i \in I} \rho_i(s_i) \in \Gamma(X, \mathscr{F}),$$

which makes sense since the sum is locally finite. For $x \in Z$, we have $s_i(x) = t(x)$ for every $i \neq 0$, and thus $s(x) = t(x)$. This proves the surjectivity of the map $\Gamma(X, \mathscr{F}) \to \Gamma(Z, \mathscr{F}|_Z)$, and shows that fine sheaves are soft.

Proposition 14.9. *Let $\mathscr{F}$ be a fine sheaf on a paracompact Hausdorff space X. Then $H^i(X, \mathscr{F}) = 0$ for every $i > 0$.*

We will show that the statement is true for the larger class of soft sheaves. The proof is very similar to that of Proposition 13.15; the first step is to study short exact sequences.

Lemma 14.10. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves on a paracompact space X, and if $\mathscr{F}'$ is soft, then $0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X) \to 0$ is an exact sequence of abelian groups.*

Proof. Again, we let $\alpha \colon \mathscr{F}' \to \mathscr{F}$ and $\beta \colon \mathscr{F} \to \mathscr{F}''$ denote the maps. By Lemma 13.8, it suffices to show that $\beta \colon \mathscr{F}(X) \to \mathscr{F}''(X)$ is surjective, and so we fix a global section $s'' \in \mathscr{F}''(X)$. The map being surjective locally, and X being paracompact, we can find a locally finite cover $X = \bigcup_{i \in I} U_i$ and sections $s_i \in \mathscr{F}(U_i)$ such that $\beta(s_i) = s''|_{U_i}$. Now paracompact spaces are automatically normal, and so we can find closed sets $K_i \subseteq U_i$ whose interiors still cover X. Note that the union of any number of K_i is always closed; this is a straightforward consequence of the local finiteness of the cover.

We now consider the set of all pairs (K, s), where K is a union of certain K_i (and hence closed), and $s \in \Gamma(K)$ satisfies $\beta(s) = s''|_K$. As before, every chain has a maximal element, and so Zorn's lemma guarantees the existence of a maximal element (K_{max}, s_{max}). We claim that $K_{max} = X$; in other words, that $K_i \subseteq K_{max}$ for every $i \in I$. In any case, the two sections s_i and s_{max} both map to s'' on the intersection $K_i \cap K_{max}$, and we can therefore find $s' \in \mathscr{F}'(K_i \cap K_{max})$ with the property that $\alpha(s') = (s_{max} - s_i)|_{K_i \cap K_{max}}$. But $\mathscr{F}'$ is soft by assumption, and so there exists $t' \in \mathscr{F}'(K_i)$ with $t'|_{K_i \cap K_{max}} = s'$. Then s_{max} and $s_i + \alpha(t')$ agree on the overlap $K_i \cap K_{max}$, and thus define a section of $\mathscr{F}$ on $K_i \cup K_{max}$ lifting s''. By maximality, we have $K_i \cup K_{max} = K_{max}$, and hence $K_i \subseteq K_{max}$ as claimed. □

Secondly, we need to know that the quotient of soft sheaves is soft.

Lemma 14.11. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence with $\mathscr{F}'$ and $\mathscr{F}$ soft, then $\mathscr{F}''$ is also soft.*

Proof. For any closed subset $Z \subseteq X$, we have a commutative diagram

$$\begin{tikzcd} \mathscr{F}(X) \arrow[r, two heads] \arrow[d] & \mathscr{F}''(X) \arrow[d] \\ \mathscr{F}(Z) \arrow[r, two heads] & \mathscr{F}''(Z). \end{tikzcd}$$

The surjectivity of the two horizontal maps is due to Lemma 14.10, and that of the vertical restriction map comes from the softness of $\mathscr{F}$. We conclude that $\mathscr{F}''(X) \to \mathscr{F}''(Z)$ is also surjective, proving that $\mathscr{F}''$ is soft. □

We are now ready to prove Proposition 14.9.

Proof. According to the preceding lemma, the quotient of a soft sheaf by a soft subsheaf is again soft. This fact implies that in (13.11), all the the sheaves $\mathscr{G}^j$ are also soft sheaves. Consequently, the entire diagram remains exact after taking global sections, which shows that $0 \to \mathscr{F}(X) \to \mathscr{F}^0(X) \to \mathscr{F}^1(X) \to \cdots$ is an exact sequence of abelian groups. But this means that $H^i(X, \mathscr{F}) = 0$ for $i > 0$. $\square$

Since the sheaves $\mathscr{A}^{p,q}$ admit partitions of unity, they are fine, and hence soft. Proposition 14.9 now puts the last piece into place for the proof of Theorem 14.1.

Corollary 14.12. *On a complex manifold M, we have $H^i\bigl(M, \mathscr{A}^{p,q}\bigr) = 0$ for every $i > 0$.*

Note. Underlying the proof of Theorem 14.1 is a more general principle, which you should try to prove by yourself: If $0 \to \mathscr{F} \to \mathscr{E}^0 \to \mathscr{E}^1 \to \cdots$ is a resolution of $\mathscr{F}$ by acyclic sheaves (meaning that $H^i(X, \mathscr{E}^k) = 0$ for all $i > 0$), then the complex $0 \to \mathscr{E}^0(X) \to \mathscr{E}^1(X) \to \cdots$ computes the cohomology groups of $\mathscr{F}$. This can be seen either by breaking up the long exact sequence into short exact sequences as in (13.11), or by a spectral sequence argument.

Class 15. Linear differential operators

Representatives for cohomology. We will spend the next few weeks studying Hodge theory, first on smooth manifolds, then on complex manifolds. Hodge theory tries to solve the problem of finding good representatives for classes in de Rham cohomology. Recall that if M is a smooth manifold, we have the space of smooth k-forms $A^k(M)$, and the exterior derivative d maps $A^k(M)$ to $A^{k+1}(M)$. The de Rham cohomology groups of M are

$$H^k(M, \mathbb{R}) = \frac{\ker\bigl(d\colon A^k(M) \to A^{k+1}(M)\bigr)}{\operatorname{im}\bigl(d\colon A^{k-1}(M) \to A^k(M)\bigr)}.$$

A class in $H^k(M, \mathbb{R})$ is represented by a closed k-form ω, but ω is far from unique, since $\omega + d\psi$ represents the same class for every $\psi \in A^{k-1}(M)$. (The only exception is the group $H^0(M, \mathbb{R})$, whose elements are the locally constant functions.)

From now on, we shall assume that M is compact and orientable, and let $n = \dim M$. Then $H^n(M, \mathbb{R}) \simeq \mathbb{R}$, and once we choose a Riemannian metric g on M, we have the volume form $vol(g) \in A^n(M)$; every class in $H^n(M, \mathbb{R})$ therefore does have a distinguished representative, namely a multiple of $vol(g)$. The basic idea behind Hodge theory is that, once a Riemannian metric has been chosen, the same is actually true for every cohomology class. Here is why: Recall that g defines an inner product on every tangent space $T_{\mathbb{R},p}M$. It induces an inner product on the spaces $\bigwedge^k T^*_{\mathbb{R},p}M$, and by integrating over M, we obtain an inner product on the space of forms $A^k(M)$.

Given a cohomology class in $H^k(M, \mathbb{R})$, we can then look for a representative of minimal norm. It is not clear that such a representative exists, but suppose that we have $\omega \in A^k(M)$ with $d\omega = 0$, and such that $\|\omega\| \leq \|\omega + d\psi\|$ for every $\psi \in A^{k-1}(M)$. For each $t \in \mathbb{R}$, we deduce from

$$\|\omega\|^2 \leq \|\omega + td\psi\|^2 = (\omega + td\psi, \omega + td\psi) = \|\omega\|^2 + 2t(\omega, d\psi) + t^2\|d\psi\|^2$$

that $(\omega, d\psi) = 0$ (by differentiation with respect to t). Consequently, ω has minimal size iff it is perpendicular to the space $dA^{k-1}(M)$ of d-exact forms. This shows that

ω is unique in its cohomology class, because an exact form that is perpendicular to the space of exact forms is necessarily zero.

An equivalent (but more useful) formulation is the following: Define the *adjoint* operator $d^*\colon A^k(M) \to A^{k-1}(M)$ by the condition that

$$(d^*\alpha, \beta) = (\alpha, d\beta)$$

for all $\alpha \in A^k(M)$ and all $\beta \in A^{k-1}(M)$. Then ω has minimal size iff $d^*\omega = 0$. Since also $d\omega = 0$, we can combine both conditions into one by defining the *Laplacian* $\Delta = d \circ d^* + d^* \circ d$; from

$$(\Delta\omega, \omega) = (dd^*\omega + d^*d\omega, \omega) = \|d\omega\|^2 + \|d^*\omega\|^2,$$

we see that ω is d-closed and of minimal norm iff ω is a *harmonic* form, in the sense that $\Delta\omega = 0$. To summarize:

Proposition 15.1. *Let (M, g) be a compact connected Riemannian manifold, and let $\omega \in A^k(M)$ be smooth k-form. The following conditions are equivalent:*

(1) $d\omega = 0$ and ω is of minimal norm in its cohomology class.
(2) $d\omega = 0$ and ω is perpendicular to the space of d-exact forms.
(3) $d\omega = 0$ and $d^*\omega = 0$, or equivalently, $\Delta\omega = 0$.

If ω satisfies any of these conditions, it is unique in its cohomology class, and is called a harmonic form *with respect to the given metric.*

On $\mathbb{R}^n$ with the usual Euclidean metric, $\Delta f = -\sum_i \partial^2 f/\partial x_i^2$ for $f \in A^0(M)$, which explains the terminology. In general, the Laplacian $\Delta\colon A^k(M) \to A^k(M)$ is an example of an elliptic differential operator. Before we can address the question of whether each cohomology class contains a representative of minimal norm, we need to review the basic theory of such operators.

Linear differential operators. We begin by describing local linear differential operators. Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $A(U)$ be the space of smooth real-valued functions on U. A *local linear differential operator* is a linear mapping $D\colon A(U) \to A(U)$ that can be written as a finite sum

$$f \mapsto Df = \sum_{i_1,\dots,i_n} h_{i_1,\dots,i_n} \frac{\partial^{i_1+\dots+i_n} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$$

with smooth coefficients $h_{i_1,\dots,i_n} \in A(U)$. Provided that $D \neq 0$, there is a largest integer d with $h_{i_1,\dots,i_n} \neq 0$ for some multi-index with $i_1 + \dots + i_n = d$, and we call d the *degree* of the operator D. The function

$$P\colon U \times \mathbb{R}^n \to \mathbb{R}, \quad (x, \xi) \mapsto \sum_{|I|=d} h_I \xi^I = \sum_{i_1+\dots+i_n=d} h_{i_1,\dots,i_n} \xi_1^{i_1} \cdots \xi_n^{i_n}$$

is called the *symbol* of P; it is smooth, and homogeneous in ξ of degree d. The operator P is said to be *elliptic* if $P(x, \xi) = 0$ implies that $\xi = 0$. We can easily generalize this to operators

$$D\colon A(U)^{\oplus p} \to A(U)^{\oplus q},$$

whose coefficients are now $p \times q$-matrices of smooth functions; the symbol of D is now a mapping $P\colon U \times \mathbb{R}^n \to \mathbb{R}^{p\times q}$ to the space of $p \times q$-matrices.

Definition 15.2. Let $U \subseteq \mathbb{R}^n$ be an open subset, and $D\colon A(U)^{\oplus p} \to A(U)^{\oplus q}$ be a local linear differential operator. Then D is called *elliptic* if $p = q$, and if the symbol $P(x, \xi)$ is an invertible matrix for every $x \in U$ and every nonzero $\xi \in \mathbb{R}^n$.

Example 15.3. The usual Laplace operator $\Delta\colon A(U) \to A(U)$, defined by the rule $\Delta f = \sum_{i=1}^n \partial^2 f/\partial x_i^2$, is an elliptic operator of order 2; in fact, the symbol is

$$P(x, \xi) = \xi_1^2 + \cdots + \xi_n^2,$$

which is clearly nonzero for $\xi \neq 0$.

Example 15.4. Since any smooth 1-form on U can be written as $\sum_i f_i dx_i$, we have $A^1(U) \simeq A(U)^{\oplus n}$. This means that the exterior derivative $d\colon A(U) \to A^1(U)$ is a linear differential operator.

We now extend the concept of linear differential operators to smooth manifolds. It should be clear how to define local differential operators in a coordinate chart; with the help of the chain rule, one easily verifies that the degree and ellipticity of an operator are independent of the choice of coordinate system. To globalize this notion, we look at smooth vector bundles. For $\pi\colon E \to M$ a smooth vector bundle on a manifold M, we let $A(U, E)$ be the space of sections of E over an open set $U \subseteq M$; its elements are smooth maps $s\colon U \to E$ with $\pi \circ s = \mathrm{id}_U$.

Definition 15.5. Let M be a smooth manifold of dimension n, and let $E \to M$ and $F \to M$ be two smooth vector bundles, of rank p and q, respectively. A *linear differential operator* $D\colon E \to F$ is a collection of operators $D_U\colon A(U, E) \to A(U, F)$, with the following two properties:

(1) D is compatible with restriction to smaller open sets; in other words, the diagram

$$\begin{tikzcd} A(U,E) \arrow[r, "D_U"] \arrow[d] & A(U,F) \arrow[d] \\ A(V,E) \arrow[r, "D_V"] & A(V,F) \end{tikzcd}$$

should commute for every pair of open sets $V \subseteq U$.

(2) For every point of M, there is a coordinate neighborhood U and local trivializations $E|_U \to U \times \mathbb{R}^p$ and $F|_U \to U \times \mathbb{R}^q$, such that the map $A(U)^{\oplus p} \to A(U)^{\oplus q}$ induced by D_U is a local linear differential operator.

D is called *elliptic of order* d if all the local operators are elliptic of order d.

Example 15.6. The exterior derivative $d\colon M \times \mathbb{R} \to T^*_{\mathbb{R}} M$ is a global linear differential operator.

Elliptic operators. As in the discussion above, the study of linear differential operators requires a metric. By definition, a *Euclidean metric* on a real vector bundle $E \to M$ is a collection of inner products $\langle -, - \rangle_p\colon E_p \times E_p \to \mathbb{R}$, whose values depend smoothly on $p \in M$, in the sense that $\langle s_1, s_2 \rangle$ is a smooth function on U for any two smooth sections $s_1, s_2 \in A(U, E)$.

Now suppose that (M, g) is a Riemannian manifold, with volume form $vol(g)$. We can then define a Hilbert space of square-integrable sections of E, as follows:

Call a map $s\colon M \to E$ *measurable* if, in local trivializations, it is given by Lebesgue-measurable functions. We can then define

$$\|s\|^2 = \int_M \langle s, s\rangle vol(g),$$

and denote by $L^2(M,E)$ the space of measurable sections with $\|s\|^2 < \infty$. One can verify that this condition defines a Hilbert space, with inner product

$$(s_1, s_2) = \int_M \langle s_1, s_2\rangle vol(g),$$

and that the subspace $A_0(M,E)$ of smooth sections with compact support is dense in $L^2(M,E)$.

Class 16. Fundamental theorem of elliptic operators

Definition 16.1. Let $D\colon E \to F$ be a linear differential operator between two vector bundles with Euclidean metrics. We say that $D^*\colon F \to E$ is a *formal adjoint* to D if the relation

$$(D^*t, s) = (t, Ds)$$

holds for every pair of $t \in A(M,F)$ and $s \in A(M,E)$ such that $\mathrm{Supp}(s) \cap \mathrm{Supp}(t)$ is compact.

If D^* is a formal adjoint to D, then clearly D is a formal adjoint to D^*. We say that D is *formally self-adjoint* if $D^* = D$.

Lemma 16.2. *Let $D\colon E \to F$ be a linear differential operator between two smooth vector bundles endowed with Euclidean metrics. Then D has a unique formal adjoint $D^*\colon F \to E$; moreover, if D is elliptic of order d, then so is D^*.*

Proof. Since $A_0(M,E)$ is dense in the Hilbert space $L^2(M,E)$, the uniqueness of D^* is straightforward: If D_1^* and D_2^* are two formal adjoints, then

$$(D_1^*t - D_2^*t, s) = (t, Ds) - (t, Ds) = 0$$

for every $s \in A_0(M,E)$ and every $t \in A(M,F)$. This implies $D_1^*t = D_2^*t$, and hence $D_1^* = D_2^*$.

With uniqueness in hand, the existence of D^* now becomes a local question, and so we may assume that we are dealing with a local linear differential operator $D\colon A(U)^{\oplus p} \to A(U)^{\oplus q}$ on an open subset $U \subseteq \mathbb{R}^n$. To simplify the notation, we shall assume that $p = q = 1$. For $\phi \in A(U)$, we then have

$$D\phi = \sum_I h_I \frac{\partial^{|I|}\phi}{\partial x^I},$$

where we use multi-index notation to denote the partial derivatives. Let $G(x)$ be the matrix representing the Riemannian metric; as we have seen before, the volume form is $vol(g) = \sqrt{\det G} \cdot dx_1 \wedge \dots \wedge dx_n$. The Euclidean metric on F is now represented by a single smooth function $c_F \in A(U)$, and so

$$(D\phi, \psi) = \int_U c_F (D\phi)\psi \cdot vol(g) = \sum_I \int_U \frac{\partial^{|I|}\phi}{\partial x^I} c_F \psi h_I \sqrt{\det G} \cdot d\mu,$$

where $\psi \in A_0(U)$. This makes the integrand compactly supported, and we can use integration by parts to move the partial derivatives over to the other terms. We then get

$$\begin{aligned}(D\phi, \psi) &= \sum_I (-1)^{|I|} \int_U \phi \cdot \frac{\partial^{|I|}}{\partial x^I}\Big(c_F \psi h_I \sqrt{\det G}\Big) d\mu \\ &= \int_U c_E \phi \cdot \sum_I (-1)^{|I|} c_E^{-1} \frac{\partial^{|I|}}{\partial x^I}\Big(c_F \psi h_I \sqrt{\det G}\Big) d\mu.\end{aligned}$$

It follows that the formal adjoint D^* is given by the formula

$$D^* \psi = \sum_I (-1)^{|I|} c_E^{-1} \frac{\partial^{|I|}}{\partial x^I}\Big(c_F \psi h_I \sqrt{\det G}\Big) \frac{1}{\sqrt{\det G}}.$$

After some reordering of terms, one sees that this defines a linear differential operator $D^* \colon A(U) \to A(U)$ of order d, with symbol

$$P^*(x,\xi) = \sum_{|I|=d} (-1)^{|I|} c_E^{-1} c_F h_I \xi^I = (-1)^d \frac{c_F}{c_E} P(x,\xi).$$

Now suppose that D is elliptic. The above formula shows that $P^*(x,\xi)$ is nonzero for $\xi \neq 0$, and proves that D^* is also an elliptic operator. $\square$

The fundamental theorem. The following theorem is the fundamental result about elliptic linear differential operators, and one of the big achievements of the theory of partial differential equations.

Theorem 16.3. *Let M be a compact manifold, and let $D \colon E \to F$ be a linear differential operator between two Euclidean vector bundles. If D is elliptic of order $d \geq 1$, then the kernel and cokernel of the map $D \colon A(M,E) \to A(M,F)$ are finite-dimensional vector spaces. Moreover, we have a direct sum decomposition*

$$A(M,E) = \ker\big(D \colon A(M,E) \to A(M,F)\big) \oplus \operatorname{im}\big(D^* \colon A(M,F) \to A(M,E)\big),$$

which is orthogonal with respect to the inner product on $L^2(M,E)$.

In its essence, Theorem 16.3 is a result about the solvability of certain systems of linear partial differential equations. Namely, suppose we are given $u \in A(M,F)$, and we are trying to solve the system of differential equations

$$Ds = u$$

on a compact Riemannian manifold M. When D, and hence D^*, is elliptic, we have the decomposition

$$A(M,E) = \ker D^* \oplus \operatorname{im} D,$$

and so we can *always* write $u = u_0 + Ds$ for a unique $u_0 \in \ker D^*$ and $s \in A(M,F)$. Thus the original equation has a solution precisely when u_0, the projection of u to the finite-dimensional vector space $\ker D^*$, is zero. This nice behavior is very special to elliptic differential equations.

We shall now take a look at the proof of Theorem 16.3; since the details are fairly involved, we have to limit ourselves to an outline. Furthermore, we shall only consider the case $F = E$ for ease of exposition. Throughout, we assume that (M,g) is a compact oriented Riemannian manifold of dimension n, and that $\pi \colon E \to M$ is a smooth vector bundle with a Euclidean metric.

The modern study of differential equations is based on the following idea: One first tries to find a "weak" solution for the equation in some kind of function space, using functional analysis. Such weak solutions need not be differentiable, and so the second step consists in proving that the solution is actually smooth. This is usually done incrementally, by passing through a whole scale of function spaces. In our setting, we shall begin by defining various spaces of sections for the vector bundle E that interpolate between the space of square-integrable sections $L^2(M,E)$ and the space of smooth sections $A(M,E)$.

Spaces of sections. To begin with, we let $C^p(M,E)$ denote the space of p-times continuously differentiable sections of the vector bundle E. Note that a section is smooth, and hence in $A(M,E)$, iff it belongs to $C^p(M,E)$ for every $p \in \mathbb{N}$.

We also need the notion of a weak derivative. Let $D\colon E \to F$ be a linear differential operator. By definition of the adjoint D^*, we have

$$(Ds, \phi) = (s, D^*\phi)$$

for every $s \in A(M,E)$ and every $\phi \in A(M,F)$. Now suppose that the section s only belongs to $L^2(M,E)$; then Ds does not make sense, because s might not be differentiable. On the other hand, the inner product $(s, D^*\phi)$ is still well-defined. This gives us a way to weaken the notion of a derivative. Consequently, we shall say s is *weakly differentiable* with respect to D if there exists a section $t \in L^2(M,F)$ such that

$$(t, \phi) = (s, D^*\phi)$$

is true for every $\phi \in A(M,F)$. Since $A(M,F)$ is dense in $L^2(M,F)$, such a section t is unique if it exists; we denote it by $\tilde{D}s$ and call it the *weak derivative* of s. Evidently, we have $\tilde{D}s = Ds$ as soon as $s \in C^d(M,E)$.

Definition 16.4. The *Sobolev space* $W^k(M,E)$ consists of all sections $u \in L^2(M,E)$ with the property that the weak derivative $\tilde{D}s \in L^2(M,F)$ exists for every linear differential operator D of order at most k.

We can use the L^2-norms of the various weak derivatives to define a norm $\|-\|_k$ on the Sobolev space $W^k(M,E)$. We shall give the formula in the case of an open set $U \subseteq \mathbb{R}^n$, and $E \simeq U \times \mathbb{R}^p$ is a trivial bundle with the usual Euclidean metric; the general case is obtained from this be covering the compact manifold with finitely many coordinate charts and using a partition of unity. A section $s \in A_0(U,E)$ is represented by p smooth functions $s_1, \dots, s_p \in A_0(U)$, and we can define

$$\|s\|_k^2 = \sum_{|I| \leq k} \sum_{j=1}^{n} \int_U \left| \frac{\partial^{|I|} s_j}{\partial x^I} \right|^2 d\mu.$$

The same formula is used for $s \in W^k(U,E)$, by replacing the partial derivatives $\partial^{|I|} s_j / \partial x^I$ by the corresponding weak derivatives of s_j.

The same method actually defines an inner product $(-,-)_k$, and it can be shown that $W^k(M,E)$ is a Hilbert space containing $A(M,E)$ as a dense subspace.

Class 17. Proof of the fundamental theorem

Basic facts about Sobolev spaces. The usefulness of Sobolev spaces comes from the following fundamental result; it shows that if a section $u \in L^2(M,E)$ has sufficiently many weak derivatives, then it is actually differentiable.

Theorem 17.1 (Sobolev lemma)**.** *For $k > p + n/2$, every section in $W^k(M, E)$ agrees almost everywhere with a p-times continuously differentiable section, and so we have $W^k(M, E) \hookrightarrow C^p(M, E)$. In particular, $\bigcap_{k \in \mathbb{N}} W^k(M, E) = A(M, E)$.*

The theorem is based on the more precise Sobolev inequality: there is a constant $B_k > 0$, depending only on k, such that

$$\|s\|_{C^p} \leq B_k \cdot \|s\|_k$$

holds for every $s \in A(M, E)$. Here $\|s\|_{C^p}$ is the p-th uniform norm, essentially the supremum over the absolute values of all derivatives of s of order at most p. This inequality is proved locally, on open subsets of $\mathbb{R}^n$, by a direct computation, and from there extends to compact manifolds by using a partition of unity. It implies the Sobolev lemma by the following approximation argument: Recall that since M is compact, the space of smooth sections $A(M, E)$ is dense in $W^k(M, E)$. Given any $u \in W^k(M, E)$, we find a sequence $s_i \in A(M, E)$ such that $\|u - s_i\| \to 0$. Because of the Sobolev inequality,

$$\|s_i - s_j\|_{C^p} \leq B_k \cdot \|s_i - s_j\|_k$$

goes to zero as $i, j \to \infty$, and so $\{s_i\}_i$ is a Cauchy sequence in $C^p(M, E)$. Since $C^p(M, E)$ is a Banach space with the p-th uniform norm, the sequence converges to a limit $s \in C^p(M, E)$; this means that u agrees almost everywhere with the p-times continuously differentiable section s.

Another fundamental result compares Sobolev spaces of different orders. Obviously, we have $W^{k+1}(M, E) \subseteq W^k(M, E)$, and since sections in $W^{k+1}(M, E)$ have one additional weak derivative, we expect the image to be rather small. Rellich's lemma makes this expectation precise.

Theorem 17.2 (Rellich's lemma)**.** *The inclusion $W^{k+1}(M, E) \hookrightarrow W^k(M, E)$ is a compact linear operator for every $k \in \mathbb{N}$.*

Recall that a bounded linear operator $T \colon H_1 \to H_2$ between two Banach spaces is *compact* if it maps bounded sets to precompact sets. Let $B \subseteq H_1$ denote the closed unit ball; then T is compact iff the closure of $T(B)$ in H_2 is a compact set.

The third result that we will need is special to elliptic operators. Before stating it, note that D is a linear differential operator of order d, and so it maps $C^{p+d}(M, E)$ into $C^p(M, E)$. Denoting by $\tilde{D}$ the extension of D introduced above, the definition of the Sobolev spaces also shows directly that $\tilde{D}$ maps $W^{k+d}(M, E)$ into $W^k(M, E)$. In other words, applying D or $\tilde{D}$ to a section involves a "loss" of d derivatives. The following theorem shows that when D is *elliptic*, this is true in the opposite direction as well: if $\tilde{D}u \in W^k(M, E)$, then we must have had $u \in W^{k+d}(M, E)$ to begin with.

Theorem 17.3 (Gårding's inequality)**.** *For any $u \in W^0(M, E)$ with the property that $\tilde{D}u \in W^k(M, E)$, we actually have $\tilde{D}u \in W^{k+d}(M, E)$. Moreover,*

$$\|u\|_{k+d} \leq C_k \big(\|u\|_0 + \|\tilde{D}u\|_k \big)$$

for a constant $C_k > 0$ that depends only on k.

Again, one proves the inequality first for $u \in A(M, E)$ by a local computation; the key point is that the symbol $P(x, \xi)$ is an invertible matrix for $\xi \neq 0$. Once the inequality is known, it is fairly straightforward to prove the existence of the required weak derivatives by approximation.

Sketch of proof. We can now give the proof of Theorem 16.3. Throughout, we let $D\colon E \to E$ be a linear differential operator that is elliptic of order d. To simplify the proof, we break it up into seven steps.

Step 1. A first observation is every section in $\ker \tilde{D}$ is smooth, and so $\ker \tilde{D} = \ker D$. To see this, suppose that $u \in W^0(M,E)$ satisfies $\tilde{D}u = 0$. By Gårding's inequality, we have $u \in W^{k+d}(M,E)$ for every $k \in \mathbb{N}$, and now the Sobolev lemma implies that $u \in A(M,E)$. Note that the same is true for the adjoint D^*, which is also elliptic.

Step 2. We prove that $\ker D = \ker \tilde{D}$ is a closed subspace of the Hilbert space $W^0(M,E)$. Applying Gårding's inequality to a section $u \in W^0(M,E)$ with $\tilde{D}u = 0$, we find that $\|u\|_{k+d} \leq C_k \|u\|_0$ for every $k \in \mathbb{N}$. Now suppose that we have a sequence $u_i \in \ker \tilde{D}$ that converges to some $u \in W^0(M,E)$. Then the inequality $\|u_i - u_j\|_{k+d} \leq C_k \|u_i - u_j\|_0$ shows that $\{u_i\}_i$ is a Cauchy sequence in $W^{k+d}(M,E)$, and hence that its limit $u \in W^{k+d}(M,E)$. But since $\tilde{D}\colon W^{k+d}(M,E) \to W^k(M,E)$ is bounded, it follows that $\tilde{D}u = \lim_i \tilde{D}u_i = 0$, and so $u \in \ker \tilde{D}$. We see from the proof that $\ker D$ is closed in every $W^k(M,E)$.

Step 3. Next, we show that $\ker D$ is finite-dimensional. Since $\ker D$ is a closed subspace of $W^0(M,E)$, it is itself a Hilbert space; let $B \subseteq \ker D$ be its closed unit ball. As we have seen, B is contained in the closed unit ball of $W^d(M,E)$ of radius C_0; since the inclusion $W^d(M,E) \hookrightarrow W^0(M,E)$ is compact by Rellich's lemma, it follows that B is compact. We can now apply Riesz' lemma to conclude that the dimension of $\ker D$ is finite. This was one of the assertions of Theorem 16.3.

Step 4. We show that if $u_i \in W^{k+d}(M,E)$ is a sequence with $u_i \in (\ker \tilde{D})^\perp$, such that $\tilde{D}u_i$ converges in $W^k(M,E)$, then $\|u_i\|_{k+d}$ is bounded. Suppose that this was not the case; then, after normalizing, we would be able to find a sequence $u_i \in W^{k+d}(M,E)$ with $\|u_i\|_{k+d} = 1$ and $u_i \in (\ker \tilde{D})^\perp$, such that $\tilde{D}u_i \to 0$. Since $\{u_i\}_i$ is bounded in $W^{k+d}(M,E)$, Rellich's lemma shows that it is precompact in $W^k(M,E)$, and after passage to a subsequence, we may assume that $\{u_i\}_i$ is a Cauchy sequence in $W^k(M,E)$. From Gårding's inequality

$$\|u_i - u_j\|_{k+d} \leq C_k \bigl(\|\tilde{D}u_i - \tilde{D}u_j\|_k + \|u_i - u_j\|_k \bigr),$$

we infer that the sequence is also Cauchy in $W^{k+d}(M,E)$, and therefore converges to a unique limit $u \in W^{k+d}(M,E)$. Then $\tilde{D}u = \lim_i \tilde{D}u_i = 0$; on the other hand, $u \in (\ker \tilde{D})^\perp$, and the only possible conclusion is that $u = 0$. But this contradicts the fact that $\|u_i\|_{k+d} = 1$ for all i.

Step 5. We show that $\tilde{D}\colon W^{k+d}(M,E) \to W^k(M,E)$ has closed image. So suppose that some $v \in W^k(M,E)$ belongs to the closure of the image. This means that there is a sequence $u_i \in W^{k+d}(M,E)$ such that $\tilde{D}u_i \to v$ in $W^k(M,E)$. Since $\ker \tilde{D}$ is closed in $W^{k+d}(M,E)$, we may furthermore assume that $u_i \in (\ker \tilde{D})^\perp$. By the result of Step 4, $\|u_i\|_{k+d}$ is bounded, and then arguing as before, we conclude that a subsequence converges to some $u \in W^{k+d}(M,E)$. Now clearly $v = \lim_i \tilde{D}u_i = \tilde{D}u$, and this shows that $\tilde{D}$ has closed image.

Step 6. We show that $\bigl(\tilde{D}^* W^{k+d}(M,E)\bigr)^\perp = \ker D$ in $W^k(M,E)$; here D^* is the formal adjoint of D, also an elliptic operator of order d, and $\tilde{D}^*$ denotes the corresponding weak differential operator. So suppose that a section $u \in W^k(M,E)$ is perpendicular to the image of $\tilde{D}^*$. Since $A(M,E)$ is dense, this is equivalent to the statement that $(u, D^*\phi)_k = 0$ for every $\phi \in A(M,E)$. But this means exactly that the weak derivative $\tilde{D}u$ exists and is equal to zero; now apply the result of Step 1 to conclude that $u \in A(M,E)$ is smooth and satisfies $Du = 0$.

Step 7. By the results of Step 5 and 6, we have an orthogonal decomposition

$$\begin{aligned} W^k(M,E) &= \big(\tilde{D}^* W^{k+d}(M,E)\big)^{\perp} \oplus \overline{\tilde{D}^* W^{k+d}(M,E)} \\ &= \ker D \oplus \tilde{D}^*\big(W^{k+d}(M,E)\big). \end{aligned}$$

for every $k \in \mathbb{N}$. We can now apply the Sobolev lemma to obtain the decomposition

$$A(M,E) = \ker D \oplus D^*\big(A(M,E)\big)$$

asserted in Theorem 16.3. In particular, $\operatorname{coker} D^*$ is finite dimensional; interchanging D and D^*, we obtain the same for $\operatorname{coker} D$, and this completes the proof.

Class 18. Harmonic theory

We can now return to the problem of finding canonical representatives for classes in $H^k(M,\mathbb{R})$ on a compact oriented Riemannian manifold (M,g). Following the general strategy outlined in previous lectures, we put inner products on the spaces of forms $A^k(M)$, and use these to define an adjoint d^* for the exterior derivative, and a Laplace operator $\Delta = dd^* + d^*d$.

Linear algebra. We begin by discussing some more linear algebra. Let V be a real vector space of dimension n, with inner product $g\colon V \times V \to \mathbb{R}$. (The example we have in mind is $V = T_{\mathbb{R},p}M$, with the inner product g_p coming from the Riemannian metric.) The inner product yields an isomorphism

$$\varepsilon\colon V \to V^*, \quad v \mapsto g(v,-),$$

between V and its dual space $V^* = \operatorname{Hom}(V,\mathbb{R})$. Note that if $e_1,\dots,e_n$ is an orthonormal basis for V, then $\varepsilon(e_1),\dots,\varepsilon(e_n)$ is the dual basis in V^*. We endow V^* with the inner product induced by the isomorphism ε, and then this dual basis becomes orthonormal as well.

All the spaces $\bigwedge^k V$ also acquire inner products, by setting

$$g(u_1 \wedge \dots \wedge u_k, v_1 \wedge \dots \wedge v_k) = \det\big(g(u_i,v_j)\big)_{i,j=1}^k$$

and extending bilinearly. These inner products have the property that, for any orthonormal basis $e_1,\dots,e_n \in V$, the vectors

$$e_{i_1} \wedge \dots \wedge e_{i_k}$$

with $i_1 < i_2 < \dots < i_k$ form an orthonormal basis for $\bigwedge^k V$.

Now suppose that V is in addition oriented. Recall that the fundamental element $\phi \in \bigwedge^n V$ is the unique positive vector of length 1; we have $\phi = e_1 \wedge \dots \wedge e_n$ for any positively-oriented orthonormal basis.

Definition 18.1. The *$*$-operator* is the unique linear operator $*\colon \bigwedge^k V \to \bigwedge^{n-k} V$ with the property that $\alpha \wedge *\beta = g(\alpha,\beta)\cdot\phi$ for any $\alpha,\beta \in \bigwedge^k V$.

Note that $\alpha \wedge *\beta$ belongs to $\bigwedge^n V$, and is therefore a multiple of the fundamental element ϕ. The $*$-operator is most conveniently defined using an orthormal basis $e_1,\dots,e_n$ for V: for any permutation σ of $\{1,\dots,n\}$, we have

$$e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(n)} = \operatorname{sgn}(\sigma)\cdot e_1 \wedge \dots \wedge e_n = \operatorname{sgn}(\sigma)\cdot\phi,$$

and consequently

$$\ast\left(e_{\sigma(1)}\wedge\cdots\wedge e_{\sigma(k)}\right)=\operatorname{sgn}(\sigma)\cdot e_{\sigma(k+1)}\wedge\cdots\wedge e_{\sigma(n)}. \tag{18.2}$$

This relation shows that $\ast$ takes an orthonormal basis to an orthonormal basis, and is therefore an isometry: $g(\ast\alpha,\ast\beta)=g(\alpha,\beta)$.

Lemma 18.3. *We have* $\ast\ast\,\alpha=(-1)^{k(n-k)}\alpha$ *for any* $\alpha\in\bigwedge^k V$.

Proof. Let $\alpha,\beta\in\bigwedge^k V$. By definition of the $\ast$-operator, we have

$$\begin{aligned}(\ast\ast\,\alpha)\wedge(\ast\beta)&=(-1)^{k(n-k)}(\ast\beta)\wedge(\ast\ast\,\alpha)=(-1)^{k(n-k)}g(\ast\beta,\ast\alpha)\cdot\phi\\&=(-1)^{k(n-k)}g(\alpha,\beta)\cdot\phi=(-1)^{k(n-k)}\alpha\wedge\ast\beta.\end{aligned}$$

This being true for all β, we conclude that $\ast\ast\,\alpha=(-1)^{k(n-k)}\alpha$. □

It follows that $\ast\colon\bigwedge^k V\to\bigwedge^{n-k}V$ is an isomorphism; this may be viewed as an abstract form of Poincaré duality (which says that on a compact oriented manifold, $H^k(M,\mathbb{R})\simeq H^{n-k}(M,\mathbb{R})$ for every $0\leq k\leq n$).

Inner products and the Laplacian. Let (M,g) be a Riemannian manifold that is compact, oriented, and of dimension n. At every point $p\in M$, we have an inner product g_p on the real tangent space $T_{\mathbb{R},p}M$, and therefore also on the cotangent space $T^*_{\mathbb{R},p}M$ and on each $\bigwedge^k T^*_{\mathbb{R},p}M$. In other words, each vector bundle $\bigwedge^k T^*_{\mathbb{R}}M$ carries a natural Euclidean metric. This allows us to define an inner product on the space of smooth k-forms $A^k(M)$ by the formula

$$(\alpha,\beta)_M=\int_M g\big(\alpha,\beta\big)\operatorname{vol}(g).$$

The individual $\ast$-operators $\ast\colon\bigwedge^k T^*_{\mathbb{R},p}M\to\bigwedge^{n-k}T^*_{\mathbb{R},p}M$ at each point $p\in M$ give us a a linear mapping

$$\ast\colon A^k(M)\to A^{n-k}(M).$$

By definition, we have $\alpha\wedge\ast\beta=g(\alpha,\beta)\cdot\operatorname{vol}(g)$, and so the inner product can also be expressed by the simpler formula

$$(\alpha,\beta)_M=\int_M\alpha\wedge\ast\beta.$$

It has the advantage of hiding the terms coming from the metric.

We already know that the exterior derivative d is a linear differential operator. Since the bundles in question carry Euclidean metrics, there is a unique adjoint; the $\ast$-operator allows us to write down a simple formula for it.

Proposition 18.4. *The adjoint* $d^*\colon A^k(M)\to A^{k-1}(M)$ *is given by the formula*

$$d^*=-(-1)^{n(k+1)}\ast d\,\ast.$$

Proof. Fix $\alpha\in A^{k-1}(M)$ and $\beta\in A^k(M)$. By Stokes' theorem, the integral of $d\big(\alpha\wedge\ast\beta\big)=d\alpha\wedge\ast\beta+(-1)^{k-1}\alpha\wedge d(\ast\beta)$ over M is zero, and therefore

$$(d\alpha,\beta)_M=\int_M d\alpha\wedge\ast\beta=(-1)^k\int_M\alpha\wedge d\ast\beta=(-1)^k\int_M\alpha\wedge\ast\big(\ast^{-1}d\ast\beta\big)$$

This shows that the adjoint is given by the formula $d^*\beta=(-1)^k\ast^{-1}d\ast\beta$. Since $d\ast\beta\in A^{n-k+1}(M)$, we can use the identity from Lemma 18.3 to compute that

$$d^*\beta=(-1)^k(-1)^{(n-k+1)(k-1)}\ast d\ast\beta,$$

from which the assertion follows because k^2+k is an even number. $\square$

Definition 18.5. For each $0 \leq k \leq n$, we define the *Laplace operator* $\Delta\colon A^k(M) \to A^k(M)$ by the formula $\Delta = d \circ d^* + d^* \circ d$. A form $\omega \in A^k(M)$ is called *harmonic* if $\Delta\omega = 0$, and we let $\mathcal{H}^k(M) = \ker \Delta$ be the space of all harmonic forms.

More precisely, each Δ is a second-order linear differential operator from the vector bundle $\bigwedge^k T_{\mathbb{R}}^* M$ to itself. It is easy to see that Δ is formally self-adjoint; indeed, the adjointness of d and d^* shows that

$$(\Delta\alpha, \beta)_M = (d\alpha, d\beta)_M + (d^*\alpha, d^*\beta)_M = (\alpha, \Delta\beta)_M.$$

By computing a formula for Δ in local coordinates, one shows that Δ is an elliptic operator. We may therefore apply the fundamental theorem of elliptic operators (Theorem 16.3) to conclude that the space of harmonic forms $\mathcal{H}^k(M)$ is finite-dimensional, and that we have an orthogonal decomposition

$$A^k(M) = \mathcal{H}^k(M) \oplus \operatorname{im}\bigl(\Delta\colon A^k(M) \to A^k(M)\bigr). \tag{18.6}$$

We can now state and prove the main theorem of real Hodge theory.

Theorem 18.7. *Let (M, g) be a compact and oriented Riemannian manifold. Then the natural map $\mathcal{H}^k(M) \to H^k(M, \mathbb{R})$ is an isomorphism; in other words, every de Rham cohomology class contains a unique harmonic form.*

Proof. Recall that a form ω is harmonic iff $d\omega = 0$ and $d^*\omega = 0$; this follows from the identity $(\Delta\omega, \omega)_M = \|d\omega\|_M^2 + \|d^*\omega\|_M^2$. In particular, harmonic forms are automatically closed, and therefore define classes in de Rham cohomology. We have to show that the resulting map $\mathcal{H}^k(M) \to H^k(M, \mathbb{R})$ is bijective.

To prove the injectivity, suppose that $\omega \in \mathcal{H}^k(M)$ is harmonic and d-exact, say $\omega = d\psi$ for some $\psi \in A^{k-1}(M)$. Then

$$\|\omega\|_M^2 = (\omega, d\psi)_M = (d^*\omega, \psi)_M = 0,$$

and therefore $\omega = 0$. Note that this part of the proof is elementary, and does not use any of the results from the theory of elliptic operators.

To prove the surjectivity, take an arbitrary cohomology class and represent it by some $\alpha \in A^k(M)$ with $d\alpha = 0$. The decomposition in (18.6) shows that we have

$$\alpha = \omega + \Delta\beta = \omega + dd^*\beta + d^*d\beta.$$

with $\omega \in \mathcal{H}^k(M)$ harmonic and $\beta \in A^k(M)$. Since $d\omega = 0$, we get $0 = d\alpha = dd^*d\beta$, and therefore

$$\|d^*d\beta\|_M^2 = (d^*d\beta, d^*d\beta)_M = (d\beta, dd^*d\beta)_M = 0,$$

proving that $d^*d\beta = 0$. This shows that $\alpha = \omega + dd^*\beta$, and so the harmonic form ω represents the original cohomology class. $\square$

Note. The space of harmonic forms $\mathcal{H}^k(M)$ depends on the Riemannian metric g; this is because the definition of the operators d^* and Δ involves the metric.

Class 19. Complex harmonic theory

The purpose of today's lecture is to extend the Hodge theorem to the Dolbeault cohomology groups $H^{p,q}(M)$ on a compact complex manifold M with a Hermitian metric h. Recall that this means a collection of positive definite Hermitian forms $h_p\colon T_p'M \times T_p'M \to \mathbb{C}$ on the holomorphic tangent spaces that vary smoothly with the point $p \in M$.

More linear algebra. As in the case of Riemannian manifolds, we begin by looking at a single Hermitian vector space (V, h); in our applications, $V = T'_p M$ will be the holomorphic tangent space to a complex manifold. Thus let V be a complex vector space of dimension n, and $h\colon V \times V \to \mathbb{C}$ a positive definite form that is linear in its first argument, and satisfies $h(v_2, v_1) = \overline{h(v_1, v_2)}$.

We denote the underlying real vector space by $V_{\mathbb{R}}$, noting that it has dimension $2n$. Multiplication by i defines a linear operator $J \in \operatorname{End}(V_{\mathbb{R}})$ with the property that $J^2 = -\operatorname{id}$. The complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}$ is a complex vector space of dimension $2n$; it decomposes into a direct sum

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}, \tag{19.1}$$

where $V^{1,0} = \ker(J - i\,\mathrm{id})$ and $V^{0,1} = \ker(J + i\,\mathrm{id})$ are the two eigenspaces of J. For any $v \in V_{\mathbb{R}}$, we have $v = \frac{1}{2}(v - iJv) + \frac{1}{2}(v + iJv)$; this means that the inclusion $V_{\mathbb{R}} \hookrightarrow V_{\mathbb{C}}$, followed by the projection $V_{\mathbb{C}} \twoheadrightarrow V^{1,0}$, defines an $\mathbb{R}$-linear map

$$V_{\mathbb{R}} \to V^{1,0}, \quad v \mapsto \frac{1}{2}(v - iJv)$$

which is an isomorphism of real vector spaces. This justifies identifying the original complex vector space V with the space $V^{1,0}$.

The decomposition in (19.1) induces a decomposition

$$\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \Big(\bigwedge^p V^{1,0}\Big) \otimes \Big(\bigwedge^q V^{0,1}\Big) = \bigoplus_{p+q=k} V^{p,q}, \tag{19.2}$$

and elements of $V^{p,q}$ are often said to be of *type* (p, q).

We have already seen that the Hermitian form h defines an inner product $g = \operatorname{Re} h$ on the real vector space $V_{\mathbb{R}}$. It satisfies $g(Jv_1, Jv_2) = g(v_1, v_2)$, and conversely, we can recover h from g by the formula

$$h(v_1, v_2) = g(v_1, v_2) + ig(v_1, Jv_2).$$

As usual, g induces inner products on the spaces $\bigwedge^k V_{\mathbb{R}}$, which we extend sesquilinearly to Hermitian inner products h on $\bigwedge^k V_{\mathbb{C}}$. We compute that

$$h\left(\frac{1}{2}(v_1 - iJv_1), \frac{1}{2}(v_2 - iJv_2)\right) = \frac{1}{2}\Big(g(v_1, v_2) + ig(v_1, Jv_2)\Big),$$

and so (up to the annoying factor of $1/2$) this definition is compatible with the original Hermitian inner product on V under the identification with $V^{1,0}$.

Lemma 19.3. *The decomposition in* (19.2) *is orthogonal with respect to the Hermitian inner product h.*

Recall that $V_{\mathbb{R}}$ is automatically oriented; the natural orientation is given by $v_1, Jv_1, \dots, v_n, Jv_n$ for any complex basis $v_1, \dots, v_n \in V$. It follows that if $e_1, \dots, e_n$ is any orthonormal basis of V with respect to the Hermitian inner product h, then

$$e_1, Je_1, e_2, Je_2, \dots, e_n, Je_n$$

is a positively oriented orthonormal basis for $V_{\mathbb{R}}$; in particular, the fundamental element is given by the formula $\varphi = (e_1 \wedge Je_1) \wedge \cdots \wedge (e_n \wedge Je_n)$.

As usual, we have the $*$-operator $\bigwedge^k V_{\mathbb{R}} \to \bigwedge^{2n-k} V_{\mathbb{R}}$; we extend it $\mathbb{C}$-linearly to $*\colon \bigwedge^k V_{\mathbb{C}} \to \bigwedge^{2n-k} V_{\mathbb{C}}$. Since we obtained the Hermitian inner product h on $\bigwedge^k V_{\mathbb{C}}$

by extending g linearly in the first and conjugate-linearly in the second argument, the $*$-operator satisfies the identity

$$\alpha \wedge *\overline{\beta} = h(\alpha, \beta) \cdot \varphi$$

for $\alpha, \beta \in \bigwedge^k V_{\mathbb{C}}$.

Lemma 19.4. *The $*$-operator maps $V^{p,q}$ into $V^{n-q,n-p}$, and satisfies $*^2\alpha = (-1)^{p+q}\alpha$ for any $\alpha \in V^{p,q}$.*

Proof. For $\beta \in V^{p,q}$ and $\alpha \in V^{r,s}$, we have $\alpha \wedge *\beta = h(\alpha, \overline{\beta}) \cdot \varphi = 0$ unless $p = s$ and $q = r$; this is because the decomposition by type is orthogonal (Lemma 19.3). It easily follows that $*\beta$ has type $(n-q, n-p)$. The second assertion is a restatement of Lemma 18.3, where we proved that $*^2 = (-1)^{k(2n-k)} \operatorname{id} = (-1)^k \operatorname{id}$ on $\bigwedge^k V_{\mathbb{R}}$. $\square$

The dual vector space $V_{\mathbb{R}}^* = \operatorname{Hom}(V_{\mathbb{R}}, \mathbb{R})$ also has a complex structure J, by defining $(Jf)(v) = f(Jv)$ for $f \in V_{\mathbb{R}}^*$ and $v \in V_{\mathbb{R}}$. Note that the isomorphism

$$\varepsilon \colon V_{\mathbb{R}} \to V_{\mathbb{R}}^*, \quad v \mapsto g(v, -)$$

is only conjugate-linear, since $\varepsilon(Jv) = g(Jv, -) = -g(v, J-) = -J\varepsilon(v)$.

The anti-holomorphic Laplacian. From now on, let (M, h) be a compact complex manifold M, of dimension n, with a Hermitian metric h. We then have the space $A^{p,q}(M)$ of smooth differential forms of type (p,q), and the $\bar{\partial}$-operator $\bar{\partial} \colon A^{p,q}(M) \to A^{p,q+1}(M)$. Recall that we defined the Dolbeault cohomology groups

$$H^{p,q}(M) = \frac{\ker\bigl(\bar{\partial} \colon A^{p,q}(M) \to A^{p,q+1}(M)\bigr)}{\operatorname{im}\bigl(\bar{\partial} \colon A^{p,q-1}(M) \to A^{p,q}(M)\bigr)}.$$

According to the discussion above, we have a Hermitian inner product on the space of (p,q)-forms, defined by

$$(\alpha, \beta)_M = \int_M \alpha \wedge *\overline{\beta},$$

and the complex-linear Hodge $*$-operator

$$* \colon A^{p,q}(M) \to A^{n-q,n-p}(M).$$

As in the real case, we define the adjoint $\bar{\partial}^* \colon A^{p,q}(M) \to A^{p,q-1}(M)$ by the condition that, for every $\alpha \in A^{p,q-1}(M)$ and $\beta \in A^{p,q}(M)$,

$$\bigl(\bar{\partial}\alpha, \beta\bigr)_M = \bigl(\alpha, \bar{\partial}^*\beta\bigr)_M.$$

By essentially the same calculation as in Proposition 18.4, we get an the following formula for the adjoint operator.

Proposition 19.5. *We have $\bar{\partial}^* = - * \partial *$.*

Proof. Fix $\alpha \in A^{p,q-1}(M)$ and $\beta \in A^{p,q}(M)$; then $\gamma = \alpha \wedge *\overline{\beta}$ is of type $(n, n-1)$, and so $d\gamma = \bar{\partial}\gamma$. We compute that $\bar{\partial}\gamma = \bar{\partial}\alpha \wedge *\overline{\beta} + (-1)^{p+q-1}\alpha \wedge \bar{\partial}(*\overline{\beta})$, and so it again follows from Stokes' theorem that

$$\bigl(\bar{\partial}\alpha, \beta\bigr)_M = \int_M \bar{\partial}\alpha \wedge *\overline{\beta} = (-1)^{p+q} \int_M \alpha \wedge \bar{\partial} * \overline{\beta}.$$

The adjoint is therefore given by the formula

$$\bar{\partial}^*\beta = (-1)^{p+q}\overline{\bigl(*^{-1}\bar{\partial} * \overline{\beta}\bigr)} = (-1)^{p+q} *^{-1} \partial * \beta,$$

using that $*$ is a real operator. Now $\partial{*}\beta \in A^{n-q+1,n-p}(M)$, and therefore ${*}^2\partial{*}\beta = (-1)^{2n-p-q+1}\beta$; putting things together, we find that $\bar\partial^*\beta = (-1)^{2n+1} * \partial * \beta = - * \partial * \beta$, as asserted above. $\square$

Definition 19.6. The *anti-holomorphic Laplacian* is the linear differential operator $\overline{\square}\colon A^{p,q}(M) \to A^{p,q}(M)$, defined as $\overline{\square} = \bar\partial \circ \bar\partial^* + \bar\partial^* \circ \bar\partial$. We say that a (p,q)-form ω is *$\bar\partial$-harmonic* if $\overline{\square}\omega = 0$, and let $\mathcal{H}^{p,q}(M) = \ker \overline{\square}$ denote the space of $\bar\partial$-harmonic forms.

One proves that $\overline{\square}$ is formally self-adjoint and elliptic, and that a (p,q)-form ω is $\bar\partial$-harmonic iff $\bar\partial\omega = 0$ and $\bar\partial^*\omega = 0$. In particular, any such form defines a class in Dolbeault cohomology. By a variant of the fundamental theorem on elliptic operators, we have an orthogonal decomposition

$$A^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \operatorname{im}\bigl(\overline{\square}\colon A^{p,q}(M) \to A^{p,q}(M)\bigr).$$

It implies the following version of the Hodge theorem for (p,q)-forms by the same argument as in the proof of Theorem 18.7.

Theorem 19.7. *Let (M,h) be a compact Hermitian manifold. Then the natural map $\mathcal{H}^{p,q}(M) \to H^{p,q}(M)$ is an isomorphism; in other words, every Dolbeault cohomology class contains a unique $\bar\partial$-harmonic form.*

Similarly, one can define the adjoint $\partial^*\colon A^{p,q}(M) \to A^{p-1,q}(M)$ and the holomorphic Laplacian $\square = \partial \circ \partial^* + \partial^* \circ \partial$, and get a representation theorem for the cohomology groups of the ∂-operator.

Note. In general, there is *no* relationship between the real Laplace operator $\Delta = d \circ d^* + d^* \circ d$ and the holomorphic Laplacian $\overline{\square} = \bar\partial \circ \bar\partial^* + \bar\partial^* \circ \bar\partial$. This means that a $\bar\partial$-harmonic form need not be harmonic, and in fact, not even d-closed.

One can prove that if $\omega \in A^{p,q}(M)$ is $\bar\partial$-harmonic, then $*\omega \in A^{n-q,n-p}(M)$ is again $\bar\partial$-harmonic. This observation implies the following duality theorem.

Corollary 19.8. *The $*$-operator defines an isomorphism $H^{p,q}(M) \simeq H^{n-q,n-p}(M)$.*

Class 20. Kähler manifolds

Let M be a compact complex manifold with a Hermitian metric h; then $g = \operatorname{Re} h$ also defines a Riemannian metric on the underlying smooth manifold. Consequently, we can represent any class in $H^k(M,\mathbb{R})$ by a harmonic form (in the kernel of Δ), and any class in $H^{p,q}(M)$ by a $\bar\partial$-harmonic form (in the kernel of $\overline{\square}$). As already mentioned, there is in general no relation between those two kinds of harmonic forms. For instance, a $\bar\partial$-harmonic form need not be d-closed; and conversely, if we decompose a harmonic form α by type as $\alpha = \sum_{p+q=k} \alpha^{p,q}$, then none of the $\alpha^{p,q}$ need be harmonic or $\bar\partial$-harmonic. In a nutshell, this is due to a lack of compatibility between the metric and the complex structure.

There is, however, a large class of complex manifolds on which the two theories interact very nicely: the so-called *Kähler manifolds.*

Kähler metrics. Recall that the projective space $\mathbb{P}^n$ has a very natural Hermitian metric, namely the Fubini-Study metric h_{FS}. Its associated $(1,1)$-form ω_{FS}, after pulling back via the map $q\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, is given by the formula

$$q^*\omega_{FS} = \frac{i}{2\pi}\partial\bar\partial \log\bigl(|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2\bigr).$$

The formula shows that $d\omega_{FS} = 0$, which means that ω_{FS} is a closed form. This simple condition turns out to be the key to the compatibility between the metric and the complex structure.

Definition 20.1. A Hermitian metric h on a complex manifold M is said to be *Kähler* if its associated $(1,1)$-form ω satisfies $d\omega = 0$. A *Kähler manifold* is a complex manifold M that admits at least one Kähler metric.

Any complex submanifold N of a Kähler manifold (M, h) is again a Kähler manifold; indeed, if we give N the induced metric, then $\omega_N = i^*\omega$, where $i\colon N \to M$ is the inclusion map, and so $d\omega_N = d(i^*\omega) = i^* d\omega = 0$. In particular, since $\mathbb{P}^n$ is Kähler, any projective manifold is automatically a Kähler manifold. To a large extent, this accounts for the usefulness of complex manifold theory in algebraic geometry.

We shall now look at the Kähler condition in local holomorphic coordinates $z_1, \dots, z_n$ on M. With $h_{j,k} = h(\partial/\partial z_j, \partial/\partial z_k)$, the associated $(1,1)$-form is given by the formula

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\bar{z}_k.$$

Note that the matrix with entries $h_{j,k}$ is Hermitian-symmetric, and therefore, $h_{k,j} = \overline{h_{j,k}}$. Now we compute that

$$d\omega = \frac{i}{2} \sum_{j,k,l} \frac{\partial h_{j,k}}{\partial z_l} dz_l \wedge dz_j \wedge d\bar{z}_k + \frac{i}{2} \sum_{j,k,l} \frac{\partial h_{j,k}}{\partial \bar{z}_l} dz_j \wedge d\bar{z}_k \wedge d\bar{z}_l,$$

and so $d\omega = 0$ iff $\partial h_{j,k}/\partial z_l = \partial h_{l,k}/\partial z_j$ and $\partial h_{j,k}/\partial \bar{z}_l = \partial h_{j,l}/\partial \bar{z}_k$. The second condition is actually equivalent to the first (by conjugating), and this proves that the metric h is Kähler iff

$$\frac{\partial h_{j,k}}{\partial z_l} = \frac{\partial h_{l,k}}{\partial z_j} \tag{20.2}$$

for every $j, k, l \in \{1, \dots, n\}$.

Note that the usual Euclidean metric on $\mathbb{C}^n$ has associated $(1,1)$-form

$$\frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j,$$

and is therefore Kähler. The following lemma shows that, conversely, any Kähler metric agrees with the Euclidean metric to second order, in suitable local coordinates.

Lemma 20.3. *A Hermitian metric h is Kähler iff, at every point $p \in M$, there is a holomorphic coordinate system $z_1, \dots, z_n$ centered at p such that*

$$\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j + O(|z|^2).$$

Proof. One direction is very easy: If we can find such a coordinate system centered at a point p, then $d\omega$ clearly vanishes at p; this being true for every $p \in M$, it follows that $d\omega = 0$, and so h is Kähler.

Conversely, assume that $d\omega = 0$, and fix a point $p \in M$. Let $z_1, \dots, z_n$ be arbitrary holomorphic coordinates centered at p, and set $h_{j,k} = h(\partial/\partial z_j, \partial/\partial z_k)$;

since we can always make a linear change of coordinates, we may clearly assume that $h_{j,k}(0) = \mathrm{id}_{j,k}$ is the identity matrix. Using that $h_{j,k} = \overline{h_{k,j}}$, we then have

$$h_{j,k} = \mathrm{id}_{j,k} + E_{j,k} + \overline{E_{k,j}} + O(|z|^2),$$

where each $E_{j,k}$ is a linear form in $z_1, \dots, z_n$. Since h is Kähler, (20.2) shows that $\partial E_{j,k}/\partial z_l = \partial E_{l,k}/\partial z_j$; this condition means that there exist quadratic functions $q_j(z)$ such that $E_{j,k} = \partial q_k/\partial z_j$ and $q_j(0) = 0$. Now let

$$w_k = z_k + q_k(z);$$

since the Jacobian $\partial(w_1, \dots, w_n)/\partial(z_1, \dots, z_n)$ is the identity matrix at $z = 0$, the functions $w_1, \dots, w_n$ give holomorphic coordinates in a small enough neighborhood of the point p. By construction,

$$dw_j = dz_j + \sum_{k=1}^n \frac{\partial q_j}{\partial z_k} dz_k = dz_j + \sum_{k=1}^n E_{k,j} dz_k.$$

and so we have, up to second-order terms,

$$\begin{aligned}
\frac{i}{2}\sum_{j=1}^n dw_j \wedge d\bar{w}_j &\equiv \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j + \frac{i}{2}\sum_{j,k=1}^n dz_j \wedge \overline{E_{k,j}} d\bar{z}_k + \frac{i}{2}\sum_{j,k=1}^n E_{k,j} dz_k \wedge d\bar{z}_j \\
&= \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j + \frac{i}{2}\sum_{j,k=1}^n \big(E_{j,k} + \overline{E_{k,j}}\big) dz_j \wedge d\bar{z}_k \\
&\equiv \frac{i}{2}\sum_{j,k=1}^n h_{j,k} dz_j \wedge d\bar{z}_k.
\end{aligned}$$

which shows that $\omega = \frac{i}{2}\sum_{j,k} dw_j \wedge d\bar{w}_k + O(|w|^2)$ in the new coordinate system. $\square$

This lemma is extremely useful for proving results about arbitrary Kähler metrics by only looking at the Euclidean metric on $\mathbb{C}^n$.

Kähler metrics and differential geometry. To show how the condition $d\omega = 0$ implies that the metric is compatible with the complex structure, we shall now look at some equivalent formulations of the Kähler condition.

We begin by reviewing some differential geometry. Let M be a smooth manifold, with real tangent bundle $T_{\mathbb{R}}M$, and let $T(M)$ denote the set of smooth vector fields on M. Recall that vector fields can be viewed as operators on smooth functions: if $\xi \in T(M)$, then $\xi \cdot f$ is a smooth function for any smooth function f. In local coordinates $x_1, \dots, x_n$, we can write $\xi = \sum a_i \partial/\partial x_i$, with smooth functions $a_1, \dots, a_n$, and then

$$\xi \cdot f = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}.$$

Given any two vector fields ξ and η, their commutator $[\xi, \eta] \in T(M)$ acts on smooth functions by the rule $[\xi, \eta] \cdot f = \xi \cdot (\eta f) - \eta \cdot (\xi f)$. In local coordinates, $\xi = \sum a_j \partial/\partial x_i$ and $\eta = \sum b_i \partial/\partial x_i$, and then

$$[\xi, \eta] = \sum_{i,j=1}^n \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j},$$

as a short computation will show.

Since $T_{\mathbb{R}}M$ is a vector bundle, there is no intrinsic way to differentiate its sections; this requires the additional data of a *connection*. Such a connection is a mapping

$$\nabla \colon T(M) \times T(M) \to T(M), \quad (\xi, \eta) \mapsto \nabla_\xi \eta,$$

linear in the first argument, and satisfying the Leibniz rule

$$\nabla_\xi (f \cdot \eta) = (\xi f) \cdot \eta + f \cdot \nabla_\xi \eta.$$

In other words, a connection gives a way to differentiate vector fields, and $\nabla_\xi \eta$ should be viewed as the derivative of η in the direction of ξ.

Proposition 20.4. *On a Riemannian manifold (M, g), there is a unique connection that is both compatible with the metric, in the sense that*

$$\xi \cdot g(\eta, \zeta) = g\big(\nabla_\xi \eta, \zeta\big) + g\big(\eta, \nabla_\xi \zeta\big),$$

and torsion-free, in the sense that

$$\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta].$$

This connection is known as the Levi-Cività connection *associated to the metric.*

Class 21. More on Kähler manifolds

First of all, we have to prove the existence and uniqueness of the Levi-Cività connection on a Riemannian manifold (M, g).

Proof. Let $x_1, \dots, x_n$ be local coordinates on M, and set $\partial_i = \partial/\partial x_i$; the Riemannian metric is represented by the matrix $g_{i,j} = g(\partial_i, \partial_j)$. To describe the connection, it is sufficient to know the coefficients $\Gamma^k_{i,j}$ in the expression

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^n \Gamma^k_{i,j} \partial_k.$$

The conditions above now mean the following: the connection is torsion-free iff $\Gamma^k_{i,j} = \Gamma^k_{j,i}$, and compatible with the metric iff

$$\frac{\partial g_{j,k}}{\partial x_i} = g\big(\nabla_{\partial_i} \partial_j, \partial_k\big) + g\big(\partial_j, \nabla_{\partial_i} \partial_k\big) = \sum_{l=1}^n g_{l,k} \Gamma^l_{i,j} + g_{j,l} \Gamma^l_{i,k}.$$

From these two identities, we compute that

$$\frac{\partial g_{i,j}}{\partial x_k} - \frac{\partial g_{i,k}}{\partial x_j} + \frac{\partial g_{j,k}}{\partial x_i} = 2 \sum_{l=1}^n g_{j,l} \Gamma^l_{i,k},$$

and so the coefficients are given by the formula

$$\Gamma^k_{i,j} = \frac{1}{2} \sum_{l=1}^n g^{k,l} \left(\frac{\partial g_{i,l}}{\partial x_j} - \frac{\partial g_{i,j}}{\partial x_l} + \frac{\partial g_{l,j}}{\partial x_i} \right),$$

where $g^{i,j}$ are the entries of the inverse matrix. $\square$

We come back to the case of a Hermitian manifold (M, h). At each point $p \in M$, we have an isomorphism between the real tangent space $T_{\mathbb{R},p}M$ and the real vector space underlying the holomorphic tangent space $T'_p M$. As usual, we denote by $J_p \in \operatorname{End}\big(T_{\mathbb{R},p}M\big)$ the operation of multiplying by i. We can say that the complex structure on M is encoded in the map $J \in \operatorname{End}\big(T_{\mathbb{R}}M\big)$. On the other hand, $g = \operatorname{Re} h$

defines a Riemannian metric on the underlying smooth manifold, with Levi-Civitā connection ∇.

Theorem 21.1. *Let (M,h) be a Hermitian manifold. The following two conditions are equivalent:*

(1) The metric h is Kähler.

(2) The complex structure $J \in \operatorname{End}\bigl(T_{\mathbb{R}}M\bigr)$ is flat for the Levi-Civitā connection, i.e., $\nabla_\xi(J\eta) = J\nabla_\xi\eta$ for any two smooth vector fields ξ, η on M.

Proof. It suffices to prove the identity $\nabla_\xi(J\eta) = J\nabla_\xi\eta$ at every point $p \in M$. Since the metric is Kähler, Lemma 20.3 allows us to choose local coordinates centered at p in which the h agrees with the Euclidean metric to second order. Now the identity only involves first-order derivatives of h, as is clear from the proof of Proposition 20.4; on the other hand, it is clearly true for the Euclidean metric on $\mathbb{C}^n$. It follows that the identity remains true for h at the point p. In this way, (1) implies (2).

To show that (2) implies (1), recall that the associated $(1,1)$-form $\omega = -\operatorname{Im} h$ is related to the Riemannian metric $g = \operatorname{Re} h$ by the formula $\omega(\xi,\eta) = g(J\xi,\eta)$. Since the metric is compatible with the connection, we thus have

$$\xi \cdot \omega(\eta,\zeta) = g\bigl(\nabla_\xi(J\eta),\zeta\bigr) + g\bigl(J\eta,\nabla_\xi\zeta\bigr) = \omega\bigl(\nabla_\xi\eta,\zeta\bigr) + \omega\bigl(\eta,\nabla_\xi\zeta\bigr). \tag{21.2}$$

Expressed in a coordinate-free manner, the exterior derivative $d\omega$ is given by the formula

$$\begin{aligned}(d\omega)(\xi,\eta,\zeta) = \xi\cdot\omega(\eta,\zeta) - \eta\cdot\omega(\xi,\zeta) + \zeta\cdot\omega(\xi,\eta)\\ + \omega\bigl(\xi,[\eta,\zeta]\bigr) - \omega\bigl(\eta,[\xi,\zeta]\bigr) + \omega\bigl(\zeta,[\xi,\eta]\bigr).\end{aligned}$$

After substituting (21.2) and using the identity $\nabla_\xi\eta - \nabla_\eta\xi = [\xi,\eta]$, we find that $d\omega = 0$, proving that the metric is indeed Kähler. $\square$

Our next goal is to prove that, on a Kähler manifold, the two Laplace operators are related by the formula $\Delta = 2\overline{\Box}$. This shows that the two notions of harmonic form (harmonic and $\bar\partial$-harmonic) are the same. Along the way, we shall establish several other relations between the different operators that have been introduced; these relations are collectively known as the Kähler identities.

The Kähler identities. Let (M,h) be a Kähler manifold; we refer to the associated $(1,1)$-form $\omega \in A^{1,1}(M)$ as the *Kähler form*. Since ω is real and satisfies $d\omega = 0$, it defines a class in $H^2(M,\mathbb{R})$; in the proof of Wirtinger's lemma, we have already seen that on a compact manifold, this class is nonzero because the formula $\operatorname{vol}(M) = \frac{1}{n!}\int_M \omega^{\wedge n}$ shows that ω is never exact.

Taking the wedge product with ω defines the so-called *Lefschetz operator*

$$L\colon A^k(M) \to A^{k+2}(M), \quad \alpha \mapsto \omega\wedge\alpha.$$

Since ω has type $(1,1)$, it is clear that L maps $A^{p,q}(M)$ into $A^{p+1,q+1}(M)$. Using the induced metric on the space of forms, we also define the adjoint

$$\Lambda\colon A^k(M) \to A^{k-2}(M)$$

by the condition that $g(L\alpha,\beta) = g(\alpha,\Lambda\beta)$. As usual, we obtain a formula for Λ involving the $*$-operator by noting that

$$g(L\alpha,\beta)\cdot\operatorname{vol}(g) = \omega\wedge\alpha\wedge*\beta = \alpha\wedge(\omega\wedge*\beta) = \alpha\wedge(L*\beta);$$

consequently, $\Lambda\beta = *^{-1}L*\beta = (-1)^k * L * \beta$ because $*^2 = (-1)^k\operatorname{id}$.

Theorem 21.3. *On a Kähler manifold* (M, h), *the following identities are true:*

$$[\Lambda, \bar{\partial}] = -i\partial^* \quad \textit{and} \quad [\Lambda, \partial] = i\bar{\partial}^*.$$

Since the two identities are conjugates of each other, it suffices to prove the second one. Moreover, both involve only the metric h and its first derivatives, and so they hold on a general Kähler manifold as soon as they are known on $\mathbb{C}^n$ with the Euclidean metric.

Class 22. The Kähler identities

We now turn to the proof of Theorem 21.3; as explained above, it is enough to prove the identity $[\Lambda, \partial] = i\bar{\partial}^*$ on $\mathbb{C}^n$ with the Euclidean metric h. In this metric, dz_j is orthogonal to $d\bar{z}_k$, and to dz_k for $k \neq j$, while

$$h(dz_j, dz_j) = h(dx_j + i dy_j, dx_j + i dy_j) = g(dx_j, dx_j) + g(dy_j, dy_j) = 2.$$

More generally, we have $h(dz_J \wedge d\bar{z}_K, dz_J \wedge d\bar{z}_K) = 2^{|J|+|K|}$.

To facilitate the computation, we introduce a few additional but more basic operators on the spaces $A^{p,q} = A^{p,q}(\mathbb{C}^n)$. First, define

$$e_j \colon A^{p,q} \to A^{p+1,q}, \quad \alpha \mapsto dz_j \wedge \alpha$$

as well as its conjugate

$$\bar{e}_j \colon A^{p,q} \to A^{p,q+1}, \quad \alpha \mapsto d\bar{z}_j \wedge \alpha.$$

We then have

$$L\alpha = \omega \wedge \alpha = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j \wedge \alpha = \frac{i}{2} \sum_{j=1}^n e_j \bar{e}_j \alpha.$$

Using the induced Hermitian inner product on forms, we then define the adjoint

$$e_j^* \colon A^{p,q} \to A^{p-1,q}$$

by the condition that $h(e_j \alpha, \beta) = h(\alpha, e_j^* \beta)$.

Lemma 22.1. *The adjoint* e_j^* *has the following properties:*

1. *If* $j \notin J$, *then* $e_j^*(dz_J \wedge d\bar{z}_K) = 0$, *while* $e_j^*(dz_j \wedge dz_J \wedge d\bar{z}_K) = 2 dz_J \wedge d\bar{z}_K$.
2. $e_k e_j^* + e_j^* e_k = 2\,\mathrm{id}$ *in case* $j = k$, *and* 0 *otherwise.*

Proof. By definition, we have

$$h(e_j^* dz_J \wedge d\bar{z}_K, dz_L \wedge d\bar{z}_M) = h(dz_J \wedge d\bar{z}_K, dz_j \wedge dz_L \wedge d\bar{z}_M),$$

and since dz_j occurs only in the second term, the inner product is always zero, proving that $e_j^* dz_J \wedge d\bar{z}_K = 0$. On the other hand,

$$\begin{aligned} h(e_j^* dz_j \wedge dz_J \wedge d\bar{z}_K, dz_L \wedge d\bar{z}_M) &= h(dz_j \wedge dz_J \wedge d\bar{z}_K, dz_j \wedge dz_L \wedge d\bar{z}_M) \\ &= 2h(dz_J \wedge d\bar{z}_K, dz_L \wedge d\bar{z}_M), \end{aligned}$$

which is nonzero exactly when $J = L$ and $K = M$. From this identity, it follows that $e_j^* dz_j \wedge dz_J \wedge d\bar{z}_K = 2 dz_J \wedge d\bar{z}_K$, establishing (1).

To prove (2) for $j = k$, observe that since $dz_j \wedge dz_j = 0$, we have

$$e_j^* e_j \bigl(dz_J \wedge d\bar{z}_K\bigr) = \begin{cases} 0 & \text{if } j \in J, \\ 2 dz_J \wedge d\bar{z}_K & \text{if } j \notin J, \end{cases}$$

while

$$e_j e_j^* \big(dz_J \wedge d\bar{z}_K \big) = \begin{cases} 2dz_J \wedge d\bar{z}_K & \text{if } j \in J, \\ 0 & \text{if } j \notin J. \end{cases}$$

Taken together, this shows that $e_j e_j^* + e_j^* e_j = 2\,\mathrm{id}$. Finally, let us prove that $e_k e_j^* + e_j^* e_k = 0$ when $j \neq k$. By (1), this is clearly true on $dz_J \wedge d\bar{z}_K$ in case $j \notin J$. On the other hand,

$$e_k e_j^* \big(dz_j \wedge dz_J \wedge d\bar{z}_K \big) = 2 e_k \big(dz_J \wedge d\bar{z}_K \big) = 2 dz_k \wedge dz_J \wedge d\bar{z}_K$$

and

$$e_j^* e_k \big(dz_j \wedge dz_J \wedge d\bar{z}_K \big) = e_j^* \big(dz_k \wedge dz_j \wedge dz_J \wedge d\bar{z}_K \big) = -2 dz_k \wedge dz_J \wedge d\bar{z}_K,$$

and the combination of the two proves the asserted identity. □

We also define the differential operator

$$\partial_j \colon A^{p,q} \to A^{p,q}, \quad \sum_{J,K} \varphi_{J,K} dz_J \wedge d\bar{z}_K \mapsto \sum_{J,K} \frac{\partial \varphi_{J,K}}{\partial z_j} dz_J \wedge d\bar{z}_K$$

and its conjugate

$$\bar{\partial}_j \colon A^{p,q} \to A^{p,q}, \quad \sum_{J,K} \varphi_{J,K} dz_J \wedge d\bar{z}_K \mapsto \sum_{J,K} \frac{\partial \varphi_{J,K}}{\partial \bar{z}_j} dz_J \wedge d\bar{z}_K.$$

Clearly, both commute with the operators e_j and e_j^*, as well as with each other. As before, let ∂_j^* be the adjoint of ∂_j, and $\bar{\partial}_j^*$ that of $\bar{\partial}_j$, and integration by parts proves the following lemma.

Lemma 22.2. *We have* $\partial_j^* = -\bar{\partial}_j$ *and* $\bar{\partial}_j^* = -\partial_j$.

We now turn to the proof of the crucial identity $[\Lambda, \partial] = i\bar{\partial}^*$.

Proof. All the operators in the identity can be expressed in terms of the basic ones, as follows. Firstly, $L = \frac{i}{2} \sum e_j \bar{e}_j$, and so the adjoint is given by the formula $\Lambda = -\frac{i}{2} \sum \bar{e}_j^* e_j^*$. Quite evidently, we have $\partial = \sum \partial_j e_j$ and $\bar{\partial} = \sum \bar{\partial}_j \bar{e}_j$, and after taking adjoints, we find that $\partial^* = -\sum \bar{\partial}_j e_j^*$ and that $\bar{\partial}^* = -\sum \partial_j \bar{e}_j^*$. Using these expressions, we compute that

$$\Lambda\partial - \partial\Lambda = -\frac{i}{2} \sum_{j,k} \Big(\bar{e}_j^* e_j^* \partial_k e_k - \partial_k e_k \bar{e}_j^* e_j^* \Big) = -\frac{i}{2} \sum_{j,k} \partial_k \Big(\bar{e}_j^* e_j^* e_k - e_k \bar{e}_j^* e_j^* \Big).$$

Now $\bar{e}_j^* e_j^* e_k - e_k \bar{e}_j^* e_j^* = \bar{e}_j^* (e_j^* e_k + e_k e_j^*)$, which equals $2\bar{e}_j^*$ in case $j = k$, and is zero otherwise. We conclude that

$$\Lambda\partial - \partial\Lambda = -i \sum_j \partial_j \bar{e}_j^* = i\bar{\partial}^*,$$

which is the Kähler identity we were after. □

Consequences. The Kähler identities lead to many wonderful relations between the different operators that we have introduced; here we shall give the three most important ones.

Corollary 22.3. *On a Kähler manifold, the various Laplace operators are related to each other by the formula* $\overline{\square} = \square = \frac{1}{2}\Delta$.

Proof. By definition,

$$\Delta = dd^* + d^*d = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}).$$

According to the Kähler identities in Theorem 21.3, we have $\bar{\partial}^* = i\partial\Lambda - i\Lambda\partial$, and therefore

$$\begin{aligned}\Delta &= (\partial + \bar{\partial})(\partial^* - i\Lambda\partial + i\partial\Lambda) + (\partial^* - i\Lambda\partial + i\partial\Lambda)(\partial + \bar{\partial}) \\ &= \partial\partial^* + \bar{\partial}\partial^* - i\bar{\partial}\Lambda\partial + i\bar{\partial}\partial\Lambda + \partial^*\partial + \partial^*\bar{\partial} - i\Lambda\partial\bar{\partial} + i\partial\Lambda\bar{\partial}.\end{aligned}$$

Now $\partial^*\bar{\partial} = i[\Lambda, \bar{\partial}]\bar{\partial} = -i\big(\Lambda\bar{\partial} - \bar{\partial}\Lambda\big)\bar{\partial} = i\bar{\partial}\Lambda\bar{\partial} = -\partial^*\bar{\partial}$ by the other Kähler identity. The above formula consequently therefore simplifies to

$$\begin{aligned}\Delta &= \square - i\bar{\partial}\Lambda\partial + i\bar{\partial}\partial\Lambda - i\Lambda\partial\bar{\partial} + i\partial\Lambda\bar{\partial} = \square - i\bar{\partial}\Lambda\partial - i\partial\bar{\partial}\Lambda + i\Lambda\bar{\partial}\partial + i\partial\Lambda\bar{\partial} \\ &= \square + i\partial(\Lambda\bar{\partial} - \bar{\partial}\Lambda) + i(\Lambda\bar{\partial} - \bar{\partial}\Lambda)\partial = \square + \partial\partial^* + \partial^*\partial = 2\square.\end{aligned}$$

The other formula, $\Delta = 2\overline{\square}$, follows from this by conjugation. ∎

Corollary 22.4. *On a Kähler manifold, the Laplace operator* Δ *commutes with the operators* $*$, L, *and* Λ, *and satisfies* $\Delta A^{p,q}(M) \subseteq A^{p,q}(M)$. *In particular,* $*$, L, *and* Λ *preserve harmonic forms.*

Proof. By taking adjoints, we obtain from the second identity in Theorem 21.3 that

$$-i\bar{\partial} = (i\bar{\partial}^*)^* = [\Lambda, \partial]^* = [\partial^*, L] = \partial^* L - L\partial^*.$$

Using the resulting formula $L\partial^* = \partial^* L + i\bar{\partial}$, we compute that

$$\begin{aligned}L\square &= L\partial\partial^* + L\partial^*\partial = \partial L\partial^* + \partial^* L\partial + i\bar{\partial}\partial \\ &= \partial\partial^* L + i\partial\bar{\partial} + \partial^*\partial L + i\bar{\partial}\partial = \partial\partial^* L + \partial^*\partial L = \square L.\end{aligned}$$

Therefore $[\Delta, L] = 2[\square, L] = 0$; after taking adjoints, we also have $[\Lambda, \Delta] = 0$. That Δ commutes with $*$ was shown in the exercises; finally, $\Delta = 2\overline{\square}$, and the latter clearly preserves the space $A^{p,q}(M)$. ∎

A nice consequence is that the Kähler form ω, which is naturally defined by the metric, is a harmonic form. Note that this is equivalent to the Kähler condition, since harmonic forms are always closed.

Corollary 22.5. *On a Kähler manifold, the Kähler form* ω *is harmonic.*

Proof. The constant function 1 is clearly harmonic; since $\omega = L(1)$, and since the operator L preserves harmonic functions, it follows that ω is harmonic. ∎

Class 23. The Hodge decomposition

Let M be a compact Kähler manifold, with Kähler form ω. We have seen in Corollary 22.3 that $\Delta = 2\bar{\Box}$; this implies that the Laplace operator Δ preserves the type of a form, and that a form is harmonic if and only if it is $\bar{\partial}$-harmonic. In particular, it follows that if a form $\alpha \in A^k(M)$ is harmonic, then its components $\alpha^{p,q} \in A^{p,q}(M)$ are also harmonic. Indeed, we have

$$0 = \Delta\alpha = \sum_{p+q=k} \Delta\alpha^{p,q},$$

and since each $\Delta\alpha^{p,q}$ belongs again to $A^{p,q}(M)$, we see that $\Delta\alpha^{p,q} = 0$.

Corollary 23.1. *On a compact Kähler manifold M, the space of harmonic forms decomposes by type as*

$$\mathcal{H}^k(M) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M),$$

where $\mathcal{H}^{p,q}(M)$ is the space of (p,q)-forms that are $\bar{\partial}$-harmonic (and hence also harmonic).

Since we know that each cohomology class contains a unique harmonic representative, we now obtain the famous *Hodge decomposition* of the de Rham cohomology of a compact Kähler manifold. We state it in a way that is independent of the particula Kähler metric.

Theorem 23.2. *Let M be a compact Kähler manifold. Then the de Rham cohomology with complex coefficients admits a direct sum decomposition*

$$H^k(M,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}, \tag{23.3}$$

with $H^{p,q}$ equal to the subset of those cohomology classes that contain a d-closed form of type (p,q). We have $H^{q,p} = \overline{H^{p,q}}$, where complex conjugation is with respect to the real structure on $H^k(M,\mathbb{C}) \simeq H^k(M,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$; moreover, $H^{p,q}$ is isomorphic to the Dolbeault cohomology group $H^{p,q}(M) \simeq H^q(M,\Omega_M^p)$.

Proof. Since M is a Kähler manifold, it admits a Kähler metric h, and we can consider forms that are harmonic for this metric. By Theorem 18.7, every class in $H^k(M,\mathbb{C})$ contains a unique complex-valued harmonic form α. Since $\alpha = \sum_{p+q=k} \alpha^{p,q}$, with each $\alpha^{p,q}$ harmonic and hence in $H^{p,q}$, we obtain the asserted decomposition. Note that by its very description, the decomposition does not depend on the choice of Kähler metric. Since the conjugate of a (p,q)-form is a (q,p)-form, it is clear that $\overline{H^{p,q}} = H^{q,p}$. Finally, every harmonic form is automatically $\bar{\partial}$-harmonic, and so we have $H^{p,q} \simeq \mathcal{H}^{p,q}(M) \simeq H^{p,q}(M)$ by Theorem 19.7. □

Recall the definition of the sheaf Ω_M^p holomorphic p-forms: its sections are smooth $(p,0)$-forms that can be expressed in local coordinates as

$$\alpha = \sum_{j_1 < \cdots < j_k} f_{j_1,\dots,j_k} dz_{j_1} \wedge \cdots \wedge dz_{j_k},$$

with locally defined holomorphic functions $f_{j_1,\dots,j_k}$. This expression shows that $\bar{\partial}\alpha = 0$. A useful (and surprising) fact is that on a compact Kähler manifold, any global holomorphic p-form is harmonic, and hence satisfies $d\alpha = 0$.

Corollary 23.4. *On a compact Kähler manifold M, every holomorphic form is harmonic, and so we get an embedding $H^0(M, \Omega_M^p) \hookrightarrow H^p(M, \mathbb{C})$ whose image is precisely the space $H^{p,0}$.*

Proof. If $\alpha \in A^{p,0}(M)$ is holomorphic, it satisfies $\bar{\partial}\alpha = 0$; on the other hand, $\bar{\partial}^*\alpha = 0$ since it would belong to the space $A^{p,-1}(M)$. Thus α is $\bar{\partial}$-harmonic, and hence also harmonic. □

The decomposition of the cohomology groups of M can be represented by the following picture, often called the *Hodge diamond* due to its shape.

$$
\begin{array}{ccccccc}
 & & & H^{n,n} & & & \\
 & & H^{n,n-1} & & H^{n-1,n} & & \\
 & H^{n,n-2} & & H^{n-1,n-1} & & H^{n-2,n} & \\
 \iddots & & & & & & \ddots \\
H^{n,0} & H^{n-1,1} & & \cdots\cdots\cdots & & H^{1,n-1} & H^{0,n} \\
 \ddots & & & & & & \iddots \\
 & H^{2,0} & & H^{1,1} & & H^{0,2} & \\
 & & H^{1,0} & & H^{0,1} & & \\
 & & & H^{0,0} & & &
\end{array}
$$

It has several symmetries: On the one hand, we have $H^{q,p} = \overline{H^{p,q}}$; on the other hand, the $*$-operator induces an isomorphism between $H^{p,q}$ and $H^{n-q,n-p}$.

Example 23.5. Let M be a compact Riemannan surface. Then any Hermitian metric h on M is Kähler, and so we get the decomposition

$$H^1(M, \mathbb{C}) = H^{1,0} \oplus H^{0,1},$$

with $H^{1,0} \simeq H^0(M, \Omega_M^1)$ and $H^{0,1} \simeq H^1(M, \mathscr{O}_M)$. In particular, the dimension is $\dim H^1(M, \mathbb{R}) = 2g$, where $g = \dim_{\mathbb{C}} H^0(M, \Omega_M^1)$ is the *genus*. This means that the genus is a topological invariant of M, a fact that should be familiar from the theory of Riemann surfaces.

Example 23.6. Let us consider the case of a compact connected Kähler manifold of dimension two (so $n = 2$). In that case, the Hodge diamond looks like this:

$$
\begin{array}{ccccc}
 & & H^{2,2} & & \\
 & H^{2,1} & & H^{1,2} & \\
H^{2,0} & & H^{1,1} & & H^{0,2} \\
 & H^{1,0} & & H^{0,1} & \\
 & & H^{0,0} & &
\end{array}
$$

If we let $h^{p,q} = \dim H^{p,q}(M)$, then $h^{0,0} = h^{2,2} = 1$ since M is connected. Moreover, the two symmetries mentioned above show that $h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2}$ and that

$h^{2,0} = h^{0,2}$. We also have $h^{1,1} \geq 1$, since the class of the Kähler form ω is a nonzero element of $H^{1,1}$.

Consequences of the Hodge decomposition. The Hodge decomposition theorem shows that compact Kähler manifold have various topological properties not shared by arbitrary complex manifolds.

Corollary 23.7. *On a compact Kähler manifold, the odd Betti numbers* $b_{2k+1} = \dim H^{2k+1}(M,\mathbb{R})$ *are always even.*

Proof. Indeed, $b_{2k+1} = \dim_{\mathbb{C}} H^{2k+1}(M,\mathbb{C})$. The latter decomposes as

$$H^{2k+1}(M,\mathbb{C}) = \bigoplus_{p+q=2k+1} H^{p,q},$$

and since $\dim_{\mathbb{C}} H^{p,q} = \dim_{\mathbb{C}} H^{q,p}$, we get the assertion. $\square$

Corollary 23.8. *On a compact Kähler manifold, the even Betti numbers* b_{2k} *are always nonzero.*

Proof. Since the operator $L = \omega \wedge (-)$ preserves harmonic forms, each $\omega^{\wedge k} = L^k(1)$ is harmonic; moreover, it is not zero because of Wirtinger's formula $\mathrm{vol}(M) = \frac{1}{n!}\int_M \omega^{\wedge n}$. Its cohomology class gives a nonzero element in $H^{2k}(M,\mathbb{R})$. $\square$

Another property of compact Kähler manifolds that is used very often in complex geometry is the following $\partial\bar{\partial}$-Lemma.

Proposition 23.9. *Let* M *be a compact Kähler manifold, and let* ϕ *be a smooth form that is both* ∂*-closed and* $\bar{\partial}$*-closed. If* ϕ *is also either* ∂*-exact or* $\bar{\partial}$*-exact, then it can be written as* $\phi = \partial\bar{\partial}\psi$.

Proof. We shall suppose that $\phi = \bar{\partial}\alpha$. Let $\alpha = \beta + \Delta\gamma$ be the decomposition given by (18.6), with β harmonic. We then have $2\bar{\square}\beta = \Delta\beta = 0$, and therefore $\bar{\partial}\beta = 0$. Using the previously mentioned identity $\bar{\partial}\partial^* = -\partial^*\bar{\partial}$, we compute that

$$\phi = \bar{\partial}\alpha = \bar{\partial}(2\square)\gamma = 2\bar{\partial}(\partial\partial^* + \partial^*\partial)\gamma = -2\partial\bar{\partial}(\partial^*\gamma) - 2\partial^*\bar{\partial}\partial\gamma.$$

Now $\partial\phi = 0$, and so the form $\partial^*\bar{\partial}\partial\gamma$ belongs to $\ker\partial \cap \operatorname{im}\partial^* = \{0\}$. Consequently, we have $\omega = \partial\bar{\partial}\psi$ with $\psi = -2\partial^*\gamma$. $\square$

Class 24. Examples of Kähler manifolds

We shall now look at several examples of Kähler and non-Kähler manifolds, and compute the Hodge decomposition in a few important examples.

The Hopf surface. The Hodge decomposition shows that compact Kähler manifolds are special (in their topological or cohomological properties), when compared to arbitrary compact complex manifolds. In this section, we construct an example of a compact complex manifold, the so-called *Hopf surface*, that admits no Kähler metric. Let $\mathbb{S}^3$ be the three-sphere in $\mathbb{C}^2$, defined as the set of points (z_1, z_2) such that $|z_1|^2 + |z_2|^2 = 1$. There is a diffeomorphism

$$\varphi\colon \mathbb{S}^3 \times \mathbb{R} \to \mathbb{C}^2 \setminus \{0\}, \quad \varphi(z_1, z_2, t) = \bigl(e^t z_1, e^t z_2\bigr).$$

The infinite cyclic group $\mathbb{Z}$ naturally acts on $\mathbb{S}^3 \times \mathbb{R}$, by letting

$$m \cdot (z_1, z_2, t) = (z_1, z_2, t + m)$$

for $m \in \mathbb{Z}$; since $\mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$, the quotient under this action is obviously isomorphic to the product $\mathbb{S}^3 \times \mathbb{S}^1$. The diffeomorphism φ allows us to transfer the action of $\mathbb{Z}$ on $\mathbb{S}^3 \times \mathbb{R}$ to an action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$. Explicitly, it is given by the formula

$$m \cdot (z_1, z_2) = \bigl(e^m z_1, e^m z_2\bigr).$$

The formula shows that $\mathbb{Z}$ acts by biholomorphisms; moreover, the action is clearly properly discontinuous and without fixed points. By Proposition 6.2, the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the action of $\mathbb{Z}$ is a complex manifold M. By construction, it is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$, and hence compact.

With the help of the Künneth formula from algebraic topology, we can compute the cohomology of the product $\mathbb{S}^3 \times \mathbb{S}^1$, and hence that of M. The result is that

$$b_0 = b_1 = b_3 = b_4 = 1, \quad b_2 = 0,$$

where $b_k = \dim H^k(M, \mathbb{R})$. It follows that M cannot possibly admit a Kähler metric, because ω would then define a nonzero class in $H^2(M, \mathbb{R})$, contradicting the fact that $b_2 = 0$. (Moreover, b_1 and b_3 are not even numbers.)

Complex projective space. An important example of a compact Kähler manifold is complex projective space $\mathbb{P}^n$. Its cohomology is easy to compute, using some results from algebraic topology

Lemma 24.1. *The cohomology groups of complex projective space are*

$$H^k(\mathbb{P}^n, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \textit{for } 0 \leq k \leq 2n \textit{ even,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. To save space, we omit the coefficients from cohomology groups. We prove the assertion by induction on $n \geq 0$, the case $n = 0$ being trivial (since $\mathbb{P}^0$ is a single point). Let $[z_0, z_1, \dots, z_n]$ be homogeneous coordinates on $\mathbb{P}^n$, and define $Z \subseteq \mathbb{P}^n$ as the set of points with $z_n = 0$. Clearly $Z \simeq \mathbb{P}^{n-1}$, and the complement $\mathbb{P}^n \setminus Z$ is isomorphic to $\mathbb{C}^n$, whose homology groups in positive degrees are zero. The Poincaré duality isomorphism

$$H^k(\mathbb{P}^n, Z) \simeq H_{2n-k}(\mathbb{P}^n \setminus Z) \simeq H_{2n-k}(\mathbb{C}^n)$$

now shows that $H^k(\mathbb{P}^n, Z) \simeq 0$ for $k < 2n$, while $H^{2n}(\mathbb{P}^n, Z) \simeq \mathbb{Z}$. We can then use the long exact cohomology sequence for the pair $(\mathbb{P}^n, Z)$,

$$\cdots \to H^k(\mathbb{P}^n, Z) \to H^k(\mathbb{P}^n) \to H^k(Z) \to H^{k+1}(\mathbb{P}^n, Z) \to \cdots$$

to conclude that the restriction map $H^k(\mathbb{P}^n) \to H^k(Z)$ is an isomorphism for $k \leq 2n-2$, and that $H^{2n-1}(\mathbb{P}^n) \simeq 0$. Likewise, we have

$$\cdots \to H^{2n-1}(Z) \to H^{2n}(\mathbb{P}^n, Z) \to H^{2n}(\mathbb{P}^n) \to H^{2n}(Z) \to \cdots,$$

and the terms at both ends are zero since $2n - 1 > 2 \dim Z = 2n - 2$. $\square$

Recall that the Fubini-Study metric on $\mathbb{P}^n$ is Kähler, with Kähler form ω_{FS}. We have already seen that each $L^k(1) = \omega_{FS}^{\wedge k}$ is harmonic and gives a nonzero class in $H^{2k}(\mathbb{P}^n, \mathbb{R})$. Since this class is clearly of type (k, k), we conclude that

$$H^{p,q}(\mathbb{P}^n) \simeq \begin{cases} \mathbb{C} & \text{for } 0 \leq p = q \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the Hodge diamond of $\mathbb{P}^n$ has the following shape:

$$
\begin{array}{ccccccccc}
 & & & & \mathbb{C} & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & \mathbb{C} & & 0 & & \\
 & & & & \vdots & & & & \\
0 & 0 & & & & & & 0 & 0 \\
 & & & & \vdots & & & & \\
 & & 0 & & \mathbb{C} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & \mathbb{C} & & & &
\end{array}
$$

Complex tori. Another useful class of example are complex tori. Recall that a complex torus is a quotient of $\mathbb{C}^n$ by a lattice Λ, that is, a discrete subgroup isomorphic to $\mathbb{Z}^{2n}$. We have seen that $T = \mathbb{C}^n/\Lambda$ is a compact complex manifold (since the action of Λ by translation is properly discontinuous and without fixed points). The quotient map $\pi\colon \mathbb{C}^n \to T$ is locally biholomorphic, and so we can use small open subsets of $\mathbb{C}^n$ as coordinate charts on T. With this choice of coordinates, it is easy to see that the pullback map $\pi^*\colon A^{p,q}(T) \to A^{p,q}(\mathbb{C}^n)$ is injective and identifies $A^{p,q}(T)$ with the space of smooth (p,q)-forms on $\mathbb{C}^n$ that are invariant under translation by elements of Λ.

In fact, T has a natural Kähler metric: On $\mathbb{C}^n$, we have the Euclidean metric with Kähler form $\frac{i}{2}\sum dz_j \wedge d\bar{z}_j$, where $z_1, \dots, z_n$ are the coordinate functions on $\mathbb{C}^n$. This metric is invariant under translations, and thus descends to a Hermitian metric h on T. Let ω be the associated $(1,1)$-form; since $q^*\omega = \frac{i}{2}\sum dz_j \wedge d\bar{z}_j$, it is clear that $d\omega = 0$, and so h is a Kähler metric.

Lemma 24.2. *The Laplace operator for this metric is given by the formula*

$$\Delta\Bigl(\sum \varphi_{J,K}\, dz_J \wedge d\bar{z}_K\Bigr) = \sum \Delta\varphi_{J,K}\cdot dz_J \wedge d\bar{z}_K,$$

where $\Delta\varphi = -\sum_{j=1}^n \bigl(\partial^2\varphi/\partial x_j^2 + \partial^2\varphi/\partial y_j^2\bigr)$ *is the ordinary Laplacian on smooth functions.*

Proof. The injectivity of $\pi^*\colon A^{p,q}(T) \to A^{p,q}(\mathbb{C}^n)$ allows us to do the calculation on $\mathbb{C}^n$, where the metric is the standard one. In the notation introduced during the proof of Theorem 21.3, we have

$$\Delta = 2\overline{\Box} = 2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) = 2\sum_{j,k=1}^n \bigl(\bar{\partial}_j \bar{e}_j \bar{e}_k^* \bar{\partial}_k^* + \bar{e}_k^* \bar{\partial}_k^* \bar{\partial}_j \bar{e}_j\bigr).$$

Now $\bar{\partial}_k^* = -\partial_k$, and so the summation simplifies to

$$-\sum_{j,k=1}^n \bigl(\bar{\partial}_j \bar{e}_j \bar{e}_k^* \partial_k + \bar{e}_k^* \partial_k \bar{\partial}_j \bar{e}_j\bigr) = -\sum_{j,k=1}^n \partial_k \bar{\partial}_j \bigl(\bar{e}_j \bar{e}_k^* + \bar{e}_k^* \bar{e}_j\bigr) = -2\sum_{j=1}^n \partial_j \bar{\partial}_j.$$

This means that we have

$$\Delta\left(\sum_{J,K} \varphi_{J,K}\, dz_J \wedge d\bar{z}_K\right) = -4\sum_{J,K}\sum_{j=1}^n \frac{\partial^2 \varphi_{J,K}}{\partial z_j \partial \bar{z}_j} dz_J \wedge d\bar{z}_K,$$

which gives the asserted formula because $4\partial^2\varphi/\partial z_j \partial \bar{z}_j = \partial^2\varphi/\partial x_j^2 + \partial^2\varphi/\partial y_j^2$. $\square$

The lemma shows that the space $\mathcal{H}^0(T)$ of real-valued smooth functions on T that are harmonic for the metric h can be identified with the space of harmonic functions on $\mathbb{C}^n$ that are Λ-periodic. Since T is compact, we know that $\mathcal{H}^0(T) \simeq H^0(T,\mathbb{R}) \simeq \mathbb{R}$, and so any such function is constant. This means that all harmonic forms of type (p,q) on T can be described as

$$\sum_{|J|=p} \sum_{|K|=q} a_{J,K}\, dz_J \wedge d\bar{z}_K, \tag{24.3}$$

with constants $a_{J,K} \in \mathbb{C}$. Thus if we let $V_{\mathbb{R}} = H^1(T,\mathbb{R})$, then $H^1(T,\mathbb{C}) = V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, with $V^{1,0}$ generated by $dz_1, \dots, dz_n$, and $V^{0,1}$ by their conjugates. Since any harmonic form as in (24.3) is a wedge product of forms in $V_{\mathbb{C}}$, it follows from the Hodge theorem that we have

$$H^k(T,\mathbb{C}) \simeq \bigwedge^k V_{\mathbb{C}},$$

and under this isomorphism, the Hodge decomposition of T is nothing but the abstract decomposition

$$\bigwedge^k V = \bigoplus_{p+q=k} V^{p,q}$$

into the subspaces $V^{p,q} = \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1}$. A basis for the space $V^{p,q}$ is given by the forms $dz_J \wedge d\bar{z}_K$ with $|J| = p$ and $|K| = q$. Note that we have $\dim V^{1,0} = \dim V^{0,1} = n$, and hence

$$h^{p,q} = \dim V^{p,q} = \binom{n}{p}\binom{n}{q}.$$

Example 24.4. Let T be a three-dimensional complex torus. Then the Hodge diamond of T has the following shape:

$$\begin{array}{ccccccc}
 & & & \mathbb{C} & & & \\
 & & \mathbb{C}^3 & & \mathbb{C}^3 & & \\
 & \mathbb{C}^3 & & \mathbb{C}^9 & & \mathbb{C}^3 & \\
\mathbb{C} & & \mathbb{C}^{10} & & \mathbb{C}^{10} & & \mathbb{C} \\
 & \mathbb{C}^3 & & \mathbb{C}^9 & & \mathbb{C}^3 & \\
 & & \mathbb{C}^3 & & \mathbb{C}^3 & & \\
 & & & \mathbb{C} & & &
\end{array}$$

Class 25. Hypersurfaces in projective space

As a more involved (and more useful) example, we shall describe how to compute the Hodge numbers of a hypersurface in projective space. As usual, let $[z_0, z_1, \dots, z_{n+1}]$ denote the homogeneous coordinates on $\mathbb{P}^{n+1}$. Then any homogeneous polynomial $F \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ defines an analytic subset $Z(F)$, consisting of all points where $F(z) = 0$. (Different polynomials can define the same analytic set; but if we assume that F is not divisible by the square of any nonunit, then the zero set uniquely determines F by the Nullstellensatz from algebraic geometry.) If for every $z \neq 0$, at least one of the partial derivatives $\partial F/\partial z_j$ is nonzero, then $Z(F)$ is a complex submanifold of $\mathbb{P}^{n+1}$ of dimension n by the implicit mapping theorem (stated above as Theorem 8.6).

Note. We will show later that, in fact, any complex submanifold of projective space is defined by polynomial equations; moreover, if $M \subseteq \mathbb{P}^{n+1}$ has dimension n, then $M = Z(F)$ for a homogeneous polynomial $F \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$.

From now on, we fix $F \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ with the above properties, and let $M = Z(F)$ be the corresponding submanifold of $\mathbb{P}^{n+1}$. We also let $d = \deg F$ be the degree of the hypersurface. As usual, we give M the Kähler metric induced from the Fubini-Study metric on $\mathbb{P}^{n+1}$; then ω is the restriction of ω_{FS}. Since we know that the cohomology of $\mathbb{P}^{n+1}$ is generated by powers of ω_{FS}, and since the powers of ω define nonzero cohomology classes on M, we get that the restriction map

$$H^k(\mathbb{P}^{n+1}, \mathbb{C}) \to H^k(M, \mathbb{C})$$

is injective for $0 \leq k \leq 2n$. Now it is a fact (which we might prove later on) that the map is an isomorphism for $0 \leq k < n$. This result is known as the *Lefschetz hyperplane section theorem*; it implies that the cohomology of M is isomorphic to that of projective space in all degrees except $k = n$. In the remaining case, we have

$$H^n(M, \mathbb{C}) = H^n(\mathbb{P}^{n+1}, \mathbb{C}) \oplus H_0^n(M, \mathbb{C}),$$

where $H_0^n(M, \mathbb{C})$ is the so-called *primitive cohomology* of the hypersurface M. Note that the first summand, $H^n(\mathbb{P}^{n+1}, \mathbb{C})$, will be either one-dimensional (if n is even), or zero (if n is odd).

Griffiths' formula. The Hodge decomposition theorem shows that we have

$$H_0^n(M, \mathbb{C}) = H_0^{n,0} \oplus H_0^{n-1,1} \oplus \cdots \oplus H_0^{0,n},$$

and a pretty result by P. Griffiths makes it possible to compute the dimensions of the various summands.

Theorem 25.1 (Griffiths). *Let $M \subseteq \mathbb{P}^{n+1}$ be a complex submanifold of dimension n, defined by a homogeneous polynomial $F \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ of degree d. Then*

$$H_0^{p,n-p} \simeq \frac{A^{n+1}(M, n+1-p)}{A^{n+1}(M, n-p) + dA^n(M, n-p)}, \tag{25.2}$$

where $A^k(M, \ell)$ denotes the space of rational k-forms on $\mathbb{P}^{n+1}$ with a pole of order at most ℓ along the hypersurface M, and d is the exterior derivative.

To explain Griffiths' formula, we recall that a rational $(n+1)$-form on $\mathbb{C}^{n+1}$ is an expression

$$\frac{A(z_1, \dots, z_{n+1})}{B(z_1, \dots, z_{n+1})} dz_1 \wedge \cdots \wedge dz_{n+1},$$

where $A, B \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ are polynomials, with B not identically zero. On the set of points where $B \neq 0$, this defines a holomorphic differential form, but there may be poles along the zero set of B. If we homogenize the expression (by replacing z_j with z_j/z_0 and multiplying through by a power of z_0), we see that rational $(n+1)$-forms on $\mathbb{P}^{n+1}$ can be described as

$$\frac{P(z_0, z_1, \dots, z_{n+1})}{Q(z_0, z_1, \dots, z_{n+1})} \Omega;$$

here Ω is given by the formula

$$\Omega = \sum_{j=0}^{n+1} (-1)^j z_j dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1},$$

and $P, Q \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ are homogeneous polynomials with $\deg P + (n+2) = \deg Q$. If the rational form has a pole of order at most ℓ along the hypersurface M, and no other poles, then we must have $Q = F^\ell$, and so $\deg P = \ell d - (n+2)$.

Likewise, one can prove by homogenizing rational n-forms on $\mathbb{C}^{n+1}$ that any rational n-form on $\mathbb{P}^{n+1}$ with a pole of order at most ℓ along M can be put into the form

$$\alpha = \sum_{0 \leq j < k \leq n+1} (-1)^{j+k} \frac{z_k P_j - z_j P_k}{F^\ell} dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1},$$

for homogeneous polynomials $P_j \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ of degree $\deg P_j = \ell d - (n+1)$. A short computation shows that we have

$$d\alpha = \frac{F \sum_j \frac{\partial P_j}{\partial z_j} - \ell \sum_j P_j \frac{\partial F}{\partial z_j}}{F^{\ell+1}} \Omega. \tag{25.3}$$

Returning to Griffiths' formula (25.2), every rational $(n+1)$-form with a pole of order at most $(n+1-p)$ along M can thus be written as $P\Omega/F^{n+1-p}$, with $P \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ homogeneous of degree $(n+1-p)d - (n+2)$. Writing S for the polynomial ring, and S_ℓ for the space of homogeneous polynomials of degree ℓ, we can say that

$$A^{n+1}(M, n+1-p) \simeq S_{(n+1-p)d-(n+2)},$$

by identifying the rational form $P\Omega/F^{n+1-p}$ with the homogeneous polynomial P. The formula in (25.3) shows that we have

$$A^{n+1}(M, n-p) + dA^n(M, n-p) \simeq \sum_{j=0}^{n+1} S_{(n-p)d-(n+1)} \frac{\partial F}{\partial z_j} + S_{(n-p)d-(n+2)} F.$$

The *Jacobian ideal* of the hypersurface M is the homogeneous ideal $J(F) \subseteq S$ generated by the partial derivatives of F,

$$J(F) = S\left(\frac{\partial F}{\partial z_0}, \frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_{n+1}}\right).$$

Recall that we always have $F \in J(F)$; this follows from the identity

$$G = \frac{1}{\deg G} \sum_{j=0}^{n+1} z_j \frac{\partial G}{\partial z_j}$$

for homogeneous polynomials G. With the help of the graded ring $R(F) = S/J(F)$, we can now restate Griffiths' formula for the Hodge decomposition of the primitive cohomology groups of M as follows: Suppose that $F \in \mathbb{C}[z_0, z_1, \dots, z_{n+1}]$ is an irreducible homogeneous polynomial of degree d, whose zero set $M = Z(F)$ is a submanifold of $\mathbb{P}^{n+1}$. Then the summands of the Hodge decomposition of the primitive cohomology of M can be described as

$$H_0^{p,n-p}(M) \simeq R(F)_{(n+1-p)d-(n+2)}. \tag{25.4}$$

To see this formula in action, let us compute a few examples:

Example 25.5. Let $M \subseteq \mathbb{P}^2$ be a smooth plane curve of degree d, defined by a homogeneous equation $F \in \mathbb{C}[z_0, z_1, z_2]$ with $\deg F = d$. Since $H^1(\mathbb{P}^2, \mathbb{C}) = 0$, we have $H_0^1(M) \simeq H^1(M, \mathbb{C})$ in this case. Griffiths' formula (with $n = 1$) says that

$$H^{1,0}(M) \simeq R(F)_{d-3} \simeq S_{d-3},$$

and so we find that the genus of the Riemann surface M is given by the formula

$$g = h^{1,0}(M) = \dim S_{d-3} = \binom{d-1}{2}.$$

So for instance, smooth plane cubic curves always have genus one.

Example 25.6. Let us look at the case of a *K3-surface*, namely a complex submanifold $M \subseteq \mathbb{P}^3$ defined by a homegeneous equation $F \in \mathbb{C}[z_0, z_1, z_2, z_3]$ of degree $d = 4$. Here we have $H^1(M, \mathbb{C}) \simeq H^1(\mathbb{P}^3, \mathbb{C}) = 0$, and $H^2(M, \mathbb{C}) = H^2(\mathbb{P}^3, \mathbb{C}) \oplus H_0^2(M, \mathbb{C})$. To compute the Hodge decomposition on the primitive part of the cohomology, we apply Griffiths' formula (with $n = 2$ and $d = 4$). Firstly,

$$H_0^{2,0}(M) \simeq R(F)_0 \simeq S_0,$$

and therefore $h^{2,0}(M) = 1$. Secondly,

$$H_0^{1,1}(M) \simeq R(F)_4 = S_4 \Big/ \sum_{j=1}^{3} S_1 \frac{\partial F}{\partial z_j},$$

and counting dimension, we find that

$$h^{1,1}(M) = 1 + \dim S_4 - 4 \dim S_1 = 1 + \binom{4+3}{3} - 4 \cdot \binom{1+3}{3} = 20.$$

Because we know from general principles that $h^{0,2}(M) = h^{2,0}(M)$, those two numbers suffice to write down the Hodge diamond of M, which looks like

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & 0 & & 0 & \\ \mathbb{C} & & \mathbb{C}^{20} & & \mathbb{C} \\ & 0 & & 0 & \\ & & \mathbb{C} & & \end{array}$$

Note. If the polynomial F is complicated, counting the dimension of the space $R(F)_{(n+1-p)d-(n+2)}$ may be fairly involved. Luckily, there is a shortcut: One can prove that the dimensions are the same for any irreducible homogeneous polynomial F of degree d (whose zero set is a submanifold), and so it suffices to do the computations in the easy case $F = z_0^d + z_1^d + \cdots + z_{n+1}^d$. The reason is that the space of all such polynomials (as an open subset of a complex vector space) is connected, and that the Hodge numbers $\dim H_0^{p,n-p}$ are continuous functions on that space.

Residues. The proof of Theorem 25.1 requires several results from algebraic geometry and algebraic topology that we do not have at our disposal; but we can at least describe the so-called *residue map*

$$A^{n+1}(M, n+1-p) \to H_0^{p,n-p}$$

that induces the isomorphism. Recall the notion of a residue from complex analysis: given a meromorphic function $f(z)$ on an open set U, holomorphic on $U \setminus \{z_0\}$, we write $f(z) = \sum_{j \in \mathbb{Z}} a_j (z - z_0)^j$ as a Laurent series, and then

$$\operatorname{Res}_{z_0} f = \frac{1}{2\pi i} \int_{|z-z_0|=\varepsilon} f(z) dz = a_{-1}.$$

Put differently, the residue map assigns to a meromorphic one-form $f(z)dz$ a complex number at each point where the form has a pole.

The same construction works for $M \subseteq \mathbb{P}^{n+1}$, and explains the case $p = n$ from above: there is a map Res_M from rational $(n+1)$-forms on $\mathbb{P}^{n+1}$ with a first-order pole along M to the space of holomorphic n-forms on M. Namely, around each point of the submanifold M, we can find local coordinates $t_1, \dots, t_{n+1}$ on an open neighborhood $U \subseteq \mathbb{P}^{n+1}$ such that $M \cap U$ is the subset defined by $t_1 = 0$. Given $\alpha \in A^{n+1}(M, 1)$, we then have

$$\alpha|_U = \frac{f(t_1, t_2, \dots, t_{n+1})}{t_1} dt_1 \wedge \cdots \wedge dt_{n+1}$$

for some holomorphic function f, and then the residue of $\alpha|_U$ is the holomorphic n-form $f(0, t_2, \dots, t_{n+1}) dt_2 \wedge \cdots \wedge dt_{n+2}$ on $M \cap U$. One can show that this does not depend on the choice of coordinates, and thus defines a global holomorphic n-form $\operatorname{Res}_M \alpha$ on M.

On forms with a pole of higher order, an additional step is needed. Suppose that $\alpha \in A^{n+1}(M, \ell)$ has a pole of order at most $\ell \geq 2$. In local coordinates, we can again express α in the form

$$\alpha|_U = \frac{f(t_1, t_2, \dots, t_{n+1})}{t_1^\ell} dt_1 \wedge \cdots \wedge dt_{n+1}.$$

For $\ell \geq 2$, the identity

$$d\left(\frac{f}{t_1^{\ell-1}} dt_2 \wedge \cdots \wedge dt_{n+1}\right) = -(\ell - 1)\alpha|_U + \frac{\partial f / \partial t_1}{t_1^{\ell-1}} dt_1 \wedge \cdots \wedge dt_{n+1}$$

allows us to write $\alpha|_U = \beta + d\gamma$, where β is an $(n+1)$-form, and γ an n-form, both holomorphic on $U \setminus (M \cap U)$ and with a pole of order at most $\ell - 1$. In other words, we can adjust $\alpha|_U$ by an exact form and reduce the order of the pole. To do this globally, choose an open covering $\mathbf{U}$ of M by suitable open subsets of $\mathbb{P}^{n+1}$, and let $1 = \sum_{i \in I} \rho_i$ be a partition of unity subordinate to $\mathbf{U}$. If $\alpha|_{U_i} = \beta_i + d\gamma_i$, then

$$\alpha = \sum_{i \in I} \rho_i \alpha|_{U_i} = d\left(\sum_{i \in I} \rho_i \gamma_i\right) + \sum_{i \in I} \left(\rho_i \beta_i - d\rho_i \wedge \gamma_i\right).$$

On the principle that the residue of an exact form should be zero, we can thus replace α by the second term on the right-hand side, which is a $(n+1)$-form α_1 with smooth coefficients and a pole of order at most $\ell - 1$. Note that the type of α_1 is now $(n+1, 0) + (n, 1)$, since $d\rho_i$ is no longer a holomorphic form. Continuing in this manner, we can reduce the order of the pole step-by-step until we arrive at a first-order pole where we know how to define the residue.

If we apply the above process to $\alpha \in A^{n+1}(M, n+1-p)$, then α_1 will have a pole of order at most $(n-p)$ and be of type $(n+1, 0) + (n, 1)$; eventually, we arrive at α_{n-p} which has a first-order pole and is of type $(n+1, 0) + \cdots + (p+1, n-p)$.

If we take the residue (by looking at the coefficient of dt_1/t_1 in local coordinates), we thus obtain

$$\operatorname{Res}_M \alpha \underset{\text{def}}{=} \operatorname{Res}_M \alpha_{n-p} \in A^{n,0}(M) \oplus \cdots \oplus A^{p,n-p}(M),$$

and this explains why poles of higher order give rise to forms of different types on M. One can show that the resulting form is closed and independent of the choices made; in this way, we obtain the map

$$\operatorname{Res}_M \colon \frac{A^{n+1}(M, n+1-p)}{A^{n+1}(M, n-p) + dA^n(M, n-p)} \to H_0^{p,n-p}(M).$$

The proof that it is an isomorphism is nontrivial.

Class 26. The Lefschetz decomposition

The cohomology of a compact Kähler manifold admits another decomposition, related to the Lefschetz operator L and its adjoint Λ. Recall that $L\colon A^k(M) \to A^{k+2}(M)$ is the operator defined by $L(\alpha) = \omega \wedge \alpha$, where ω is the Kähler form of the metric, and that $\Lambda = - * L *\colon A^k(M) \to A^{k-2}(M)$. Both operators commute with Δ by Lemma 22.4, and therefore take harmonic forms to harmonic forms.

Lemma 26.1. *We have* $[L, \Lambda] = (p-n)\operatorname{id}$ *on the space* $A^p(M)$.

Proof. The identity involves no derivatives of the metric, and it is therefore sufficient to prove it for the Euclidean metric on $\mathbb{C}^n$. We shall use the operators e_j and e_j^* introduced during the proof of Theorem 21.3. As shown there, $L = \frac{i}{2}\sum e_j \bar{e}_j$ and $\Lambda = \frac{i}{2}\sum e_j^* \bar{e}_j^*$, and so we have

$$L\Lambda - \Lambda L = -\frac{1}{4}\sum_{j,k=1}^n \Big(e_k \bar{e}_k e_j^* \bar{e}_j^* - e_j^* \bar{e}_j^* e_k \bar{e}_k \Big) = \frac{1}{4}\sum_{j,k=1}^n \Big(e_k e_j^* \bar{e}_k \bar{e}_j^* - e_j^* e_k \bar{e}_j^* \bar{e}_k \Big)$$

For $j = k$, we can use the identity $e_j e_j^* + e_j^* e_j = 2\operatorname{id}$ to compute that

$$\begin{aligned}\sum_{j=1}^n \Big(e_j e_j^* \bar{e}_j \bar{e}_j^* - e_j^* e_j \bar{e}_j^* \bar{e}_j \Big) &= \sum_{j=1}^n \Big(e_j e_j^* \bar{e}_j \bar{e}_j^* - (2\operatorname{id} - e_j e_j^*)(2\operatorname{id} - \bar{e}_j \bar{e}_j^*) \Big) \\ &= 2\sum_{j=1}^n \Big(e_j e_j^* + \bar{e}_j \bar{e}_j^* - 2\operatorname{id} \Big).\end{aligned}$$

On the other hand, we have $e_j e_k^* + e_k^* e_j = 0$ if $j \neq k$, and therefore

$$\sum_{j=1}^n \Big(e_k e_j^* \bar{e}_k \bar{e}_j^* - e_j^* e_k \bar{e}_j^* \bar{e}_k \Big) = \sum_{j=1}^n \Big(e_k e_j^* \bar{e}_k \bar{e}_j^* - e_k e_j^* \bar{e}_k \bar{e}_j^* \Big) = 0.$$

Combining the two individual calculations, we find that

$$L\Lambda - \Lambda L = \frac{1}{2}\sum_{j=1}^n \Big(e_j e_j^* + \bar{e}_j \bar{e}_j^* - 2\operatorname{id} \Big) = \frac{1}{2}\sum_{j=1}^n \Big(e_j e_j^* + \bar{e}_j \bar{e}_j^* \Big) - n\operatorname{id}.$$

Now $e_j e_j^*$ acts as multiplication by 2 on $dz_J \wedge d\bar{z}_K$ whenever $j \in J$, and otherwise it is zero; the same is true for $\bar{e}_j \bar{e}_j^*$. Consequently, the operator $[L, \Lambda]$ multiplies $dz_J \wedge d\bar{z}_K$ by the integer $|J| + |K| - n$, as asserted. $\square$

We shall usually denote the commutator $[L, \Lambda]$ by H; in other words, $H\alpha = (k-n)\alpha$ for $\alpha \in A^k(M)$.

The Lefschetz decomposition. The adjointness of L and Λ means that we have an orthogonal decomposition

$$A^k(M) = \ker\bigl(\Lambda\colon A^k(M) \to A^{k-2}(M)\bigr) \oplus \operatorname{im}\bigl(L\colon A^{k-2}(M) \to A^k(M)\bigr). \tag{26.2}$$

Definition 26.3. A form $\alpha \in A^k(M)$ is called *primitive* if $\Lambda\alpha = 0$.

Thus every $\alpha \in A^k(M)$ can be written in the form $\alpha = \alpha_0 + L\beta$, with $\alpha_0 \in A^k(M)$ primitive and $\beta \in A^{k-2}(M)$. Note that α_0 is uniquely determined by α, but the same may not be true for β. In any case, we can repeat this process for β, until we arrive at an expression of the form

$$\alpha = \alpha_0 + L\alpha_1 + L^2\alpha_2 + \cdots + L^{\lfloor k/2 \rfloor}\alpha_{\lfloor k/2 \rfloor}, \tag{26.4}$$

where each $\alpha_j \in A^{k-2j}(M)$ is primitive. We would like to know that the forms α_j in this decomposition are uniquely determined by α, and to that end, we first prove the following lemma.

Lemma 26.5. *Let $\alpha \in A^{n-\ell}(M)$ be a primitive form. Then for any $k \geq 1$,*

$$\Lambda L^k \alpha = k(\ell - k + 1) L^{k-1}\alpha.$$

In particular, every primitive form in degree above n (i.e., with $\ell < 0$) is zero.

Proof. We have $\Lambda\alpha = 0$ and $H\alpha = -\ell\alpha$, which explains our choice of indexing. The stated formula remains true for $k = 0$; we prove the general case by induction on $k \geq 0$. Using that $L\Lambda - \Lambda L = H$, we have

$$\begin{aligned} \Lambda L^{k+1}\alpha &= (L\Lambda - H)L^k\alpha = Lk(\ell - k + 1)L^{k-1}\alpha - (-\ell + 2k)L^k\alpha \\ &= (k\ell - k^2 + k + \ell - 2k)L^k\alpha = (k+1)(\ell - k)L^k\alpha, \end{aligned}$$

which is the desired formula for $k + 1$.

Now suppose that we had a primitive form α with $\ell < 0$. Since $A^k(M) = 0$ once $k > 2n$, we clearly have $L^{n+1}\alpha = 0$. Let $k \in \mathbb{N}$ be the smallest integer such that $L^k\alpha = 0$. Since the coefficient $k(\ell - k + 1)$ in our identity is nonzero for $k > 0$, we would have $L^{k-1}\alpha = 0$ if $k > 0$; thus $k = 0$, which means that $\alpha = 0$. $\square$

Corollary 26.6. *If $\alpha \in A^{n-\ell}(M)$ is primitive and nonzero, then $L^{\ell+1}\alpha = 0$, while*

$$\alpha,\ L\alpha,\ L^2\alpha,\ \ldots,\ L^\ell\alpha$$

are all nonzero.

Proof. By the identity in Lemma 26.5, $\Lambda L^{\ell+1}\alpha = 0$. This means that $L^{\ell+1}\alpha$ is primitive and satisfies $H\alpha = (-\ell + 2\ell + 2)\alpha = (\ell + 2)\alpha$, and therefore has to be zero. On the other hand, we have $\Lambda^\ell L^\ell \alpha = (\ell!)^2\alpha$, and this shows that $L^\ell\alpha \neq 0$. $\square$

We can now show that the forms α_j in the decomposition (26.4) are uniquely determined.

Proposition 26.7. *Every form $\alpha \in A^k(M)$ admits a unique decomposition*

$$\alpha = \sum_{j=\max(k-n,0)}^{\lfloor k/2 \rfloor} L^j \alpha_j$$

where each $\alpha_j \in A^{k-2j}(M)$ is primitive.

Proof. We have $H\alpha_j = -(2j+n-k)\alpha_j$; the range of the summation is explained by the fact that $L^j\alpha_j$ can only be nonzero if $j \leq 2j+n-k$, or equivalently, if $j \geq k-n$. Now (26.4) shows that any α admits a decomposition of this kind.

To establish the uniqueness, it suffices to show that if $\alpha = 0$, then each $\alpha_j = 0$. We can apply the operator Λ to the decomposition and use the identity in Lemma 26.5 to obtain

$$0 = \sum_{j=\max(k-n,0)}^{\lfloor k/2 \rfloor} j(j+n-k+1)L^{j-1}\alpha_j.$$

Since the coefficients are nonzero for $j > 0$, we conclude inductively that $\alpha_j = 0$, except possibly for α_0 (which only appears if $k \leq n$). But we already know that $\alpha_0 = 0$ because $A^k(M) = \ker\Lambda \oplus \operatorname{im} L$. □

Class 27. More on the Lefschetz decomposition

The decomposition so far was on the level of forms. Now we use the fact that M is a compact Kähler manifold, and so every class in $H^k(M,\mathbb{C})$ is uniquely represented by a complex-valued harmonic form. Since both L and Λ preserve harmonic forms, we obtain the following *Lefschetz decomposition* of the cohomology of M.

Theorem 27.1. *Let M be a compact Kähler manifold with Kähler form ω, and let L and Λ be the corresponding operators. Then every cohomology class $\alpha \in H^k(M,\mathbb{C})$ admits a unique decomposition*

$$\alpha = \sum_{j=\max(k-n,0)}^{\lfloor k/2 \rfloor} L^j\alpha_j,$$

with $\alpha_j \in H^{k-2j}(M,\mathbb{C})$ primitive, i.e., $\Lambda\alpha_j = 0$.

The decomposition is compatible with the Hodge decomposition, in the following sense: ω is a $(1,1)$-form, and so $LA^{p,q}(M) \subseteq A^{p+1,q+1}(M)$ and $\Lambda A^{p,q}(M) \subseteq A^{p-1,q-1}(M)$. Thus the Hodge components of a primitive form are again primitive, and if α belongs to $A^{p,q}(M)$, then each α_j belongs to $A^{p-j,q-j}(M)$.

A useful consequence of the Lefschetz decomposition is the following result, commonly known as the *Hard Lefschetz Theorem.*

Corollary 27.2. *For $k \leq n$, the operator $L^{n-k} \colon H^k(M,\mathbb{C}) \to H^{2n-k}(M,\mathbb{C})$ is an isomorphism.*

Proof. The surjectivity follows from Theorem 27.1: if $\beta \in H^{2n-k}(M,\mathbb{C})$, then

$$\beta = \sum_{j=\max(n-k,0)}^{\lfloor n-k/2 \rfloor} L^j\beta_j \in \operatorname{im} L^{n-k}.$$

To prove the injectivity, suppose that $\alpha \in H^k(M,\mathbb{C})$ satisfies $L^{n-k}\alpha = 0$. Again using the decomposition coming from the theorem, we then have

$$0 = \sum_{j=\max(k-n,0)}^{\lfloor k/2 \rfloor} L^{n-k+j}\alpha_j = \sum_{i=\max(n-k,0)}^{\lfloor n-k/2 \rfloor} L^i\alpha_{i+k-n},$$

having put $i = n-k+j$; now the uniqueness of the decomposition shows that all $\alpha_j = 0$, and hence that $\alpha = 0$. □

Representation theory. The relation $[L, \Lambda] = (k-n)\,\mathrm{id}$ on $A^k(M)$ can be interpreted in terms of representation theory of Lie algebras. Recall that the Lie algebra $\mathfrak{sl}_2$ consists of all 2×2-matrices of trace zero, with Lie bracket given by the commutator $[A, B] = AB - BA$. As a vector space, $\mathfrak{sl}_2$ is three-dimensional, and the three matrices

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a natural basis. An easy computation shows that

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

A *representation* of the Lie algebra $\mathfrak{sl}_2$ is a linear map $\rho\colon \mathfrak{sl}_2 \to \mathrm{End}(V)$ to the endomorphisms of a vector space V, such that $\rho\bigl([A, B]\bigr) = \rho(A)\rho(B) - \rho(B)\rho(A)$. Equivalently, it consists of three linear operators $\rho(E)$, $\rho(F)$, and $\rho(H)$ on V, subject to the three commutator relations above.

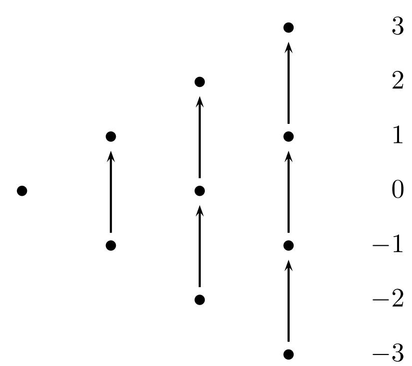
Lemma 27.3. *The operators L, Λ, and H, with $H = (k-n)\,\mathrm{id}$ on $A^k(M)$, determine a representation of $\mathfrak{sl}_2$ on the vector space $A^*(M) = \bigoplus_{k=0}^{2n} A^k(M)$.*

Proof. By Lemma 26.1, $[L, \Lambda] = H$; on the other hand, if $\alpha \in A^k(M)$, then we have

$$[H, L]\alpha = H(\omega \wedge \alpha) - \omega \wedge (k-n)\alpha = 2\omega \wedge \alpha = 2L\alpha,$$

and likewise $[H, \Lambda]\alpha = -2\Lambda\alpha$. $\square$

Now it is a general fact in representation theory that any finite-dimensional representation of $\mathfrak{sl}_2$ decomposes into direct sum of irreducible representations. Each irreducible representation in turn is generated by a primitive vector $v \in V$, satisfying $Fv = 0$ and $Hv = -\ell v$, and consists of the vectors v, Ev, $E^2 v$, $\dots$, $E^\ell v$. Note that these are all eigenvectors for H, with eigenvalues $-\ell$, $-\ell+2$, $-\ell+4$, $\dots$, ℓ, respectively. Thus a typical representation has the following form:



Each column stands for one irreducible representation; the arrows correspond to the action of E, and the integers indicate the weight of the corresponding vectors, meaning the eigenvalue of H. This picture gives a vivid illustration of the Lefschetz decomposition and the Hard Lefschetz Theorem.

The Hodge-Riemann bilinear relations. If $\alpha \in A^{p,q}(M)$ is primitive, then we have seen that $L^{n-k+1}\alpha = 0$, while $L^{n-k}\alpha \neq 0$ (here $k = p+q$). Observe that $L^{n-k}\alpha$ is a form of type $(p+n-k, q+n-k) = (n-q, n-p)$, and that the same is true for $*\alpha$. The following result shows that the two forms are the same, up to a certain factor.

Lemma 27.4. *Let $\alpha \in A^{p,q}(M)$ be a primitive form, meaning that $\Lambda\alpha = 0$. Then*

$$*\alpha = (-1)^{k(k+1)/2} i^{p-q} \frac{L^{n-k}}{(n-k)!}\alpha,$$

where $k = p+q \leq n$.

The lemma is very useful for describing the inner product $(\alpha,\beta)_M = \int_M \alpha \wedge *\overline{\beta}$ on the space of forms more concretely. Fix $0 \leq k \leq n$, and define a bilinear form on the space $A^k(M)$ by the formula

$$Q(\alpha,\beta) = (-1)^{k(k-1)/2} \cdot (L^{n-k}\alpha,\beta)_M = (-1)^{k(k-1)/2} \int_M \omega^{n-k} \wedge \alpha \wedge \beta.$$

It is easy to see that $Q(\beta,\alpha) = (-1)^k Q(\alpha,\beta)$, and so Q is either linear or antilinear, depending on the parity of k. Now suppose that $\alpha, \beta \in A^{p,q}(M)$ are both primitive forms, with $p+q=k$. By virtue of Lemma 27.4, we then have

$$(\alpha,\beta)_M = \int_M \alpha \wedge *\overline{\beta} = \frac{(-1)^{k(k+1)/2} i^{q-p}}{(n-k)!} \int_M \alpha \wedge L^{n-k}\overline{\beta} = \frac{i^{p-q}}{(n-k)!} Q(\alpha,\overline{\beta}).$$

As a consequence, we obtain the so-called *Hodge-Riemann bilinear relations.*

Theorem 27.5. *The bilinear form $Q(\alpha,\beta) = (-1)^{k(k-1)/2} \int_M \omega^{n-k} \wedge \alpha \wedge \beta$ has the following two properties:*

1. *In the Hodge decomposition of $H^k(M,\mathbb{C})$, the subspaces $H^{p,q}$ and $H^{p',q'}$ are orthogonal to each other unless $p=p'$ and $q=q'$.*
2. *For any nonzero primitive $\alpha \in H^{p,q}$, we have $i^{p-q}Q(\alpha,\overline{\alpha}) > 0$.*

Example 27.6. Let us consider the case of a compact Kähler surface M (so $n = \dim M = 2$). Here the Hodge decomposition takes the form

$$H^2(M,\mathbb{C}) = H^{2,0} \oplus \left(H_0^{1,1} \oplus \mathbb{C}\omega\right) \oplus H^{0,2},$$

with $H_0^{1,1} = \ker(\Lambda \colon H^{1,1} \to H^{0,0})$ the primitive cohomology. According to the bilinear relations, the form $\int_M \alpha \wedge \beta$ is positive definite on $\mathbb{C}\omega$ and on the subspace $H^{2,0} \oplus H^{0,2}$; on the other hand, it is negative definite on the primitive subspace $H_0^{1,1}$. Put differently, the quadratic form $Q(\alpha) = \int_M \alpha \wedge \alpha$ has signature $(1,N)$ on the space $H^{1,1}(M)$, where $N = \dim H_0^{1,1}$, a result known as the *Hodge index theorem* for surfaces.

The proof of Lemma 27.4 requires a somewhat lengthy computation (or some knowledge of representation theory), and so we shall only look at the special case $k = n = 2$. Here the assertion is that $*\alpha = -i^{p-q}\alpha$ for any $\alpha \in A^{p,q}(M)$ with $\Lambda\alpha = 0$ and $p+q=2$. As usual, it suffices to prove this for the Euclidean metric on $\mathbb{C}^2$, and so we may assume that $\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ and hence $vol(g) = -\frac{1}{4} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$. The six 2-forms $dz_1 \wedge dz_2$, $dz_1 \wedge d\bar{z}_1$, $dz_1 \wedge d\bar{z}_2$, $dz_2 \wedge d\bar{z}_1$, $d\bar{z}_2 \wedge d\bar{z}_2$, and $dz_2 \wedge d\bar{z}_2$ are pairwise orthogonal, and each have norm $h(\alpha,\alpha) = 4$.

Recall that the $*$-operator was defined by the condition that $\alpha \wedge *\overline{\beta} = h(\alpha,\beta) vol(g)$. We distinguish two cases, based on the type of the form α.

The first case is that $\alpha \in A^{2,0}(M)$, which means that it is automatically primitive. Because of the orthogonality, $*(dz_1 \wedge dz_2)$ must be a multiple of $dz_1 \wedge dz_2$; to see which, we use that $h(dz_1 \wedge dz_2, dz_1 \wedge dz_2) = 4$, and hence

$$\begin{aligned}(dz_1 \wedge dz_2) \wedge (d\bar{z}_1 \wedge d\bar{z}_2) &= -dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 = 4vol(g) \\ &= h(dz_1 \wedge dz_2, dz_1 \wedge dz_2)vol(g),\end{aligned}$$

proving that $*(dz_1 \wedge dz_2) = dz_1 \wedge dz_2$. Since we can write $\alpha = f dz_1 \wedge dz_2$, we now have $*\alpha = \alpha$ as claimed.

The second case is that $\alpha \in A^{1,1}(M)$; here α is primitive iff $L\alpha = 0$. Writing

$$\alpha = f_{1,1}dz_1 \wedge d\bar{z}_1 + f_{2,2}dz_2 \wedge d\bar{z}_2 + f_{1,2}dz_1 \wedge d\bar{z}_2 + f_{2,1}dz_2 \wedge d\bar{z}_1,$$

we have $\omega \wedge \alpha = \frac{i}{2}(f_{1,1} + f_{2,2})dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$, and hence $f_{1,1} + f_{2,2} = 0$. As above, we compute that $*(dz_1 \wedge d\bar{z}_1) = dz_2 \wedge d\bar{z}_2$ because

$$(dz_1 \wedge d\bar{z}_1) \wedge (d\bar{z}_2 \wedge dz_2) = h(dz_1 \wedge d\bar{z}_1, dz_1 \wedge d\bar{z}_1)vol(g).$$

On the other hand, $*(dz_1 \wedge d\bar{z}_2) = -dz_1 \wedge d\bar{z}_2$ because

$$(dz_1 \wedge d\bar{z}_2) \wedge (-d\bar{z}_1 \wedge dz_2) = h(dz_1 \wedge d\bar{z}_2, dz_1 \wedge d\bar{z}_2)vol(g).$$

Similar computations for the other two basic forms show that we have

$$*\alpha = f_{1,1}dz_2 \wedge d\bar{z}_2 + f_{2,2}dz_1 \wedge d\bar{z}_1 - f_{1,2}dz_1 \wedge d\bar{z}_2 - f_{2,1}dz_2 \wedge d\bar{z}_1 = -\alpha,$$

which is the desired result.

Class 28. Holomorphic vector bundles

Let M be a complex manifold. Recall that a holomorphic vector bundle of rank r is a complex manifold E, together with a holomorphic mapping $\pi\colon E \to M$, such that the following two conditions are satisfied:

(1) For each point $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is a $\mathbb{C}$-vector space of dimension r.
(2) For every $p \in M$, there is an open neighborhood U and a biholomorphism
$$\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^r$$
mapping E_p into $\{p\} \times \mathbb{C}^r$, such that the composition $E_p \to \{p\} \times \mathbb{C}^r \to \mathbb{C}^r$ is an isomorphism of $\mathbb{C}$-vector spaces.

For two local trivializations (U_α, ϕ_α) and (U_β, ϕ_β), the composition $\phi_\alpha \circ \phi_\beta^{-1}$ is of the form $(\mathrm{id}, g_{\alpha,\beta})$ for a holomorphic mapping

$$g_{\alpha,\beta}\colon U_{\alpha,\beta} \to \mathrm{GL}_r(\mathbb{C}).$$

As we have seen, these transition functions satisfy the compatibility conditions

$$\begin{aligned} g_{\alpha,\beta} \cdot g_{\beta,\gamma} \cdot g_{\gamma,\alpha} &= \mathrm{id} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma, \\ g_{\alpha,\alpha} &= \mathrm{id} \quad \text{on } U_\alpha; \end{aligned}$$

conversely, every collection of transition functions determines a holomorphic vector bundle. Also recall that a holomorphic section of the vector bundle is a holomorphic mapping $s\colon M \to E$ such that $\pi \circ s = \mathrm{id}$; locally, such a section is described by holomorphic functions $s_\alpha\colon U_\alpha \to \mathbb{C}^r$, subject to the condition that $g_{\alpha,\beta} \cdot s_\beta = s_\alpha$ on $U_\alpha \cap U_\beta$.

Definition 28.1. A *morphism* between two holomorphic vector bundles $\pi\colon E \to M$ and $\pi'\colon E' \to M$ is a holomorphic mapping $f\colon E \to E'$ satisfying $\pi' \circ f = \pi$, such that the restriction of f to each fiber is a linear map $f_p\colon E_p \to E'_p$. If each f_p is an isomorphism of vector spaces, then f is said to be an *isomorphism*.

Example 28.2. The trivial vector bundle of rank r is the product $M \times \mathbb{C}^r$. A vector bundle E is trivial if it is isomorphic to the trivial bundle. Equivalently, E is trivial if it admits r holomorphic sections $s_1, \dots, s_r$ whose values $s_1(p), \dots, s_r(p)$ give a basis for the vector space E_p at each point $p \in M$.

Given a holomorphic vector bundle $\pi\colon E \to M$, we let $A(U, E)$ denote the space of smooth sections of E over an open set $U \subseteq M$. Likewise, $A^{p,q}(U, E)$ denotes the space of (p, q)-forms with coefficients in E; in a local trivialization $\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$, these are given by r-tuples $\omega_\alpha \in A^{p,q}(U)^{\oplus r}$, subject to the relation

$$\omega_\alpha = g_{\alpha,\beta} \cdot \omega_\beta$$

on $U_\alpha \cap U_\beta$. As usual, they can also be viewed as sections of a sheaf $\mathscr{A}^{p,q}(E)$.

Example 28.3. Say L is a line bundle (so $r = 1$), which means that the transition functions $g_{\alpha,\beta} \in \mathscr{O}_M^*(U_\alpha \cap U_\beta)$ are holomorphic functions. In this case, a (p, q)-form with coefficients in L is nothing but a collection of smooth forms $\omega_\alpha \in A^{p,q}(U_\alpha)$, subject to the condition that $\omega_\alpha = g_{\alpha,\beta}\omega_\beta$. Said differently, the individual forms do not agree on the intersections between the open sets (as they would for a usual (p, q)-form), but differ by the factor $g_{\alpha,\beta}$. One can view this as a kind of "twisted" version of (p, q)-forms.

Hermitian metrics and the Chern connection. Recall that for a smooth function $f \in A(U)$, the exterior derivative df is a smooth 1-form on U. Since M is a complex manifold, we have $d = \partial + \bar\partial$, and correspondingly, $df = \partial f + \bar\partial f$. Because of the Cauchy-Riemann equations, f is holomorphic if and only if $\bar\partial f \in A^{0,1}(U)$ is zero.

For a holomorphic vector bundle $E \to M$, there similarly exists an operator $\bar\partial\colon A(M, E) \to A^{0,1}(M, E)$, with the property that a smooth section s is holomorphic iff $\bar\partial s = 0$. To construct this $\bar\partial$-operator, note that in a local trivialization $\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$, smooth sections of E are given by smooth mappings $s_\alpha\colon U_\alpha \to \mathbb{C}^r$; we may then define $\bar\partial s_\alpha = (\bar\partial s_{\alpha,1}, \dots, \bar\partial s_{\alpha,r})$, which is a vector of length r whose entries are $(0, 1)$-forms. On the overlap $U_\alpha \cap U_\beta$ between two trivializations, we have $s_\alpha = g_{\alpha,\beta} \cdot s_\beta$, and therefore

$$\bar\partial s_\alpha = g_{\alpha,\beta} \cdot \bar\partial s_\beta$$

because the entries of the $r \times r$-matrix $g_{\alpha,\beta}$ are holomorphic functions. This shows that if $s \in A(U, E)$, then $\bar\partial s$ is a well-defined element of $A^{0,1}(U, E)$.

On the other hand, this method cannot be used to define analogues of d or $\bar\partial$, since the corresponding derivatives of the $g_{\alpha,\beta}$ do not vanish. The correct generalization of d, as it turns out, is that of a connection on E. As in differential geometry, a *connection* on a complex vector bundle is a mapping

$$\nabla\colon T(M) \times A(M, E) \to A(M, E)$$

that associates to a smooth tangent vector field ξ and a smooth section s another smooth section $\nabla_\xi s$, to be viewed as the derivative of s along ξ. The connection is required to be $A(M)$-linear in its first argument and to satisfy the Leibniz rule

$$\nabla_\xi(fs) = (\xi f) \cdot s + f\nabla_\xi s$$

for any smooth function f. Given a local trivialization $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^r$, we have r distinguished holomorphic sections $s_1, \dots, s_r$ of E, corresponding to the coordinate vectors on $\mathbb{C}^r$. We can then represent the action of the connection as

$$\nabla s_j = \sum_{k=1}^{r} \theta_{j,k} \otimes s_k$$

for certain $\theta_{j,k} \in A^1(U)$; this shorthand notation means that

$$\nabla_\xi s_j = \sum_{k=1}^{r} \theta_{j,k}(\xi) s_k.$$

Because of the Leibniz rule, the 1-forms $\theta_{j,k}$ uniquely determine the connection.

As in differential geometry, it is necessary to choose a metric on the vector bundle before one has a canonical connection. We have already encountered the following notion for the holomorphic tangent bundle $T'M$.

Definition 28.4. A *Hermitian metric* on a complex vector bundle $\pi\colon E \to M$ is a collection of Hermitian inner products $h_p\colon E_p \times E_p \to M$ that vary smoothly with $p \in M$, in the sense that $h(s_1, s_2)$ is a smooth function for any two smooth sections $s_1, s_2 \in A(M, E)$.

Given a local trivialization $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^r$ of the vector bundle as above, we describe the Hermitian metric h through its coefficient matrix, whose entries

$$h_{j,k} = h(s_j, s_k)$$

are smooth functions on U. We have $h_{k,j} = \overline{h_{j,k}}$, and the matrix is positive definite.

It turns out that, once we have chosen a Hermitian metric on E, there is a unique connection compatible with the metric and the complex structure on E. To define it, we observe that the complexified tangent bundle splits as $T_{\mathbb{C}}M = T'M \oplus T''M$ into the holomorphic and antiholomorphic tangent bundles. Correspondingly, we can split any connection on E as $\nabla = \nabla' + \nabla''$, with $\nabla'\colon T'(M) \times A(M, E) \to A(M, E)$ and $\nabla''\colon T''(M) \times A(M, E) \to A(M, E)$.

Proposition 28.5. *Let E be a holomorphic vector bundle with a Hermitian metric h. Then there exists a unique connection that is compatible with the metric, in the sense that for every smooth tangent vector field ξ, we have*

$$\xi \cdot h(s_1, s_2) = h(\nabla_\xi s_1, s_2) + h(s_1, \nabla_\xi s_2),$$

and compatible with the complex structure, in the sense that

$$\nabla''_\xi s = (\bar{\partial} s)(\xi)$$

for any smooth section ξ of the anti-holomorphic tangent bundle $T''M$.

This connection is called the *Chern connection* of the holomorphic vector bundle E; one usually summarizes the second condition by writing $\nabla'' = \bar{\partial}$.

Proof. To prove the uniqueness, suppose that we have such a connection ∇; we will find a formula for the coefficients $\theta_{j,k}$ in terms of the metric. So let $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^r$ be a local trivialization of the vector bundle, and let $s_1, \dots, s_r$ denote the corresponding holomorphic sections of E over U. The Hermitian metric is described by its coefficient matrix, whose entries $h_{j,k} = h(s_j, s_k)$ are smooth functions on U.

The second condition means that $\nabla'' s_j = \bar\partial s_j = 0$ because each s_j is holomorphic, and so we necessarily have

$$\nabla s_j = \nabla' s_j = \sum_{k=1}^{r} \theta_{j,k} \otimes s_k$$

with $(1,0)$-forms $\theta_{j,k} \in A^{1,0}(U)$ that uniquely determine the connection. By the first condition,

$$dh_{j,k} = h(\nabla s_j, s_k) + h(s_j, \nabla s_k) = \sum_{l=1}^{r} \bigl(h_{l,k}\theta_{j,l} + h_{j,l}\overline{\theta_{k,l}} \bigr),$$

and this identity shows that $\partial h_{j,k} = \sum h_{l,k}\theta_{j,l}$ and $\bar\partial h_{j,k} = \sum h_{j,l}\overline{\theta_{k,l}}$ (which is the conjugate of the former). If we let $h^{j,k}$ denote the entries of the inverse matrix, it follows that

$$\theta_{j,k} = \sum_{l=1}^{r} h^{l,k} \partial h_{j,l},$$

which proves the uniqueness of the Chern connection. Conversely, we can use this formula to define the connection locally; because of uniqueness, the local definitions have to agree on the intersections of different open sets, and so we get a globally defined connection on E. □

Example 28.6. One should think of the Chern connection ∇ as a replacement for the exterior derivative d, and of ∇' as a replacement for ∂; in this way, the identity $\nabla = \nabla' + \bar\partial$ generalizes the formula $d = \partial + \bar\partial$. In fact, d is the Chern connection on the trivial bundle $E = M \times \mathbb{C}$ (whose smooth sections are the smooth functions) for the Hermitian metric induced by the standard metric on $\mathbb{C}$.

Class 29. Holomorphic line bundles

We will be especially interested in the case $r = 1$, that is, in holomorphic *line bundles*. Local trivializations now take the form $\phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$, and consequently, a holomorphic line bundle can be described by a collection of holomorphic functions $g_{\alpha,\beta} \in \mathscr{O}_M^*(U_\alpha \cap U_\beta)$ that satisfy the cocycle condition $g_{\alpha,\beta} g_{\beta,\gamma} = g_{\alpha,\gamma}$. A line bundle is trivial, meaning isomorphic to $M \times \mathbb{C}$, precisely when it admits a nowhere vanishing section; in that case, we have $g_{\alpha,\beta} = s_\beta / s_\alpha$, with $s_\alpha \in \mathscr{O}_M^*(U_\alpha)$. If we only consider line bundles that are trivial on a fixed open cover $\mathbf{U}$, then the set of isomorphism classes of such line bundles is naturally in bijection with the Čech cohomology group $H^1(\mathbf{U}, \mathscr{O}_M^*)$. Likewise, the set of isomorphism classes of arbitrary line bundles is in bijection with the group $H^1(M, \mathscr{O}_M^*)$.

Example 29.1. The *tensor product* of two holomorphic line bundles L' and L'' is the holomorphic line bundle $L = L' \otimes L''$ with transition functions $g_{\alpha,\beta} = g'_{\alpha,\beta} g''_{\alpha,\beta}$. This operation corresponds to multiplication in the group $H^1(M, \mathscr{O}_M^*)$.

Example 29.2. The *dual* of a holomorphic line bundle L is the holomorphic line bundle L^{-1} with transition functions $g_{\alpha,\beta}^{-1}$. Since $L \otimes L^{-1}$ is isomorphic to the trivial bundle $M \times \mathbb{C}$, we see that L^{-1} is the inverse of L in the group $H^1(M, \mathscr{O}_M^*)$.

Example 29.3. Let $D \subseteq M$ be a hypersurface in M, that is, an analytic subset of dimension $n-1$ that is locally defined by the vanishing of a single holomorphic function. Then there is a holomorphic line bundle $\mathscr{O}_X(-D)$, whose sections over an open set U are all holomorphic functions $f \in \mathscr{O}_X(U)$ that vanish along $U \cap D$. To compute the transition functions, suppose that we have $U_\alpha \cap D = Z(f_\alpha)$, where each f_α is not divisible by the square of any nonunit, and hence unique up to multiplication by units. It follows that the ratios $g_{\alpha,\beta} = f_\beta / f_\alpha$ are nowhere vanishing holomorphic functions on $U_\alpha \cap U_\beta$.

Let us describe Hermitian metrics and the Chern connection in the case of a holomorphic line bundle. A local trivialization $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}$ of the line bundle is the same as a nonvanishing holomorphic section $s \in A(U, L)$, and so a Hermitian metric h on L is locally described by a single smooth function $h = h(s,s)$ with values in the positive real numbers.

The Chern connection is determined by its action on s, and if we put $\nabla s = \theta \otimes s$, then we have seen that $\theta \in A^{0,1}(U)$ because $\nabla'' s = \bar\partial s = 0$. The other defining property of the Chern connection,

$$\partial h + \bar\partial h = dh(s,s) = h(\nabla s, s) + h(s, \nabla s) = h\theta + h\overline{\theta}$$

shows that we have $\theta = h^{-1}\partial h = \partial \log h$. The $(1,1)$-form $\Theta = \bar\partial\theta = -\partial\bar\partial \log h$ is called the *curvature* of the connection.

Lemma 29.4. *Let h be a Hermitian metric on a holomorphic line bundle $L \to M$.*

(1) The curvature form $\Theta_L \in A^{1,1}(M)$ is globally well-defined.
(2) $\partial\Theta_L = \bar\partial\Theta_L = 0$, and the class of Θ_L in $H^{1,1}(M)$ does not depend on h.
(3) With the induced metric on L^{-1}, we have $\Theta_{L^{-1}} = -\Theta_L$.
(4) With the induced metric on $L_1 \otimes L_2$, we have $\Theta_{L_1 \otimes L_2} = \Theta_{L_1} + \Theta_{L_2}$.

Proof. In a local trivialization $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}$, we have $\Theta = -\partial\bar\partial \log h(s,s)$, where s is the distinguished holomorphic section determined by ϕ. For a second trivialization ϕ', we have $s' = fs$ for some $f \in \mathscr{O}_M^*(U)$, and hence $h(s', s') = |f|^2 h(s,s)$. It follows that

$$\Theta' = -\partial\bar\partial\bigl(|f|^2 + h(s,s)\bigr) = -\partial\bar\partial(f\overline{f}) + \Theta = \Theta,$$

and so Θ is independent of the local trivializations. Moreover, the local formula clearly shows that $\partial\Theta = \bar\partial\Theta = 0$, and so Θ defines a class in the Dolbeault cohomology group $H^{1,1}(M)$.

To prove that $[\Theta]$ does not depend on h, note that since the fibers of L are one-dimensional, any other choice of Hermitian metric has to differ from h by multiplication by a positive real-valued function $\psi \in A(M)$. We then have $-\partial\bar\partial \log(\psi h) = \Theta + \bar\partial\partial \log\psi$, and so both forms differ by a $\bar\partial$-exact form, and hence define the same cohomology class. The remaining two assertions are left as an exercise. □

Note. More generally, suppose that we have a connection $\nabla\colon \mathscr{A}(E) \to \mathscr{A}^1(E)$ on a holomorphic vector bundle E. It induces a mapping $\nabla\colon \mathscr{A}^1(E) \to \mathscr{A}^2(E)$, by requiring that the product rule

$$\nabla(\alpha \otimes s) = d\alpha \otimes s - \alpha \wedge \nabla s$$

be satisfied for smooth forms $\alpha \in A^1(M)$ and smooth sections $s \in A(M, E)$. The composition $\nabla^2\colon \mathscr{A}(E) \to \mathscr{A}^2(E)$ is called the *curvature* of the connection. From

$$\nabla^2(fs) = \nabla\bigl(df \otimes s + f\nabla s\bigr) = -df \wedge \nabla s + df \wedge \nabla s + f \cdot \nabla^2 s,$$

we see that ∇^2 is an $A(M)$-linear operator, and hence described in local trivializations by an $r \times r$-matrix of 2-forms. Now suppose that (E, h) is a holomorphic vector bundle with a Hermitian metric, and let ∇ be its Chern connection. As we have seen, the connection is locally given by the formula

$$\nabla s_j = \sum_{k=1}^{r} \theta_{j,k} \otimes s_k$$

and so we have

$$\begin{aligned}\nabla^2 s_j &= \sum_{k=1}^{r} \bigl(d\theta_{j,k} \otimes s_k - \theta_{j,k} \wedge \nabla s_k\bigr) = \sum_{k=1}^{r} d\theta_{j,k} \otimes s_k - \sum_{k,l=1}^{r} \theta_{j,k} \wedge \theta_{k,l} \otimes s_l \\ &= \sum_{k=1}^{r} \left(d\theta_{j,k} - \sum_{l=1}^{r} \theta_{j,l} \wedge \theta_{l,k} \right) \otimes s_k = \sum_{k=1}^{r} \Theta_{j,k} \otimes s_k.\end{aligned}$$

To describe more concretely the forms $\Theta_{j,k} \in A^2(U)$, we note that the Chern connection satisfies $\partial h_{j,k} = \sum h_{l,k}\theta_{j,l}$; from this, we obtain

$$0 = \partial^2 h_{j,k} = \sum_{l=1}^{r} \bigl(\partial h_{l,k} \wedge \theta_{j,l} + h_{l,k}\partial\theta_{j,l}\bigr) = \sum_{l,m=1}^{r} h_{m,k}\theta_{l,m} \wedge \theta_{j,l} + \sum_{l=1}^{r} h_{l,k}\partial\theta_{j,l}.$$

It follows that we have $\Theta_{j,k} = \bar{\partial}\theta_{j,k}$, which are therefore $(1,1)$-forms. Just as in the case of line bundles, one can show that the curvature is a globally defined $(1,1)$-form with coefficients in the bundle $\operatorname{Hom}(E,E)$.

It follows from Lemma 29.4 that we have a group homomorphism

$$H^1(M, \mathscr{O}_M^*) \to H^{1,1}(M)$$

that associates to a holomorphic line bundle the cohomology class of its curvature form Θ_L (with respect to an arbitrary Hermitian metric). On the other hand, the exponential sequence $0 \to \mathbb{Z}_M \to \mathscr{O}_M \to \mathscr{O}_M^* \to 0$ gives us a long exact sequence, part of which reads

$$H^1(M, \mathscr{O}_M) \longrightarrow H^1(M, \mathscr{O}_M^*) \longrightarrow H^2(M, \mathbb{Z}_M) \longrightarrow H^2(M, \mathscr{O}_M)$$

As mentioned earlier, the sheaf cohomology group $H^2(M, \mathbb{Z}_M)$ is isomorphic to the singular cohomology group $H^2(M, \mathbb{Z})$, which in turn maps to the de Rham cohomology group $H^2(M, \mathbb{C})$. The class in $H^2(M, \mathbb{Z})$ associated to (the isomorphism class of) a holomorphic line bundle L is called the first *Chern class* of L, and is denoted by $c_1(L)$. The following lemma shows that $c_1(L)$ can also be computed from Θ_L.

Lemma 29.5. *We have* $c_1(L) = \frac{i}{2\pi}[\Theta_L]$, *as elements of* $H^2(M, \mathbb{C})$.

Examples. On $\mathbb{P}^n$, we have the tautological line bundle $\mathscr{O}_{\mathbb{P}^n}(-1)$, described as follows: by definition, each point of $\mathbb{P}^n$ corresponds to a line in $\mathbb{C}^{n+1}$, which we take to be the fiber of $\mathscr{O}_{\mathbb{P}^n}(-1)$ over that point. In other words, the fiber of $\mathscr{O}_{\mathbb{P}^n}(-1)$ over the point $[z_0, z_1, \dots, z_n]$ is the line $\mathbb{C} \cdot (z_0, z_1, \dots, z_n)$. This makes $\mathscr{O}_{\mathbb{P}^n}(-1)$ a subbundle of the trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$, and gives it a natural Hermitian metric (induced from the standard metric on the trivial bundle).

To compute the associated curvature form $\Theta \in A^{1,1}(\mathbb{P}^n)$, let $U_0 \simeq \mathbb{C}^n$ be one of the standard open sets; then $[1, z_1, \dots, z_n] \mapsto (1, z_0, \dots, z_n)$ defines a holomorphic section s_0 of our line bundle on U_0, with norm

$$h_0 = h(s_0, s_0) = 1 + |z_1|^2 + \cdots + |z_n|^2.$$

Consequently,

$$\frac{i}{2\pi}\Theta_0 = -\frac{i}{2\pi}\partial\bar{\partial}\log h_0 = -\frac{i}{2\pi}\partial\bar{\partial}\log\bigl(1+|z_1|^2+\cdots+|z_n|^2\bigr) = -\omega_{FS}|_{U_0}$$

is the negative of the Fubini-Study form. In particular, this shows that ω_{FS} equals the first Chern class of the dual line bundle $\mathscr{O}_{\mathbb{P}^n}(1)$.

For another example, let M be a complex manifold of dimension n with a Hermitian metric h; in other words, h is a Hermitian metric on the holomorphic tangent bundle $T'M$. Now consider the so-called *canonical bundle* Ω_M^n, whose sections over an open set U are the holomorphic n-forms on U. Locally, any such section can be written in the form $f(z)dz_1\wedge\cdots\wedge dz_n$, with f holomorphic. Since Ω_M^n can be viewed as the n-th wedge power of the dual of $T'M$, it inherits a Hermitian metric. In local coordinates $z_1,\dots,z_n$, we define as usual a matrix H with entries

$$h_{j,k} = h\bigl(\partial/\partial z_j, \partial/\partial z_k\bigr),$$

and then $h^{j,k} = h(dz_j, dz_k)$ are the entries of the inverse matrix H^{-1}. The induced Hermitian metric on Ω_M^n satisfies

$$h(dz_1\wedge\cdots\wedge dz_n, dz_1\wedge\cdots\wedge dz_n) = \det H^{-1} = -\det H,$$

and therefore its curvature form is given by

$$\Theta = -\partial\bar{\partial}\log h(dz_1\wedge\cdots\wedge dz_n, dz_1\wedge\cdots\wedge dz_n) = \partial\bar{\partial}\log(\det H).$$

Class 30. Hodge theory for holomorphic line bundles

We begin by proving Lemma 29.5 from last time, namely that the image of the first Chern class $c_1(L)$ in $H^2(M,\mathbb{C})$ is represented by the $(1,1)$-form $\frac{i}{2\pi}\Theta$.

Proof. The main issue is to transform the element $c_1(L)$ from a class in Čech cohomology to a class in de Rham cohomology. Since we did not prove Theorem 13.20, we will just go through the procedure here without justifying it. We begin by covering M by open sets U_α over which L is trivial. Choose holomorphic functions $f_{\alpha,\beta}\in\mathscr{O}_M(U_\alpha\cap U_\beta)$ lifting the transition functions $g_{\alpha,\beta}$ under the map exp, meaning that $g_{\alpha,\beta} = e^{2\pi i f_{\alpha,\beta}}$; then $c_1(L)$ is the class of the 2-cocycle

$$c_{\alpha,\beta,\gamma} = f_{\beta,\gamma} - f_{\alpha,\gamma} + f_{\alpha,\beta}.$$

To turn this cocycle into a class in de Rham cohomology, let $1 = \sum\rho_\alpha$ be a partition of unity subordinate to the cover. Since $c_{\alpha,\beta,\gamma}$ are locally constant, $df_{\alpha,\beta}$ is a 1-cocycle for the sheaf $\mathscr{A}_M^1$. Using the partition of unity, we define

$$\varphi_\alpha = \sum_\gamma \rho_\gamma\cdot df_{\gamma,\alpha}\in A^1(U_\alpha)$$

which is easily seen to satisfy $df_{\alpha,\beta} = \varphi_\beta - \varphi_\alpha$. Thus the forms $d\varphi_\alpha\in A^2(U_\alpha)$ agree on the overlaps between open sets, and thus define a global 2-form that is closed and represents the image of $c_1(L)$ in $H^2(M,\mathbb{C})$.

Now choose a Hermitian metric h on L, and let $h_\alpha = h(s_\alpha, s_\alpha)$ be the resulting local functions. From the relation $s_\beta = g_{\alpha,\beta}s_\alpha$, we find that $h_\beta = |g_{\alpha,\beta}|^2 h_\alpha$, and hence

$$\theta_\beta - \theta_\alpha = \partial\log h_\beta - \partial\log h_\alpha = \partial\log\bigl(|g_{\alpha,\beta}|^2\bigr) = \frac{dg_{\alpha,\beta}}{g_{\alpha,\beta}} = 2\pi i\cdot df_{\alpha,\beta}.$$

This means that we have

$$\varphi_\alpha = \frac{i}{2\pi} \sum_\gamma \rho_\gamma(\theta_\alpha - \theta_\gamma) = \frac{i}{2\pi}\theta_\alpha - \psi,$$

where $\psi = \frac{i}{2\pi}\sum \rho_\gamma \theta_\gamma \in A^1(M)$. Remembering that $\Theta_\alpha = \bar\partial\theta_\alpha = d\theta_\alpha$, we now get

$$d\varphi_\alpha = \frac{i}{2\pi}\Theta_\alpha + d\psi,$$

which shows that $\frac{i}{2\pi}\Theta_L$ represents the same cohomology class as $c_1(L)$. $\square$

Note. Since Θ_L is a form of type $(1,1)$, the first Chern class $c_1(L)$ is an example of a *Hodge class*: a class in $H^{2p}(M,\mathbb{Z})$ whose image in $H^{2p}(M,\mathbb{C})$ belongs to the subspace $H^{p,p}$ in the Hodge decomposition. In fact, the kind of argument just given proves that any Hodge class in $H^2(M,\mathbb{Z})$ is the first Chern class of a holomorphic line bundle on M, a fact that is known as the Lefschetz $(1,1)$-theorem. To see how this works, consider the diagram

$$\begin{tikzcd} H^1(M,\mathscr{O}_M^*) \arrow[r, "c_1"] & H^2(M,\mathbb{Z}) \arrow[r] \arrow[d] & H^2(M,\mathscr{O}_M) \\ & H^2(M,\mathbb{C}) \arrow[ur] & \end{tikzcd}$$

in which the first row is exact (as part of a long exact sequence). Under the assumption that M is a Kähler manifold, we have the Hodge decomposition $H^2(M,\mathbb{C}) = H^{2,0}\oplus H^{1,1}\oplus H^{0,2}$; moreover, $H^2(M,\mathscr{O}_M) \simeq H^{0,2}$, and under this identification, the diagonal map in the diagram is the projection map. Note that any $\alpha \in H^2(M,\mathbb{C})$ in the image of $H^2(M,\mathbb{Z})$ is real, and hence satisfies $\alpha^{2,0} = \overline{\alpha^{0,2}}$. Thus a class in $H^2(M,\mathbb{Z})$ is the first Chern class of a holomorphic line bundle iff its image in $H^2(M,\mathbb{C})$ is of type $(1,1)$.

Harmonic forms. Let L be a holomorphic line bundle on a compact complex manifold M. Since the $\bar\partial$-operator $\bar\partial\colon A^{p,q}(M,L) \to A^{p,q+1}(M,L)$ satisfies $\bar\partial\circ\bar\partial = 0$, we can define the Dolbeault cohomology groups of L as

$$H^{p,q}(M,L) = \frac{\ker\bigl(\bar\partial\colon A^{p,q}(M,L) \to A^{p,q+1}(M,L)\bigr)}{\operatorname{im}\bigl(\bar\partial\colon A^{p,q-1}(M,L) \to A^{p,q}(M,L)\bigr)}.$$

Note that the usual Dolbeault cohomology is the special case of the trivial bundle $M\times\mathbb{C}$. As before, the Dolbeault complex

$$0 \longrightarrow \mathscr{A}^{p,0}(L) \xrightarrow{\bar\partial} \mathscr{A}^{p,1}(L) \xrightarrow{\bar\partial} \mathscr{A}^{p,2}(L) \longrightarrow \cdots \longrightarrow \mathscr{A}^{p,n}(L) \longrightarrow 0$$

resolves the sheaf $\Omega_M^p\otimes L$ of holomorphic p-forms with coefficients in L, and since each $\mathscr{A}^{p,q}(L)$ is a fine sheaf, we find that

$$H^{p,q}(M,L) \simeq H^q\bigl(M,\Omega_M^p\otimes L\bigr)$$

computes the sheaf cohomology groups of $\Omega_M^p\otimes L$. We would now like to generalize the Hodge theorem to this setting, and show that cohomology classes in $H^{p,q}(M,L)$ can be represented by harmonic forms.

We proceed along the same lines as before, and so the first step is to define Hermitian inner products on the spaces $A^{p,q}(M,L)$. To do that, choose a Hermitian metric h on the complex manifold M, and let g be the corresponding Riemannian metric and ω the associated $(1,1)$-form. (For the time being, it is not necessary to assume that h is a Kähler metric.) We also choose a Hermitian metric h_L on the holomorphic line bundle L. With both metrics in hand, we can define a Hermitian inner product on $A^{p,q}(M,L)$; writing a typical element as $\alpha \otimes s$, with $\alpha \in A^{p,q}(M)$ and $s \in A(M,L)$ smooth, we set

$$(\alpha_1 \otimes s_1, \alpha_2 \otimes s_2)_L = \int_M h(\alpha_1, \alpha_2) h_L(s_1, s_2) vol(g).$$

We then let $\bar{\partial}^* \colon A^{p,q}(M,L) \to A^{p,q-1}(M,L)$ be the adjoint of $\bar{\partial}$ with respect to the inner product, and define the Laplace operator

$$\overline{\Box} = \bar{\partial} \circ \bar{\partial}^* + \bar{\partial}^* \circ \bar{\partial} \colon A^{p,q}(M,L) \to A^{p,q}(M,L).$$

A local calculation (made easy by the fact that L is locally trivial) shows that $\overline{\Box}$ is an elliptic operator of order two. Thus if we let $\mathcal{H}^{p,q}(M,L) = \ker \overline{\Box}$ denote the space of $\bar{\partial}$-*harmonic* forms with coefficients in L, we get

$$\mathcal{H}^{p,q}(M,L) \simeq H^{p,q}(M,L)$$

by applying the general theorem about elliptic operators (Theorem 16.3).

Class 31. The Kodaira vanishing theorem

The most important consequence of being able to represent classes in $H^{p,q}(M,L)$ by $\bar{\partial}$-harmonic forms is the famous Kodaira vanishing theorem; roughly speaking, it says that if L is "positive" (in the way that the line bundle $\mathscr{O}_{\mathbb{P}^n}(1)$ on projective space is positive), then its Dolbeault cohomology vanishes for $p+q > n$.

To see in what sense the line bundle $\mathscr{O}_{\mathbb{P}^n}(1)$ is positive, recall that its curvature form satisfies $\frac{i}{2\pi}\Theta = \omega_{FS}$, which is the Kähler form of the Fubini-Study metric. The following definition generalizes this situation.

Definition 31.1. A holomorphic line bundle $L \to M$ is called *positive* if its first Chern class $c_1(L)$ can be represented by a closed $(1,1)$-form Ω whose associated Hermitian form is positive definite.

More concretely, what this means is that if we write $\Omega = \frac{i}{2}\sum_{j,k} f_{j,k} dz_j \wedge d\bar{z}_k$ in local coordinates $z_1, \dots, z_n$, then the Hermitian matrix with entries $f_{j,k}$ should be positive definite. We express this more concisely by saying that Ω is a *positive* form. Of course, such a form Ω is the associated $(1,1)$-form of a Hermitian metric on M, and since $d\Omega = 0$, this metric is Kähler. In particular, if there exists a positive line bundle on M, then M is necessarily a Kähler manifold.

Here is the precise statement of Kodaira's vanishing theorem.

Theorem 31.2. *Let L be a positive line bundle on a compact complex manifold M. Then $H^{p,q}(M,L) = 0$ whenever $p+q > n$.*

Generalized Kähler identities and the proof. Throughout, we fix a compact complex manifold M and a positive line bundle L on it. As mentioned above, there is a Kähler metric h on M whose associated $(1,1)$-form ω represents $c_1(L)$, and we assume from now on that M has been given that metric. The following lemma allows us to choose a compatible Hermitian metric h_L on the line bundle L, with the property that $\frac{i}{2\pi}\Theta_L = \omega$ is the Kähler form.

Lemma 31.3. *Let L be a positive line bundle on a compact Kähler manifold M, and suppose that ω is a closed $(1,1)$-form that represents $c_1(L)$. Then there is a (essentially unique) Hermitian metric on L whose curvature satisfies $\frac{i}{2\pi}\Theta = \omega$.*

Proof. Choose an arbitrary Hermitian metric h_0 on L, and let $\Theta_0 \in A^{1,1}(M)$ be the associated curvature form. Then both $\frac{i}{2\pi}\Theta_0$ and ω represent the first Chern class of L, and so their difference is a $(1,1)$-form that is both closed and $\bar{\partial}$-exact. By the $\partial\bar{\partial}$-Lemma (see Proposition 23.9), there exists a smooth real-valued function $\psi \in A(M)$ such that

$$\omega = \frac{i}{2\pi}\Theta_0 + \frac{i}{2\pi}\partial\bar{\partial}\psi.$$

Now define a new Hermitian metric on L by setting $h_L = e^{-\psi}h_0$. We then have

$$\Theta_L = -\partial\bar{\partial}\log h = \Theta_0 + \partial\bar{\partial}\psi,$$

and hence $\frac{i}{2\pi}\Theta_L = \omega$ as asserted. □

The Hermitian metric h_L on the line bundle L also gives rise to the Chern connection $\nabla\colon A(M,L) \to A^1(M,L)$. We have $\nabla = \nabla' + \nabla''$, and by definition of the Chern connection, $\nabla'' = \bar{\partial}$. To emphasize the analogy with the case of usual forms, we shall write ∂ instead of ∇' throughout this section. We then get operators

$$\bar{\partial}\colon A^{p,q}(M,L) \to A^{p,q+1}(M,L) \quad \text{and} \quad \partial\colon A^{p,q}(M,L) \to A^{p+1,q}(M,L)$$

by enforcing the Leibniz rule. Note that we have $\bar{\partial}\circ\bar{\partial} = 0$ and $\partial\circ\partial = 0$; on the other hand, $\partial\circ\bar{\partial} + \bar{\partial}\circ\partial$ is not usually zero, but is related to the curvature of L. (In the case of the trivial bundle $L = M \times \mathbb{C}$, the curvature is zero, which explains why we have $\partial\circ\bar{\partial} + \bar{\partial}\circ\partial = 0$.)

Lemma 31.4. *If $\Theta_L \in A^{1,1}(M)$ denotes the curvature form of the metric h_L, then we have $\partial\bar{\partial} + \bar{\partial}\partial = \Theta_L$.*

Proof. From the definition of the curvature form, we have $\Theta_L = \nabla^2 = (\partial + \bar{\partial})\circ(\partial + \bar{\partial})$, and so the identity follows. To illustrate what is going on, here is a more concrete proof. Let $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}$ be a local trivialization of L, and let s be the corresponding holomorphic section of L over U. As usual, write $\nabla s = \theta \otimes s$, and since we are dealing with the Chern connection, we have $\theta = \partial \log h_L(s,s)$.

For any $\alpha \in A^{p,q}(U)$, we have $\bar{\partial}(\alpha\otimes s) = (\bar{\partial}\alpha)\otimes s$ by definition of the $\bar{\partial}$-operator; on the other hand,

$$\begin{aligned}\partial(\alpha\otimes s) &= \nabla'(\alpha\otimes s) = (\partial\alpha)\otimes s + (-1)^{p+q}\alpha\wedge\nabla' s \\ &= (\partial\alpha)\otimes s + (-1)^{p+q}(\alpha\wedge\theta)\otimes s = (\partial\alpha + \theta\wedge\alpha)\otimes s.\end{aligned}$$

Consequently,

$$\begin{aligned}(\partial\bar{\partial} + \bar{\partial}\partial)(\alpha\otimes s) &= \partial\big(\bar{\partial}\alpha\otimes s\big) + \bar{\partial}\big((\partial\alpha + \theta\wedge\alpha)\otimes s\big) \\ &= (\partial\bar{\partial}\alpha + \theta\wedge\bar{\partial}\alpha)\otimes s + (\bar{\partial}\partial\alpha + \bar{\partial}\theta\wedge\alpha - \theta\wedge\bar{\partial}\alpha)\otimes s \\ &= (\bar{\partial}\theta\wedge\alpha)\otimes s = (\Theta_L\wedge\alpha)\otimes s.\end{aligned}$$

□

As usual, we let ∂^* and $\bar{\partial}^*$ denote the adjoint operators of ∂ and $\bar{\partial}$, with respect to the inner product introduced last time. To make this more explicit, let us describe the operator ∂^* in a local trivialization $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}$. If s denotes the corresponding holomorphic section, then any element of $A^{p,q}(U,L)$ can be written as $\alpha\otimes s$ for a unique $\alpha \in A^{p,q}(U)$. Fix $\beta \in A^{p,q}(U)$ with compact support. By definition of the adjoint, we have

$$\big(\partial^*(\alpha\otimes s), \beta\otimes s\big)_L = \big(\alpha\otimes s, \partial(\beta\otimes s)\big)_L.$$

We already computed that $\partial(\beta \otimes s) = (\partial\beta + \theta \wedge \beta) \otimes s$, where $\theta = \partial \log f = f^{-1}\partial f$ and $f = h_L(s,s)$ is smooth and positive real-valued. Consequently,

$$\begin{aligned}
\big(\alpha \otimes s, \partial(\beta \otimes s)\big)_L &= \int_M h\big(\alpha, \partial\beta + \theta \wedge \beta\big) h_L(s,s)\, vol(g) \\
&= \int_M h\big(\alpha, f\partial\beta + \partial f \wedge \beta\big)\, vol(g) = \int_M h\big(\alpha, \partial(f\beta)\big)\, vol(g).
\end{aligned}$$

This latter is the usual inner product between α and $\partial(f\beta)$, and therefore equals

$$\int_M h\big(\partial^*\alpha, f\beta)\big)\, vol(g) = \int_M h(\partial^*\alpha, \beta) h_L(s,s)\, vol(g) = \big((\partial^*\alpha) \otimes s, \beta\big)_L.$$

The conclusion is that $\partial^*(\alpha \otimes s) = (\partial^*\alpha) \otimes s$.

Lastly, we extend the usual Lefschetz operator $L(\alpha) = \omega \wedge \alpha$ to forms with coefficients in the line bundle by the rule

$$L(\alpha \otimes s) = (\omega \wedge \alpha) \otimes s.$$

Likewise, we define $\Lambda(\alpha \otimes s) = (\Lambda\alpha) \otimes s$. It is not hard to see that $\Lambda \colon A^{p,q}(M,L) \to A^{p-1,q-1}(M,L)$ is the adjoint of $L \colon A^{p,q}(M,L) \to A^{p+1,q+1}(M,L)$ with respect to the inner product introduced above.

For the proof, we only need two identities between this bewildering number of operators, and we have already proved both of them. Firstly, note that $\Theta_L = -2\pi i\omega$, and so we can restate the formula in Lemma 31.4 as $\partial\bar{\partial} + \bar{\partial}\partial = -2\pi i L$. After taking adjoints, we obtain the first important identity:

$$\partial^*\bar{\partial}^* + \bar{\partial}^*\partial^* = 2\pi i\Lambda. \tag{31.5}$$

Moreover, the fact that ∂^* is locally given by $\partial^*(\alpha \otimes s) = (\partial^*\alpha) \otimes s$ shows that the Kähler identity $[\Lambda, \bar{\partial}] = -i\partial^*$ generalizes to this setting of L-valued forms, giving us the second important identity:

$$\Lambda\bar{\partial} - \bar{\partial}\Lambda = -i\partial^* \tag{31.6}$$

as operators on the space $A^{p,q}(M,L)$. We are now ready to prove Theorem 31.2

Proof. Since M is compact, we can represent classes in $H^{p,q}(M,L)$ by $\bar{\partial}$-harmonic forms, and so it suffices to prove that any $\alpha \in \mathcal{H}^{p,q}(M,L)$ with $p + q > n$ has to be zero. Since α is $\bar{\partial}$-harmonic, we have $\bar{\partial}\alpha = 0$ and $\bar{\partial}^*\alpha = 0$. Now we use the two identities (31.5) and (31.6) to compute the norm of $\Lambda\alpha$. This goes as follows:

$$\begin{aligned}
(\Lambda\alpha, \Lambda\alpha)_L &= \frac{i}{2\pi}\big(\Lambda\alpha, (\bar{\partial}^*\partial^* + \partial^*\bar{\partial}^*)\alpha\big)_L = \frac{i}{2\pi}\big(\Lambda\alpha, \bar{\partial}^*\partial^*\alpha\big)_L \\
&= \frac{i}{2\pi}\big(\bar{\partial}\Lambda\alpha, \partial^*\alpha\big)_L = \frac{i}{2\pi}\big((\bar{\partial}\Lambda - \Lambda\bar{\partial})\alpha, \partial^*\alpha\big)_L \\
&= \frac{i}{2\pi}(i\partial^*\alpha, \partial^*\alpha)_L = -\frac{1}{2\pi}(\partial^*\alpha, \partial^*\alpha)_L.
\end{aligned}$$

Since we are dealing with an inner product, it follows that both sides have to be zero; in particular, $\Lambda\alpha = 0$, and so α is primitive. But we have already seen that there are no nonzero primitive forms in degree above n, and so if $p + q > n$, we get that $\alpha = 0$, as claimed. □

Note. Note that the proof depends on the identity $\partial\bar{\partial} + \bar{\partial}\partial = -2\pi i\omega$, which holds because the first Chern class of L is representable by a Kähler form. It is in this way that the positivity of the line bundle gives us the additional minus sign, which is crucial to the proof.

Since $H^{p,q}(M, L)$ computes the sheaf cohomology groups of $\Omega_M^p \otimes L$, we can also conclude the following.

Corollary 31.7. *If L is a positive line bundle on a compact complex manifold M, then $H^q(M, \Omega_M^p \otimes L) = 0$ for $p+q > n$. In particular, we have $H^q(M, \Omega_M^n \otimes L) = 0$ for every $q > 0$.*

Class 32. The Kodaira embedding theorem, Part 1

Recall that every complex submanifold of projective space is a Kähler manifold: a Kähler metric is obtained by restricting the Fubini-Study to the submanifold. Our next goal is to describe exactly which compact Kähler manifolds are *projective*, i.e., can be embedded into projective space as submanifolds. A necessary condition for M to be projective is the existence of a positive line bundle; indeed, if $M \subseteq \mathbb{P}^N$ is a submanifold, then the restriction of $\mathscr{O}_{\mathbb{P}^N}(1)$ to M is clearly a positive line bundle, since its first Chern class is represented by the restriction of ω_{FS} to M. That this condition is also sufficient is the content of the famous *Kodaira embedding theorem*: a compact complex manifold is projective if and only if it possesses a positive line bundle. In the next few lectures, we will use the Kodaira vanishing theorem to prove this result.

Maps to projective space. We begin by looking at the relationship between holomorphic line bundles and maps to projective space. Suppose then that we have a holomorphic map $f \colon M \to \mathbb{P}^N$ from a compact complex manifold to projective space. We say that f is *nondegenerate* if the image $f(M)$ is not contained in any hyperplane of $\mathbb{P}^N$. It is clearly sufficient to understand nondegenerate maps, since if f is degenerate, it is really a map from M into a projective space of smaller dimension.

Given a nondegenerate map $f \colon M \to \mathbb{P}^N$, we obtain a holomorphic line bundle $L = f^* \mathscr{O}_{\mathbb{P}^N}(1)$, the *pullback* of $\mathscr{O}_{\mathbb{P}^N}(1)$ via the map f. The fiber of L at some point $p \in M$ is defined to be the fiber of $\mathscr{O}_{\mathbb{P}^N}(1)$ at the image point $f(p)$, in other words, $L_p = \mathscr{O}_{\mathbb{P}^N}(1)_{f(p)}$. More concretely, recall that the line bundle $\mathscr{O}_{\mathbb{P}^N}(1)$ is given by the transition functions $g_{j,k} = z_k/z_j$ with respect to the standard open cover of $\mathbb{P}^N$ by the open sets $U_j = \bigl\{ [z] \in \mathbb{P}^N \bigm| z_j \neq 0 \bigr\}$. We may then define L as being the line bundle with transition functions $g_{j,k} \circ f$ on the open cover $f^{-1}(U_j)$ of M. Now every section of $\mathscr{O}_{\mathbb{P}^N}(1)$ defines, by pulling back, a section of L on M, and the resulting map

$$H^0\bigl(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1)\bigr) \to H^0(M, L)$$

is injective since f is nondegenerate. Note that we have $\dim H^0\bigl(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1)\bigr) = N + 1$.

Conversely, suppose that we have a holomorphic line bundle L on M, together with a subspace $V \subseteq H^0(M, L)$ that is *base-point free*. By this we mean that at every point $p \in M$, there should be a holomorphic section $s \in V$ that does not vanish at the point p (and hence generates the one-dimensional vector space L_p). We can then construct a holomorphic mapping from M to projective space as follows: Let $N = \dim V - 1$, choose a basis $s_0, s_1, \dots, s_N \in V$, and define

$$f \colon M \to \mathbb{P}^N, \quad f(p) = \bigl[s_0(p), s_1(p), \dots, s_N(p)\bigr].$$

That is to say, at each point of M, at least one of the sections, say s_0, is nonzero; in some neighborhood U of the point, we can then $s_j = f_j s_0$ for $f_j \in \mathscr{O}_M(U)$ holomorphic. On that open set U, the mapping f is then given by the formula $f(p) = [1, f_1(p), \dots, f_N(p)] \in \mathbb{P}^N$.

Note. A more invariant description of the map f is the following: Let $\mathbb{P}(V)$ be the set of codimension 1 subspaces of V; any such is the kernel of a linear functional on V, unique up to scaling, and so $\mathbb{P}(V)$ is naturally isomorphic to the projective space of lines through the origin in V^*. From this point of view, the mapping $f\colon M \to \mathbb{P}(V)$ takes a point $p \in M$ to the subspace $V(p) = \{\, s \in V \mid s(p) = 0 \,\}$. Since V is assumed to be base-point free, $V(p) \subseteq V$ is always of codimension 1, and so the mapping is well-defined.

The two processes above are clearly inverse to each other, and so we obtain the following result: nondegenerate holomorphic mappings $f\colon M \to \mathbb{P}^N$ are in one-to-one correspondence with base-point free subspaces $V \subseteq H^0(M, L)$ of dimension $N+1$. In particular, any holomorphic line bundle L whose space of global sections $H^0(M, L)$ is base-point free defines a holomorphic mapping

$$\varphi_L\colon M \to \mathbb{P}^N,$$

where $N = \dim H^0(M, L) - 1$. We abbreviate this by saying that L is base-point free; alternatively, one says that L is *globally generated*, since it implies that the restriction mapping $H^0(M, L) \to L_p$ is surjective for each point $p \in M$.

Example 32.1. Consider the line bundle $\mathscr{O}_{\mathbb{P}^1}(k)$ on the Riemann sphere $\mathbb{P}^1$. We have seen in the exercises that its space of sections is isomorphic to the space of homogeneous polynomials of degree k in $\mathbb{C}[z_0, z_1]$. What is the corresponding map to projective space? If we use the monomials $z_0^k, z_0^{k-1} z_1, \dots, z_0 z_1^{k-1}, z_1^k$ as a basis, we see that the line bundle is base-point free, and that the map is given by

$$\mathbb{P}^1 \to \mathbb{P}^k, \quad [z_0, z_1] \mapsto [z_0^k, z_0^{k-1} z_1, \dots, z_0 z_1^{k-1}, z_1^k].$$

It is easy to see that this is an embedding; the image is the so-called *rational normal curve* of degree k.

Example 32.2. More generally, the line bundle $\mathscr{O}_{\mathbb{P}^n}(k)$ embeds $\mathbb{P}^n$ into the larger projective space $\mathbb{P}^N$, where $N = \binom{n+k}{n} - 1$; this is the so-called *Veronese embedding.*

The Kodaira embedding theorem. For a line bundle L and a positive integer k, we let $L^k = L \otimes L \otimes \dots \otimes L$ be the k-fold tensor product of L. We can now state Kodaira's theorem in a more precise form.

Theorem 32.3. *Let M be a compact complex manifold, and let L be a positive line bundle on M. Then there is a positive integer k_0 with the following property: for every $k \geq k_0$, the line bundle L^k is base-point free, and the holomorphic mapping φ_{L^k} is an embedding of M into projective space.*

In general, suppose that L is a base-point free line bundle on M; let us investigate under what conditions the corresponding mapping $\varphi\colon M \to \mathbb{P}^N$ is an embedding. Clearly, the following two conditions are necessary and sufficient:

(a) φ is injective: if $p, q \in M$ are distinct points, then $\varphi(p) \neq \varphi(q)$.
(b) At each point $p \in M$, the differential $\varphi_*\colon T'_p M \to T'_{\varphi(p)} \mathbb{P}^N$ is injective.

Indeed, since M is compact, the map φ is automatically open, and so the first condition implies that φ is a homeomorphism onto its image $\varphi(M)$. The second condition, together with the implicit function theorem, can then be used to show that the inverse map φ^{-1} is itself holomorphic, and hence that φ is an embedding.

We shall now put both conditions in a more intrinsic form that only refers to the line bundle L and its sections. As above, let $s_0, s_1, \dots, s_N$ be a basis for the space of sections $H^0(M, L)$. Then (a) means that, for any two distinct points $p, q \in M$, the two vectors $\bigl(s_0(p), s_1(p), \dots, s_N(p)\bigr)$ and $\bigl(s_0(q), s_1(q), \dots, s_N(q)\bigr)$ should be linearly independent. Equivalently, the restriction map

$$H^0(M, L) \to L_p \oplus L_q$$

that associates to a section s the pair of values $(s(p), s(q))$ should be surjective. If this is satisfied, one says that L *separates points*.

Consider now the other condition. Fix a point $p \in M$, and suppose for simplicity that $s_0(p) \neq 0$. In a neighborhood of p, we then have $s_j = f_j s_0$ for holomorphic functions $f_1, \dots, f_N$, and (b) is saying that the matrix of partial derivatives

$$\begin{pmatrix} \partial f_1/\partial z_1 & \partial f_1/\partial z_2 & \cdots & \partial f_1/\partial z_n \\ \partial f_2/\partial z_1 & \partial f_2/\partial z_2 & \cdots & \partial f_2/\partial z_n \\ \vdots & \vdots & & \vdots \\ \partial f_N/\partial z_1 & \partial f_N/\partial z_2 & \cdots & \partial f_N/\partial z_n \end{pmatrix}$$

should have rank n at the point p. Another way to put this is that the holomorphic 1-forms $df_1, df_2, \dots, df_N$ should span the holomorphic cotangent space $T_p^{1,0}M$. More intrinsically, we let $H^0(M, L)(p)$ denote the space of sections that vanish at p. We can write any such section as $s = f s_0$, with f holomorphic in a neighborhood of p and satisfying $f(p) = 0$. Then $df(p) \otimes s_0$ is a well-defined element of the vector space $T_p^{1,0}M \otimes L_p$, independent of the choice of s_0; in these terms, condition (b) is equivalent to the surjectivity of the linear map

$$H^0(M, L)(p) \to T_p^{1,0}M \otimes L_p.$$

If this holds, one says that L *separates tangent vectors*.

Since our main tool is a vanishing theorem, it is useful to notice that both conditions can also be stated using the language of sheaves. For any point $p \in M$, we define $\mathscr{I}_p$ as the sheaf of all holomorphic functions on M that vanish at the point p. Likewise, we let $\mathscr{I}_p(L)$ denote the sheaf of holomorphic sections of L that vanish at p, and note that it is a subsheaf of the sheaf $\mathscr{O}_M(L)$ of all holomorphic sections of L. We then have an exact sequence of sheaves

$$0 \longrightarrow \mathscr{I}_p(L) \longrightarrow \mathscr{O}_M(L) \longrightarrow L_p \longrightarrow 0,$$

where we consider L_p as a sheaf supported at the point p (meaning that for any open set $U \subseteq M$, we have $L_p(U) = L_p$ if $p \in U$, and zero otherwise). The relevant portion of the long exact sequence of cohomology groups is

$$0 \longrightarrow H^0\bigl(M, \mathscr{I}_p(L)\bigr) \longrightarrow H^0(M, L) \longrightarrow L_p \longrightarrow H^1\bigl(M, \mathscr{I}_p(L)\bigr),$$

and so the surjectivity of the restriction map would follow from the vanishing of the group $H^1\bigl(M, \mathscr{I}_p(L)\bigr)$. The problem is that, unless M is a Riemann surface, this is not the cohomology group of a holomorphic line bundle, and so the Kodaira vanishing theorem does not apply to it. To overcome this difficulty, we shall use the device of blowing up: it replaces a point (codimension n) with a copy of $\mathbb{P}^{n-1}$ (codimension $n-1$), and thus allows us to work with line bundles.

Class 33. The Kodaira embedding theorem, Part 2

We continue working towards the proof of Theorem 32.3. As before, M will be a compact complex manifold, and L a holomorphic line bundle on M. We have seen that, because of the exact sequence

$$0 \longrightarrow H^0\big(M, \mathscr{I}_p(L)\big) \longrightarrow H^0(M, L) \longrightarrow L_p \longrightarrow H^1\big(M, \mathscr{I}_p(L)\big),$$

one can show that L is base-point free by proving that the cohomology group $H^1\big(M, \mathscr{I}_p(L)\big)$ vanishes for every $p \in M$. We cannot do this directly, since $\mathscr{I}_p(L)$ is not a line bundle; instead, we use the trick of blowing up the point. Today, we shall study global properties of the blow up $\operatorname{Bl}_p M$ that are necessary for the proof.

Blowing up. Let M be a complex manifold of dimension n. The blow-up of M at a point p is another complex manifold $\operatorname{Bl}_p M$, in which the point is replaced by a copy of $\mathbb{P}^{n-1}$. This so-called exceptional divisor E is basically the projective space of lines in $T'_p M$, and should be thought of as parametrizing directions from p into M. Recall the construction of $\operatorname{Bl}_p M$. First, we defined the blow-up of $\mathbb{C}^n$ at the origin as

$$\operatorname{Bl}_0 \mathbb{C}^n = \big\{ \big(z, [a]\big) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \;\big|\; z \text{ lies on the line } \mathbb{C} \cdot a \big\}.$$

The first projection $\pi\colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$ is an isomorphism outside the origin, and $\pi^{-1}(0)$ is a copy of $\mathbb{P}^{n-1}$. For any open set $D \subseteq \mathbb{C}^n$ containing the origin, we then define $\operatorname{Bl}_0 D$ as $\pi^{-1}(D)$. Finally, given a point p on an arbitrary complex manifold M, we choose a coordinate chart $\phi\colon U \to D$ around p, with $D \subseteq \mathbb{C}^n$ an open polydisk, and construct the complex manifold $\operatorname{Bl}_p M$ by gluing together $M \setminus \{p\}$ and $\operatorname{Bl}_0 D$ according to the map ϕ.

We now have to undertake a more careful study of the blow-up. From now on, we set $\tilde{M} = \operatorname{Bl}_p M$, and let $\pi\colon \operatorname{Bl}_p M \to M$ be the blow-up map. The exceptional divisor $E = \pi^{-1}(p)$ is a complex submanifold of $\tilde{M}$ of dimension $n-1$. We briefly recall why. The statement only depends on a small open neighborhood of E in $\tilde{M}$, and so it suffices to prove this for the exceptional divisor in $\operatorname{Bl}_0 \mathbb{C}^n$. Here, we have the second projection $q\colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{P}^{n-1}$, and so we get n natural coordinate charts $V_j = q^{-1}(U_j)$ (where U_j is the set of points $[a] \in \mathbb{P}^{n-1}$ with $a_j \neq 0$). These are given by

$$\mathbb{C}^n \to V_j, \quad (b_1, \dots, b_n) \mapsto \big(b_j a, [a]\big)$$

where $a = (b_1, \dots, b_{j-1}, 1, b_{j+1}, \dots, b_n)$. In these charts, the map π takes the form

$$\pi(b_1, \dots, b_n) = \big(b_j b_1, \dots, b_j b_{j-1}, b_j, b_j b_{j+1}, \dots, b_j b_n\big),$$

and so the exceptional divisor $E \cap U_j$ is exactly the submanifold defined by the equation $b_j = 0$.

Since E has dimension $n-1$, it determines a holomorphic line bundle $\mathscr{O}_{\tilde{M}}(-E)$, whose sections over any open set $U \subseteq \tilde{M}$ are those holomorphic functions on U that vanish along $U \cap E$. To simplify the notation, we write $\mathscr{O}_E(1)$ for the image of $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$ under the isomorphism $E \simeq \mathbb{P}^{n-1}$.

Lemma 33.1. *The restriction of $\mathscr{O}_{\tilde{M}}(-E)$ to the exceptional divisor is isomorphic to $\mathscr{O}_E(1)$.*

Proof. The statement only depends on a small neighborhood of E in $\tilde{M}$, and we may therefore assume that we are dealing with the blowup of $\mathbb{C}^n$ at the origin. We have seen in the exercises that the second projection $q\colon \operatorname{Bl}_0 \mathbb{C}^n \to \mathbb{P}^{-1}$ is the holomorphic line bundle $\mathscr{O}_{\mathbb{P}^{n-1}}(-1)$. The exceptional divisor is precisely the image of the zero section, and by another exercise, its line bundle is isomorphic to $q^*\mathscr{O}_{\mathbb{P}^{n-1}}(1)$. Obviously, the restriction of this line bundle to the exceptional divisor is now $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$, as claimed. □

To simplify the notation a little, we shall write $[-E]$ for the line bundle $\mathscr{O}_{\tilde{M}}(-E)$, and $[E]$ for its dual. As usual, we also let $[E]^k$ be the k-fold tensor product of $[E]$ with itself. Lastly, we write K_M for the canonical bundle Ω_M^n. In order to apply the Kodaira vanishing theorem on $\tilde{M}$, we need to now how the canonical bundle $K_{\tilde{M}}$ is related to K_M.

Lemma 33.2. *The canonical bundle of $\tilde{M}$ satisfies $K_{\tilde{M}} \simeq \pi^* K_M \otimes [E]^{n-1}$.*

Proof. To show the gist of the statement, we shall only prove this in the case $M = \mathbb{C}^n$ and $\tilde{M} = \operatorname{Bl}_0 \mathbb{C}^n$. With $z_1, \dots, z_n$ the usual coordinate system on $\mathbb{C}^n$, the canonical bundle Ω_M^n is trivial, generated by the section $dz_1 \wedge \cdots \wedge dz_n$. To prove the lemma, it is enough to show that the line bundle $K_{\tilde{M}} \otimes [-E]^{n-1}$ is trivial on $\tilde{M}$. Note that its holomorphic sections are holomorphic n-forms that vanish at least to order $n-1$ along E.

Consider the pullback $\pi^*(dz_1 \wedge \cdots \wedge dz_n)$. In one of the n open sets V_j that cover the blow-up, the exceptional divisor is defined by the equation $b_j = 0$, and the map π is given by the formula $\pi(b_1, \dots, b_n) = (b_j b_1, \dots, b_j b_{j-1}, b_j, b_j b_{j+1}, \dots, b_j b_n)$. Consequently, we have

$$\begin{aligned} \pi^*(dz_1 \wedge \cdots \wedge dz_n) &= d(b_j b_1) \wedge \cdots \wedge d(b_j b_{j-1}) \wedge db_j \wedge d(b_j b_{j+1}) \wedge \cdots \wedge d(b_j b_n) \\ &= b_j^{n-1} db_1 \wedge \cdots \wedge db_n, \end{aligned}$$

and so $\pi^*(dz_1 \wedge \cdots \wedge dz_n)$ is a global section of $K_{\tilde{M}} \otimes [-E]^{n-1}$. The above formula shows that, moreover, it generates said line bundle on each open set V_j, and so the line bundle is indeed trivial. □

Lemma 33.3. *Let L be a positive line bundle on M. Then for sufficiently large k, the line bundle $\tilde{L}^k \otimes [-E]$ is again positive.*

Proof. Recall that a real $(1,1)$-form α is said to be positive if $\alpha(\xi, \bar{\xi}) > 0$ for every nonzero tangent vector $\xi \in T_p' M$. A holomorphic line bundle is positive if it admits a Hermitian metric for which the real $(1,1)$-form $\frac{i}{2\pi}\Theta$ is positive.

We give the pullback line bundle $\tilde{L} = \pi^* L$ the induced Hermitian metric. Since L is positive, its first Chern class $\omega = \frac{i}{2\pi}\Theta_L$ is a positive form, and so $\frac{i}{2\pi}\Theta_{\tilde{L}} = \pi^*\omega$ is positive outside the exceptional divisor E. At points of E, however, the form $\pi^*\omega$ fails to be positive—more precisely, we have $(\pi^*\omega)(\xi, \bar{\xi}) = 0$ for any ξ that is tangent to E—because the restriction of $\tilde{L}$ to E is trivial. The idea is to construct a Hermitian metric h_E on $[-E]$ which is positive in the directions tangent to E; by choosing $k \gg 0$, we can then make sure that $\Omega_k = \pi^*\omega + \frac{i}{2\pi}\Theta_E$, which represents the first Chern class of $\tilde{L}^k \otimes [-E]$, is a positive form on $\tilde{M}$.

To construct that metric, let U be an open neighborhood of the point p, isomorphic to an open polydisk $D \subseteq \mathbb{C}^n$, and let $z_1, \dots, z_n$ be the resulting holomorphic coordinate system centered at p. Then $U_1 = \pi^{-1}(U)$ is isomorphic to $\operatorname{Bl}_0 D$, the blow-up of the origin in D, which we originally constructed as a submanifold of the product $D \times \mathbb{P}^{n-1}$. We may thus view U_1 itself as being a submanifold of $U \times \mathbb{P}^{n-1}$; under this identification, the line bundle $[-E]$ is isomorphic to the pullback of $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$ by the map $q\colon U_1 \to \mathbb{P}^{n-1}$. The latter has a canonical metric, and so we get a Hermitian metric h_1 on the restriction of $[-E]$ to the open set U_1. Note that $i/2\pi$ times its curvature form is equal to the pullback $q^*\omega_{FS}$ of the Fubini-Study from $\mathbb{P}^{n-1}$.

Let $M^* = M \setminus \{p\}$; by construction, the map π is an isomorphism between $U_2 = \tilde{M} \setminus E$ and M^*, and since $[-E]$ is trivial on the complement of E, it has a distinguished nowhere vanishing section s_E over U_2, corresponding to the constant function $1 \in \mathscr{O}_M(M^*)$. We can thus put a Hermitian metric h_2 on the restriction of $[-E]$ to U_2, by declaring the pointwise norm of s_E to be 1. Now let $\rho_1 + \rho_2 = 1$ be a partition of unity subordinate to the open cover U, M^*, and define a Hermitian metric on $[-E]$ by setting

$$h_E = (\rho_1 \circ \pi)h_1 + (\rho_2 \circ \pi)h_2.$$

This is well-defined, and indeed a Hermitian metric (because the convex combination of two Hermitian inner products on a vector space is again a Hermitian inner product).

To complete the proof, we have to argue that $\Omega_k = \frac{i}{2\pi}\Theta_E + k\pi^*\omega$ is a positive form if $k \gg 0$. First consider the open set $U_1 = \pi^{-1}(U)$ containing the exceptional divisor. For any $k > 0$, the form $k \cdot pr_1^*\omega + pr_2^*\omega_{FS}$ on the product $U \times \mathbb{P}^{n-1}$ is clearly positive. In a sufficiently small neighborhood V of the exceptional divisor (namely outside the support of $\rho_2 \circ \pi$), Ω_k is the restriction of that form to the submanifold U_1, and is therefore positive as well. On the other hand, the complement $\tilde{M} \setminus V$ of that neighborhood is a compact set in $\tilde{M} \setminus E$, on which $\frac{i}{2\pi}\Theta_E$ is bounded and $\pi^*\omega$ is positive. By taking k sufficiently large, we can therefore make Ω_k be positive on $\tilde{M} \setminus V$ as well. $\square$

Class 34. The Kodaira embedding theorem, Part 3

The two conditions. We now come to the proof of Theorem 32.3. We continue to let M be a compact complex manifold, and $L \to M$ a positive line bundle. In order to prove the embedding theorem, we have to show that for $k \gg 0$, the following three things are true:

1. The line bundle L^k is base-point free, and therefore defines a holomorphic mapping $\varphi_{L^k}\colon M \to \mathbb{P}^{N_k}$, where $N_k = \dim H^0(M, L^k) - 1$. Equivalently, for every point $p \in M$, the restriction map $H^0(M, L^k) \to L_p^k$ is surjective.
2. The mapping φ_{L^k} is injective; equivalently, for every pair of distinct points $p, q \in M$, the restriction map $H^0(M, L^k) \to L_p^k \oplus L_q^k$ is surjective.
3. The mapping φ_{L^k} is an immersion, which means that its differential is injective; equivalently, the map $H^0(M, L^k)(p) \to T_p^{1,0}M \otimes L_p^k$ is surjective at every point $p \in M$.

In each of the three cases, the strategy is to blow up the point (or points) in question, and to reduce the surjectivity to the vanishing of some cohomology group on the blow-up. We then show that, after choosing $k \gg 0$, the group is question is zero by Kodaira's theorem.

We shall break the proof down into four steps, which are fairly similar to each other.

Step 1. To show that L^k is base-point free for $k \gg 0$, we begin by proving that for every fixed point $p \in M$, the map $H^0(M, L^k) \to L_p^k$ is surjective once k is large. Let $\pi\colon \tilde{M} \to M$ denote the blow-up of M at the point p, and let $E = \pi^{-1}(p)$ be the exceptional divisor. Let $i\colon E \hookrightarrow \tilde{M}$ be the inclusion map, and let $\tilde{L} = \pi^* L$ be the pullback of the line bundle. Every section of L on M defines by pullback a section of $\tilde{L} = \pi^* L$ on $\tilde{M}$. The resulting linear map

$$H^0(M, L^k) \to H^0(\tilde{M}, \tilde{L}^k)$$

is an isomorphism by Hartog's theorem. Indeed, suppose that $\tilde{s}$ is a global section of $\tilde{L}^k$. Since $\tilde{M} \setminus E \simeq M^*$, the restriction of $\tilde{s}$ to $\tilde{M} \setminus E$ gives a holomorphic section of L^k over M^*. If $n \geq 2$, then Hartog's theorem shows that this section extends holomorphically over the point p, proving that $\tilde{s}$ is in the image of $H^0(M, L^k)$. (If $n = 1$, we have $\tilde{M} = M$ and $E = \{p\}$, and so the statement is trivial.)

Now clearly a section of L^k vanishes at the point p iff the corresponding section of $\tilde{L}^k$ vanishes along the exceptional divisor E; in other words, we have a commutative diagram

$$\begin{tikzcd} H^0(M, L^k) \arrow[r] \arrow[d, equal] & L_p^k \arrow[d, equal] \\ H^0(\tilde{M}, \tilde{L}^k) \arrow[r] & H^0(E, i^* \tilde{L}_p^k). \end{tikzcd}$$

Note that $i^* \tilde{L}^k \simeq \mathscr{O}_E \otimes \tilde{L}_p^k$, since the restriction of $\tilde{L}^k$ to the exceptional divisor is the trivial line bundle with fiber $\tilde{L}_p^k$. It is therefore sufficient to prove that, on $\tilde{M}$, the restriction map $H^0(\tilde{M}, \tilde{L}^k) \to H^0(E, i^* \tilde{L}^k)$ is surjective.

Because of the long exact cohomology sequence

$$H^0(\tilde{M}, \tilde{L}^k) \longrightarrow H^0(E, i^* \tilde{L}^k) \longrightarrow H^1\big(\tilde{M}, \tilde{L}^k \otimes [-E]\big),$$

the surjectivity is a consequence of $H^1\big(\tilde{M}, \tilde{L}^k \otimes [-E]\big) \simeq 0$. This will follow from the Kodaira vanishing theorem, provided we can show that

$$\tilde{L}^k \otimes [-E] \simeq K_{\tilde{M}} \otimes P_k$$

for some positive line bundle P_k. By Lemma 33.2, we have $K_{\tilde{M}} \simeq \pi^* K_M \otimes [E]^{n-1}$, and so

$$P_k \simeq \pi^*\big(L^k \otimes K_M^{-1}\big) \otimes [-E]^n.$$

Now fix a sufficiently large integer ℓ, with the property that $L^\ell \otimes K_M^{-1}$ is positive. By Lemma 33.3, there exists an integer m_0 such that the line bundle $\tilde{L}^m \otimes [-E]$ is positive for $m \geq m_0$. But then

$$\pi\big(L^\ell \otimes K_M^{-1}\big) \otimes \big(\tilde{L}^m \otimes [-E]\big)^n \simeq \pi^*\big(L^{mn+\ell} \otimes K_M^{-1}\big) \otimes [-E]^n$$

is positive, and so it suffices to take $k \geq m_0 n + \ell$. With this choice of k, we have

$$H^1\big(\tilde{M}, \tilde{L}^k \otimes [-E]\big) \simeq H^1\big(\tilde{M}, K_{\tilde{M}} \otimes P_k\big) \simeq 0,$$

which vanishes by Theorem 31.2 because P_k is a positive line bundle. So if $k \geq m_0 n + \ell$, then the restriction map $H^0(M, L^k) \to L_p^k$ is surjective.

Unfortunately, the value of m_0 might depend on the point $p \in M$ that we started from. To show that one value works for all points $p \in M$, we use a compactness argument. Namely, if $H^0(M, L^k) \to L_p^k$ is surjective at some point $p \in M$, it means that L^k has a section that does not vanish at p. The same section is nonzero at nearby points, and so the restriction map is surjective on some neigborhood of the point. We can therefore cover M by open sets U_i, such that the restriction map is surjective for $k \geq k_i$. By compactness, finitely many of these open sets cover M, and if we let k_0 be the maximum of the corresponding k_i, then we get surjectivity at all points for $k \geq k_0$. We have now shown that the mapping φ_{L^k} is well-defined and holomorphic for sufficiently large values of k.

Step 2. Exactly the same proof shows that, given any pair of distinct points $p, q \in M$, the restriction map $H^0(M, L^k) \to L_p^k \oplus L_q^k$ is surjective for $k \gg 0$. We only need to let $\pi \colon \tilde{M} \to M$ be the blow-up of M at both points, and $E = \pi^{-1}(p) \cup \pi^{-1}(q)$ the union of the two exceptional divisors (which is still a submanifold of dimension $n-1$). If $i \colon E \hookrightarrow \tilde{M}$ denotes the inclusion, then it suffices to prove the surjectivity of

$$H^0(\tilde{M}, \tilde{L}^k) \to H^0(E, i^* \tilde{L}^k),$$

which holds for the same reason as before once $k \gg 0$. Note that the value of k now depends on the pair of points $p, q \in M$; but this time, we cannot use the same compactness proof because $M \times M \setminus \Delta$ is no longer compact. We will deal with this issue in the last step of the proof.

Step 3. Next, we prove that for a fixed point $p \in M$, the map

$$H^0(M, L^k)(p) \to T_p^{1,0} M \otimes L_p^k$$

becomes surjective if $k \gg 0$. Here $H^0(M, L^k)(p)$ denotes the space of sections of L^k that vanish at the point p. Again let $\pi \colon \tilde{M} \to M$ be the blow-up of M at the point p, let $i \colon E \hookrightarrow \tilde{M}$ be the inclusion of the single exceptional divisor, and let $\tilde{L} = \pi^* L$ be the pullback of our positive line bundle. This time, we use the commutative diagram

$$\begin{tikzcd}
H^0(M, L^k)(p) \arrow[r] \arrow[d, equal] & T_p^{1,0} M \otimes L_p^k \arrow[d, equal] \\
H^0\bigl(M, \tilde{L}^k \otimes [-E]\bigr) \arrow[r] & H^0\bigl(E, i^* \tilde{L}^k \otimes [-E]\bigr).
\end{tikzcd}$$

Note that the restriction of $\tilde{L}^k \otimes [-E]$ to the exceptional divisor is isomorphic to $\mathscr{O}_E(1) \otimes \tilde{L}_p^k$, and so its space of global sections is $H^0\bigl(E, \mathscr{O}_E(1)\bigr) \otimes \tilde{L}_p^k$. Sections of $\mathscr{O}_E(1)$ are linear forms in the variables $z_1, \dots, z_n$, which exactly correspond to the holomorphic cotangent space $T_p^{1,0} M$.

In other words, it is now sufficient to prove the surjectivity of

$$H^0\bigl(\tilde{M}, \tilde{L}^k \otimes [-E]\bigr) \to H^0\bigl(E, \mathscr{O}_E(1)\bigr) \otimes \tilde{L}_p^k,$$

for which we may use the exact sequence

$$H^0\bigl(\tilde{M}, \tilde{L}^k \otimes [-E]\bigr) \longrightarrow H^0\bigl(E, i^* \tilde{L}^k \otimes [-E]\bigr) \longrightarrow H^1\bigl(\tilde{M}, \tilde{L}^k \otimes [-E]^2\bigr).$$

To prove the vanishing of the group $H^1\bigl(\tilde{M}, \tilde{L}^k \otimes [-E]^2\bigr)$, we argue as before to obtain

$$\tilde{L}^k \otimes [-E]^2 \simeq K_{\tilde{M}} \otimes Q_k$$

for a positive line bundle Q_k, once $k \geq (n-1)m_0 + \ell$. The required vanishing then follows from Theorem 31.2. Again, note that the lower bound on k may depend on the point $p \in M$.

Step 4. To finish the proof, we have to argue that there is a single integer k_0, such that (a) and (b) hold for all points $p, q \in M$ once $k \geq k_0$. We shall prove this by using the compactness of the product $M \times M$.

Recall that (b) holds at some point $p_0 \in M$ iff the differential of the mapping φ_{L^k} is injective. By basic calculus, this implies that φ_{L^k} is injective in a small neighborhood of p_0, and so (a) and (b) are both true for all (p, q) with $p \neq q$ that belong to a small neighborhood of $(p_0, p_0) \in M \times M$. On the other hand, Step 3 shows that (a) holds in a neighborhood of every pair (p, q) with $p \neq q$. It follows that we can cover $M \times M$ by open subsets V_i, on each of which (a) and (b) are true once $k \geq k_i$. By compactness, finitely many of those open sets cover the product, and so we again obtain a single value of k_0 such that φ_{L^k} is an embedding for $k \geq k_0$. This completes the proof of the Kodaira embedding theorem.

Class 35. Complex tori and Riemann's criterion

In algebraic geometry, a line bundle is called *very ample* if φ_L is an embedding; L is called *ample* if L^k is very ample for $k \gg 0$. Thus what we have shown is: a line bundle L on a compact Kähler manifold M is positive iff it is ample. Thus for the complex geometer, ampleness corresponds to positivity of curvature, in the sense that $\frac{i}{2\pi}\Theta$ is a positive form.

Example 35.1. During the proof of Theorem 32.3, we have seen that if $\pi \colon \operatorname{Bl}_p M \to M$ is the blow-up of M at some point p, and if L is a positive line bundle on M, then $\pi^* L^k \otimes [-E]$ is a positive line bundle on $\operatorname{Bl}_p M$ for $k \gg 0$. It follows that if the manifold M is projective, the blow-up $\operatorname{Bl}_p M$ is also projective. Since the latter was defined by gluing, this is not at all obvious.

The Kodaira embedding theorem can be restated to provide a purely cohomological criterion for a compact Kähler manifold to be projective.

Proposition 35.2. *Let M be a compact Kähler manifold. Then M is projective if, and only if, there exists a closed positive $(1,1)$-form $\omega \in A^2(M)$ whose cohomology class $[\omega]$ is rational, i.e., belongs to the subspace $H^2(M, \mathbb{Q}) \subseteq H^2(M, \mathbb{C})$.*

Proof. If M is projective, then we can take for ω the restriction of the Fubini-Study form from projective space. We will prove the converse by showing that M has a positive line bundle. After multiplying ω by a positive integer, we can assume that $[\omega]$ belongs to the image of the map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C})$. As M is Kähler, we have $H^2(M, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$, and as previously explained, the exact sequence

$$H^1(M, \mathscr{O}_M) \longrightarrow H^1(M, \mathscr{O}_M^*) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathscr{O}_M)$$

shows that $[\omega]$ is the first Chern class of a holomorphic line bundle L on M. By construction, L is positive (since its first Chern class is represented by the positive form ω), and so M is projective by Theorem 32.3. $\square$

In certain cases, the criterion can be used directly to prove projectivity. A very useful one is the following.

Corollary 35.3. *If a compact Kähler manifold M satisfies $H^2(M, \mathscr{O}_M) \simeq 0$, then it is necessarily projective.*

Proof. Fix some Kähler metric h_0 on M, and let ω_0 be the Kähler form. Then ω_0 is a closed positive $(1,1)$-form whose cohomology class belongs to $H^2(M,\mathbb{R})$. We can represent classes in $H^2(M,\mathbb{C})$ uniquely by harmonic forms (with respect to the metric h_0), with classes in $H^2(M,\mathbb{R})$ represented by real forms. Moreover, the inner product $(\alpha,\beta)_M$ that we previously defined gives us a way to measure distances in $H^2(M,\mathbb{C})$. By assumption, the two subspaces $H^{0,2}$ and $H^{2,0}$ in the Hodge decomposition are both zero, and so $H^2(M,\mathbb{C}) = H^{1,1}$. Now the space of rational classes $H^2(M,\mathbb{Q})$ is dense in $H^2(M,\mathbb{R})$, and so for any $\varepsilon > 0$, there exists a harmonic $(1,1)$-form ω with rational cohomology class satisfying $\|\omega - \omega_0\|_M < \varepsilon$. Now the point is that, M being compact, any such ω that is sufficiently close to ω_0 will still be positive (because the condition of being positive definite is stable under small perturbations). We can then conclude by the criterion in Proposition 35.2. $\square$

Example 35.4. A *Calabi-Yau manifold* is a compact Kähler manifold M whose canonical bundle K_M is isomorphic to the trivial line bundle, and on which the cohomology groups $H^q(M,\mathscr{O}_M)$ for $1 \leq q \leq \dim M - 1$ vanish. If $\dim M \geq 3$, then such an M can always be embedded into projective space.

Example 35.5. Any compact Riemann surface is projective. (This can of course be proved more easily by other methods.)

Complex tori. A nice class of compact Kähler manifolds is that of complex tori, which meant quotients of the form $T = \mathbb{C}^n/\Lambda$, for Λ a lattice in $\mathbb{C}^n$. In the exercises, we have seen that the standard metric on V descends to a Kähler metric on T. To illustrate the usefulness of Kodaira's theorem, we shall settle the following question: when is a complex torus T projective?

Example 35.6. Everyone knows that elliptic curves (the case $n = 1$) can always be embedded into $\mathbb{P}^2$ as cubic curves.

The following theorem, known as Riemann's criterion, gives a necessary and sufficient condition for T to be projective.

Theorem 35.7. *Let $T = \mathbb{C}^n/\Lambda$ be a complex torus. Then T is projective if, and only if, there exists a positive definite Hermitian bilinear form $h\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, whose imaginary part $\operatorname{Im} h$ takes integral values on $\Lambda \times \Lambda$.*

Proof. In fact, the stated condition is equivalent to the existence of a closed positive $(1,1)$-form on T whose cohomology class is integral; the proof is therefore mostly an exercise in translation.

To begin with, recall that if we let $V_{\mathbb{R}} = H^1(T,\mathbb{R})$, then the complexification decomposes as $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$; as we saw in the proof of Lemma 24.2, a basis for the space $V^{1,0}$ is given by the images of the forms $dz_1, \dots, dz_n$. Since $\mathbb{C}^n$ is its own holomorphic tangent space, this means that $V^{1,0}$ is naturally isomorphic to the dual vector space of $\mathbb{C}^n$. The Hodge decomposition of the cohomology of T is given by the spaces $V^{p,q} = \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1} \subseteq H^{p+q}(T,\mathbb{C})$. Thus a closed $(1,1)$-form ω on T is the same thing as an element of the space $V^{1,1} = V^{1,0} \otimes V^{0,1}$, which is the same thing as a Hermitian bilinear form h on $\mathbb{C}^n$ (because $V^{0,1}$ is the complex conjugate of $V^{1,0}$). Also, ω is clearly positive iff h is positive definite.

What does it mean for ω to be integral? The first homology group $H_1(T,\mathbb{Z})$ is isomorphic to the lattice Λ—indeed, every $\lambda \in \Lambda$ defines an element in homology, namely the image in T of the line segment connecting 0 and λ. By the universal coefficients theorem, $V_{\mathbb{R}} \simeq \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R})$, with similar isomorphisms for the higher cohomology groups. In particular, ω belongs to the image of $H^2(T, \mathbb{Z})$ iff $h(\lambda_1, \lambda_2) \in \mathbb{Z}$ for every $\lambda_1, \lambda_2 \in \Lambda$. This concludes the proof. $\square$

Class 36. Kähler manifolds and projective manifolds

At this point, a few words about the nature of projective manifolds are probably in order. Most compact Kähler manifolds are not projective, and the subset of those that are is quite small. To see why this should be, let us consider the space $H_{\mathbb{R}}^{1,1}$, the intersection of $H^{1,1}$ and $H^2(M, \mathbb{R})$ inside $H^2(M, \mathbb{C})$. It consists of those real cohomology classes that can be represented by a closed form of type $(1,1)$. We say that a class $\alpha \in H_{\mathbb{R}}^{1,1}$ is a *Kähler class* if it can be represented by a closed positive $(1,1)$-form. The set of all such forms is a cone (since it is closed under addition, and under multiplication by positive real numbers), the so-called *Kähler cone* of the manifold M. Now in order for M to be projective, the Kähler cone has to contain at least one nonzero rational class. But the space $H_{\mathbb{Q}}^{1,1} = H^2(M, \mathbb{Q}) \cap H^{1,1}$ of rational classes is a discrete subset of $H_{\mathbb{R}}^{1,1}$, and in general, it is unlikely that the Kähler cone will intersect it nontrivially.

Example 36.1. Consider again the case of K3-surfaces, that is, compact Kähler surfaces whose Hodge diamond looks like

$$
\begin{array}{ccccc}
 & & \mathbb{C} & & \\
 & 0 & & 0 & \\
\mathbb{C} & & \mathbb{C}^{20} & & \mathbb{C} \\
 & 0 & & 0 & \\
 & & \mathbb{C} & &
\end{array}
$$

When discussing Griffith's theorem, we saw that nonsingular quartic hypersurfaces in $\mathbb{P}^3$ are K3-surfaces. The space of homogeneous polynomials of degree 4 has dimension $\binom{4+3}{3} = 35$, and so nonsingular quartic hypersurfaces are naturally parametrized by an open subset in $\mathbb{P}^{34}$. On the other hand, the automorphism group of $\mathbb{P}^3$ has dimension 15, and if we take its action into account, we find that this particular class of K3-surfaces forms a 19-dimensional family.

In the theory of deformations of complex manifolds, it is shown that there is a 20-dimensional manifold P that parametrizes all possible K3-surfaces (20 being the dimension of $H^{1,1}$). Now what about projective K3-surfaces? They form a dense subset of P, consisting of countably many analytic subsets of dimension 19. So, just as in the case of those K3-surfaces that can be realized as quartic surfaces in $\mathbb{P}^3$, projective K3-surfaces always come in 19-dimensional families; but altogether, they are still a relatively sparse subset of the space of all K3-surfaces.

Why are the subsets corresponding to projective K3-surfaces all of dimension 19? The answer has to do with the Hodge decomposition on $H^2(M, \mathbb{C})$. Let us fix some projective K3-surface M_0, and consider those M that are close to M_0 on the moduli space P. It is possible to identify the cohomology group $H^2(M, \mathbb{Z})$ with $H^2(M_0, \mathbb{Z})$, and hence $H^2(M, \mathbb{C})$ with $H^2(M_0, \mathbb{C})$. We can then think of the Hodge decomposition on $H^2(M, \mathbb{C})$ as giving us a decomposition of the fixed 22-dimensional vector space $H^2(M_0, \mathbb{C})$ into subspaces of dimension 1, 20, and 1. (This is an example of a so-called variation of Hodge structure.)

M_0 being projective, there exists $\omega_0 \in H^2(M_0,\mathbb{Z})$ whose class in $H^2(M_0,\mathbb{C})$ is represented by a closed positive $(1,1)$-form. Through the isomorphism $H^2(M,\mathbb{Z}) \simeq H^2(M_0,\mathbb{Z})$, we get a class $\omega_M \in H^2(M,\mathbb{Z})$ on every nearby K3-surface M. If M is to remain projective, then this class should still be of type $(1,1)$, which means that its image in $H^{0,2}(M)$ should be zero. Since $\dim H^{0,2}(M) = 1$, this is one condition, and so the set of M where $\omega_M \in H^{1,1}(M)$ will be a hypersurface in P (positivity is automatic if M is close to M_0).

A complex torus without geometry. To illustrate how far a general compact Kähler manifold is from being projective, we shall now look at an example of a two-dimensional complex torus T in which the only analytic subsets are points and T itself. In contrast to this, a submanifold of projective space always has a very rich geometry, since there are many analytic subsets obtained by intersecting with various linear subspaces of projective space. The torus T in the example (due to Zucker) can therefore not be embedded into projective space.

Let $V = \mathbb{C}\oplus\mathbb{C}$, with coordinates (z,w), and let $J\colon V \to V$ be the complex-linear mapping defined by $J(z,w) = (iz,-iw)$. Let $\Lambda \subseteq V$ be a lattice with the property that $J(\Lambda) = \Lambda$, and form the 2-dimensional complex torus $T = V/\Lambda$. Then J induces an automorphism of T, and we refer to T as a *J-torus*. Any lattice of this type can be described by a basis of the form v_1, v_2, Jv_1, Jv_2, and is thus given by a 2×4-matrix

$$\begin{pmatrix} a & b & ia & ib \\ c & d & -ic & -id \end{pmatrix}$$

with complex entries. Here $a,b,c,d \in \mathbb{C}$ need to be chosen such that the four column vectors of the matrix are linearly independent over $\mathbb{R}$, but are otherwise arbitrary. In this way, we have a whole four-dimensional family of J-tori.

Lemma 36.2. *If we let $f = a\bar{d} - b\bar{c}$, then both the real and the imaginary part of $\theta = f^{-1} dz \wedge d\bar{w}$ are closed $(1,1)$-forms with integral cohomology class.*

Proof. Both the real and the imaginary part of θ are closed forms of type $(1,1)$, because $\operatorname{Re}\theta = \frac{1}{2}(\theta + \overline{\theta})$ and $\operatorname{Im}\theta = \frac{1}{2i}(\theta - \overline{\theta})$. As explained before, we have $\Lambda = H_1(T,\mathbb{Z})$, and so to show that a closed form defines an integral cohomology class, it suffices to evaluate it on vectors in Λ. If we substitute (u_1,v_1) and (u_2,v_2) into the form $dz \wedge d\bar{w}$, we obtain $u_1\overline{v_2} - u_2\overline{v_1}$. The 16 evaluations of $dz\wedge d\bar{w}$ can thus be summarized by the matrix computation

$$\begin{pmatrix} -\bar{c} & a \\ -\bar{d} & b \\ -i\bar{c} & ia \\ -i\bar{d} & ib \end{pmatrix} \begin{pmatrix} a & b & ia & ib \\ \bar{c} & \bar{d} & i\bar{c} & i\bar{d} \end{pmatrix} = \begin{pmatrix} 0 & a\bar{d}-b\bar{c} & 0 & i(a\bar{d}-b\bar{c}) \\ b\bar{c}-a\bar{d} & 0 & i(b\bar{c}-a\bar{d}) & 0 \\ 0 & i(a\bar{d}-b\bar{c}) & 0 & b\bar{c}-a\bar{d} \\ i(b\bar{c}-a\bar{d}) & 0 & a\bar{d}-b\bar{c} & 0 \end{pmatrix},$$

which proves that all values of θ on $\Lambda\times\Lambda$ are contained in the set $\{0,\pm 1,\pm i\}$. $\square$

Now let $\alpha = \operatorname{Re}\theta$ and $\beta = \operatorname{Im}\theta$; both are closed $(1,1)$-forms with integral cohomology class. Our next goal is to show that, for a generic lattice Λ (corresponding to a generic choice of $a,b,c,d \in \mathbb{C}$), these are the only cohomology classes that are both integral and of type $(1,1)$.

Lemma 36.3. *If the lattice Λ is generic, then $H^2(T,\mathbb{Z}) \cap H^{1,1}(T) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$.*

Proof. Let e_1, e_2, e_3, e_4 be the four basis vectors of Λ, and let $e_1^*, e_2^*, e_3^*, e_4^* \in H^1(T, \mathbb{Z})$ be the dual basis. According to the calculation above, we then have

$$\alpha = e_1^* \wedge e_2^* - e_3^* \wedge e_4^* \qquad \text{and} \qquad \beta = e_1^* \wedge e_4^* - e_2^* \wedge e_3^*.$$

We can now write any element in $H^2(T, \mathbb{Z})$ in the form

$$\varphi = \sum_{1 \leq j < k \leq 4} u_{j,k} e_j^* \wedge e_k^*,$$

where the six coefficients $u_{j,k}$ are integers. In order for this form to be of type $(1,1)$, what has to happen is that $dz \wedge dw \wedge \varphi = 0$. For every choice of integers $u_{j,k}$, this is a polynomial equation in the four complex numbers a, b, c, d.

What are those equations? By a computation similar to the above, one has

$$\begin{pmatrix} -c & a \\ -d & b \\ ic & ia \\ id & ib \end{pmatrix} \begin{pmatrix} a & b & ia & ib \\ c & d & -ic & -id \end{pmatrix} = \begin{pmatrix} 0 & ad-bc & -2iac & -i(ad+bc) \\ bc-ad & 0 & -i(ad+bc) & -2ibd \\ 2iac & i(ad+bc) & 0 & ad-bc \\ i(ad+bc) & 2ibd & bc-ad & 0 \end{pmatrix},$$

from which it follows that

$$dz \wedge dw = (ad-bc)(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*) - i(ad+bc)(e_1^* \wedge e_4^* + e_2^* \wedge e_3^*) - 2iac e_1^* \wedge e_3^* - 2ibd e_2^* \wedge e_4^*.$$

After simplifying the resulting formulas, we find that $dz \wedge dw \wedge \varphi = C e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*$, where the coefficient is given by

$$C = (ad-bc)(u_{3,4} + u_{1,2}) - i(ad+bc)(u_{2,3} + u_{1,4}) + 2iac u_{2,4} + 2ibd u_{1,3}.$$

To complete the proof, we have to show that for a general choice of $(a, b, c, d) \in \mathbb{C}^4$, the equation $C = 0$ can only be satisfied if φ is a linear combination of α and β.

By subtracting suitable multiples of α and β, we may assume that $u_{3,4} = u_{2,3} = 0$. We are then left with the equation

$$(ad-bc)u_{1,2} - i(ad+bc)u_{1,4} + 2iac u_{2,4} + 2ibd u_{1,3} = 0.$$

If we now set $a = xb$ and $c = yd$, and choose $x, y \in \mathbb{C}$ algebraically independent over $\mathbb{Q}$, we arrive at

$$(x-y)u_{1,2} - i(x+y)u_{1,4} + 2ixy u_{2,4} + 2i u_{1,3},$$

which clearly has no nontrivial solution in integers $u_{1,2}, u_{1,4}, u_{2,4}, u_{1,3}$. This proves that each of the polynomial equations above defines a proper analytic subset of $\mathbb{C}^4$, and consequently of measure zero. We have countably many of these sets (parametrized by the choice of $u_{j,k}$), and it follows that the set of parameters $(a, b, c, d) \in \mathbb{C}^4$ for which the corresponding J-torus satisfies $H^{1,1}(T) \cap H^2(T, \mathbb{Z}) \neq \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ has measure zero. □

From now on, we let T be a generic J-torus in the sense of Lemma 36.3. Recall that J defines an automorphism of T. It is easy to see that we have $J^*\theta = f^{-1}(idz) \wedge (id\bar{w}) = -\theta$, and hence $J^*\alpha = -\alpha$ and $J^*\beta = -\beta$. Since T is generic, we conclude that $J^*\varphi = -\varphi$ for every class $\varphi \in H^2(T, \mathbb{Z}) \cap H^{1,1}(T)$.

Lemma 36.4. *If T is a generic J-torus, then T contains no analytic subsets of dimension one.*

Proof. We will first show that T contains no one-dimensional complex submanifolds. Suppose to the contrary that $C \subseteq T$ was such a submanifold. Integration over C defines a cohomology class $[C] \in H^2(T, \mathbb{Z}) \cap H^{1,1}(T)$, and by the calculation above, we have $[J^{-1}C] = J^*[C] = -[C]$. This shows that $[C] + [J^{-1}C] = 0$. But such an identity is impossible on a compact Kähler manifold: letting ω be the Kähler form of the natural Kähler metric on T, the integral

$$\int_T \omega \wedge \big([C] + [J^{-1}C]\big) = \int_C \omega|_C + \int_{J^{-1}C} \omega|_{J^{-1}C} = \operatorname{vol}(C) + \operatorname{vol}(J^{-1}C)$$

is the volume of the two submanifolds with respect to the induced metric, and hence positive. This is a contradiction, and so it follows that T cannot contain any one-dimensional submanifolds.

Similarly, if $Z \subseteq T$ is a one-dimensional analytic subset, one can show that integration over the set of smooth points of Z (the complement of a finite set of points) defines a cohomology class $[Z] \in H^2(T, \mathbb{Z}) \cap H^{1,1}(T)$, whose integral against the Kähler form ω is positive. As before, we conclude that there cannot be such analytic subsets in a generic J-torus T. □

Class 37. The Levi extension theorem

To conclude our discussion of the class of compact Kähler manifolds that can be embedded into projective space, we will prove Chow's theorem: every complex submanifold of $\mathbb{P}^n$ is defined by polynomial equations, and hence an algebraic variety. We will deduce this from an extension theorem for analytic sets, known as the *Levi extension theorem.* First, recall a basic definition from earlier in the semester: a closed subset Z of a complex manifold M is said to be analytic if, for every point $p \in Z$, there are locally defined holomorphic functions $f_1, \dots, f_r \in \mathscr{O}_M(U)$ such that $Z \cap U = Z(f_1, \dots, f_r)$ is the common zero set.

Here is the statement of the extension theorem (first proved in this form by the two German mathematicians Remmert and Stein).

Theorem 37.1. *Let M be a connected complex manifold of dimension n, and let $Z \subseteq M$ be an analytic subset of codimension at least $k+1$. If $V \subseteq M \setminus Z$ is an analytic subset of codimension k, then the closure $\overline{V}$ in M remains analytic.*

Example 37.2. Recall the following special case of Hartog's theorem: if f is a holomorphic function on $M \setminus \{p\}$, and if $\dim M \geq 2$, then f extends to a holomorphic function on M. In the same situation, Levi's theorem shows that if $V \subseteq M \setminus \{p\}$ is an analytic subset of codimension 1, then its closure $\overline{V}$ is analytic in M. The Levi extension theorem may thus be seen as a generalization of Hartog's theorem from holomorphic functions to analytic sets.

We begin the proof by making several reductions. In the first place, it suffices to prove the statement under the additional assumption that $Z \subseteq M$ is a submanifold of codimension $\geq k$. The general case follows from this by the following observation: by one of the exercises, the set of singular points of Z (i.e., those points where Z is not a submanifold of M) is contained in a proper analytic subset Z_1. Similarly, the set of singular points of Z_1 is contained in a proper analytic subseteq $Z_2 \subset Z_1$. Thus we have a chain $Z = Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ of closed analytic sets, with each $Z_j \setminus Z_{j+1}$ a complex submanifold of codimension $\geq k$ in M. Since there can be no infinite strictly decreasing chains of analytic sets, we have $Z_{r+1} = \emptyset$ for some $r \in \mathbb{N}$.

We may now extend V successively over the submanifolds $Z_j \setminus Z_{j+1}$, by first taking the closure of V in $M \setminus Z_1$, then in $M \setminus Z_2$, and so on.

In the second place, the definition of analytic sets is local, and so we only need to show that $\overline{V}$ is analytic in a neighborhood of any of its points. We may therefore assume in addition that M is a polydisk in $\mathbb{C}^n$ containing the origin, and that $0 \in Z$. After a suitable change of coordinates, we can furthermore arrange that the submanifold Z is of the form $z_1 = z_2 = \cdots = z_{k+1} = 0$.

Thus the general case of Levi's theorem is reduced to the following local statement.

Proposition 37.3. *Let $D \subseteq \mathbb{C}^n$ be a polydisk containing the origin, and let $Z = Z(z_1, \dots, z_{k+1})$. If V is an analytic subset of $D \setminus Z$ of codimension k, then $\overline{V}$ is an analytic subset of D.*

For simplicity, we shall only give the proof in the case $k = 1$ and $n = 2$. Exactly the same argument works for $k = 1$ and arbitrary n, except that the notation becomes more cumbersome; to prove the general case, one needs to know slightly more about the local structure of analytic sets than we have proved.

To fix the notation, let us say that $D = \Delta^2$ is the set of points $(z, w) \in \mathbb{C}^2$ with $|z| < 1$ and $|w| < 1$, and that Z consists of the point $(0, 0)$. Furthermore, V is an analytic subset of $D \setminus \{(0, 0)\}$ of dimension one, and we may clearly choose the coordinate system in such a way that the line $z = 0$ is not contained in V. We will prove the theorem by explicitly constructing a holomorphic function $H \in \mathscr{O}(D)$ whose zero locus is $\overline{V}$.

Let $D' = \Delta^* \times \Delta$ be the set of points in D where $z \neq 0$. We first want to show that $V' = V \cap D'$ is defined by the vanishing of a single holomorphic function on D'. Consider the associated line bundle $\mathscr{O}_{D'}(-V')$. We already know that $H^1(D', \mathscr{O}) \simeq 0$ and $H^2(D', \mathbb{Z}) \simeq 0$, and so the long exact sequence coming from the exponential sequence shows that $H^1(D', \mathscr{O}^*) \simeq 0$. We conclude that the line bundle $\mathscr{O}_{D'}(-V')$ is trivial, and hence that there is a holomorphic function $h \in \mathscr{O}(D')$ whose zero set is the divisor V'. The rest of the proof consists in suitably extending h to a holomorphic function H on a neighborhood of the origin in D.

Since V does not contain the line $z = 0$, the intersection $V \cap Z(z)$ consists of a discrete set of points in the punctured disk $0 < |w| < 1$. We may thus find a small circle, say of radius $\varepsilon > 0$, that does not meet any of these points. By continuity, the set of points (z, w) with $|z| \leq \delta$ and $|w| = \varepsilon$ will not meet V, provided that we choose $\delta > 0$ sufficiently small.

Now we claim that V intersects each vertical disk in the same number of points. For fixed z with $0 < |z| \leq \delta$, that number is given by the integral

$$d(z) = \frac{1}{2\pi i} \int_{|w|=\varepsilon} \frac{1}{h(z, w)} \frac{\partial h(z, w)}{\partial w} dw \in \mathbb{Z},$$

which counts the zeros of the holomorphic function $h(z, -)$ inside the disk $|w| < \varepsilon$. Since $d(z)$ is continuous and integer-valued, it has to be constant; let $d = d(0)$ be the constant value.

For fixed z with $0 < |z| \leq \delta$, we let $w_1(z), \dots, w_d(z)$ be the w-coordinates of the intersection points (in any order). The power sums

$$\sum_{j=1}^{d} w_j(z)^k = \frac{1}{2\pi i} \int_{|w|=\varepsilon} \frac{w^k}{h(z, w)} \frac{\partial h(z, w)}{\partial w} dw$$

are evidently holomorphic functions of z as long as $0 < |z| < \delta$. By Newton's identities, the same is therefore true for the elementary symmetric functions $\sigma_k(z)$. On the other hand, $|\sigma_k(z)|$ is clearly bounded by the quantity $\binom{d}{k} \cdot \varepsilon^k$, and therefore extends to a holomorphic function on the set $|z| < \delta$ by Riemann's theorem.

If we now define

$$H(z, w) = w^d - \sigma_1(z) w^{d-1} + \sigma_2(z) w^{d-2} + \cdots + (-1)^d \sigma_d(z),$$

then H is a holomorphic function for $|z| < \delta$ and $|w| < \varepsilon$, whose roots for fixed $z \neq 0$ are exactly the points $w_1(z), \dots, w_d(z)$. Its zero set $Z(H)$ is a closed analytic set which, by construction, contains all points of V that satisfy $0 < |z| < \delta$ and $|w| < \varepsilon$. It is then not hard to see that $Z(H) = \overline{V}$, proving that $\overline{V}$ is indeed analytic.

Class 38. Chow's theorem

We now want to show that complex submanifolds (and, more generally, analytic subsets) of projective space are algebraic varieties. As usual, we use homogeneous coordinates $[z_0, z_1, \dots, z_n]$ on projective space. For a homogeneous polynomial $F \in \mathbb{C}[z_0, z_1, \dots, z_n]$, the condition $F(z)$ is invariant under scaling (since $F(\lambda z) = \lambda^{\deg F} F(z)$ holds); this means that any collection $F_1, \dots, F_k$ of homogeneous polynomials defines a closed subset $Z = Z(F_1, \dots, F_k)$ of $\mathbb{P}^n$. It is clearly analytic; in fact, its intersection with each of the standard open subsets $U_0, U_1, \dots, U_n$ is defined by polynomial functions.

Definition 38.1. An analytic subset $Z \subseteq \mathbb{P}^n$ is said to be a *projective algebraic variety* if it is of the form $Z(F_1, \dots, F_k)$ for some collection of homogeneous polynomials.

The following result, known as Chow's theorem, shows that any analytic subset of $\mathbb{P}^n$ is actually a projective algebraic variety.

Theorem 38.2. *If $Z \subseteq \mathbb{P}^n$ is an analytic set, then there exist homogeneous polynomials $F_1, \dots, F_k \in \mathbb{C}[z_0, z_1, \dots, z_n]$ such that $Z = Z(F_1, \dots, F_k)$.*

Proof. The proof consists in a simple, but very clever, application of the Levi extension theorem. If $Z = \emptyset$, then we may take $F_1 = 1$; to exclude this trivial case, we assume from now on that $Z \neq \emptyset$. Recall that by definition of $\mathbb{P}^n$ as a quotient, we have the holomorphic quotient map $q \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. The preimage $V = q^{-1}(Z)$ is therefore an analytic subset of $\mathbb{C}^{n+1} \setminus \{0\}$. Note that each component of V has dimension at least 1, since it has to be closed under rescaling the coordinates by $\mathbb{C}^*$. This means that the codimension of V is at most n; on the other hand, the codimension of the origin in $\mathbb{C}^{n+1}$ is $n + 1$. We may thus apply Theorem 37.1 and conclude that the closure $\overline{V}$ is an analytic subset of $\mathbb{C}^{n+1}$. Observe (and this is important) that $\overline{V}$ is a *cone*: for $z \in \overline{V}$ and $\lambda \in \mathbb{C}$, we also have $\lambda z \in \overline{V}$.

It remains to produce polynomial equations that define Z. Let $\mathscr{O}_{n+1}$ be the local ring at the origin in $\mathbb{C}^{n+1}$ (its elements are germs of holomorphic functions, or equivalently, convergent power series), and let $I \subseteq \mathscr{O}_{n+1}$ be the ideal of germs of holomorphic functions that vanish on $\overline{V}$. Any $f \in I$ can be written as a convergent power series in $z = (z_0, z_1, \dots, z_n)$ in some neighborhood of the origin; thus

$$f(z) = \sum_{j=0}^{\infty} f_j(z),$$

with $f_j \in \mathbb{C}[z_0, z_1, \dots, z_n]$ homogeneous of degree j. The fact that $\overline{V}$ is a cone now implies that $f_j \in I$. To see why, fix a point $z \in \overline{V}$; for $|\lambda| < 1$, we then have

$$0 = f(\lambda z) = \sum_{j=0}^{\infty} f_j(\lambda z) = \sum_{j=0}^{\infty} \lambda^j f_j(z).$$

Since $f(\lambda z)$ is holomorphic in λ, the identity theorem shows that we have $f_j(z) = 0$ for all $j \geq 0$. Consequently, $f_j \in I$ as claimed. It follows that I is generated by homogeneous polynomials.

By Theorem 4.1, the ring $\mathscr{O}_{n+1}$ is Noetherian, and so I is finitely generated. This means that there are finitely many homogeneous polynomials $F_1, \dots, F_k \in \mathbb{C}[z_0, z_1, \dots, z_n]$ such that $I = (F_1, \dots, F_k)$; it is then obvious that we have $Z = Z(F_1, \dots, F_k)$. $\square$

Combining Chow's theorem and the Kodaira embedding theorem, we obtain the following corollary.

Corollary 38.3. *If a compact complex manifold M carries a positive line bundle, then M is isomorphic to a projective algebraic variety.*

As a matter of fact, any globally defined analytic object on projective space is algebraic; this is the content of the so-called *GAGA theorem* of Serre. More precisely, Serre's theorem asserts that there is an equivalence of categories between coherent analytic sheaves and coherent algebraic sheaves on $\mathbb{P}^n$. We will discuss coherent sheaves in more detail next week.

Example 38.4. For a simple example, consider holomorphic line bundles on $\mathbb{P}^n$. The exponential sequence

$$H^1(\mathbb{P}^n, \mathscr{O}) \longrightarrow H^1(\mathbb{P}^n, \mathscr{O}^*) \longrightarrow H^2(\mathbb{P}^n, \mathbb{Z}) \longrightarrow H^2(\mathbb{P}^n, \mathscr{O})$$

shows that the group of line bundles is isomorphic to $\mathbb{Z}$ (the two cohomology groups on the left and right vanish by Lemma 24.1). Thus every holomorphic line bundle is of the form $\mathscr{O}_{\mathbb{P}^n}(d)$ for some $d \in \mathbb{Z}$. These line bundles are actually *algebraic*, because they are locally trivial on the standard open cover of $\mathbb{P}^n$, with transition functions given by polynomials.

The following lemma shows how Hartog's theorem can be used to prove that every holomorphic section of the line bundle $\mathscr{O}_{\mathbb{P}^n}(d)$ on projective space is given by a homogeneous polynomial of degree d.

Lemma 38.5. *For $d \geq 0$, the space of global sections of the line bundle $\mathscr{O}_{\mathbb{P}^n}(d)$ is isomorphic to the space of homogeneous polynomials of degree d in $\mathbb{C}[z_0, z_1, \dots, z_n]$.*

Proof. With respect to the standard open cover $U_0, U_1, \dots, U_n$, the transition functions of the line bundle $\mathscr{O}_{\mathbb{P}^n}(-1)$ are given by z_j/z_k; hence those of $\mathscr{O}_{\mathbb{P}^n}(d)$ are z_k^d/z_j^d. A global section of the line bundle is a collection of holomorphic functions $f_j \in \mathscr{O}_{\mathbb{P}^n}(U_j)$ such that $f_j = z_k^d/z_j^d f_k$ on $U_j \cap U_k$. As before, let $q \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the quotient map, and put $f_j' = f_j \circ q$, which is defined and holomorphic on the set where $z_j \neq 0$. The relation above becomes

$$z_j^d f_j' = z_k^d f_k',$$

which shows that $z_j^d f_j'$ is the restriction of a holomorphic function on $\mathbb{C}^{n+1} \setminus \{0\}$. By Hartog's theorem, said function extends holomorphically to all of $\mathbb{C}^{n+1}$, and so we obtain some $F \in \mathscr{O}(\mathbb{C}^{n+1})$ with the property that $F = z_j^d f_j'$ for $z_j \neq 0$. For $\lambda \in \mathbb{C}^*$ and $z \neq 0$, we now have (for some $0 \leq j \leq n$)

$$F(\lambda z) = (\lambda z_j)^d f_j'(\lambda z) = \lambda^d z_j^d f_j'(z) = \lambda^d F(z),$$

and by expanding F into a convergent power series, we see that F has to be a homogeneous polynomial of degree d. We conclude that $f_j = F/z_j^d$, proving that the original section of $\mathscr{O}_{\mathbb{P}^n}(d)$ is indeed given by a homogeneous polynomial. The converse is obvious. □

Class 39. Coherent analytic sheaves and Oka's theorem

We will now leave the world of compact Kähler manifolds, and turn to another important class of complex manifolds, the so-called Stein manifolds. (These play the same role in complex geometry as affine varieties do in algebraic geometry.) Their theory is tightly interwoven with the theory of coherent analytic sheaves, and so we discuss that first.

Coherent analytic sheaves. When we studied the local properties of holomorphic functions, we showed that $\mathscr{O}_n$, the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$, is Noetherian. This means that every ideal of $\mathscr{O}_n$ is finitely generated. From this result, we deduced that any analytic set Z containing $0 \in \mathbb{C}^n$ is locally defined by finitely many holomorphic functions: let $I_Z \subseteq \mathscr{O}_n$ be the ideal of functions vanishing on Z; because I_Z is finitely generated, there is an open neighborhood D of the origin, and holomorphic functions $f_1, \dots, f_k \in \mathscr{O}(D)$ whose germs generate I_Z, such that $Z \cap D = Z(f_1, \dots, f_k)$.

The Noetherian property only gives information about Z at the origin, though. For instance, suppose that $g \in \mathscr{O}(D)$ is another holomorphic function that vanishes on Z. The germ of g belongs to the ideal I_Z, and hence we have $g = a_1 f_1 + \cdots + a_k f_k$ in the ring $\mathscr{O}_n$; but since $a_1, \dots, a_n$ may only be defined on a much smaller open neighborhood of the origin, this relation does not describe g on the original open set D.

In fact, this stronger finiteness property is true: there exist holomorphic functions $b_1, \dots, b_k \in \mathscr{O}(D)$ with the property that $g = a_1 f_1 + \cdots + b_k f_k$. The natural setting for such questions is the theory of analytic sheaves.

Definition 39.1. An *analytic sheaf* $\mathscr{F}$ on a complex manifold M is a sheaf of abelian groups, such that for every open set $U \subseteq M$, the group of sections $\mathscr{F}(U)$ is a module over the ring of holomorphic functions $\mathscr{O}_M(U)$, in a way that is compatible with restriction.

Of course, $\mathscr{O}_M$ itself is an analytic sheaf. Here are two other classes of examples:

Example 39.2. Let $Z \subseteq M$ be an analytic subset. Consider the sheaf $\mathscr{I}_Z$, whose sections over an open set $U \subseteq M$ are those holomorphic functions in $\mathscr{O}_M(U)$ that vanish along the intersection $U \cap Z$. Evidently, $\mathscr{I}_Z$ is an analytic sheaf, known as the *ideal sheaf* of the analytic set Z. Questions about holomorphic functions that vanish along Z are then really questions about this analytic sheaf.

Example 39.3. Let $p\colon E \to M$ be a holomorphic vector bundle of rank r. Consider the sheaf $\mathscr{E}$ of holomorphic sections of E; by definition, $\mathscr{E}(U)$ consists of all holomorphic mappings $s\colon U \to E$ with the property that $p \circ s = \mathrm{id}_U$. Then $\mathscr{E}$ is again an analytic sheaf. As a matter of fact, $\mathscr{E}$ is an example of a *locally free sheaf*: if $p^{-1}(U) \simeq U \times \mathbb{C}^r$, then the restriction of $\mathscr{E}$ to U is isomorphic to $\mathscr{O}_U^{\oplus r}$.

If $\mathscr{F}$ is an analytic sheaf, then at every point $p \in M$, the stalk

$$\mathscr{F}_p = \lim_{U \ni p} \mathscr{F}(U)$$

is a module over the local ring $\mathscr{O}_{M,p}$. For instance, the stalk of the ideal sheaf $\mathscr{I}_Z$ is the ideal in the local ring $\mathscr{O}_{M,p}$ defined by the analytic set Z. The question that we were discussing a few moments ago now leads to the following definition.

Definition 39.4. An analytic sheaf $\mathscr{F}$ is said to be *locally finitely generated* if every point of M has an open neighborhood U with the following property: there are finitely many sections $s_1, \dots, s_k \in \mathscr{F}(U)$ that generate the stalk $\mathscr{F}_p$ at every point $p \in U$.

The sections $s_1, \dots, s_k \in \mathscr{F}(U)$ determine a morphism $\mathscr{O}_U^{\oplus k} \to \mathscr{F}|_U$ of analytic sheaves, and the condition on the stalks is equivalent to the surjectivity of that morphism. In general, the kernel of this morphism may not itself be locally finitely generated.

Definition 39.5. An analytic sheaf $\mathscr{F}$ is said to be *coherent* if, in addition to being locally finitely generated, it satisfies the following condition: locally on M, there exists an exact sequence of analytic sheaves

$$\mathscr{O}_U^{\oplus p} \xrightarrow{F} \mathscr{O}_U^{\oplus q} \longrightarrow \mathscr{F}|_U \longrightarrow 0,$$

where F is some $q \times p$-matrix of holomorphic functions on U.

The following result, known as *Oka's theorem*, is fundamental in the theory of coherent analytic sheaves.

Theorem 39.6. *If $F\colon \mathscr{O}_M^{\oplus p} \to \mathscr{O}_M^{\oplus q}$ is a morphism of analytic sheaves, then the kernel of F is locally finitely generated.*

To illustrate the statement, suppose that $D \subseteq \mathbb{C}^n$ is an open set, and $f_1, \dots, f_k \in \mathscr{O}(D)$ are holomorphic functions. The kernel of $f\colon \mathscr{O}(D)^{\oplus k} \to \mathscr{O}(D)$ consists of all the relations between $f_1, \dots, f_k$, that is, of all k-tuples of holomorphic functions $a_1, \dots, a_k$ such that $a_1 f_1 + \dots + a_k f_k = 0$. Oka's theorem is the assertion that finitely many of these k-tuples generate all the relations.

Theorem 39.7. *If $Z \subseteq M$ is an analytic set in a complex manifold M, then the sheaf of ideals $\mathscr{I}_Z$ is coherent.*

The following lemma can be proved directly from the definition of coherence, with some diagram chasing.

Lemma 39.8. *The kernel and cokernel of any morphism between coherent sheaves is again coherent.*

Lemma 39.9. *Let $\mathscr{F}$ be a coherent analytic sheaf. If sections $s_1, \dots, s_k \in \mathscr{F}(U)$ generate the stalk of $\mathscr{F}$ at some point $p_0 \in M$, then they generate the stalks $\mathscr{F}_p$ at all nearby points.*

Proof. The sections $s_1, \dots, s_k$ determine a morphism $\phi\colon \mathscr{O}_U^{\oplus k} \to \mathscr{F}$, and by the preceding lemma, $\operatorname{coker} \phi$ is again a coherent analytic sheaf. That the sections generate the stalk of $\mathscr{F}$ at the point p_0 means exactly than $(\operatorname{coker} \phi)_{p_0} \simeq 0$. The problem is thus reduced to the following simpler statement: if $\mathscr{G}$ is a coherent analytic sheaf whose stalk at some point $p_0 \in M$ is isomorphic to zero, then the same is true at all points in a neigborhood of p_0.

The proof is easy. Indeed, $\mathscr{G}$ being coherent, there is an exact sequence

$$\mathscr{O}_U^{\oplus p} \xrightarrow{F} \mathscr{O}_U^{\oplus q} \longrightarrow \mathscr{G}|_U \longrightarrow 0$$

on some neighborhood U of the point p_0. Now $\mathscr{G}_{p_0} \simeq 0$ means that the matrix F, whose entries are holomorphic functions on U, has maximal rank at the point p_0; in other words, at least one of its $q \times q$-minors does not vanish at the point p_0. But then the same minor is nonzero on some open neighborhood V of p_0, proving that $\mathscr{G}|_V \simeq 0$ as claimed. □

Grauert's theorem. We close this brief overview of the theory of coherent sheaves by stating one of the most important results, namely Grauert's theorem. Let $f\colon X \to Y$ be a holomorphic mapping between complex manifolds. For any analytic sheaf $\mathscr{F}$ on X, one can define the so-called *direct image sheaf* $f_*\mathscr{F}$; for $U \subseteq Y$, the sections of this sheaf are given by $(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}U)$.

Example 39.10. We always have a morphism of sheaves of rings $\mathscr{O}_Y \to f_*\mathscr{O}_X$. Indeed, for any open set $U \subseteq Y$, composition with the holomorphic mapping f defines a ring homomorphism $\mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}U)$.

The morphism $\mathscr{O}_Y \to f_*\mathscr{O}_X$ can be used to give any direct image sheaf $f_*\mathscr{F}$ the structure of an analytic sheaf on Y. Grauert's theorem gives the condition for the direct image of a coherent sheaf to be coherent.

Theorem 39.11. *If $f\colon X \to Y$ is a proper holomorphic mapping, meaning that the preimage of every compact set is compact, then for every coherent analytic sheaf $\mathscr{F}$ on X, the direct image sheaf $f_*\mathscr{F}$ is again coherent.*

On a compact complex manifold M, the trivial mapping to a point is proper; a special case of Grauert's theorem is the following finiteness result.

Corollary 39.12. *On a compact complex manifold M, the space of global sections of any coherent analytic sheaf is a finite-dimensional complex vector space.*

Proof. Let $f\colon M \to pt$ map M to a point. Then $f_*\mathscr{F}$ is nothing but the complex vector space $\mathscr{F}(M)$, which is coherent iff it is finite-dimensional. Since M is compact, f is proper, and so the result follows from Grauert's theorem. □

Of course, this need not be true if M is not compact: for instance, on the complex manifold $M = \mathbb{C}$, the sheaf of holomorphic functions $\mathscr{O}$ is coherent; but the space of its global sections is the space of all entire functions, and thus very far from being finite-dimensional.

Class 40. Stein manifolds

Cohomology of analytic sheaves. Recall that for any sheaf of abelian groups $\mathscr{F}$ on a topological space X, we defined cohomology groups $H^i(X, \mathscr{F})$ by the following procedure: $\mathscr{F}$ has a natural resolution

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varepsilon} \mathscr{F}^0 \xrightarrow{d^0} \mathscr{F}^1 \xrightarrow{d^1} \mathscr{F}^2 \xrightarrow{d^2} \cdots,$$

the so-called *Godement resolution*, by flabby sheaves. Here $\mathscr{F}^0$ is the sheaf of discontinuous sections of $\mathscr{F}$, then $\mathscr{F}^1$ is the sheaf of discontinuous sections of

$\operatorname{coker} \varepsilon$, and so on. By definition, $H^i(X, \mathscr{F})$ is the i-th cohomology group of the complex

$$0 \longrightarrow \mathscr{F}^0(X) \longrightarrow \mathscr{F}^1(X) \longrightarrow \mathscr{F}^2(X) \longrightarrow \cdots .$$

Now suppose that X is a complex manifold, and $\mathscr{F}$ an analytic sheaf. In this case, each sheaf $\mathscr{F}^i$ in the Godement resolution is again an analytic sheaf, and so each abelian group $\mathscr{F}^i(X)$ is a module over the ring $\mathscr{O}_X(X)$ of holomorphic functions on X. Consequently, the cohomology groups $H^i(X, \mathscr{F})$ are themselves $\mathscr{O}_X(X)$-modules.

Theorem 40.1. *If X is a compact complex manifold, and $\mathscr{F}$ a coherent analytic sheaf, then each cohomology group $H^i(X, \mathscr{F})$ is a finite-dimensional complex vector space.*

When doing Hodge theory on compact Kähler manifolds, we have already seen special cases of this result: for $\mathscr{F} = \mathscr{O}_X$, or $\mathscr{F} = \Omega^1_X$, etc.

Stein manifolds. On certain complex manifolds, the higher cohomology groups of every coherent analytic sheaf are trivial.

Example 40.2. If D is a polydisk in $\mathbb{C}^n$ (or $\mathbb{C}^n$ itself), then $H^i(D, \mathscr{F}) = 0$ for every coherent analytic sheaf and every $i > 0$. Because of Dolbeault's theorem, we already know that this is true for the sheaf of holomorphic functions $\mathscr{O}$. To extend the result to arbitrary coherent analytic sheaves, one proves that $\mathscr{F}$ admits a finite resolution of the form

$$0 \longrightarrow \mathscr{O}^{\oplus p_r} \longrightarrow \mathscr{O}^{\oplus p_{r-1}} \longrightarrow \cdots \longrightarrow \mathscr{O}^{\oplus p_1} \longrightarrow \mathscr{O}^{\oplus p_0} \longrightarrow \mathscr{F} \longrightarrow 0;$$

the result follows from this by purely formal reasoning.

Definition 40.3. A *Stein manifold* is a complex manifold M with the property that $H^i(M, \mathscr{F}) = 0$ for every coherent analytic sheaf $\mathscr{F}$ and every $i > 0$.

Example 40.4. Any complex submanifold of a Stein manifold is again a Stein manifold. In particular, any complex submanifold of $\mathbb{C}^n$ is Stein. The proof goes as follows: Let $i \colon N \hookrightarrow M$ denote the inclusion map. Then one can show that for any coherent analytic sheaf $\mathscr{F}$ on N, the direct image $i_* \mathscr{F}$ is again coherent. (This is a special case of Grauert's theorem, but much easier to prove.) Moreover, one has $H^i(N, \mathscr{F}) \simeq H^i(M, i_* \mathscr{F})$, and this obviously implies that N is a Stein manifold.

Example 40.5. Any non-compact Riemann surface is known to be a Stein manifold by a theorem of Behnke and Stein.

Example 40.6. If M is a Stein manifold, then any covering space of M is again a Stein manifold.

A Stein manifold always has a very rich function theory, since the vanishing of higher cohomology groups of coherent sheaves makes it easy to construct holomorphic functions. To illustrate this, let M be a Stein manifold, and let $\mathscr{O}_M(M)$ be the ring of its global holomorphic functions. We shall prove that any holomorphic function on an analytic subset can be extended to all of M.

Lemma 40.7. *Let $Z \subseteq M$ be an analytic subset of a Stein manifold M, and let f be a holomorphic function on Z. Then there exist $g \in \mathscr{O}_M(M)$ with the property that $g(z) = f(z)$ for every $z \in Z$.*

Proof. Let $\mathscr{I}_Z$ denote the coherent ideal sheaf of the analytic subset Z. We have an exact sequence

$$0 \longrightarrow \mathscr{I}_Z \longrightarrow \mathscr{O}_M \longrightarrow i_* \mathscr{O}_Z \longrightarrow 0,$$

in which $\mathscr{O}_Z$ denotes the sheaf of holomorphic functions on Z. Passing to cohomology, we find that $\mathscr{O}_M(M) \to \mathscr{O}_Z(Z) \to H^1(M, \mathscr{I}_Z)$ is exact. Now $\mathscr{I}_Z$ is a coherent sheaf, and therefore $H^1(M, \mathscr{I}_Z) = 0$ because M is a Stein manifold. It follows that the restriction map $\mathscr{O}_M(M) \to \mathscr{O}_Z(Z)$ is surjective. □

In particular, since every pair of points $p, q \in M$ determines an analytic subset $\{p, q\}$, we see that holomorphic functions on a Stein manifold separate points.

Corollary 40.8. *In a Stein manifold, every compact analytic subset is finite.*

Proof. A holomorphic function on a compact analytic set is locally constant. □

Lemma 40.9. *At every point $p \in M$, there exist holomorphic functions $f_1, \dots, f_n \in \mathscr{O}_M(M)$ that define local holomorphic coordinates in a neighborhood of the point.*

Proof. Let $z_1, \dots, z_n \in \mathscr{O}_M(U)$ be local holomorphic coordinates, centered at the point p. If we denote by $\mathscr{I}$ the ideal sheaf of the point p, then we have $z_j \in \mathscr{I}(U)$. The quotient sheaf $\mathscr{I}/\mathscr{I}^2$ is supported at the point p, and is in fact an n-dimensional complex vector space, spanned by the images of $z_1, \dots, z_n$. Any $f \in \mathscr{I}(M)$ may be expanded on U into a convergent power series of the form

$$f(z) = \sum_{|I| \geq 1} a_I z^I,$$

and the vector determined by f is nothing but the linear part $a_1 z_1 + \dots + a_n z_n$.

By the same argument as before, the short exact sequence

$$0 \longrightarrow \mathscr{I}^2 \longrightarrow \mathscr{I} \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow 0,$$

together with the vanishing of the cohomology group $H^1(M, \mathscr{I}^2)$, proves that the restriction map $\mathscr{I}(M) \to \mathscr{I}/\mathscr{I}^2$ is surjective. We may therefore find a holomorphic function f_j whose image in $\mathscr{I}/\mathscr{I}^2$ equals z_j; it follows that the Jacobian matrix $\partial(f_1, \dots, f_n)/\partial(z_1, \dots, z_n)$ is the identity matrix at the point p, and so $f_1, \dots, f_n$ define a local holomorphic coordinate system by the implicit mapping theorem. □

The Oka principle. A basic idea in the theory of Stein manifolds is the following so-called *Oka principle*: On a Stein manifold, any problem that can be formulated in terms of cohomology has only topological obstructions. Said differently, such a problem has a holomorphic solution if and only if it has a continuous solution.

Example 40.10. Consider again the exponential sequence

$$H^1(M, \mathscr{O}_M) \longrightarrow H^1(M, \mathscr{O}_M^*) \longrightarrow H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathscr{O}_M).$$

Since the higher cohomology groups of the sheaf $\mathscr{O}_M$ are zero, it follows that the space of line bundles on M is isomorphic to the group $H^2(M, \mathbb{Z})$. In other words, every integral second cohomology class is the first Chern class of a holomorphic line bundle; unlike the case of compact Kähler manifolds, there are no conditions of type.

A far more powerful theorem along these lines has been proved by Grauert.

Theorem 40.11. *Let M be a Stein manifold, and $E \to M$ a holomorphic vector bundle. If E is topologically trivial, then it is also holomorphically trivial.*

This is a very striking result. Suppose that $E \simeq M \times \mathbb{C}^r$ as topological vector bundles; this means that E admits r continuous sections that are linearly independent at each point $p \in M$. Grauert's theorem says that, in this case, E also has r holomorphic sections with the same property.

Class 41. The embedding theorem

We have seen that every complex submanifold of $\mathbb{C}^n$ is a Stein manifold. In fact, the converse is also true—this is the content of the famous embedding theorem for Stein manifolds.

Theorem 41.1. *Let M be an n-dimensional Stein manifold. Then there exists a proper holomorphic embedding $i\colon M \hookrightarrow \mathbb{C}^{2n+1}$, and so M is biholomorphic to a complex submanifold of $\mathbb{C}^{2n+1}$.*

The proof works by constructing sufficiently many holomorphic functions on M to give a proper holomorphic embedding into $\mathbb{C}^N$ for some large integer N. As long as $N > 2n+1$, one can show that projection from a generic point outside of M still embeds the manifold into $\mathbb{C}^{N-1}$. In this way, one can reduce the dimension of the ambient space to $2n+1$.

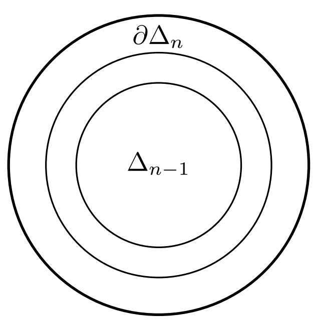
Example 41.2. $\mathbb{C}^*$ is a Stein manifold, and may be embedded into $\mathbb{C}^2$ by the (polynomial) mapping $t \mapsto (t, t^{-1})$.

Example 41.3. Every non-compact Riemann surface is a one-dimensional Stein manifold, and can therefore be embedded into $\mathbb{C}^3$. A famous unsolved problem is whether there always exists an embedding into $\mathbb{C}^2$.

Embedding the unit disk. Rather than describe the proof of the embedding theorem in general, let us focus on a specific example: the unit disk Δ in the complex plane. We already know that Δ is a Stein manifold, and the embedding theorem claims that Δ is isomorphic to a closed submanifold of $\mathbb{C}^3$. To verify this claim, we shall now construct a (more or less explicit) embedding $i\colon \Delta \hookrightarrow \mathbb{C}^3$.

The unit disk is already embedded into the complex plane $\mathbb{C}$, of course, but as an open subset, not as a *closed* complex submanifold. In order for $i(\Delta)$ to be a submanifold of $\mathbb{C}^3$ of that type, it is necessary for the embedding i to be *proper*, which is to say that $i(z)$ should go to infinity as z approaches the boundary of Δ.

For $n \geq 1$, let Δ_n denote the open disk of radius $1 - 1/(n+1)$, centered at the origin (and set $\Delta_0 = \emptyset$).



We shall define two holomorphic functions $f, g \in \mathscr{O}(\Delta)$ in such a way that $|f(z)| + |g(z)| \geq n$ for every $z \in \Delta_n \setminus \Delta_{n-1}$. We may then set

$$i \colon \Delta \to \mathbb{C}^3, \quad i(z) = \bigl(z, f(z), g(z)\bigr).$$

This mapping will be a holomorphic embedding (because of the first coordinate), and also proper (because of the second and third coordinate), and its image $i(\Delta)$ is therefore a complex submanifold of $\mathbb{C}^3$, biholomorphic to the unit disk.

We begin by constructing $f \in \mathscr{O}(\Delta)$ with the property that $|f(z)| > n+1$ on $\partial\Delta_n$. To that end, we inductively define a sequence of holomorphic functions $f_1, f_2, \dots$, such that

$$|f_n(z)| \geq \sum_{k=1}^{n-1} |f_k(z)| + n + 2 \qquad \text{for } z \in \partial\Delta_n,$$

while

$$|f_n(z)| \leq 2^{-n} \qquad \text{for } z \in \Delta_{n-1}.$$

Suppose that we already have $f_1, \dots, f_{n-1}$. We may then take f_n to be a monomial of the form

$$f_n(z) = (\alpha_n z)^{\beta_n}.$$

We first choose $(n+1)/n < \alpha_n < n/(n-1)$, to guarantee that $|\alpha_n z|$ is less than 1 on Δ_{n-1}, and greater than 1 on $\partial\Delta_n$, and then take β_n large enough to satisfy both conditions. If we now put

$$f(z) = \sum_{n=1}^{\infty} f_n(z),$$

then $f \in \mathscr{O}(\Delta)$ because the series converges uniformly on compact subsets of Δ. Moreover, for $z \in \partial\Delta_n$, we have

$$|f(z)| \geq |f_n(z)| - \sum_{k=1}^{n-1} |f_k(z)| - \sum_{k=n+1}^{\infty} |f_k(z)| \geq n + 2 - \sum_{k=n+1}^{\infty} 2^{-k} > n+1,$$

as desired.

Of course, the absolute value of $f(z)$ is large only on the circles $\partial\Delta_n$; on the open annuli between them, there will be other points where $|f(z)|$ is small. In fact, we know from complex analysis that any holomorphic function $f \colon \Delta \to \mathbb{C}$ has open image, and is therefore never proper. To overcome this problem, let

$$E_n = \bigl\{\, z \in \Delta_n \setminus \Delta_{n-1} \bigm| |f(z)| \leq n \,\bigr\}.$$

We now construct a second function $g \in \mathscr{O}(\Delta)$, with the property that $|g(z)| \geq n$ on E_n. It will then be the case that $|f(z)| + |g(z)| \geq n$ on $\Delta_n \setminus \Delta_{n-1}$, which is what we need.

Observe that E_n is a compact subset of Δ, due to the fact that $|f(z)| > n+1$ on $\partial\Delta_n$. Moreover, E_n is clearly disjoint from the compact set $\overline{\Delta}_{n-1}$. Proceeding by induction, we shall again define a sequence of holomorphic functions $g_1, g_2, \dots$, such that

$$|g_n(z)| \geq \sum_{k=1}^{n-1} |g_k(z)| + n + 1 \qquad \text{for } z \in E_n,$$

while

$$|g_n(z)| \leq 2^{-n} \qquad \text{for } z \in \Delta_{n-1}.$$

Suppose that we already have $g_1, \dots, g_{n-1}$. Let M_n denote the supremum of $\sum_{k=1}^{n-1} |g_k(z)| + n + 1$ over the compact set E_n. Define a holomorphic function h_n on a small open neighborhood of $E_n \cup \overline{\Delta}_{n-1}$, by letting h_n be equal to $M_n + 2^{-n}$ near E_n, and equal to 0 near $\overline{\Delta}_{n-1}$. By the Runge approximation theorem, we may find a holomorphic function $g_n \in \mathscr{O}(\Delta)$ that approximates h_n to within 2^{-n} on the compact set $E_n \cup \overline{\Delta}_{n-1}$; this choice of g_n has the desired properties.

We may now set $g(z) = \sum_{n=1}^{\infty} g_n(z)$, which is again holomorphic, and satisfies $|g(z)| > n$ for every $z \in E_n$ by the same reasoning as before. It follows that $|f(z)| + |g(z)| \geq n$ on the annulus $\Delta_n \setminus \Delta_{n-1}$, and this proves that the mapping

$$\Delta \to \mathbb{C}^2, \quad z \mapsto \big(f(z), g(z)\big)$$

is indeed proper.

Note. It is possible to do better and embed the unit disk into $\mathbb{C}^2$. For example, Alexander has shown that the mapping $\big(\lambda, \lambda'/\lambda(1-\lambda)\big)$ from the upper halfplane to $\mathbb{C}^2$ descends to a proper holomorphic embedding of Δ. Here $\lambda \colon \mathbb{H} \to \mathbb{C} \setminus \{0, 1\}$ is the elliptic modular function; its basic property is that the elliptic curve $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ has Weierstraß form $y^2 = x(x-1)(x-\lambda(\tau))$.