ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 16, nº 4 (1989), p. 579-601.

<http://www.numdam.org/item?id=ASNSP_1989_4_16_4_579_0>

A Construction of Unitary Representations in Parabolic Rank Two

M.W. BALDONI-SILVA - A.W. KNAPP (*)

0. - Introduction

Some years ago, B. Speh and the second author [7] investigated unitary representations of $G = SU(N, 2)$ that arise as Langlands quotients of the nonunitary principal series. The nonunitary principal series consists of the induced representations

$$U(MAN, \sigma, \nu) = \operatorname{ind}_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1), \tag{0.1}$$

where MAN is a minimal parabolic subgroup. For this G, M is a (compact) double cover of the unitary group $U(N-2)$ and A is Euclidean of dimension 2. To visualize matters, we fix an irreducible representation σ of M and consider the two-dimensional picture of A parameters e^{ν}, where ν is a real-valued linear functional on the Lie algebra $\mathcal{A}$ of A. In interesting cases, the typical picture of unitary points (i.e., those points for which the corresponding Langlands quotient can be made unitary) within the positive Weyl chamber, is as in Figure 1.

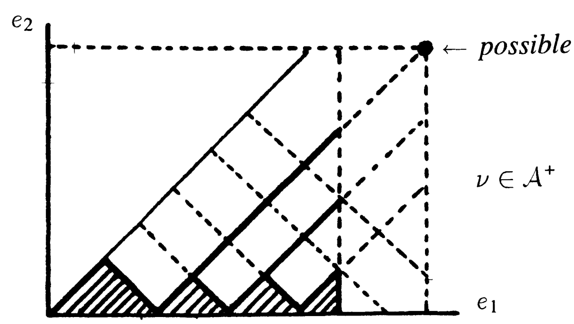


Figure 1: *Typical picture of unitary points in* $SU(N, 2)$

Pervenuto alla Redazione il 4 Febbraio 1989.

(*) Supported by National Science Foundation Grants MCS 80-01854, DMS 85-01793, and DMS 87-11593.

There are some two-dimensional regions of unitary points along the horizontal axis, there are some upward-sloping diagonal lines, and there is sometimes one isolated point.

Our concern here is with the lines of unitarity in the picture. The two-dimensional regions are a rank-one phenomenon, and we can regard them as already understood in an inductive argument. What about the lines? The proof in [7] that the lines give unitary points uses the fact that the lines correspond to irreducible degenerate series; the argument for irreducibility is rather complicated and shows little hope of generalization.

In the present paper, we shall show that the unitarity along these lines is indeed a rank-one phenomenon, provided we take into account some formal properties of intertwining operators. As we shall see, the argument generalizes to other real-rank-two groups and also to other parabolic-rank-two situations. In some cases, the argument produces isolated unitary points, as well as lines. The main results are Theorems 4.11 and 5.2.

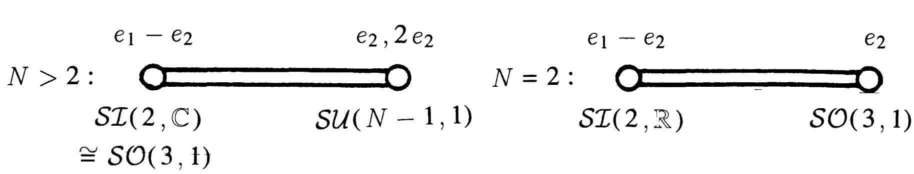
However, when we try to generalize the argument to parabolic rank n, we find that it does not seem to handle $(n-1)$-dimensional sets in parabolic rank n, just one-dimensional sets and certain isolated points. This failure raises a question that we mention in Section 7.

We use the following notational conventions throughout. The group G is a connected linear semisimple Lie group with maximal compact subgroup K. Subgroups are denoted by upper-case Latin letters, and their Lie algebras are denoted by the corresponding lower-case German letters.

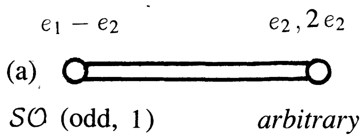
We are indebted to O.S. Rothaus for supplying a proof of Lemma 1.2 for us, to Chin-Han Sah for helping us to cope with some of the intricacies of $SO(n,1)$ in Section 4, and to H. Schlichtkrull for giving us independent confirmation of some of the unitarity we discovered.

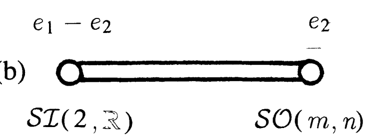
1. - Oversimplified idea

In $SU(N,2)$ the configuration of roots of $(\mathcal{G},\mathcal{A})$ is of type $(BC)_2$, for $N>2$, and may be taken to be of type B_2 for $N=2$. We give the precise diagrams below.



In the parabolic-rank-two generalization, we assume that G is linear, that MAN is a cuspidal parabolic subgroup of G, that all roots of $(\mathfrak{g}, \mathfrak{a})$ are useful in the sense of [4], and that the corresponding configuration of roots is as in (1.1a) or (1.1b).

(1.1) (a) 

(b) 

We work with induced representations (0.1) in which σ is a discrete series or (nonzero) limit of discrete series representation of M, A has dimension 2, and ν is a real-valued linear functional on $\mathfrak{a}$ in the closed positive Weyl chamber.

Configuration (1.1a) arises with MAN minimal parabolic in G when G is locally $SU(N,2)$, $Sp(N,2)$, $SO^*(10)$, or the real form of E_6 whose symmetric space is Hermitian. It arises with MAN general whenever A has dimension 2 and G has restricted roots of type $(BC)_n$ for some n; it can occur in other situations as well. The prototype of configuration (1.1b) is a certain parabolic-rank-two situation with $G = \widetilde{SO}(m+1, n+1)$; the parabolic subgroup is minimal when $G = \widetilde{SO}(N,2)$. Our detailed discussion will concentrate on (1.1a); adjustments to the arguments that allow one to handle (1.1b) will be discussed in Section 5.

The Weyl group $W(\mathfrak{a})$ for the parabolic is the usual 8-element group for our root system, and we let s_0 be its long element, which acts on $\mathfrak{a}$ by -1. We shall see in Lemma 4.1 for configuration (1.1a) that s_0 fixes the class of σ.

Then unitarity of the Langlands quotient of (0.1) is detected by the standard intertwining operator of [9] corresponding to s_0. This operator maps $U(MAN,\sigma,\nu)$ to $U(MAN,\sigma,-\nu)$. It is Hermitian on each K type, and the Langlands quotient is unitary if this operator is semidefinite. This operator factors as a product $ABCD$ in a fashion that mirrors a decomposition of s_0 as a product of four simple reflections. Here A and C are disguised intertwining operators for the subalgebra attached to $e_1 - e_2$ [SO (odd, 1) and $S\ell(2,\mathbb{R})$ in the two cases of (1.1)], and B and D are disguised intertwining operators attached to $(e_2, 2e_2)$ or to e_2. We shall make use (on each K type) of the following fact from linear algebra.

PROPOSITION 1.1. *Suppose that A, B, C, and D are n-by-n complex matrices with $ABCD$ Hermitian. Suppose further that*

(1) *B and D are Hermitian positive semidefinite*

(2) *$A = PC^*$ with P Hermitian positive semidefinite.*

Then $ABCD$ is positive semidefinite.

In fact, $ABCD = P(C^*BC)D$. So the proposition reduces to the following lemma, which will be proved in Section 2.

LEMMA 1.2. *If R, S, T are positive semidefinite Hermitian n-by-n matrices and RST is Hermitian, then RST is positive semidefinite.*

In our application B and D are intertwining operators for the parabolic-rank-one situation built from $(e_2, 2e_2)$ or from e_2. Positivity for them corresponds to unitarity of certain Langlands quotients obtained from maximal parabolic subgroups, and this unitarity is understood completely [1]. Thus we know exactly when (1) holds in the proposition.

In considering condition (2), we must take into account slight differences between (1.1a) and (1.1b). For now, we consider (1.1a). Condition (2) for (1.1a) is a condition only on A and C, which are operators for SO(odd, 1), and these operators are well understood. (For formulas on all K types, see Klimyk-Gavrilik [3, p. 54].) The following properties of the operators and corresponding representations (nonunitary principal series) will be relevant for us in the situation (1.1a):

(i) The K types have multiplicity one. This is well known and follows from Frobenius reciprocity and the branching theorem for $\widetilde{SO}(n) \to \widetilde{SO}(n-1)$. As a consequence, the intertwining operator is a scalar for each K type and each ν. As ν varies through real values, this scalar moves through real multiples of some complex number. (This last fact can be read off from Klimyk-Gavrilik [3] and presumably can be derived directly from Theorem 5.1 of [2]).

(ii) The total operator (on all K types) has a kernel at ν if and only if (0.1) is reducible at ν. This is part of the Langlands theory; see Theorem 7.24 of [6].

(iii) Reducibility of (0.1) can occur only at integral points of the infinitesimal character. See Proposition 6.1 of Speh-Vogan [12].

(iv) For fixed M parameter and varying A parameter, Figure 2 gives the connection between reducibility of (0.1) and singularities of the infinitesimal character at points where the infinitesimal character is integral. This will be proved in Section 3.

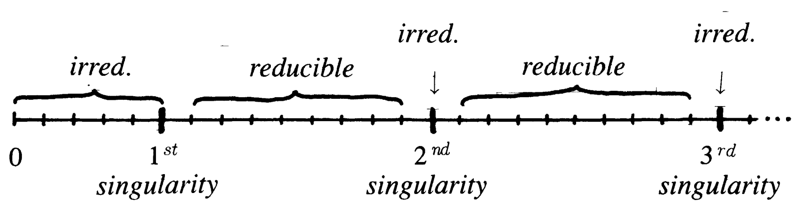


Figure 2: *Reducibility points in* $\widetilde{SO}$ (odd, 1)

(v) In Figure 2 on any interval between two consecutive singularities of the infinitesimal character, the kernel of the intertwining operator at integral points decreases as one moves to the right. This can be read off from Klimyk-Gavrilik [3]. We do not have an abstract proof, but possibly one can be constructed with the aid of Zuckerman's translation functors [14].

Let us see how these properties are relevant for condition (2). Let A^{SO} be the standard intertwining operator for $\widetilde{SO}$ (odd, 1) attached to $e_1 - e_2$, and let α be the positive restricted root for this group. If we write $\nu = ae_1 + be_2$, then the operators A and C are essentially

$$A \approx A^{SO}\left(s_\alpha, \sigma_0, \frac{1}{2}(a-b)\alpha\right),$$

$$C \approx A^{SO}\left(s_\alpha, s_\alpha\sigma_0, \frac{1}{2}(a+b)\alpha\right),$$

for a suitable σ_0 constructed from σ. Since ν is real,

$$C^* \approx A^{SO}\left(s_\alpha, \sigma_0, \frac{1}{2}(a+b)\alpha\right) = A^{SO}\left(s_\alpha, \sigma_0, \left[\frac{1}{2}(a-b)+b\right]\alpha\right).$$

The upward-sloping lines in Figure 1 are the set of ν's where the factor A of $ABCD$ has a kernel. Each line has $\frac{1}{2}(a-b)$ constant and equal to the value at one of the reducibility points in Figure 2. Fix attention on one such line.

As b varies (increasing from 0), the argument of C^* increases from its starting value, which is the argument for A. In terms of Figure 2, the argument of C^* is moving to the right from the initial reducibility point. According to property (v) above, ker $A \supseteq$ ker C^* until the argument of C^* gets to a singularity of the infinitesimal character.

Let us see the effect on the scalar function in (i). On a K type where A is 0, certainly A is a nonnegative multiple of C^*. On a K type where A is not 0, the operator $A^{SO}\left(s_\alpha, \sigma_0, \left[\frac{1}{2}(a-b)+t\right]\alpha\right)$ is not 0 for $0 \le t \le b$, because of the inclusion of kernels of (v). By (i) this operator is a real multiple of A on the K type, and continuity says that it is a positive multiple. We conclude that $A = PC^*$ for an operator P that is a nonnegative function of ν on each K type, this condition holding until the argument of C^* gets to a singularity of the infinitesimal character. Beyond that, the equality $A = PC^*$ extends until the next integral point, by (iv) and continuity.

In other words, the two conditions in Proposition 1.1 are satisfied for the intertwining operator $ABCD$ in configuration (1.1a) provided

(1) the horizontal and vertical coordinates of ν (namely ae_1 and be_2) correspond to unitary points in the appropriate parabolic-rank-one situations

(2a) ν is on an upwardly sloping line corresponding to reducibility coming from ker A nontrivial

(2b) the parameters of A and C^* for the group $\widetilde{SO}$ (odd, 1) are not so far apart that they are separated by a singularity of the infinitesimal character, followed by a reducibility point.

Then Proposition 1.1 says that $ABCD$ is semidefinite, and the corresponding Langlands quotient must be unitary.

For the special case of $SU(N,2)$, the group $\widetilde{SO}$ (odd, 1) is $SL(2,\mathbb{C})$. There is at most one singularity of the infinitesimal character in Figure 2. Thus there are only two intervals in the picture. Since the argument of A corresponds to a reducibility point, the arguments of A and C^* both correspond to the right-hand interval. Thus (2b) is automatic above. So when (1) and (2a) are satisfied, we have unitarity. This fact accounts for the lines of unitarity in Figure 1.

In the situation (1.1b), the above argument needs some minor modifications and additional hypotheses. We defer discussion of these points to Section 5.

2. - Proof of linear algebra lemma

In this section we prove Lemma 1.2. The argument is due to O.S. Rothaus.

Let $K = \ker T$, and write $T = U^2$ with U Hermitian. Define U' to be U^{-1} on $K^\perp$ and to be 0 on K. Then $TU' = U$, and $U'RSU = U'RSTU'$. Since $RST(K) = 0 \subseteq K$, $RST(K^\perp) \subseteq K^\perp$. Therefore $U'(RST)U'$ leaves stable both K and $K^\perp$. On $K^\perp$ it has the same signature as RST, while on K it is 0 (and so is RST). Thus $U'RST'U'$ has the same signature as RST.

We claim that the nonzero eigenvalues of $U'RSU(= U'RSTU')$ are a subset of the eigenvalues of RS. In fact, $RST(K^\perp) \subseteq K^\perp$ and T carries $K^\perp$ onto $K^\perp$ (being invertible there). So $RS(K^\perp) \subseteq K^\perp$. Thus we can write $U'(RS)U$ in block form as

$$\begin{array}{cc} K^\perp & K \end{array}$$

$$\begin{pmatrix} u^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} m_1 & * \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} u^{-1}m_1 & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u^{-1}m_1 u & 0 \\ 0 & 0 \end{pmatrix},$$

and its nonzero eigenvalues are those for m_1.

To complete the proof, it is enough to show that all eigenvalues of RS are ≥ 0. This is well known. We simply write $S = V^2$ with V Hermitian. Then $RS = (RV)V$ has the same eigenvalues as VRV, which is positive semidefinite.

3. - Reducibility in $\widetilde{SO}$ (odd, 1)

In this section we prove property (iv) of Section 1 for the nonunitary principal series of $\widetilde{SO}$ (odd, 1).

PROPOSITION 3.1. *For $\widetilde{SO}$ (odd, 1), let α be the positive restricted root, let σ be the M parameter, and let the integral points for the A parameter be $\nu = (c+n)\alpha$, with c fixed (equal to 0 or $\frac{1}{2}$) and n varying in $\mathbb{Z}$. Let n_0 be the least $n \in \mathbb{Z}$ such that $c+n \geq 0$ and such that the infinitesimal character of $U(MAN, \sigma, (c+n)\alpha)$ is singular. For $n \in \mathbb{Z}$ and $c+n \geq 0$, $U(MAN, \sigma, (c+n)\alpha)$ is irreducible if $n \leq n_0$ or if the infinitesimal character is singular, and it is reducible otherwise.*

PROOF. For $n < n_0$, this result follows from Proposition 6.1 of Speh-Vogan [12]. Next suppose that $(c+n_1)\alpha$ gives a singular point. Then the Plancherel factor $p_\sigma(\nu)$ of Knapp-Stein [8] is 0 at $\nu = (c+n_1)\alpha$ (see p. 543 of that paper). If w is a representative in K of the reflection s_α, then Proposition 27 and Theorem 4 of [8] give

$$A(w^{-1}, w\sigma, -\nu)A(w, \sigma, \nu) = Cp_\sigma(\nu)^{-1}I \tag{3.1}$$

as an identity of meromorphic functions. It follows that $A(w^{-1}, w\sigma, -\nu)$ has a pole at $\nu = (c+n_1)\alpha$. This pole can only be simple, according to Theorem 3 of [8]. Since the right side of (3.1) has a pole (on all functions $\not\equiv 0$) at $\nu = (c+n_1)\alpha$, we see that $A(w, \sigma, (c+n_1)\alpha)$ has kernel 0. Then the Langlands theory says that $U(MAN, \sigma, (c+n_1)\alpha)$ is irreducible.

Finally suppose that $(c+n_1)\alpha$ is not a singular point and that $n_1 > n_0$. The previous paragraph shows that $A(w^{-1}, w\sigma, -\nu)$ has a pole at $\nu = (c+n_0)\alpha$. We shall deduce from this fact that $A(w^{-1}, w\sigma, -\nu)$ has a pole at $\nu = (c+n_1)\alpha$. Assuming this conclusion temporarily, we examine (3.1). The right side is regular at $\nu = (c+n_1)\alpha$, since $(c+n_1)\alpha$ is not a singular point of the infinitesimal character. It follows that $A(w, \sigma, (c+n_1)\alpha)$ has a nonzero kernel. Since $A(w, \sigma, (c+n_1)\alpha)$ cannot be the 0 operator, $U(MAN, \sigma, (c+n_1)\alpha)$ is reducible.

To get at the pole, it is enough to show that if $A(w^{-1}, w\sigma, -\nu)$ has a pole at $\nu = (c+n)\alpha$, then it has a pole also at $\nu = (c+n+1)\alpha$. We refer to pages 524 and 550 of [8]. In the notation of that paper, we expand

$$e^{(1+z)\rho H(y)} f(K(y)) = f(1) + g_1(z, y) + \cdots + g_{2n}(z, y) + R_n(z, y), \tag{3.2}$$

where g_j is entire in z and α-homogeneous of degree j in y. Then a pole occurs at $z = -\frac{j}{p+2q}$ (which corresponds to $\nu = -\frac{1}{2}j\alpha$) if and only if $\sigma^{-1}(yw)g_j(z, y)$ has mean value nonzero at $z = -\frac{j}{p+2q}$. When there is a pole, we can multiply f by a function on K such that the new expansion (3.2) is $\|y\|^2$ times the old one, except for the remainder; here $\|y\|^2$ is the Euclidean norm. Unwinding the notation, we find that the new f fives us a pole at $z=-\frac{j+2}{p+2q}$, which corresponds to $\nu=-\frac{1}{2}(j+2)\alpha$.

4. - Precise results

The treatment in Section 1 is an oversimplification in two ways. The first way is hidden by the vague description of what intertwining operator from [9] we are actually using. As we shall see, there are three slightly different kinds of operators coming into play. Moreover the multiplicative relations that the operators satisfy usually mirror multiplication only in the normalizer $N_K(\mathcal{A})$, not in its quotient $W(\mathcal{A})$.

The other way that oversimplification occurs is in the passage from operators for G to operators for a subgroup $\widetilde{SO}$ (odd, 1). When intertwining operators for G are realized in the noncompact picture in the sense of [9], the operators for G corresponding to simple reflections are tensor products of operators for $\widetilde{SO}$ (odd, 1) and identity operators. But it is necessary to use the compact picture in the present investigation, and the relevant formula ((7.10) of [9]) is not very helpful for deducing condition (2) in Proposition 1.1.

So let us be more precise. In this section we work exclusively with the configuration (1.1a). For any $\mathcal{A}$ root β, we let $\mathcal{G}_\beta$ be the corresponding root space, and we put

$$\mathcal{G}^{(\beta)}=\mathcal{M}\oplus\mathbb{R}H_\beta\oplus\sum_{t\neq 0}\mathcal{G}_{t\beta}.$$

Situation (1.1a) is to mean that G is connected linear, that the $\mathcal{A}$ roots form a system of type $(BC)_2$, and that $\mathcal{G}^{(e_1-e_2)}$ is the direct sum of ideals $\mathcal{SO}\oplus\mathcal{SO}'$ with $H_\beta\in\mathcal{SO}$ and $\mathcal{SO}\cong\mathcal{SO}$ (odd, 1).

In this situation, Lemma 2.7 of [5] shows that $2e_1$ and $2e_2$ extend to real roots (relative to a Cartan subalgebra $\mathcal{A}\oplus\mathcal{B}$ in which $\mathcal{B}$ is a compact Cartan subalgebra of $\mathcal{M}$). Then Lemma 7.8 of [5] says that the reflections s_{2e_1} and s_{2e_2} have representatives w_1 and w_2 in $N_K(\mathcal{A})$ that commute with each other and centralize the identity component M_0. We fix w_1 and w_2 with these properties. Also we fix w_{12} in $N_K(\mathcal{A})$ to be a representative of $s_{e_1-e_2}$.

LEMMA 4.1. *In the situation* (1.1a), *let σ be a discrete series or (nonzero) limit of discrete series representation of M. Then $w_1\sigma\cong\sigma$ and $w_2\sigma\cong\sigma$. Moreover σ extends to a representation (on the same Hilbert space) of the group generated by M, w_1, and w_2.*

PROOF. To see that $w_1\sigma\cong\sigma$, we apply Lemma 10.3 of [10]. Since $2e_1$ is a real root, Lemma 3.8b of [5] says that s_{2e_1} fixes the Harish-Chandra parameter of σ. The central character of σ has domain the group generated by the center Z_{M_0} and the elements γ_{2e_1} and γ_{2e_2} defined in (2.3) of [10], according to (1.6) and Lemma 2.1b of [10]. Since w_1 centralizes these generators, w_1 fixes the central character. Then Lemma 10.3 of [10] says $w_1\sigma\cong\sigma$. Similarly $w_2\sigma\cong\sigma$.

Since $w_2\sigma \cong \sigma$, Lemma 7.9 of [9] says that σ extends to a representation σ_2 of the group M_2 generated by M and w_2. Meanwhile the same lemma allows us to define $\sigma(w_1)$ so as to extend σ to the group generated by M and w_1. We shall invoke the lemma a third time to extend σ_2 to a representation of the group generated by M_2 and w_1. To do so, we are to prove that $w_1\sigma_2 \cong \sigma_2$. Put $E = \sigma(w_1)$. For m in M, we have

$$w_1\sigma_2(m) = \sigma_2(w_1^{-1}mw_1) = \sigma(w_1^{-1}mw_1) = E^{-1}\sigma(m)E = E^{-1}\sigma_2(m)E, \tag{4.1a}$$

and for w_2, we have

$$\begin{aligned} w_1\sigma_2(w_2) &= \sigma_2(w_1^{-1}w_2w_1) && \\ &= \sigma_2(w_2) && \text{since } w_1w_2 = w_2w_1 \\ &= \sigma(w_2) && \\ &= \sigma(w_1)^{-1}\sigma(w_2)\sigma(w_1) && \text{by an argument below} \\ &= E^{-1}\sigma_2(w_2)E. && \end{aligned} \tag{4.1b}$$

For (4.1b) we have used that $\sigma(w_1)$ and $\sigma(w_2)$ commute; this follows since $\sigma(w_1)$ and $\sigma(w_2)$ are in the commuting algebra of $\sigma|_{M_0}$, which is commutative by Lemma 4.4 of [5]. From (4.1), $w_1\sigma_2 \cong \sigma_2$. Thus σ extends to the group generated by M, w_1, and w_2, as asserted.

Invoking Lemma 4.1, we fix an extension of σ and hence consistent definitions of $\sigma(w_1)$ and $\sigma(w_2)$. We use unnormalized intertwining operators as follows.

If $S = MAN$ is our parabolic subgroup and $S' = MAN'$ is an associated parabolic and if the induced representations act by left translations, we define a formal expression by

$$A(S' : S : \sigma : \nu)f(x) = \int_{\overline{N}\cap N'} f(x\overline{n})d\overline{n}, \tag{4.2}$$

where $\overline{N}$ is opposite to N. This operator formally satisfies

$$U(S', \sigma, \nu)A(S' : S : \sigma : \nu) = A(S' : S : \sigma : \nu)U(S, \sigma, \nu)$$

and is made rigorous in [9]. For w in $N_K(\mathfrak{A})$, let

$$R(w)f(x) = f(xw) \tag{4.3}$$

and

$$A_S(w, \sigma, \nu) = R(w)A(w^{-1}Sw : S : \sigma : \nu). \tag{4.4}$$

This operator formally satisfies

$$U(S, w\sigma, w\nu)A_S(w, \sigma, \nu) = A_S(w, \sigma, \nu)U(S, \sigma, \nu).$$

When ν is real-valued and is in the positive Weyl chamber,

$$\sigma(w_1 w_2)A_S(w_1 w_2, \sigma, \nu) \tag{4.5}$$

is the operator that decides the unitarity of the Langlands quotient of $U(S, \sigma, \nu)$, according to Theorem 16.6 of [6].

LEMMA 4.2. *In situation* (1.1a),

$$\sigma(w_1 w_2)A_S(w_1 w_2, \sigma, \nu) = ABCD,$$

where

$$\begin{aligned} A^* &= R(w_{12})A(w_{12}^{-1} S w_{12} : S : \sigma : \bar{\nu}) \\ B &= (w_{12}\sigma)(w_2)A_S(w_2, w_{12}\sigma, w_{12}w_2\nu) \\ C &= R(w_{12})A(w_{12}^{-1} S w_{12} : S : \sigma : w_2\nu) \\ D &= \sigma(w_2)A_S(w_2, \sigma, \nu). \end{aligned}$$

Here $(w_{12}\sigma)(w_2) = \sigma(w_{12}^{-1} w_2 w_{12})$, *and the adjoint is taken K type by K type relative to the $L^2(K)$ inner product for the induced space.*

PROOF. The elements w_1 and $w_{12}^{-1} w_2 w_{12}$ both represent s_{2e_1} in $W(A)$. Since (4.5) is independent of the representative of $s_{2e_1}s_{2e_2}$, (4.5) is

$$\begin{aligned} &= \sigma\left[(w_{12}^{-1} w_2 w_{12})w_2\right] A_S\left[(w_{12}^{-1} w_2 w_{12})w_2, \sigma, \nu\right] \\ &= [\sigma(w_{12}^{-1} w_2 w_{12})A_S(w_{12}^{-1} w_2 w_{12}, \sigma, \nu)]D \end{aligned}$$

by Lemma 13.1 of [9].

If we expand the factor in brackets first by (4.4), then by Theorem 7.6 of [9], and finally by Proposition 7.1 of [9], we find that it equals ABC above. It is easy to check that $E = \sigma(w_{12}^{-1} w_2 w_{12})$ satisfies

$$w_2(w_{12}\sigma) = E^{-1}(w_{12}\sigma)E$$

and that

$$E^2 = (w_{12}\sigma)(w_2^2),$$

and then Lemma 7.9 of [9] says we may take $(w_{12}\sigma)(w_2) = E$. This proves the lemma.

LEMMA 4.3. *In situation* (1.1a), *the operators B and D of Lemma 4.2 are Hermitian (on each K type).*

PROOF. Proposition 7.10 of [9] gives

$$[\sigma(w_2)A_S(w_2,\sigma,\nu)]^* = \sigma(w_2^{-1})A_S(w_2^{-1},\sigma,-w_2\overline{\nu}).$$

Since w_2 and w_2^{-1} both represent s_{2e_2}, the right side is

$$\begin{aligned} &= \sigma(w_2)A_S(w_2,\sigma,-w_2\overline{\nu}) \\ &= \sigma(w_2)R(w_2)A(s_{2e_2}^{-1}Ss_{2e_2} : S : \sigma : w_1\overline{\nu}). \end{aligned} \tag{4.6}$$

In a suitable open subset of $(\mathcal{A}')^{\mathbb{C}}$, Theorem 6.6 of [9] makes sense of (4.2) as

$$A(s_{2e_2}^{-1}Ss_{2e_2} : S : \sigma : \nu)f(k) = \int_{N_{-2e_2}} e^{-(\nu+\rho)H(\overline{n})}\sigma\,[\mu(\overline{n})]^{-1}\,f\,[k\mathcal{K}(\overline{n})]\,d\overline{n}. \tag{4.7}$$

Here ρ is half the sum of the positive $\mathcal{A}$ roots (with multiplicities) and $\overline{n} = \mathcal{K}(\overline{n})\mu(\overline{n})e^{H(\overline{n})}n$ is a decomposition of $\overline{n}$ according to $KMAN$. In (4.7), $H(\overline{n})$ lies in $\mathbb{R}H_{2e_2}$, and (4.7) is unchanged if we replace ν by $w_1\nu$. Thus (4.6) on an open set of $(\mathcal{A}')^{\mathbb{C}}$ is

$$= \sigma(w_2)A_S(w_2,\sigma,\overline{\nu}),$$

and this equality must extend to other ν's by analytic continuation. Hence D is Hermitian. A similar argument proves B is Hermitian.

The above discussion handles the first sense in which Section 1 oversimplifies matters. To handle the second sense, we rewrite the intertwining operators using Frobenius reciprocity as in Section 5 of [7]. Let τ be an irreducible representation of K acting in a space V^τ, let V^σ be the space on which σ operates, and let $\mathbf{V}_\sigma^\tau$ be the subspace of the induced space transforming under K according to τ. If v is in V^τ and E is in $Hom_{K\cap M}(V^\tau, V^\sigma)$, we define

$$\varphi_{v,E}(k) = E(\tau(k)^{-1}v), \qquad k \in K.$$

Then $\varphi_{v,E}$ is in $\mathbf{V}_\sigma^\tau$, and the resulting bilinear mapping on pairs (v, E) gives us a linear map

$$\Phi : V^\tau \otimes_{\mathbb{C}} Hom_{K\cap M}(V^\tau, V^\sigma) \to \mathbf{V}_\sigma^\tau \tag{4.8}$$

that is known to be a linear isomorphism onto. (See Wallach [13], p. 270).

The space V^σ comes equipped with an inner product $\langle\cdot,\cdot\rangle_\sigma$. Let $\langle\cdot,\cdot\rangle_\tau$ be a K-invariant inner product on V^τ. If E is in $Hom_{K\cap M}(V^\tau, V^\sigma)$, we then have a well defined adjoint E^* in $Hom_{K\cap M}(V^\sigma, V^\tau)$, and we define an inner product on $Hom_{K\cap M}(V^\tau, V^\sigma)$ by $\langle E', E\rangle = \mathrm{d}_\tau^{-1}\mathrm{Tr}(E^*E')$, where d_τ is the degree of τ. The inner products on V^τ and $Hom_{K\cap M}(V^\tau, V^\sigma)$ yield a canonical inner product on $V^\tau \otimes_{\mathbb{C}} Hom_{K\cap M}(V^\tau, V^\sigma)$.

LEMMA 4.4. *The mapping Φ of* (4.8) *respects inner products.*

PROOF. Let $\{v_i\}$ be an orthonormal basis of V^τ. Then

$$\langle \varphi_{v',E'}, \varphi_{v,E}\rangle_{L^2(K,V^\sigma)} = \int_K \langle E'\tau(k)^{-1}v', E\tau(k)^{-1}v\rangle_\sigma \, dk$$

$$= \int_K \langle (E^*E')\tau(k)^{-1}v', \tau(k)^{-1}v\rangle_\tau \, dk$$

$$= \sum_{i,j} \int_K \langle (E^*E')v_i, v_j\rangle_\tau \langle \tau(k)^{-1}v', v_i\rangle_\tau \overline{\langle \tau(k)^{-1}v, v_j\rangle_\tau} \, dk$$

$$= \sum_{i,j} d_\tau^{-1} \langle (E^*E')v_i, v_j\rangle_\tau \langle v', v\rangle_\tau \overline{\langle v_i, v_j\rangle_\tau}$$

by Schur orthogonality

$$= d_\tau^{-1} \mathrm{Tr}(E^*E)\langle v', v\rangle_\tau$$

$$= \langle v' \otimes E', v \otimes E\rangle$$

as required.

The next lemma moves our intertwining operators from the image of Φ to the domain. If the parabolic subgroup $S = MAN$ is minimal, the formula simplifies to the one in Wallach [13, p. 270].

LEMMA 4.5. *If β is an S-simple A root, then the map Φ in* (4.8) *satisfies*

$$\Phi^{-1}A(s_\beta^{-1}Ss_\beta : S : \sigma : \nu)\Phi = I \otimes a(\tau : \sigma : \beta : \nu), \tag{4.9a}$$

where $a(\tau : \sigma : \beta : \nu)$ is analytically continued from

$$a(\tau : \sigma : \beta : \nu)(E) = \int_{\overline{N}_\beta} e^{-(\nu+\rho_\beta)H(\overline{n})}\sigma(\mu(\overline{n}))^{-1}E\tau(\mathcal{K}(\overline{n}))^{-1} d\overline{n}. \tag{4.9b}$$

Here ρ_β is half the sum of the A roots $t\beta$ with $t > 0$ (counting multiplicities) and $g = \mathcal{K}(g)\mu(g)e^{H(g)}n$ is a decomposition of g according to $G = KMAN$.

REMARK. The operator $a(\tau : \sigma : \beta : \nu)$ is in

$$End_{\mathbb{C}}(Hom_{K\cap M}(V^\tau, V^\sigma)).$$

Identity (4.9b) can be written more concretely as

$$A(s_\beta^{-1}Ss_\beta : S : \sigma : \nu)\varphi_{v,E} = \varphi_{v,a(\tau:\sigma:\beta:\nu)(E)}.$$

The proof of the lemma is a routine calculation starting from (4.2) with $f = \varphi_{v,E}$ and taking into account that $\rho = \rho_\beta$ on $\mathbb{R}H_\beta$ (see Proposition 1.2 of [9]).

Next we relate the operators A, B, C, D of Lemma 4.2 to operators in a parabolic-rank-one setting. For each $\mathcal{A}$ root β, let $G_0^{(\beta)}$ be the analytic subgroup of G with Lie algebra $\mathfrak{g}^{(\beta)}$, and let $G^{(\beta)} = G_0^{(\beta)}M$. If β is S-positive, put $S^{(\beta)} = S \cap G^{(\beta)}$, and form the induced representations

$$U^{(\beta)}(S^{(\beta)}, \sigma, \nu) = \operatorname{ind}_{S^{(\beta)}}^{G^{(\beta)}}(\sigma \otimes \exp(\nu|_{\mathbb{R}H_\beta}) \otimes 1).$$

These representations depend on ν only through $\nu|_{\mathbb{R}H_\beta}$.

Let us study the intertwining operator D of Lemma 4.2 given by

$$D = \sigma(w_2)A_S(w_2, \sigma, \nu) \quad \text{for } G.$$

Let D_2 be the corresponding operator for $G^{(2e_2)}$ given by

$$D_2 = \sigma(w_2)A_{S^{(2e_2)}}(w_2, \sigma, \nu) \quad \text{for } G^{(2e_2)}.$$

Fix an irreducible representation τ of K. Under the correspondence Φ in (4.8), Lemma 4.5 and arguments in [7] give us

$$D \text{ on } V^\tau \longleftrightarrow \left[E \to \sigma(w_2)a(\tau : \sigma : 2e_2 : \nu)(E)\tau(w_2)^{-1}\right];$$

we have dropped from the notation the tensor product with the identity. Put $K_2 = K \cap G^{(2e_2)}$, and let $\tau|_{K_2}$ decompose into irreducible representations as $\tau|_{K_2} = \sum \oplus \tau_j$. Let V_j^τ be the subspace of V^τ in which τ_j operates, and let $i_j : V^{\tau_j} \to V^\tau$ be the j^{th} injection. For each τ_j we have, in obvious notation,

$$D_2 \text{ on } V^{\tau_j} \longleftrightarrow \left[E_2 \to \sigma(w_2)a_2(\tau_j : \sigma : 2e_2 : \nu)(E_2)\tau_j(w_2)^{-1}\right]. \tag{4.10}$$

LEMMA 4.6. *If $\nu|_{\mathbb{R}H_{2e_2}}$ is real and is in the positive Weyl chamber for $G^{(2e_2)}$ and is such that the Langlands quotient of $U^{(2e_2)}(S^{(2e_2)}, \sigma, \nu)$ is unitary, then the operator D of Lemma* 4.2 *is semidefinite. Similarly if $w_2\nu|_{\mathbb{R}H_{2e_1}}$ is real and is in the positive Weyl chamber for $G^{(2e_1)}$ and is such that the Langlands quotient of $U^{(2e_1)}(S^{(2e_1)}, \sigma, w_2\nu)$ is unitary, then the operator B of Lemma* 4.2 *is semidefinite.*

PROOF FOR D. The assumed unitarity means that D_2 is semidefinite, say positive semidefinite. By Lemma 4.4 the operators on the right side of (4.10) are positive semidefinite for all j. Let E be given in $Hom_{K\cap M}(V^\tau, B^\sigma)$, and put $E_j = E \circ i_j$. Then E_j is in $Hom_{K_2\cap M}(V^{\tau_j}, V^\sigma)$, and the given positivity means that

$$\langle \sigma(w_2)a_2(\tau_j : \sigma : 2e_2 : \nu)(E_j)\tau_j(w_2)^{-1}, E_j\rangle \geq 0,$$

i.e.,

$$\operatorname{Tr}\left[E_j^* \sigma(w_2) a_2(\tau_j : \sigma : 2e_2 : \nu)(E_j)\tau_j(w_2)^{-1}\right] \geq 0. \tag{4.11}$$

If $j \neq k$, then

$$\operatorname{Tr}\left[E_k^* \sigma(w_2) a_2(\tau_j : \sigma : 2e_2 : \nu)(E_j)\tau_j(w_2)^{-1}\right] = 0, \tag{4.12}$$

since the operator in question maps V^{τ_j} to V^{τ_k}. Summing (4.11) on j, taking (4.12) into account, and sorting matters out, we obtain

$$\operatorname{Tr}\left[E^* \sigma(w_2) a(\tau : \sigma : 2e_2 : \nu)(E)\tau(w_2)^{-1}\right] \geq 0.$$

By Lemma 4.4, D is positive semidefinite.

PROOF FOR B. The assumptions imply that $w_{12}w_2\nu|_{\mathbb{R}H_{2e_2}}$ is real, is in the positive Weyl chamber for $G^{(2e_2)}$, and is such that $U^{(2e_2)}\left(S^{(2e_2)}, w_{12}\sigma, w_{12}w_2\nu\right)$ is unitary. Then the proof for D shows that B is semidefinite.

Next let us study the operators A and C of Lemma 4.2. The condition we seek for Proposition 1.1 is that there must be a positive semidefinite P with $A = PC^*$, i.e., $A^* = CP$. Taking the formulas of Lemma 4.2 into account, we are looking for circumstances when

$$A(w_{12}^{-1}Sw_{12} : S : \sigma : \nu) = A(w_{12}^{-1}Sw_{12} : S : \sigma : w_2\nu)P$$

with ν real-valued. The operators here all carry $\mathbf{V}_\sigma^\tau$ into itself if τ is a K type of $U(S, \sigma, \nu)$. Translating this equation by Φ in (4.8) and applying Lemma 4.5, we see that we are seeking an operator $d(\nu)$ carrying $Hom_{K\cap M}(V^\tau, V^\sigma)$ into itself such that

$$a(\tau : \sigma : e_1 - e_2 : \nu) = a(\tau : \sigma : e_1 - e_2 : w_2\nu)d(\nu). \tag{4.13}$$

Let $\alpha = e_1 - e_2$. First we study equation (4.13) in $G^{(\alpha)}$. Let $K^{(\alpha)} = K \cap G^{(\alpha)}$, etc., and let τ_0 be a $K^{(\alpha)}$ type of $U^{(\alpha)}(S^{(\alpha)}, \sigma, \nu)$. Recall the decomposition $\mathcal{G}^{(\alpha)} = \mathcal{SO} \oplus \mathcal{SO}'$, with $\mathcal{SO}$ and $\mathcal{SO}'$ ideals in $\mathcal{G}^{(\alpha)}$; here $\mathcal{SO} \cong \mathcal{SO}$ (odd, 1). Since $\mathcal{SO}' \subseteq \mathcal{M} \subseteq \mathcal{G}^{(\alpha)}$, we have

$$\mathcal{M} = (\mathcal{M} \cap \mathcal{SO}) \oplus \mathcal{SO}'.$$

On the level of connected groups, we write correspondingly

$$M_0 = M^{SO}M^{SO'} \tag{4.14}$$

as a commuting product, possibly not direct. The group M normalizes each factor of (4.14) since M normalizes $\mathcal{G}_\alpha$ and $\mathcal{G}_{-\alpha}$, which generate $\mathcal{M} \cap \mathcal{SO}$. Theorem 1.1 of [10] allows us to write

$$\sigma|_{M_0} = \sigma_1 \oplus \cdots \oplus \sigma_L \tag{4.15}$$

with each σ_j irreducible on M_0 and the various σ_j inequivalent. Since σ is irreducible and (4.15) is the primary decomposition of $\sigma|_{M_0}$, it follows that

$$\sigma_j \cong m_j\sigma_1 \quad \text{for some } m_j \in M. \tag{4.16}$$

Meanwhile the irreducibility of σ_j and the compactness of M_0^{SO} imply

$$\sigma_j \cong \sigma_j^{SO} \otimes \sigma_j^{SO'} \tag{4.17}$$

with σ_j^{SO} and $\sigma_j^{SO'}$ irreducible on M^{SO} and $M^{SO'}$, respectively. Since the element m_j in (4.16) normalizes M^{SO} and $M^{SO'}$, (4.17) gives

$$\sigma_j^{SO} \cong m_j\sigma_1^{SO} \qquad \text{and} \qquad \sigma_j^{SO'} \cong m_j\sigma_1^{SO'}. \tag{4.18}$$

LEMMA 4.7. *Up to inner automorphisms, the only nontrivial outer automorphism of* $\mathcal{SO}$ (odd, 1) *is the Cartan involution.*

PROOF. Let φ be an automorphism of $\mathcal{SO} = \mathcal{SO}$ (odd, 1), and let $\mathcal{SO} = \mathcal{T} \oplus \mathcal{P}$ be a Cartan decomposition. Since any two $\mathcal{T}$'s are conjugate, we may assume $\varphi(\mathcal{T}) = \mathcal{T}$. If B is the Killing form, it follows that φ carries the orthocomplement (with respect to B) of $\mathcal{T}$ into itself. Thus $\varphi(\mathcal{P}) = \mathcal{P}$. Let $B_\theta(X,Y) = -B(X,\theta Y)$, where θ is the Cartan involution. Since φ leaves $\mathcal{T}$ and $\mathcal{P}$ stable, φ is an isometry of $\mathcal{SO}$ with respect to B_θ. In particular, $\varphi|_{\mathcal{P}}$ is in the orthogonal group $O(\mathcal{P})$. Since $\mathcal{SO}$ is $\mathcal{SO}$ (odd, 1), θ has determinant -1 on $\mathcal{P}$. Possibly composing φ with θ, we may assume $\varphi|_{\mathcal{P}}$ is in $SO(\mathcal{P})$. Since $\mathrm{Ad}(K)|_{\mathcal{P}} = SO(\mathcal{P})$, we can choose $k \in K$ with $\mathrm{Ad}(k)|_{\mathcal{P}} = \varphi|_{\mathcal{P}}$. Then $\mathrm{Ad}(k)^{-1}\varphi$ is 1 on $\mathcal{P}$ and maps $\mathcal{T}$ to itself. Since $[\mathcal{P},\mathcal{P}] = \mathcal{T}$, $\mathrm{Ad}(k)^{-1}\varphi = 1$ on $\mathcal{SO}$. Thus $\varphi = \mathrm{Ad}(k)$.

In terms of (4.18), Lemma 4.7 implies that each σ_j^{SO} is equivalent with σ_1^{SO} or with its Weyl group transform $s_\alpha\sigma_1^{SO}$. Thus the space V^σ on which σ operates splits as

$$V^\sigma = (V_1 \otimes V_1') \oplus (V_2 \otimes V_2'), \tag{4.19}$$

with M^{SO} operating on V_1 and V_2 by σ_1^{SO} and $s_\alpha\sigma_1^{SO}$, respectively, and with $M^{SO'}$ operating in V_1' and V_2'. (When $s_\alpha\sigma_1^{SO} \cong \sigma_1^{SO}$, we just write $V^\sigma = V_1 \otimes V_1'$).

Next consider the irreducible $K^{(\alpha)}$ type τ_0. Intersecting $\mathcal{G}^{(\alpha)} = \mathcal{SO} \oplus \mathcal{SO}'$ with $\mathcal{T}^{(\alpha)}$ gives us

$$\mathcal{T}^{(\alpha)} = \mathcal{T}^{SO} \oplus \mathcal{T}^{SO'}.$$

On the level of connected groups, we have a commuting product decomposition

$$K_0^{(\alpha)} = K^{SO}K^{SO'}, \tag{4.20}$$

possibly not direct. The group $K^{(\alpha)}$ normalizes each factor of (4.20) since $K^{(\alpha)}$ is generated by $K^{(\alpha)} \cap M$ and $K_0^{(\alpha)}$. We write

$$\tau_0|_{K_0^{(\alpha)}} = \sum n_j \tau_{0,j} \tag{4.21}$$

with the $\tau_{0,j}$ irreducible and inequivalent, and (4.20) gives us

$$\tau_{0.j} = \tau_{0.j}^{SO} \otimes \tau_{0.j}^{SO'}.$$

Since τ_0 is irreducible and (4.21) is a primary decomposition, we obtain

$$\tau_{0,j} \cong k_j \tau_{0,1} \quad \text{for some} \quad k_j \in K^{(\alpha)}.$$

Since k_j normalizes each factor of (4.20), we conclude

$$\tau_{0.j}^{SO} \cong k_j \tau_{0.1}^{SO} \quad \text{and} \quad \tau_{0.j}^{SO'} \cong k_j \tau_{0.1}^{SO'}. \tag{4.22}$$

LEMMA 4.8. *Every automorphism of* $\mathcal{SO}$ (odd) *is inner.*

PROOF. The argument is in the same style as Lemma 4.7. Write $\mathcal{SO}(2n+1) = \mathcal{SO}(2n) \oplus i\mathcal{P}$ with $i\mathcal{P}$ built from the last row and column. Given an automorphism φ, we can compose with an inner automorphism so that φ leaves $\mathcal{SO}(2n)$ and $i\mathcal{P}$ stable. Moreover φ will act as an orthogonal transformation of $i\mathcal{P}$. The matrix $\text{diag}(1, \cdots, 1, -1, -1)$ gives an inner automorphism on $\mathcal{SO}(2n+1)$ and acts on $i\mathcal{P}$ with determinant -1 since dim $\mathcal{P}$ is even. If φ has determinant -1 on $i\mathcal{P}$, we can compose with this inner automorphism and make it have determinant $+1$. Once φ acts as a rotation on $i\mathcal{P}$, φ has to be inner by the same argument as in Lemma 4.8.

In terms of (4.22), Lemma 4.8 implies that each $\tau_{0,j}^{SO}$ is equivalent with $\tau_{0,1}^{SO}$. Thus the space V^{τ_0} on which τ_0 operates splits as

$$V^{\tau_0} \cong V_0 \otimes V_0' \tag{4.23}$$

with K^{SO} operating by $\tau_{0,1}^{SO}$ in V_0 and with $K^{SO'}$ operating in V_0'.

LEMMA 4.9. *Within* $G^{(\alpha)}$, *let* τ_0 *be a* $K^{(\alpha)}$ *type of* $U^{(\alpha)}(S^{(\alpha)}, \sigma, \nu)$. *Then the endomorphism* $a^{(\alpha)}(\tau_0 : \sigma : \alpha : \nu)$ *of* $Hom_{K^{(\alpha)}}(\underbrace{V^{\tau_0}, V^{\sigma}})$ *is scalar, and the scalar is obtained as follows: Within the group* $\widetilde{SO}$ (odd, 1), *let* σ_1^{SO} *be the representation of* M^{SO} *in* (4.17), *and let* $\tau_{0,1}^{SO}$ *be the representation of* K^{SO} *in* (4.22). *Then the scalar is the scalar by which the operator* $A(s_\alpha^{-1} S^{SO} s_\alpha : S^{SO} : \sigma_1^{SO} : \nu)$ *operates on the* K^{SO} *type* $\tau_{0,1}^{SO}$. *(Recall property* (i) *in Section 1.)*

REMARK. Note that σ_1^{SO} is determined only up to the operation of the Weyl group reflection s_α.

PROOF. For E in $Hom_{K^{(\alpha)}}(V^{\tau_0}, V^{\sigma})$, we compute $a^{(\alpha)}(\tau_0 : \sigma : \alpha : \nu)$ from (4.9b). The group $\overline{N}_\alpha$ is contained in $K_0^{(\alpha)}A^{(\alpha)}N^{(\alpha)}$ (with no need for M), and so $\sigma(\mu(\overline{n}))$ can be taken to be 1. Thus

$$a^{(\alpha)}(\tau_0 : \sigma : \alpha : \nu) = \text{ right by } \int_{\overline{N}_\alpha} e^{-(\nu+\rho_\alpha)H(\overline{n})}\tau_0(\mathcal{K}(\overline{n}))^{-1}d\overline{n}. \tag{4.24}$$

By (4.23), this expression is conjugate (by a transformation of V^{τ_0}) to

$$\text{right by } \left[\int_{\overline{N}_\alpha} e^{-(\nu+\rho_\alpha)H(\overline{n})}\tau_{0,1}^{SO}(\mathcal{K}(\overline{n}))^{-1}d\overline{n} \otimes 1 \right].$$

The integral here commutes with $\tau_{0,1}^{SO}(M^{SO})$. Since the M^{SO} types have multiplicity one, the integral is diagonal and is scalar on each M^{SO} type. On the other hand, E is in $Hom_{K_0^{(\alpha)}}(V^{\tau_0}, V^{\sigma})$ and is 0 on the M^{SO} types that are incompatible with σ. By (4.19), the operator acts on the direct sum of two spaces as a scalar in each, the two spaces corresponding to σ_1^{SO} and $s_\alpha\sigma_1^{SO}$. The scalar in each case is the value of the $\widetilde{SO}$ (odd, 1) intertwining operator on the K^{SO} type $\tau_{0,1}^{SO}$, by Lemma 4.5. But these two scalars are the same, as it is seen from the bottom display on p. 22 of [9]. This proves the lemma.

COROLLARY 4.10. *Within $G^{(\alpha)}$, let $\tau_0, \tau_{0,1}^{SO}$ and σ_1^{SO} be as in Lemma 4.9. Suppose that $\nu = t_0\alpha$, with $t_0 > 0$, is a reducibility point for $U^{SO}(S^{SO}, \sigma_1^{SO}, t_0\alpha)$. Then there is a meromorphic complex-valued function $d_0(z)$ with the following properties:*

(i) $a^{(\alpha)}(\tau_0 : \sigma : \alpha : t_0\alpha) = d_0(z)a^{(\alpha)}(\tau_0 : \sigma : \alpha : z\alpha)$

(ii) *$d_0(t)$ takes values in $\mathbb{R} \cup \{\infty\}$ for t real*

(iii) *$d_0(t)$ is finite and ≥ 0 for those $t > t_0$ such that there do not exist t_1 and t_2, with $t_0 < t_1 < t_2 < t$, for which $U^{SO}(S^{SO}, \sigma_1^{SO}, t_1\alpha)$ has singular infinitesimal character and $U^{SO}(S^{SO}, \sigma_1^{SO}, t_2\alpha)$ is reducible.*

PROOF. This is immediate from Lemma 4.9 and the properties of $\widetilde{SO}$ (odd, 1) listed in Section 1.

Corollary 4.10 addresses equation (4.13) in the group $G^{(\alpha)}$. We return to equation (4.13) in G. Let $\tau|_{K^{(\alpha)}} = \sum \tau_j$ be a decomposition into irreducibles, let τ_j act on V^{τ_j}, and let i_j and p_j be the j^{th} injection and j^{th} projection for the direct sum decomposition $V^\tau = \sum V^{\tau_j}$. The functional ν varies over a two-parameter family, and we freeze one of the parameters by insisting that the projection of ν, in the direction of α, be $t_0\alpha$, with t_0 as in Corollary 4.10. Let d_j be the function provided by Corollary 4.10 for the $K^{(\alpha)}$ type τ_j, and write, in obvious notation, $d_j(\nu)$ in place of $d_j(z)$. Then the corollary gives us

$$a^{(\alpha)}(\tau_j : \sigma : \alpha : \nu) = a^{(\alpha)}(\tau_j : \sigma : \alpha : w_2\nu)d_j(w_2\nu).$$

For E in $Hom_{K\cap M}(V^\tau, V^\sigma)$, $E\circ i_j$ is in $Hom_{K^{(\alpha)}\cap M}(V^{\tau_j}, V^\sigma)$. Then we have

$$a(\tau:\sigma:\alpha:\nu)(E)=\sum_j a^{(\alpha)}(\tau_j:\sigma:\alpha:\nu)(E\circ i_j)\circ p_j.$$

Define $\tilde{d}(\nu)$ in $Hom_{\mathbb{C}}(V^\tau, V^\tau)$ by

$$\tilde{d}(\nu)=\sum i_j d_j(w_2\nu)p_j.$$

Then we find

$$\begin{aligned}
&a(\tau:\sigma:\alpha:w_2\nu)^{-1}a(\tau:\sigma:\alpha:\nu)(E)\\
&=\sum_{j,k} a^{(\alpha)}(\tau_j:\sigma:\alpha:w_2\nu)^{-1}\left([a^{(\alpha)}(\tau_k:\sigma:\alpha:\nu)(E\circ i_k)\circ p_k]\circ i_j\right)\circ p_j\\
&=\sum_j a^{(\alpha)}(\tau_j:\sigma:\alpha:w_2\nu)^{-1}a^{(\alpha)}(\tau_j:\sigma:\alpha:\nu)(E\circ i_j)\circ p_j\\
&=\sum_j d_j(w_2\nu)(E\circ i_j)\circ p_j\\
&=(\text{right by } \tilde{d}(\nu))(E).
\end{aligned}$$

Consequently $d(\nu)=(\text{right by } \tilde{d}(\nu))$ is in $Hom_{K\cap M}(V^\tau, V^\sigma)$ and satisfies (4.13). Evidently $d(\nu)$ is Hermitian when it is finite, and (iii) in Corollary 4.10 tells us conditions under which it is positive semidefinite. We summarize as follows.

THEOREM 4.11. *In situation* (1.1a), *as defined at the beginning of Section 4, let σ be a discrete series or (nonzero) limit of discrete series representation, and let σ_1^{SO} be associated to σ and $\widetilde{SO}$* (odd, 1) *by* (4.17). *With ν in the closed positive Weyl chamber, write $\nu=ae_1+be_2$, and assume that $U(S,\sigma,0)$ is irreducible in G and*

$$U^{SO}\left(S^{SO},\sigma_1^{SO},\frac{1}{2}(a-b)(e_1-e_2)\right)$$

is reducible in $\widetilde{SO}$ (odd, 1). *Further assume that ν has the property that there do not exist t_1 and t_2, with $\frac{1}{2}(a-b)<t_1<t_2<\frac{1}{2}(a+b)$, such that $U^{SO}(S^{SO},\sigma_1^{SO},t_1(e_1-e_2))$ has singular infinitesimal character and $U^{SO}(S^{SO},\sigma_1^{SO},t_2(e_1-e_2))$ is reducible. Then the operators A and C of Lemma 4.2 satisfy $A=PC^*$ with P positive semidefinite. If, in addition, the Langlands quotients of $U^{(2e_2)}(S^{(2e_2)},\sigma,be_2)$ and $U^{(2e_1)}(S^{(2e_1)},\sigma,ae_1)$ are unitary, then the Langlands quotient of $U(S,\sigma,\nu)$ is unitary.*

PROOF. The assertions about A and C have just been proved above in the setting of the domain of Φ. Lemma 4.4 allows us to transfer them to the induced space. The hypotheses about $U^{(2e_2)}$ and $U^{(2e_1)}$ imply that B and D in Lemma 4.2 are semidefinite, according to Lemma 4.6. By Proposition 1.1, $ABCD$ is semidefinite. But Theorem 16.6 of [6] and Lemma 4.2 show that $ABCD$ is the operator that decides the unitarity of $U(S, \sigma, \nu)$.

5. - Configuration (1.1b)

Previous sections have dealt largely with configuration (1.1a) for the roots of $(\mathcal{G}, \mathcal{A})$. The above arguments need some changes - and some additional hypotheses - in the case of (1.1b). In (1.1b), $S\mathcal{I}(2,\mathbb{R})$ replaces $S\mathcal{O}$ (odd, 1). The list of properties of intertwining operators and nonunitary principal series for $SL(2,\mathbb{R})$ is almost the same as for $\widetilde{SO}$ (odd, 1); this time the properties are more elementary.

(i) The K types have multiplicity one.

(ii) The total operator has a kernel at ν if and only if (0.1) is reducible at ν.

(iii) The representation σ is one-dimensional, and we write it as I or sgn. If $\sigma = I$, then reducibility occurs at odd multiples of the restricted root α. If $\sigma =$ sgn, then reducibility occurs at even multiples of α.

(iv) The infinitesimal character is singular only at $\nu = 0$.

(v) On any interval in the positive Weyl chamber, the kernel of the intertwining operator at reducibility points decreases as one moves to the right.

Lemma 4.1 does not extend to (1.1b) without modification. For one thing, e_1 and e_2 need not be real roots, and for another thing, $e_1 - e_2$ and $e_1 + e_2$ have become real roots. Thus we shall incorporate the conclusion of Lemma 4.1 into our hypotheses.

Then everything through Lemma 4.6 works without further change. Again we study (4.13). Taking $\alpha = e_1 - e_2$, we first study equation (4.13) in $G^{(\alpha)}$. Since α is real, the element γ_α lies in the center of M. Thus $\sigma(\gamma_\alpha)$ is scalar, necessarily ± 1. Since $\{1, \gamma_\alpha\}$ is the M group for the $SL(2,\mathbb{R})$ subgroup, σ yields a well defined one-dimensional representation σ^{SL} of $\{1, \gamma_\alpha\}$ by restriction. We do not need Lemma 4.7.

Next we trace through what happens to the $K^{(\alpha)}$ type τ_0. On the level of connected groups, we have a commuting product decomposition $K_0^{(\alpha)} = K^{SL}K^{SL'}$ in obvious notation. The group K^{SL} is a circle, with one-dimensional irreducible representations parametrized by integers N. An outer automorphism of K^{SL} interchanges N and $-N$. Now the $SL(2,\mathbb{R})$ intertwining operators are the same for N as for $-N$, by (5.8) of [7], for example. So we get an analog of Lemma 4.9 as follows.

LEMMA 5.1. *Within $G^{(\alpha)}$, let τ_0 be a $K^{(\alpha)}$ type of $U^{(\alpha)}(S^{(\alpha)}, \sigma, \nu)$. Then the endomorphism $a^{(\alpha)}(\tau_0 : \sigma : \alpha : \nu)$ of $Hom_{K^{(\alpha)} \cap M}(V^{\tau_0}, V^{\sigma})$ is scalar, and the scalar is obtained as follows: Within the group $SL(2,\mathbb{R})$, let σ^{SL} be the* *representation of* $\{1, \gamma_\alpha\}$ *obtained by restriction from* σ, *and let* N *or* $(N, -N)$ *be the parameter(s) of the representation(s) of* K^{SL} *obtained from* $\tau_0|_{K_0^{(\alpha)}}$. *Then the scalar is the scalar by which the operator* $A(s_\alpha^{-1} S^{SL} s_\alpha : S^{SL} : \sigma^{SL} : \nu)$ *operates on the* K^{SL} *type(s) with parameter(s)* N *or* $(N, -N)$.

Consequently we obtain a function $d_0(z)$ just as in Corollary 4.10. Property (iii) simplifies in the corollary to say that $d_0(t)$ is finite and ≥ 0 for $t > t_0$, since singular infinitesimal character cannot occur in $SL(2, \mathbb{R})$ for $t > 0$.

The remainder of the argument requires no change, and we arrive at the following result.

THEOREM 5.2. *In situation* (1.1b), *let* σ *be a discrete series or (nonzero) limit of discrete series representation of* M. *Let* w_1 *and* w_2 *be representatives in* $N_K(\mathcal{A})$ *of* s_{e_1} *and* s_{e_2}, *and assume that* $w_1\sigma \cong \sigma \cong w_2\sigma$ *in such a way that* σ *extends from* M *to the group generated by* M, w_1, *and* w_2. *Let* σ^{SL} *be the restriction of* σ *to* $\{1, \gamma_{e_1 - e_2}\}$. *With* ν *in the closed positive Weyl chamber, write* $\nu = ae_1 + be_2$, *and assume that* $U(S, \sigma, 0)$ *is irreducible in* G *and*

$$U^{SL}\left(S^{SL}, \sigma^{SL}, \frac{1}{2}(a-b)(e_1 - e_2)\right)$$

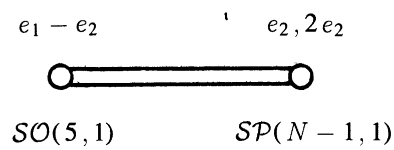
is reducible in $SL(2, \mathbb{R})$. *Then the operators* A *and* C *of Lemma* 4.2 *satisfy* $A = PC^*$ *with* P *positive semidefinite. If, in addition, the Langlands quotients of* $U^{(e_2)}(S^{(e_2)}, \sigma, be_2)$ *and* $U^{(e_1)}(S^{(e_1)}, \sigma, ae_1)$ *are unitary, then the Langlands quotient of* $U(S, \sigma, \nu)$ *is unitary.*

6. - Examples

1) $SU(N, 2)$. The diagram of roots of $(\mathcal{G}, \mathcal{A})$ appears at the beginning of Section 1, and a qualitative picture of unitarity appears in Figure 1. There is reducibility along the diagonal lines below the diagonal, and Theorem 4.11 addresses what happens along the upward-sloping diagonals. At the end of Section 1, we observed that the numbers t_1 and t_2 cannot exist in the theorem, and thus $A = PC^*$ all the way along the upward-sloping diagonals. The theorem goes on to say that we get unitarity at a point on such a diagonal whenever the horizontal and vertical coordinates correspond to unitary points in subgroups that are essentially $SU(N-1, 1)$.

2) $\widetilde{SO}(N, 2)$. The diagram of roots of $(\mathcal{G}, \mathcal{A})$ is as in (1.1b) with $m = N-1$ and $n = 1$, and a qualitative picture of unitarity is as in Figure 1, except that the outlined rectangle is now square. Theorem 5.2 implies that $A = PC^*$ all the way along the upward-sloping diagonals. The theorem goes on to say that we get unitarity at any point on such a diagonal whenever the horizontal and vertical coordinates correspond to unitary points in subgroups that are essentially $SO(N-1, 1)$.

3) $Sp(N,2)$. The diagram of roots of $(\mathcal{G},\mathcal{A})$ is



Figures 3 and 4 give two illustrations of what can happen.

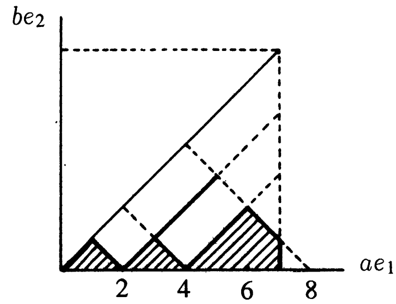


Figure 3: *Unitary points in* $Sp(6,2)$ *with minimal* K *type* $2e_8 + 2e_7$

In Figure 3, a downward-sloping diagonal terminates unitarity along the upward-sloping diagonals. Imagine inserting the missing downward-sloping diagonal that meets the e_1 axis at $6e_1$. This diagonal meets the upward-sloping diagonals at parameters that correspond to t_1 in Theorem 4.11. The downward-sloping diagonal that terminates unitarity crosses where the parameters are t_2. So the theorem gives no further unitarity along the upward-sloping diagonals. It is known that this picture is sharp.

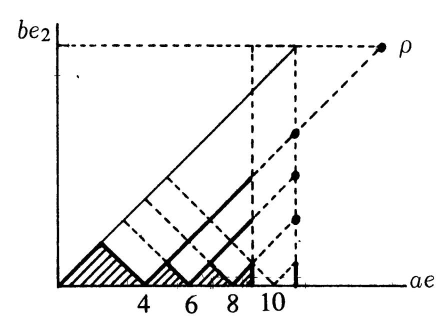


Figure 4: *Known unitary points in* $Sp(6,2)$ *for* σ *trivial*

In Figure 4, one has $A = PC^*$ all the way along the upward-sloping diagonals. Thus Theorem 4.11 says we get unitarity at any point on such a diagonal whenever the horizontal and vertical coordinates correspond to unitary points for the spherical case in $Sp(5,1)$. Since the trivial representation is isolated as a unitary representation in $Sp(5,1)$, the effect is that Theorem 4.11 gives us several isolated unitary representations in Figure 4. At our urging, H. Schlichtkrull has obtained this unitarity in the spherical case by other methods [11].

Figure 4 illustrates something else. The relationship $A = PC^*$ is valid within the triangles along the e_1 axis. So once again we get unitarity when the horizontal and vertical coordinates correspond to unitary points in $Sp(5,1)$. The result is an interval of unitarity at the bottom of the vertical line of isolated unitary representations.

7. - Concluding remarks

Not all parabolic-rank-two situations, even with $(BC)_2$ or B_2 as $(\mathcal{G}, \mathcal{A})$ root diagram, lead to configurations as in (1.1). Such parabolic-rank-two situations perhaps exhibit totally new phenomena. We do not know. What we have investigated is whether there is *some* parabolic-rank-n generalization of our theorems. All of our investigations suggest that the multiplicity-one behavior that we used for $\widetilde{SO}$ (odd, 1) or $SL(2,\mathbb{R})$ is vital. Meanwhile Lemma 1.2 is special to three Hermitian operators and does not extend to arbitrarily long products.

Thus any generalization of our results whose techniques are not too far removed from ours ought to use Proposition 1.1 in a situation where A and C correspond to a real-rank-one group. When B or D corresponds to a parabolic-rank-*two* situation, the generic sets of unitary points, for large ν, are one-dimensional, and we find that Proposition 1.1 is at best giving us one-dimensional sets of unitary points for large ν. This phenomenon persists if we induct on the parabolic rank. When all coordinates are large, we get only one-dimensional sets of unitary points (as well as isolated points).

Now one might except from Figure 1 that there are $(n-1)$-dimensional sets of unitary points in parabolic rank n if all coordinates are large. Yet our construction gives only one-dimensional sets. This situation suggests the question:

In parabolic rank n (say with $(\mathcal{G}, \mathcal{A})$ roots of type $(BC)_n$), are there sets of unitary points with dimension > 1 and with all coordinates large?

No examples come to mind.

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