Lecture 11 - Tensors as maps, dual spaces, transformation properties, alternating tensors, and wedge products

Feb 25, 2009

1 Tensors as maps

Let $V$ be a vector space. We define $V^*$ to be the vector space of linear maps $V \to \mathbb{R}$, but we also know that (in the finite dimensional case at least) the space $V$ is the space of maps $V^* \to \mathbb{R}$. Likewise, we can consider elements of $\otimes^{0,k} V$ to be $k$-fold linear maps $V \times \cdots \times V \to \mathbb{R}$:

$$T \in \otimes^{0,k} V \text{ given by } T = T_{i_1 i_2 \ldots i_k} v^{i_1} \otimes v^{i_2} \otimes \cdots \otimes v^{i_k}$$

$$T(v^{(1)}, \ldots, v^{(k)}) = T_{i_1 i_2 \ldots i_k} v^{i_1} \otimes v^{i_2} \otimes \cdots \otimes v^{i_k} (v^{(1)}, \ldots, v^{(k)})$$

$$= T_{i_1 i_2 \ldots i_k} v^{i_1}(v^{(1)}) v^{i_2}(v^{(2)}) \cdots v^{i_k}(v^{(k)}),$$

and elements of $\otimes^{k,0} V$ to be $k$-fold linear maps $V^* \times \cdots \times V^* \to \mathbb{R}$:

$$T \in \otimes^{k,0} V \text{ given by } T = T^{i_1 i_2 \ldots i_k}_{i} v^{i_1}_i \otimes v^{i_2}_i \otimes \cdots \otimes v^{i_k}_i$$

$$T(v^{(1)}_i, v^{(2)}_i, \ldots, v^{(k)}_i) = T^{i_1 i_2 \ldots i_k}_{i} v^{i_1}_i \otimes v^{i_2}_i \otimes \cdots \otimes v^{i_k}_i (v^{(1)}_i, \ldots, v^{(k)}_i)$$

$$= T^{i_1 i_2 \ldots i_k}_{i} v^{i_1}_i(v^{(1)}_i) v^{i_2}_i(v^{(2)}_i) \cdots v^{i_k}_i(v^{(k)}_i).$$

Finally, it is possible to regard any element $T \in \otimes^{r,s} V$ as a map $T : V^* \times \cdots \times V^* \times V \times \cdots \times V \to \mathbb{R}$. For example, an element $T \in \otimes^{1,2} V$, given by

$$T = T^{i}_{jk} v^{i} \otimes v^{j} \otimes v^{k}$$

can be considered to be a map $V^* \times V \times V \to \mathbb{R}$:

$$T(v^*, w, x) = T^{i}_{jk} v^i(v^*) v^j(w) v^k(x),$$

where $v^* \in V^*$ and $w, x \in V$. 

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2 An addition to the Einstein notation

We introduce another feature of Einstein notation. Recall that we defined isomorphisms $V^* \otimes V \otimes V^* \approx V \otimes V^* \otimes V^*$, etc. However, it is sometimes important to preserve the order of the tensor products. As a point of fact,

$$T = v^* \otimes w \otimes x^* \quad \text{and} \quad S = w \otimes v^* \otimes x^*$$

are different tensors. This is encoded in the Einstein notation by preserving the ordering of the indices:

$$T = T_{i_j k} v_i \otimes v_j \otimes v_k$$

and

$$S = S_{j_i k} v_j \otimes v_i \otimes v_k$$

are in different tensor spaces. In fact,

$$T : V \times V^* \times V \to \mathbb{R},$$

whereas

$$S : V^* \times V \times V \to \mathbb{R}.$$  

3 Dual spaces

If $V^*$ is dual to $V$ and $V$ is dual to $V^*$, what is the dual to $\otimes^{r,s} V$? It is $\otimes^{s,r} V$.

Given a tensor $T^{r,s} \in \otimes^{r,s} V$, we can consider it to be a linear map $\otimes^{s,r} V \to \mathbb{R}$. On decomposable elements of $\otimes^{s,r} V$ we define this by

$$T \left( v^{(i_1)} \otimes v^{(i_2)} \otimes \cdots \otimes v^{(i_r)} \otimes v^{(j_1)} \otimes v^{(j_2)} \otimes \cdots \otimes v^{(j_s)} \right)$$

$$\triangleq T \left( v_{(j_1)}, v_{(j_2)}, \ldots, v_{(j_s)}, v_{(i_1)}, v_{(i_2)}, \ldots, v_{(i_r)} \right),$$

and extending linearly.

4 Transformation properties

Let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ be different bases for $V$, with $A = A_{(f_i) \rightarrow (e_i)}$ the the transition matrix between them. Let $\{e^1, \ldots, e^n\}$ and $\{f^1, \ldots, f^n\}$ be the respective dual bases, with transition maps $B = B_{(f^i) \rightarrow (e^i)}$. We have

$$f_i = A_{ji} e_j \quad \text{and} \quad f^i = B_{ji} e^j.$$
As was discussed in Lecture 8, section 4, we have the relation \( B = A^{-1} \). That is to say,

\[
B^i_k A^k_j = \delta^i_j \quad \text{and} \quad A^i_k B^k_j = \delta^i_j.
\]

A tensor \( T \in \bigotimes^{1,1} V \), for instance, may have the expression

\[
T = T^i_j e_i \otimes e^j
\]
in the \( \{e_i\} - \{e^j\} \) basis. Its expression in the \( \{f_i\} - \{f^j\} \) basis is given by

\[
T = T^i_j e_i \otimes e^j
= T^i_j (A^k_i f_k) \otimes (B^l_j f^l)
= (A^k_i B^l_j T^i_l) f_k \otimes f^l.
\]

Likewise for elements of any of the spaces \( \bigotimes^{r,s} V \).

## 5 Symmetric and alternating tensors

A tensor \( T : V \times \cdots \times V \to \mathbb{R} \) (\( r \) many \( V \)'s) is called a **symmetric tensor** if

\[
T (v^{(1)}, \ldots, v^{(i)}, v^{(i+1)}, \ldots, v^{(r)}) = T (v^{(1)}, \ldots, v^{(i+1)}, v^{(i)}, \ldots, v^{(r)})
\]

That is, if you interchanging any two consecutive entries leaves the tensor unchanged. A tensor \( T : V \times \cdots \times V \to \mathbb{R} \) (\( r \) many \( V \)'s) is called an **alternating or antisymmetric tensor** if

\[
T (v^{(1)}, \ldots, v^{(i)}, v^{(i+1)}, \ldots, v^{(r)}) = -T (v^{(1)}, \ldots, v^{(i+1)}, v^{(i)}, \ldots, v^{(r)})
\]

That is, if interchanging any two consecutive entries introduces a minus sign.

This leads us to two new definitions.

**Definition** The space \( \bigotimes^s V^* \subset \bigotimes^{0,s} V \) is the space of symmetric tensors of the type \( V \times \cdots \times V \to \mathbb{R} \) (\( s \) many \( V^* \)'s).

**Definition** The space \( \bigwedge^s V^* \subset \bigotimes^{0,s} V \) is the space of alternating tensors of the type \( V \times \cdots \times V \to \mathbb{R} \) (\( s \) many \( V^-V')\).

**Example** Let \( V \) be a vectors space with basis \( \{v_i\} \) and dual basis \( \{v^i\} \). Let

\[
T = v^1 \otimes v^2 + v^2 \otimes v^1
S = v^1 \otimes v^2 - v^2 \otimes v^1
U = v^1 \otimes v^2
W = v^1 \otimes v^1.
\]

Then \( T \) and \( W \) are symmetric tensors, \( S \) is an antisymmetric tensor, and \( U \) is neither symmetric nor antisymmetric.
6 The $\text{Alt}$ map and the wedge product

There is a canonical way of transforming any tensor $T \in \bigotimes^{0,s} V$ into an alternating tensor, given by the $\text{Alt}$ map:

$$\text{Alt} : \bigotimes^{0,s} V \to \bigwedge^s V^*$$

$$\text{Alt}(T)(v_1, \ldots, v_s) = \frac{1}{s!} \sum_{\pi \in \text{Sym}(s)} (-1)^{\left|\pi\right|} T(v_{\pi(1)}, \ldots, v_{\pi(s)}).$$

For instance, if $T \in \bigotimes^{0,2} V$, then

$$\text{Alt}(T)(v, w) = \frac{1}{2} (T(v, w) - T(w, v)).$$

If $T \in \bigotimes^{0,3} V$, then

$$\text{Alt}(T)(v, w, x) = \frac{1}{6} (T(v, w, x) - T(v, x, w) - T(w, v, x) - T(w, x, v) + T(x, v, w) - T(x, w, v)).$$

Since $\text{Alt}(T)$ is itself a tensor, we should be able to express in terms of a basis. For instance if $T = v_1 \otimes v_2$ then

$$\text{Alt}(T) = \frac{1}{2} (v_1 \otimes v_2 - v_2 \otimes v_1)$$

and if $T = v_1 \otimes v_2 \otimes v_3$, then

$$\text{Alt}(T) = \frac{1}{6} (v_1 \otimes v_2 \otimes v_3 - v_1 \otimes v_3 \otimes v_2 - v_2 \otimes v_1 \otimes v_3 + v_2 \otimes v_3 \otimes v_1 + v_3 \otimes v_1 \otimes v_2 - v_3 \otimes v_2 \otimes v_1).$$

**Theorem 6.1** The map $\text{Alt} : \bigotimes^{0,s} V \to \bigwedge^s V^*$ is onto, and linear (meaning $\text{Alt}(\alpha T + S) = \alpha \text{Alt}(T) + \text{Alt}(S)$). If $T \in \bigotimes^{0,s} V$, then $\text{Alt}(\text{Alt}(T)) = \text{Alt}(T)$.

\[\square\]

Given two alternating tensors, $T \in \bigwedge^n V^*$ and $S \in \bigwedge^m V^*$, the wedge product $T \wedge S$ of $T$ and $S$ is defined to be

$$T \wedge S \equiv \text{Alt}(T \otimes S).$$

Notice that $T \wedge S \in \bigwedge^{n+m} V^*$.

**Theorem 6.2** If $T \in \bigwedge^n V^*$ and $S \in \bigwedge^m V^*$, then $T \wedge S = (-1)^{nm} S \wedge T$.

**Example.** Express $v^1 \wedge v^2$ as a tensor.

**Solution:**

$$v^1 \wedge v^2 = \text{Alt}(v^1 \otimes v^2) = \frac{1}{2} (v^1 \otimes v^2 - v^2 \otimes v^1).$$