1 Metrics

Let $V$ be a vector space with basis $\{v_1, \ldots, v_n\}$. Assume $V$ is endowed with an inner product $g \in \otimes^2 V^\ast$. That is, $g$ is given by

$$g = g_{ij} v^i \otimes v^j$$

where $g$ satisfies

- Symmetry: $g(v, w) = g(w, v)$ for any $v, w \in V$. In other words, the matrix $g_{ij}$ is symmetric: $g_{ij} = g_{ji}$.
- Nondegeneracy: if $0 \neq v \in V$, then there is some $\bar{v} \in V$ so that $g(v, \bar{v}) \neq 0$.

An inner products is often called a metric.

2 The musical isomorphisms

Given a basis $\{v_i\} \subset V$, we have discussed the existence of a dual basis $\{v^i\} \subset V^\ast$. One might be tempted to think that this leads to an isomorphism $V \to V^\ast$, but any such attempt to define such an isomorphism will be dependent on the basis that has been chosen.

If the vector space has a metric $g$, there is a natural (that is, basis-independent) isomorphism

$$\flat : V \to V^\ast.$$ 

This is defined by

$$\flat(v) \in V^\ast$$
$$\flat(v)(w) = g(v, w).$$
Usually this is denoted more simply by
\[ v \in V \mapsto v_\flat \in V^* \]
\[ v_\flat (\cdot) = g(\cdot, v) \]
\[ v_\flat (w) = g(v, w) \quad \text{for} \quad w \in V. \]

and the like. The fact that this is an isomorphism is equivalent to the nondegeneracy of the metric (homework problem). The inverse of the "♭" isomorphism is the "♯" isomorphism
\[ \sharp : V^* \to V \quad \text{is given by} \quad \sharp = \flat^{-1}. \]

Given \( f \in V^* \), we have
\[ \sharp(f) \in V, \quad \text{often denoted} \quad f^\sharp \in V. \]

It is easy to show (homework problem) that \( f^\sharp \in V \) is characterized by
\[ g(f^\sharp, v) = f(v). \]

## 3 The metric on the dual space

If \( g = g_{ij} v^i \otimes v^j \) is a metric on \( V \), we can define, in a natural (that is to say, basis-free) way, a metric on the dual space \( V^* \). Given \( f, g \in V^* \), we define
\[ g(f, g) = g(f^\sharp, g^\sharp) \]
(recalling that \( f^\sharp, g^\sharp \in V \) and \( g : V \times V \to \mathbb{R} \)). Considering \( g \) as a map \( V^* \times V^* \to \mathbb{R} \), we can write
\[ g = g^{ij} v_i \otimes v_j. \]

It is possible to prove that the matrix \( g^{ij} \) is the inverse of the matrix \( g_{ij} \) (homework problem). That is to say, it holds that
\[ g^{ik} g_{kj} = \delta^i_j. \]

## 4 The musical isomorphisms in component form (raising and lowering indices)

Given a basis \( \{v_i\} \subset V \) and its dual basis \( \{v^i\} \subset V^* \), how can we express the musical isomorphisms? Assume
\[ v = a^i. \]
is a vector (recall this is shorthand for $v = \alpha^i v_i$). How can we find the components of the covector $v_\flat = \alpha^i$? (We are NOT assuming that the numbers $\alpha^i$ are the same as the numbers $\alpha_i$.) By the definition of $v_\flat$, we have

$$v_\flat(v_j) = g(v, v_j) = \alpha^i g(v_i, v_j) = \alpha^i g_{kl} \delta^k_i \delta^l_j = \alpha^i g_{ij}$$

But of course also

$$v_\flat(v_j) = \alpha_i \delta^i_j = \alpha_j.$$

Therefore $\alpha_j = \alpha^i g_{ij}$.

This procedure is often called lowering the index.

Now we describe the $\sharp$ isomorphism in components. Let $f = f_i$ be a covector (recall that this means $f = f_i v^i$). We define the numbers $f^i$ by $f^i = f_i$. Using the definition of $f^i$, we have

$$f(v_j) = g(f^i, v_j) = g(f^i v_i, v_j) = f^i g_{ij}$$

Thus we can implicitly define $f^i$ by the relationship

$$f^i g_{ij} = f_j.$$

Recalling that $g^{ij}$ is the inverse of $g_{ij}$, we have

$$f^i g_{ij} g^{jk} = f_j g^{jk}$$

This means that $f^i = f^k v_i \in V$. This procedure is often called raising the index.

5 Raising and lowering tensor indices

Given an arbitrary tensor, for example $T = T^i_{\, j} \in \otimes^{1,1} V$, we can raise or lower its indices. For example, the corresponding $T_{ij} \in \otimes^{0,2} V$ is given by

$$T_{ij} = T^k_{\, j} g_{ki}$$

and the corresponding tensor $T^{ij} \in \otimes^{2,0} V$ is given by

$$T^{ij} = T^i_{\, k} g^{kj}.$$