Lecture 13 - Vectors as directional derivatives

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1 Coordinates

Let $M$ be some space, say Euclidean $n$-space, Minkowski $1+n$-space, or the like. Coordinates are functions on the space $M$ that assign to each point some unique set of numbers. It is important to understand that a given space $M$ is not a vector space, and coordinates are not basis vectors of any kind. Coordinates are functions, pure and simple.

2 Vectors, tangent spaces, and the tangent bundle

Intuitively, a vector is a magnitude and a direction. This is not a rigorous definition, however. A concept that can be made precise is the notion of the derivative of a function along a curve. To define this concept, let $p \in M$ be a point, let $f : M \to \mathbb{R}$ be a function, and let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve parameterized by $\tau \in (-\epsilon, \epsilon)$ with $\gamma(0) = p$. Then the derivative of $f$ along $\gamma$ at $p$ is defined to be

$$\frac{d}{d\tau} \bigg|_p f \triangleq \lim_{h \to 0} \frac{f(\gamma(h)) - f(\gamma(0))}{h}.$$  

One computes this expression using partial derivatives: if $\{x^1, \ldots, x^n\}$ are coordinates on $M$, we can write $f = f(x^1, \ldots, x^n)$ and compute

$$\frac{d}{d\tau} f = \frac{dx^1}{d\tau} \frac{\partial}{\partial x^1} f + \ldots + \frac{dx^n}{d\tau} \frac{\partial}{\partial x^n} f,$$

that is, the operator $\frac{d}{d\tau}$ is a linear combination of the operators $\frac{\partial}{\partial x^i}$.

We have not defined the term "vector" yet, but intuitively two paths $\gamma(\tau)$ and $\tilde{\gamma}(\tilde{\tau})$ which pass through the point $p$ posses the same velocity vector at $p$ if $\frac{dx^i}{d\tau} = \frac{dx^i}{d\tilde{\tau}}$, which is to say that $\frac{d}{d\tau} = \frac{d}{d\tilde{\tau}}$. 

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Our intuitive notion of vectors seems to coincide with the mathematically precise notion of directional derivatives. Thus we say \( v \) is a vector based at \( p \in M \) if \( v_p \) is a linear combination of the directional derivatives \( \partial / \partial x^i \):

\[
v = v^i \frac{\partial}{\partial x^i} \bigg|_p.
\]

Note that we are justified in say that the partials \( \partial / \partial x^i \) are directional derivatives: \( \partial / \partial x^i \) is obtained by varying \( x^i \) and fixing all other coordinates.

The tangent space at \( p \), denoted \( T_p M \), is defined to be the vector space of all vectors based at \( p \).

The tangent bundle of \( M \), denoted \( TM \), is defined to be the collection of all tangent spaces \( T_p M \) based at all points \( p \) of \( M \).

### 3 Change of coordinates

If the coordinate functions are changed, it is important to know how to change the basis vectors of each tangent space \( T_p M \). Let \( \{ x^1, \ldots, x^n \} \) and \( \{ y^1, \ldots, y^n \} \) be two coordinate systems on \( M \). We have the relationship

\[
\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}.
\]

For example, if \( r, \theta \) are the so-called polar coordinates on the Euclidean plane and \( x = r \cos \theta \), \( y = r \sin \theta \) are the corresponding rectangular coordinates, we have

\[
\frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} = \cos(\theta) \frac{\partial}{\partial x} + \sin(\theta) \frac{\partial}{\partial y} = \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial y}
\]

\[
\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} = -r \sin(\theta) \frac{\partial}{\partial x} + r \cos(\theta) \frac{\partial}{\partial y} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}
\]