Lecture 8 - Vector spaces and their Duals, II

2/18/09

This lecture completes our formal discussion of dual spaces.

1 The double dual, V**

The space V* is defined to be the space of linear operators on V. Of course, V* is a vector space itself, so also has a dual, denoted V**, called the “double-dual” of V.

But as a matter of fact, one can consider elements of V to act on elements of V*: there is an map

\[ \mathcal{N} : V \hookrightarrow V^{**} \]

given by

\[ \mathcal{N}(v) \in V^{**} \]
\[ \mathcal{N}(v)(f) \triangleq f(v) \text{ for any } f \in V^*. \]

**Theorem 1.1** If V is a finite dimensional vector space, then the map \( \mathcal{N} : V \rightarrow V^{**} \) is a a vector space isomorphism.

**Pf** Homework problem 3.6. \( \square \)

Often we drop the “\( \mathcal{N} \)” from the notation, and just consider elements \( v \in V \) to act on elements \( f \in V^* \) directly:

\[ v \in V^{**} \text{ acts on } V^* \text{ by} \]
\[ v(f) \triangleq f(v) \text{ for any } f \in V^*. \]
2 Change-of-basis matrices

Consider two bases \( \{e_1, \ldots, e_n\} \) and \( \{f_1, \ldots, f_n\} \) for the vector space \( V \). Any vector \( v \in V \) can be expressed as a column vector in either system, though the column vector will be different. If one knows the vector for \( v \) in the \( e_i \) system, how can \( v \) be expressed in the \( f_i \) system?

One has to know the relation between the two systems. Define the numbers \( A_j^i \) implicitly by
\[
e_j = A_j^i f_i.
\]

Then, for example,
\[
e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{\{e_i\}} = \begin{pmatrix} A_1^1 \\ A_2^1 \\ \vdots \\ A_n^1 \end{pmatrix}_{\{f_i\}}.
\]
\[
e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}_{\{e_i\}} = \begin{pmatrix} A_j^1 \\ \vdots \\ A_j^{j-1} \\ A_j^j \\ A_j^{j+1} \\ \vdots \\ A_j^n \end{pmatrix}_{\{f_i\}}.
\]

Therefore, a vector \( v = \alpha^i e_i \) can be expressed
\[
v = \alpha^j e_j = \alpha^j A_j^i f_i = \begin{pmatrix} A_1^1 \alpha^j \\ A_2^1 \alpha^j \\ \vdots \\ A_n^1 \alpha^j \\ A_1^2 \alpha^j \\ \vdots \\ A_n^2 \alpha^j \\ \vdots \\ A_1^n \alpha^j \\ A_2^n \alpha^j \\ \vdots \\ A_n^n \alpha^j \end{pmatrix}_{\{f_i\}}.
\]
The final vector is just the matrix multiplication
\[
\begin{pmatrix} A_1^1 \alpha^j \\ A_2^1 \alpha^j \\ \vdots \\ A_n^1 \alpha^j \\ \vdots \\ A_1^n \alpha^j \\ A_2^n \alpha^j \\ \vdots \\ A_n^n \alpha^j \end{pmatrix}_{\{f_i\}} = \begin{pmatrix} A_1^1 & A_1^2 & \cdots & A_1^n \\ A_2^1 & A_2^2 & \cdots & A_2^n \\ \vdots & \vdots & \ddots & \vdots \\ A_n^1 & A_n^2 & \cdots & A_n^n \end{pmatrix}_{\{f_i\} \rightarrow \{e_i\}} \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^n \end{pmatrix}_{\{e_i\}}.
\]

Notice the subscript \( \{f_i\} \leftarrow \{e_i\} \) on the matrix. It is used to indicate what bases \( A \) transitions between. The transformation from the \( f_i \) to the \( e_i \) basis is given by the inverse matrix:
\[
A_{\{e_i\} \rightarrow \{f_i\}} = (A_{\{f_i\} \rightarrow \{e_i\}})^{-1}.
\]

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3 Active vs. Passive transformations

There are always two ways to think about an operator $A : V \to V$. A so-called active transformation uses a fixed coordinate system, and performs a transformation of the underlying space. A so-called passive transformation just changes the basis vectors and leaves the underlying space fixed. However these are conceptual differences only: any given operator can be interpreted in either way.

Let’s illustrate this with an example. Let $V = \mathbb{R}^2$ with standard basis $e_1 = \hat{i}$, $e_2 = \hat{j}$. Let $A$ be given by

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$ 

Thought of as an active transformation, this is a rotation of space counterclockwise through an angle of $\theta$.

On the other hand, consider another basis $f_1 = \cos(\theta)\hat{i} - \sin(\theta)\hat{j}$, $f_2 = \sin(\theta)\hat{i} + \cos(\theta)\hat{j}$. Then $A$ is just the change-of-basis matrix from the $e_i$ to the $f_i$ bases.

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}_{\{f_i\}\to\{e_i\}}.$$ 

Note that the new basis $f_i$ is a rotation of the old basis $e_i$ through a clockwise angle of $\theta$.

Thus the matrix $A$ can be considered to be either a transformation of the underlying space (an active transformation, in this case counterclockwise rotation by $\theta$) or as a change of basis that leaves the vector space unchanged (a passive transformation, in this case a clockwise rotation of the basis vectors by $\theta$).

4 Actions on the dual space

Let $A : V \to V$ be a linear operator. We have not defined any kind of action of $A$ on the dual space $V^*$. But, as we shall see, there should be such an action.

To see this, consider $A$ to be a passive transformation, changing from, say, the $\{e_i\} \subset V$ to the $\{f_i\} \subset V$ basis. Let $f \in V^*$ be a linear functional, and let $w \in V$ be a vector. Let $w_{\{e_i\}}$, $f_{\{e_i\}}$ be their expressions in the $\{e_i\}$ basis, and $w_{\{f_i\}}$, $f_{\{f_i\}}$ be their expressions in the $\{f_i\}$ basis. We have

$$w_{\{f_i\}} = A_{\{f_i\}\to\{e_i\}}w_{\{e_i\}}.$$ 

Now, since $f_{\{e_i\}}$ and $f_{\{f_i\}}$ are the same covector regardless of its expression in either basis, it must have the same action on $w$, regardless of basis. Letting $A(f)$ indicate the action of
A on f, we therefore must have

\[ A(f)(A(w)) = f(w). \]

\[ (A(f))_{\{f_i\}} \cdot (A(w))_{\{f_i\}} = (A(f))_{\{f_i\}} \cdot A_{\{f_i\} \rightarrow \{e_i\}} w_{\{e_i\}}. \quad (Matrix \ multiplication) \]

Thus it must be the case that

\[ (A(f))_{\{f_i\}} = f_{\{e_i\}} \cdot (A_{\{f_i\} \rightarrow \{e_i\}})^{-1} \quad (Matrix \ multiplication) \]

Expressing this abstractly (that is, without necessarily choosing a basis),

*Given f : V → R, we have A(f) : V → R, given by*

\[ A(f)(v) = f(A^{-1}v) \quad \text{for any} \; v \in V. \]