JOURNAL OF FUNCTIONAL ANALYSIS 82, 151–236 (1989)

Intertwining Operators and Unitary Representations,

M. W. BALDONI-SILVA*

Dipartimento di Matematica, Università degli Studi di Trento, 38050 Povo (TN), Italy

AND

A. W. KNAPP†

Department of Mathematics, State University of New York, Stony Brook, New York 11794

Communicated by L. Gross

Received October 2, 1987

For a linear semisimple Lie group with a compact Cartan subgroup, the authors obtain formulas for the action of intertwining operators on certain subspaces of standard induced representations. These formulas provide explicit limitations on the pool of candidates for irreducible unitary representations, since the only possible invariant inner product is given by such an intertwining operator.

The problem of classifying the irreducible unitary representations of a semisimple Lie group comes down to this: The Langlands classification describes the wider class of irreducible "admissible" representations, and one simply has to decide which irreducible admissible representations admit invariant Hermitian inner products. In fact, the problem is even more concrete than this description makes it sound. The Langlands classification realizes representations on quotients of L^2 spaces, and the L^2 norm provides a noninvariant inner product for reference. There is at most one invariant Hermitian form (up to scalars), and this form is given in terms of the L^2 inner product by an explicit intertwining operator. (See [12, Chapts. 14 and 16], for details.) To decide unitarity, it is enough to determine whether this intertwining operator is semidefinite.

* Partially supported by National Science Foundation Grants MCS 81-08814 (Institute for Advanced Study, Princeton, NJ) and DMS 85-01793 (Cornell University).

† Supported by a Guggenheim Fellowship and by National Science Foundation Grants MCS 80-01854, DMS 85-01793, and DMS 87-11593.

0022-1236/89 $3.00

Although the operator is given by an explicit integral or singular integral, attempts at evaluating the integral or deciding the operator's signature directly have been largely unsuccessful. In [1, 2], we introduced two related techniques to help compute a small part of the signature. As a result we obtained necessary conditions for unitarity. The techniques are based on an old idea that has been used extensively by Klimyk, often in collaboration with Gavrilik, for particular classical groups (see, e.g., [10]): We take advantage of the intertwining property of the operator to relate the behavior on one subspace to that on another.

In four papers [2–5], we combined known results with formulas obtained from these techniques to classify irreducible unitary representations in certain situations ($SU(N, 2)$, some other groups of real rank two, and Langlands quotients obtained from maximal parabolic subgroups). Although the four papers contain the full combinatorial arguments necessary to derive the classifications from the formulas, they do not contain the proofs of the formulas themselves.

In the present paper we begin the derivations of these formulas. We shall see that the formulas are rather complicated to derive from the techniques

TABLE I

Locations of Proofs of Announced Results

Announced result	Location of proof
Theorem 1 of [1]	Corollary 5.2 and Remarks Proposition 3.4 and Corollary 3.6
Theorem 2 of [1]	Corollary 5.2 and Theorem 7.2
Theorem 3 of [1]	Corollary 3.6
Theorem 4 of [1]	Proposition 3.4 and Remarks Conclusion (b) of [1] not completely proved
Theorem 5 of [1]	Theorem 3.3
Theorem 6 of [1]	[5]
Theorem 7 of [1]	[5]
Lemma 2.1 of [2]	Corollary 1.6
Theorem 2.2 of [2]	Theorem 5.1 and Remarks
Theorem 2.3 of [2]	Theorem 6.4 and Remarks
Theorem 2.4 of [2]	Theorem 10.1 and Remarks
Theorem 2.5 of [2]	Theorem 2.1 and Remarks
Proposition 3.1 of [5]	Theorem 1.3
Theorem 3.2 of [5]	Corollary 5.2 and Remarks
Theorem 3.4 of [5]	[14]
Proposition 3.5 of [5]	Corollary 7.3
Proposition 3.6 of [5]	Corollary 8.2
Proposition 3.7 of [5]	Corollary 8.3
Proposition 3.8 of [15]	Theorem 9.1

of [1, 2], and we shall be content for now with handling only those formulas that are applicable when the underlying semisimple group has a compact Cartan subgroup. In a sequel we shall derive the formulas that are applicable when there is no compact Cartan subgroup.

The derivations begin in Section 5. In Sections 1–3 we develop some preliminary material on decompositions of tensor products and on multiplicity questions, and in Section 4 we review the techniques introduced in [1, 2].

Contents.

1. Occurrence of K Types in a Tensor Product

Let G be a linear connected reductive Lie group, and let K be a maximal compact subgroup. We denote Lie algebras by corresponding lower-case German letters, and we write $\mathbb{C}$ as a superscript to indicate complexification. Let θ be a Cartan involution of $\mathfrak{g}$ with respect to $\mathfrak{k}$, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as the corresponding Cartan decomposition. We fix on $\mathfrak{g}$ a nondegenerate symmetric bilinear form B_0 invariant under θ such that ad $\mathfrak{g}$ acts by skew transformations, B_0 is negative on $\mathfrak{k} \times \mathfrak{k}$, B_0 is positive on $\mathfrak{p} \times \mathfrak{p}$, and $B_0(\mathfrak{k}, \mathfrak{p}) = 0$. We extend B_0 to $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$ so as to be complex bilinear.

In this paper, we shall assume that rank G = rank K and that $\mathfrak{g}$ has no simple factor of type G_2. We fix a maximal torus B in K, and then B is a Cartan subgroup of G. Let $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ be the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{b}^{\mathbb{C}}$, and let Δ_K and Δ_n be the subsets of compact and noncompact roots, respectively. Roots are real-valued on $i\mathfrak{b}$. The form B_0 induces an inner product $\langle \cdot, \cdot \rangle$ on the set of linear functionals on $\mathfrak{b}^{\mathbb{C}}$ that are real-valued on $i\mathfrak{b}$, and we write $\mu' \perp \mu''$ if $\langle \mu', \mu'' \rangle = 0$. If μ' and μ'' are roots, we write $\mu' \perp\perp \mu''$ (and say μ' is *strongly orthogonal* to μ'') if neither $\mu' + \mu''$ nor $\mu' - \mu''$ is in $\Delta \cup \{0\}$.

Starting in Section 2, we shall work with a specific choice of Δ^+, the set of positive roots within Δ, and we shall let $\Delta_K^+ = \Delta_K \cap \Delta^+$. But for now let us suppose that Δ_K^+ is any positive system for $\Delta_K = \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$. If W_K denotes the Weyl group of Δ_K and if μ' is a linear functional on $\mathfrak{b}^{\mathbb{C}}$ that is real-valued on $i\mathfrak{b}$, then there exists $w \in W_K$ such that $w\mu'$ is Δ_K^+ dominant, and we write $(\mu')^\vee$ for this dominant form. If Λ' is Δ_K^+ dominant and is (analytically) integral (i.e., if $\exp \Lambda'$ is well defined on B), we let $\tau_{\Lambda'}$ be an irreducible representation of K with highest weight Λ'. We shall regard $\mathfrak{p}^{\mathbb{C}}$ as a representation of K under $\mathrm{Ad}(K)$; the weights of $\mathfrak{p}^{\mathbb{C}}$ are the members of Δ_n, each with multiplicity one.

PROPOSITION 1.1. *Let Λ' be integral and Δ_K^+ dominant. Then*

(a) *every irreducible constituent of $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ has highest weight of the form $\Lambda' + \beta$ with β in Δ_n;*

(b) *every irreducible constituent of $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ has multiplicity one.*

This is well known and follows from Problems 13 and 14 on p. 111 of [12]. In Theorem 1.3 below we shall describe which forms $\Lambda' + \beta$ actually do arise as highest weights in $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$. But first we need a lemma. If Λ' is dominant integral, we let $\Delta_{K,\Lambda'}$ be the subset of roots in Δ_K orthogonal to Λ'. This is a root system, and the simple roots of $\Delta_{K,\Lambda'}^+ = \Delta_{K,\Lambda'} \cap \Delta_K^+$ are simple in Δ_K^+ since Λ' is dominant. Let $W_{K,\Lambda'}$ be the Weyl group of $\Delta_{K,\Lambda'}$; this is the subgroup of W_K fixing Λ', by Chevalley's Lemma [12, p. 81]. If μ' is merely integral, we let $\Delta_{K,\mu'}$ be the subset of roots in Δ_K orthogonal to μ'; this is a root system but is not necessarily generated by Δ_K^+ simple roots.

LEMMA 1.2. *Let Λ' be integral and Δ_K^+ dominant, and let β be in Δ_n. Then $(\Lambda' + \beta)^\vee$ is of the form $\Lambda' + \beta'$ with β' in Δ_n, and β' is obtained constructively as follows: Let β_1 be the result of making β dominant for $\Delta_{K,\Lambda'}^+$ (by means of $W_{K,\Lambda'}$). Then exactly one of the following things happens:*

(a) *$\Lambda' + \beta_1$ is Δ_K^+ dominant, and $\beta' = \beta_1$.*

(b) *There exists a Δ_K^+ simple root γ with $2\langle \Lambda', \gamma\rangle/|\gamma|^2 = +1$ and $2\langle \beta_1, \gamma\rangle/|\gamma|^2 = -2$. In this case let β_2 be the short noncompact root $\beta_1 + \gamma$. Then β' is the result of making β_2 dominant for $\Delta_{K,\Lambda'}^+$ (by means of $W_{K,\Lambda'}$).*

Remark. If β is short, then so is β_1, and $2\langle \beta_1, \gamma\rangle/|\gamma|^2 = -2$ is impossible. Thus the process above stops with $(\Lambda' + \beta)^\vee = \Lambda' + \beta_1$.

Proof. If γ is Δ_K^+ simple, we write

$$\frac{2\langle \Lambda' + \beta_1, \gamma\rangle}{|\gamma|^2} = \frac{2\langle \Lambda', \gamma\rangle}{|\gamma|^2} + \frac{2\langle \beta_1, \gamma\rangle}{|\gamma|^2}.$$

For γ in $\Delta_{K,\Lambda'}^+$, both terms on the right are $\geqslant 0$. For γ not in $\Delta_{K,\Lambda'}^+$, the first term is $\geqslant 1$, and the sum on the right can be <0 only if the two terms on the right are $+1$ and -2, respectively, since $\mathfrak{g}$ has no G_2 factors. Thus $\Lambda' + \beta_1$ is Δ_K^+ dominant unless the condition in (b) holds.

In any event, let $\beta_1 = w\beta$ with w in $W_{K,\Lambda'}$. Then $w\Lambda' = \Lambda'$, so that $w(\Lambda' + \beta) = \Lambda' + \beta_1$. If $\Lambda' + \beta_1$ is Δ_K^+ dominant, we thus conclude $\beta' = \beta_1$.

Now suppose the condition in (b) holds. Let s_γ be reflection in γ. Then the inner products in (b) force

$$s_\gamma(\Lambda' + \beta_1) = (\Lambda' - \gamma) + (\beta_1 + 2\gamma) = \Lambda' + (\beta_1 + \gamma) = \Lambda' + \beta_2,$$

so that $\Lambda' + \beta$ is conjugate to $\Lambda' + \beta_2$ by W_K. If we now repeat the proof, replacing β by β_2 throughout, then the remark shows that $(\Lambda' + \beta_2)^\vee = \Lambda' + \beta'$ with β' equal to the result of making β_2 dominant for $\Delta^+_{K,\Lambda'}$.

THEOREM 1.3. *Let Λ' be integral and Δ^+_K dominant, let β be a noncompact root, and suppose $\Lambda' + \beta$ is Δ^+_K dominant. Then $\tau_{\Lambda'+\beta}$ fails to occur in $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ if and only if there exists a (necessarily short) Δ^+_K simple root γ such that γ is in $\Delta^+_{K,\Lambda'}$ and that $\gamma \perp \beta$ but $\gamma \perp\!\!\!/\!\!\!\perp \beta$.*

Proof. If μ' is an integral form, we define sgn μ' as follows. If $\mu' \perp \gamma$ for some γ in Δ_K, we take sgn $\mu' = 0$. Otherwise, there is a unique w in W_K such $w\mu'$ is Δ^+_K dominant, and we take sgn $\mu' =$ sgn w. From [9, p. 142] or [12, p. 112], we have

$$\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}} = \sum_{\beta' \in \Delta_n} \operatorname{sgn}(\Lambda' + \beta' + \delta_K)\, \tau_{(\Lambda' + \beta' + \delta_K)^\vee - \delta_K}, \tag{1.1}$$

where δ_K is half the sum of the members of Δ^+_K.

Now suppose $\tau_{\Lambda'+\beta}$ does not occur in $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$. By (1.1), there exists $\beta' \neq \beta$ in Δ_n such that

$$(\Lambda' + \beta + \delta_K)^\vee - \delta_K = (\Lambda' + \beta' + \delta_K)^\vee - \delta_K. \tag{1.2}$$

Since δ_K is algebraically integral, it will be helpful to regard $(\Lambda' + \beta + \delta_K)^\vee$ as the highest weight of a representation of a finite cover of K. Since $\Lambda' + \beta$ is Δ^+_K dominant, (1.2) implies that $\Lambda' + \beta' + \delta_K$ is a weight of $\tau_{\Lambda'+\beta+\delta_K}$. Hence

$$(\Lambda' + \beta + \delta_K) - (\Lambda' + \beta' + \delta_K) = \sum_{\gamma \in \Delta^+_K} n_\gamma \gamma, \qquad n_\gamma = \text{integer} \geqslant 0 \tag{1.3}$$

and

$$|\Lambda' + \beta + \delta_K|^2 = |\Lambda' + \beta' + \delta_K|^2. \tag{1.4}$$

Expanding (1.4) gives

$$2\langle \Lambda' + \delta_K, \beta - \beta' \rangle = |\beta'|^2 - |\beta|^2.$$

Since $\Lambda' + \delta_K$ is Δ^+_K dominant and nonsingular, substitution from (1.3) shows that $|\beta'|^2 - |\beta|^2 > 0$. Thus β' is long and β is short. Hence we have

$$|\beta|^2 = 2\langle \Lambda' + \delta_K, \beta - \beta' \rangle = \sum_{\gamma \in \Delta_K^+} n_\gamma 2\langle \Lambda' + \delta_K, \gamma \rangle$$
$$\geqslant \sum_{\gamma \in \Delta_K^+} n_\gamma \left(\frac{2\langle \delta_K, \gamma \rangle}{|\gamma|^2} \right) |\gamma|^2 \geqslant \sum_{\gamma \in \Delta_K^+} n_\gamma |\gamma|^2,$$

and equality can hold throughout only if $n_\gamma > 0$ implies γ is orthogonal to Λ' and is Δ_K^+ simple. Since β is short, we conclude that equality holds throughout, that there is only one nonzero term on the right, and that its γ is short and has $n_\gamma = 1$. Thus $\beta - \beta' = \gamma$ with γ short and simple in $\Delta_{K,\Lambda'}^+$. Since the long root β' is the difference of the short roots β and γ, β and γ must be orthogonal but not strongly orthogonal. This completes the proof in one direction.

Conversely suppose that there exists a short Δ_K^+ simple root γ such that γ is in $\Delta_{K,\Lambda'}^+$ and that $\gamma \perp \beta$ but $\gamma \not\perp\!\!\!\perp \beta$. Put $\beta' = \beta - \gamma$, and let s_γ be reflection in γ. Then

$$s_\gamma(\Lambda' + \beta + \delta_K) = \Lambda' + \beta + (\delta_K - \gamma) = \Lambda' + \beta' + \delta_K$$

shows that (1.2) holds and that $\operatorname{sgn}(\Lambda' + \beta' + \delta_K) = -1$.

In (1.1), suppose that another term, say the β'' term, has

$$(\Lambda' + \beta + \delta_K)^\vee - \delta_K = (\Lambda' + \beta'' + \delta_K)^\vee - \delta_K$$

and $\beta'' \neq \beta$. Running through the argument in the direct part of the proof and then the argument in the above paragraph, we see that $\operatorname{sgn}(\Lambda' + \beta'' + \delta_K) = -1$. Therefore the total coefficient of $\tau_{(\Lambda' + \beta + \delta_K)^\vee - \delta_K}$ in (1.1) is $\leqslant 0$ and must be 0.

COROLLARY 1.4. *Let Λ' be integral and Δ_K^+ dominant, let β be a noncompact root, and suppose $\Lambda' + \beta$ is Δ_K^+ dominant. Then $\tau_{\Lambda' + \beta}$ occurs in $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ if either*

(a) *all noncompact roots have the same length or*

(b) *β is long.*

THEOREM 1.5. *Let μ' be integral, let $\Lambda' = (\mu')^\vee$, let β be a noncompact root, and suppose $\tau_{(\mu' + \beta)^\vee}$ occurs in $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$. Let $E = E_{(\mu' + \beta)^\vee}$ be the projection of $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ on the $\tau_{(\mu' + \beta)^\vee}$ subspace (along the subspaces for the other K types). If v' is a nonzero weight vector for $\tau_{\Lambda'}$ with weight μ' and if X_β is a nonzero root vector for β, then $E(v' \otimes X_\beta)$ is nonzero.*

Proof. For much of the proof, we shall assume that $\mu' = \Lambda'$, i.e., that μ' is Δ_K^+ dominant. First suppose $\Lambda' + \beta$ is Δ_K^+ dominant. Let $v'' = \sum v_i \otimes X_{\beta_i}$ be a nonzero highest weight vector for the $\tau_{\Lambda' + \beta}$ subspace of $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$. We may assume that each v_i is a nonzero weight vector of weight $\Lambda'+\beta-\beta_i$ and that these weights are decreasing as i increases. Suppose $v_1\neq cv'$. Choose γ in Δ_K^+ with $\tau_{\Lambda'}(X_\gamma)v_1\neq 0$. Then

$$\begin{aligned}0&=(\tau_{\Lambda'}\otimes\mathrm{Ad})(X_\gamma)\Big(\sum v_i\otimes X_{\beta_i}\Big)\\&=\sum\tau_{\Lambda'}(X_\gamma)\,v_i\otimes X_{\beta_i}+\sum v_i\otimes[X_\gamma,X_{\beta_i}].\end{aligned}$$

The only term on the right side with first component of weight $\geqslant\Lambda'+\beta-\beta_1+\gamma$ is $\tau_{\Lambda'}(X_\gamma)\,v_1\otimes X_{\beta_1}$. Therefore $\tau_{\Lambda'}(X_\gamma)\,v_1=0$, contradiction. We conclude that

$$v''=cv'\otimes X_\beta+\sum_{i\geqslant 2}v_i\otimes X_{\beta_i}\tag{1.5}$$

with $c\neq 0$.

Let us introduce a K-invariant inner product for $\tau_{\Lambda'}$ and take the induced inner product on $\tau_{\Lambda'}\otimes\mathfrak{p}^{\mathbb{C}}$. Then $E=E_{(\Lambda'+\beta)^\vee}$ is an orthogonal projection. If $E(v'\otimes X_\beta)=0$, then (1.5) gives

$$v''=Ev''=E\Big(\sum_{i\geqslant 2}v_i\otimes X_{\beta_i}\Big).$$

Thus

$$\|v''\|=\Big\|E\Big(\sum_{i\geqslant 2}v_i\otimes X_{\beta_i}\Big)\Big\|\leqslant\Big\|\sum_{i\geqslant 2}v_i\otimes X_{\beta_i}\Big\|.$$

Since the two terms on the right side of (1.5) are orthogonal and c is $\neq 0$, this inequality gives a contradiction.

This completes the proof if $\Lambda'+\beta$ is Δ_K^+ dominant. The next case to consider is that $\Lambda'+s\beta$ is Δ_K^+ dominant for some s in $W_{K,\Lambda'}$. Let w be a representative of s in K. If $E(v'\otimes X_\beta)=0$, then also

$$\begin{aligned}0=\tau_{(\Lambda'+\beta)^\vee}(w)\,E(v'\otimes X_\beta)&=E((\tau_{\Lambda'}\otimes\mathrm{Ad})(w)(v'\otimes X_\beta))\\&=E(\tau_{\Lambda'}(w)\,v'\otimes\mathrm{Ad}(w)\,X_\beta).\end{aligned}\tag{1.6}$$

Since s is in $W_{K,\Lambda'}$, $\tau_{\Lambda'}(w)\,v'$ is a nonzero vector of weight $s\Lambda'=\Lambda'$ in the space for $\tau_{\Lambda'}$; thus $\tau_{\Lambda'}(w)\,v'$ is a nonzero multiple of v'. Moreover, $\mathrm{Ad}(w)\,X_\beta$ is a nonzero multiple of $X_{s\beta}$. Thus (1.6) gives us $E(v'\otimes X_{s\beta})=0$, in contradiction to the result of the previous paragraph.

This completes the proof if $\Lambda'+s\beta$ is Δ_K^+ dominant for some s in $W_{K,\Lambda'}$. Now we consider general β. Choose s in $W_{K,\Lambda'}$ such that $s\beta$ is $\Delta_{K,\Lambda'}^+$ dominant. The previous paragraph and Lemma 1.2 show that we are done unless β is long and there exists a short Δ_K^+ simple root γ such that

$2\langle \Lambda', \gamma\rangle/|\gamma|^2 = +1$ and $2\langle s\beta, \gamma\rangle/|\rangle|^2 = -2$. The root $\beta_2 = s\beta + \gamma$ is short, and the result of the previous paragraph shows that $E(v' \otimes X_{s\beta+\gamma}) \neq 0$. Since v' is a highest weight vector for $\tau_{\Lambda'}$, we have

$$\tau_{(\Lambda'+\beta)^\vee}(X_\gamma)\, E(v' \otimes X_{s\beta}) = E(\tau_{\Lambda'}(X_\gamma)\, v' \otimes X_{s\beta} + v' \otimes \operatorname{ad}(X_\gamma)\, X_{s\beta})$$
$$= E(v' \otimes \operatorname{ad}(X_\gamma)\, X_{s\beta}).$$

The right side is a nonzero multiple of $E(v' \otimes X_{s\beta+\gamma})$, which we know to be nonzero. Therefore $E(v' \otimes X_{s\beta})$ on the left side is nonzero. Arguing with a representative w of s as in (1.6), we see that $E(v' \otimes X_\beta) \neq 0$.

Finally in the general case in which μ' is not necessarily Δ_K^+ dominant, we introduce a new positive system for Δ_K so that μ' is dominant, and then the theorem reduces to the case that has already been proved.

COROLLARY 1.6. *Let μ' be integral, let $\Lambda' = (\mu')^\vee$, let β be a noncompact root, and suppose either that all noncompact roots have the same length or that β is long. If v' is a nonzero weight vector for $\tau_{\Lambda'}$ with weight μ', and if X_β is a nonzero root vector for β, then $E_{(\mu'+\beta)^\vee}(v' \otimes X_\beta)$ is nonzero.*

Proof. This follows from Theorem 1.5 and Corollary 1.4.

2. Some Identities for Half Sums of Roots

Our chief objective in this paper will be to obtain explicit formulas for the effect of standard intertwining operators on certain K types of induced representations. In this section we introduce the representations to be studied, note how to compute their minimal K types, and establish some identities for half sums of roots.

We shall introduce our notation in the full generality that we need. But we shall state and prove the main result of this section, Theorem 2.1, only for the case of representations induced from maximal parabolic subgroups. However, we have used Theorem 2.1 in [2] in more generality; thus we shall indicate at the end of this section the simple modifications in the statement and proof that give the generalization quoted in [2].

We continue with the notation of Section 1. In particular, $\mathfrak{g}$ has a compact Cartan subalgebra $\mathfrak{b}$, Δ is $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, and Δ_K and Δ_n are the subsets of compact and noncompact roots, respectively. By [8, pp. 155–156], we select root vectors X_β for β in Δ in such a way that

$$B_0(X_\beta, X_{-\beta}) = 2/|\beta|^2 \tag{2.1a}$$

and

$$\theta \bar{X}_\beta = -X_{-\beta}. \tag{2.1b}$$

Then it follows that H_β, defined by

$$H_\beta = [X_\beta, X_{-\beta}], \tag{2.2}$$

satisfies $\beta(H_\beta) = 2$ and that

$$\begin{aligned} &X_\beta + X_{-\beta},\ i(X_\beta - X_{-\beta}) && \text{are in } \mathfrak{g} \text{ if } \beta \text{ is noncompact} \\ &X_\beta - X_{-\beta},\ i(X_\beta + X_{-\beta}) && \text{are in } \mathfrak{g} \text{ if } \beta \text{ is compact.} \end{aligned} \tag{2.3}$$

The Hermitian form

$$\langle X, Y \rangle = -B_0(X, \theta \bar{Y}) \tag{2.4}$$

is a positive definite inner product on $\mathfrak{g}^{\mathbb{C}}$ that is invariant under $\mathrm{Ad}(K)$. If (τ, V) is any finite-dimensional representation of K and if $\langle \cdot, \cdot \rangle$ is a positive definite K-invariant Hermitian inner product on V, then $\tau(X)^* = -\tau(X)$ for X in $\mathfrak{k}$, and it follows that

$$\tau(X)^* = -\tau(\bar{X}) = -\tau(\theta \bar{X}) \qquad \text{for} \quad X \in \mathfrak{k}^{\mathbb{C}}.$$

From this identity we readily find that

$$\tau(H_\beta)^* = \tau(H_\beta) \qquad \text{for} \quad \beta \in \Delta \tag{2.5a}$$

$$\tau(X_\gamma)^* = \tau(X_{-\gamma}) \qquad \text{for} \quad \gamma \in \Delta_K \tag{2.5b}$$

$$\tau[X_\beta, X_{\beta'}]^* = -\tau[X_{-\beta}, X_{-\beta'}] \qquad \text{for} \quad \beta \in \Delta_K, \beta' \in \Delta_K \ \text{or } \beta \in \Delta_n, \beta' \in \Delta_n. \tag{2.5c}$$

For functions on G, we use vector field notation for differentiation, letting $Xf(g) = (d/dt) f((\exp tX)^{-1} g)|_{t=0}$ if X is in $\mathfrak{g}$. If X and Y are in $\mathfrak{g}$ and if $Z = X + iY$, we let $Zf = Xf + iYf$. Then

$$Z\bar{f} = \overline{\bar{Z}f}. \tag{2.6}$$

Fix a nonempty ordered set $\alpha_1, ..., \alpha_l$ of noncompact roots that are *superorthogonal* in the sense that no nontrivial linear combination of the α_j is a root. (Often we shall be interested in the case $l = 1$, and then we write α for α_1.) Define

$$\mathfrak{a} = \sum_{j=1}^{l} \mathbb{R}(X_{\alpha_j} + X_{-\alpha_j}),$$

and use the lexicographic ordering from the ordered basis

$$X_{\alpha_1} + X_{-\alpha_1}, ..., X_{\alpha_l} + X_{-\alpha_l}$$

to define a notation of positivity. Using this $\mathfrak{a}$ and this notion of positivity, we can construct a parabolic subgroup MAN in the usual way, and MAN will be *cuspidal* in the sense that rank $M = \operatorname{rank}(K \cap M)$. Let ρ be half the sum, with multiplicities counted, of the roots of $(\mathfrak{g}, \mathfrak{a})$ that are positive relative to N. We write $\tilde{\alpha}_j$ for the Cayley transform of α_j. Then $\tilde{\alpha}_j(X_{\alpha_j} + X_{-\alpha_j}) = 2$.

Let $\mathfrak{b}_-$ be the common kernel in $\mathfrak{b}$ of the α_j's. Then $\mathfrak{b}_-$ is a compact Cartan subalgebra of $\mathfrak{m}$, and

$$\Delta_- = \{\gamma \in \Delta \mid \gamma \perp \alpha_j \text{ for all } j\}$$

may be identified as the root system of $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}_-^{\mathbb{C}})$. [Caution: The root vectors for the roots of $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}_-^{\mathbb{C}})$ are not the X_γ's for $\gamma \in \Delta_-$ but are Cayley transforms (relative to $\alpha_1, ..., \alpha_l$) of the X_γ's.]

We fix a discrete series or (nonzero) limit of discrete series representation σ of M. Let $M^\# = M_0 Z_M$, the product of the identity component and the center of M. By (12.82) and Proposition 12.32 of [12], σ is induced from a discrete series or limit $\sigma^\#$ of $M^\#$ acting in a Hilbert space $V^{\sigma^\#}$.

Now Lemma 12.30 of [12] shows that $M^\# = M_0 F$, where F is the finite abelian group generated by the elements $\gamma_\beta = \exp \pi i H_\beta$ with β a real root of $\Delta(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{a} \oplus \mathfrak{b}_-)^{\mathbb{C}})$. Since the Cayley transformed $\tilde{\alpha}_j$'s span the real roots and since the α_j's are assumed superorthogonal, the only β's that are relevant are $\beta = \tilde{\alpha}_j$. Thus $\sigma^\#$ is determined by its Harish-Chandra parameter $(\lambda_0, (\Delta_-)^+)$ and its scalar value on each element $\gamma_{\tilde{\alpha}_j}$ of Z_M. We write γ_{α_j} for $\gamma_{\tilde{\alpha}_j}$.

Let λ be the minimal $(K \cap M^\#)$ type of $\sigma^\#$ given on $\mathfrak{b}_-$ by

$$\lambda = \lambda_0 - \delta_{-,c} + \delta_{-,n},$$

where $\delta_{-,c}$ and $\delta_{-,n}$ are the respective half sums of the positive M-compact and M-noncompact roots of Δ_-. Following the procedure of [11], we introduce a positive system Δ^+ containing $(\Delta_-)^+$ such that each α_j is simple for Δ^+. (The condition on the α_j's in [11] is weaker than simplicity of the α_j's, but the α_j's turn out to be simple here because of the superorthogonality.) Let $\Delta_K^+ = \Delta^+ \cap \Delta_K$ and $\Delta_n^+ = \Delta^+ \cap \Delta_n$, and let δ, δ_K, δ_n, and δ_- be the half sums of the members of Δ^+, Δ_K^+, Δ_n^+, and $(\Delta_-)^+$, respectively.

We shall study the family of induced representations

$$U(\nu) = U(MAN, \sigma, \nu) = \operatorname{ind}_{MAN}^G(\sigma \otimes e^\nu \otimes 1), \tag{2.7}$$

where ν is a complex-valued linear functional on $\mathfrak{a}$ and the induction is normalized so that imaginary ν yields unitary $U(\nu)$. We regard the induced representation as acting on functions by the left regular representation.

The main result of [11] is that the highest weights of the minimal K types of $U(\nu)$ are given by all Δ_K^+ dominant expressions of the form

$$\Lambda = \lambda - \sum_{j=1}^{l} \frac{\langle 2\delta_K, \alpha_j \rangle}{|\alpha_j|^2} \alpha_j + \mu, \tag{2.8}$$

where μ is given by $\sum_{j=1}^{l} s_j \alpha_j$ with each s_j equal to $\pm\frac{1}{2}\alpha_j$ or to 0, depending (in a nontrivial fashion) on the value of $\sigma^{\#}(\gamma_{\alpha_j})$. Moreover at least one choice of the system of signs in the $\pm\frac{1}{2}\alpha_j$ terms of μ gives a Δ_K^+ dominant Λ. We fix such a choice of μ and hence Λ. From [11] an alternative formula for Λ is

$$\Lambda = \lambda_0 + \delta - 2\delta_K - \sum_{j=1}^{l} \tfrac{1}{2}\alpha_j + \mu. \tag{2.9}$$

It is clear that $\Lambda|_{\mathfrak{b}_-} = \lambda$. The first part of the proof of the minimal K type formula that appears on pp. 629–631 of [12] shows that a highest weight vector for τ_Λ in $U(\nu)$ is highest of type τ_λ for $K \cap M_0$ and that the value of $\tau_\Lambda(\gamma_{\alpha_j})$ on a highest weight vector is the same as the scalar value of $\sigma^{\#}(\gamma_{\alpha_j})$. (To see these facts, it is necessary to supplement the arguments of [12] with the observation that $\mathscr{S} = \operatorname{span} \tau_\Lambda(K_r)\, v_\Lambda$ in [12] is one-dimensional. In fact, G_r is locally a product of $SL(2, \mathrm{R})$'s since the α_j's are superorthogonal. Hence K_r is abelian and is contained in B.)

It will be important to us that τ_Λ has multiplicity one in $U(\nu)$. This is a theorem of Vogan [19]. In our situation the arguments in [12] reduce this assertion to the corresponding result in a product of $SL(2, \mathbb{R})$'s, where it is well known.

For the remainder of this section, we shall take $l=1$ and write $\alpha = \alpha_1$. Later we shall state a version of Theorem 2.1 valid for general l. Define

$$\begin{aligned} \nu_0^+ &= 1 + \frac{2\langle \mu, \alpha \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta \text{ and } \langle \Lambda, \beta - \alpha \rangle = 0\} \\ &\quad + \#\left\{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta,\ |\beta|^2 < |\alpha|^2, \frac{2\langle \Lambda, \beta - \alpha \rangle}{|\beta - \alpha|^2} = +1\right\} \end{aligned} \tag{2.10a}$$

$$\begin{aligned} \nu_0^- &= 1 - \frac{2\langle \mu, \alpha \rangle}{|\alpha|^2} + 2\#\{\beta \in \Delta_n^+ \mid \beta + \alpha \in \Delta \text{ and } \langle \Lambda, \beta + \alpha \rangle = 0\} \\ &\quad + \#\left\{\beta \in \Delta_n^+ \mid \beta + \alpha \in \Delta,\ |\beta|^2 < |\alpha|^2, \frac{2\langle \Lambda, \beta + \alpha \rangle}{|\beta + \alpha|^2} = +1\right\}. \end{aligned} \tag{2.10b}$$

THEOREM 2.1. *Let μ' be an integral form on* $\mathfrak{b}$, *and define* (*with the signs taken consistently throughout*)

$$\begin{aligned}
\mathrm{I} = \rho(X_\alpha + X_{-\alpha}) &+ \frac{2\langle \mu', \pm\alpha\rangle}{|\alpha|^2} \\
&- \#\left\{\beta \in \Delta_n \mid \beta \mp \alpha \in \Delta, |\beta|^2 < |\alpha|^2, \frac{2\langle \mu', \beta \mp \alpha\rangle}{|\beta \mp \alpha|^2} = 1\right\} \\
&- 2\#\left\{\beta \in \Delta_n \mid \beta \mp \alpha \in \Delta, |\beta|^2 \geqslant |\alpha|^2, \frac{2\langle \mu', \beta \mp \alpha\rangle}{|\beta \mp \alpha|^2} \geqslant 1\right\} \\
&- 2\#\left\{\beta \in \Delta_n \mid \beta \mp \alpha \in \Delta, |\beta|^2 < |\alpha|^2, \frac{2\langle \mu', \beta \mp \alpha\rangle}{|\beta \mp \alpha|^2} \geqslant 2\right\}
\end{aligned}$$

$$\begin{aligned}
\mathrm{II} = \rho(X_\alpha + X_{-\alpha}) &- \frac{2\langle \mu' \pm \alpha, \pm\alpha\rangle}{|\alpha|^2} \\
&- \#\left\{\beta \in \Delta_n \mid \beta \pm \alpha \in \Delta, |\beta|^2 < |\alpha|^2, \frac{2\langle \mu' \pm \alpha, \beta \pm \alpha\rangle}{|\beta \pm \alpha|^2} = 1\right\} \\
&- 2\#\left\{\beta \in \Delta_n \mid \beta \pm \alpha \in \Delta, |\beta|^2 \geqslant |\alpha|^2, \frac{2\langle \mu' \pm \alpha, \beta \pm \alpha\rangle}{|\beta \pm \alpha|^2} \geqslant 1\right\} \\
&- 2\#\left\{\beta \in \Delta_n \mid \beta \pm \alpha \in \Delta, |\beta|^2 < |\alpha|^2, \frac{2\langle \mu' \pm \alpha, \beta \pm \alpha\rangle}{|\beta \pm \alpha|^2} \geqslant 2\right\}.
\end{aligned}$$

Then

(a) $\mathrm{I} + \mathrm{II} = 2\#\{\beta \in \Delta_n \mid \beta \mp \alpha \in \Delta, |\beta|^2 > |\alpha|^2, \mu' \perp \beta \mp \alpha\}$,

(b) $\mathrm{I} + \mathrm{II} = 0$ *if* μ' *is* Δ_K^+ *dominant and* $\tau_{(\mu' \pm \alpha)^\vee}$ *occurs in* $\tau_{\mu'} \otimes \mathfrak{p}^{\mathbb{C}}$,

(c) $\mathrm{I} = -\mathrm{II} = v_0^\pm$ *if* $\mu' = \Lambda$ *and* $\tau_{(\Lambda \pm \alpha)^\vee}$ *occurs in* $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$.

Proof of (a). Put $\mu'_\gamma = 2\langle \mu', \gamma\rangle / |\gamma|^2$. The sum $\mathrm{I} + \mathrm{II} - 2\rho(X_\alpha + X_{-\alpha})$ is

$$\begin{aligned}
= -2 &- \#\{\beta \in \Delta_n \mid \beta \mp \alpha \in \Delta, |\beta|^2 < |\alpha|^2, \mu'_{\beta \mp \alpha} = 1\} \\
&- \#\{\beta \in \Delta_n \mid \beta \pm \alpha \in \Delta, |\beta|^2 < |\alpha|^2, (\mu' \pm \alpha)_{\beta \pm \alpha} = 1\} \\
&- 2\#\{\beta \in \Delta_n \mid \beta \mp \alpha \in \Delta, |\beta|^2 \geqslant |\alpha|^2, \mu'_{\beta \mp \alpha} \geqslant 1\} \\
&- 2\#\{\beta \in \Delta_n \mid \beta \mp \alpha \in \Delta, |\beta|^2 < |\alpha|^2, \mu'_{\beta \mp \alpha} \geqslant 2\} \\
&- 2\#\{\beta \in \Delta_n \mid \beta \pm \alpha \in \Delta, |\beta|^2 \geqslant |\alpha|^2, (\mu' \pm \alpha)_{\beta \pm \alpha} \geqslant 1\} \\
&- 2\#\{\beta \in \Delta_n \mid \beta \pm \alpha \in \Delta, |\beta|^2 < |\alpha|^2; (\mu' \pm \alpha)_{\beta \pm \alpha} \geqslant 2\}.
\end{aligned}$$

On the second and sixth lines, we have $(\pm\alpha)_{\beta \pm \alpha} = 2$. On the fifth line, we have $(\pm\alpha)_{\beta \pm \alpha} = 0$ when $|\beta|^2 > |\alpha|^2$ and $(\pm\alpha)_{\beta \pm \alpha} = 1$ when $|\beta|^2 = |\alpha|^2$. Taking these relations into account and replacing β by $-\beta$ in the same lines, we find that the above expression is

$$\begin{aligned}
&= -2 - \#\ \{\beta\in\Delta_n \mid \beta\mp\alpha\in\Delta,\ |\beta|^2<|\alpha|^2,\ \mu'_{\beta\mp\alpha}=1\} \\
&\quad - \#\ \{\beta\in\Delta_n \mid \beta\mp\alpha\in\Delta,\ |\beta|^2<|\alpha|^2,\ \mu'_{\beta\mp\alpha}=1\} \\
&\quad - 2\#\ \{\beta\in\Delta_n \mid \beta\mp\alpha\in\Delta,\ |\beta|^2\geqslant|\alpha|^2,\ \mu'_{\beta\mp\alpha}\geqslant 1\} \\
&\quad - 2\#\ \{\beta\in\Delta_n \mid \beta\mp\alpha\in\Delta,\ |\beta|^2<|\alpha|^2,\ \mu'_{\beta\mp\alpha}\geqslant 2\} \\
&\quad - [2\#\ \{\beta\in\Delta_n \mid \beta\mp\alpha\in\Delta,\ |\beta|^2>|\alpha|^2,\ \mu'_{\beta\mp\alpha}\leqslant -1\} \\
&\quad + 2\#\ \{\beta\in\Delta_n \mid \beta\mp\alpha\in\Delta,\ |\beta|^2=|\alpha|^2,\ \mu'_{\beta\mp\alpha}\leqslant 0\}] \\
&\quad - 2\#\ \{\beta\in\Delta_n \mid \beta\mp\alpha\in\Delta,\ |\beta|^2<|\alpha|^2,\ \mu'_{\beta\mp\alpha}\leqslant 0\} \\
&= -2 - 2\#\ \{\beta\in\Delta_n \mid \beta\mp\alpha\in\Delta\} \\
&\quad + 2\#\ \{\beta\in\Delta_n \mid \beta\mp\alpha\in\Delta,\ |\beta|^2>|\alpha|^2,\ \mu'_{\beta\mp\alpha}=0\}. \qquad (2.11)
\end{aligned}$$

Now

$$2\rho(X_\alpha+X_{-\alpha}) = \frac{2\langle 2\rho,\tilde{\alpha}\rangle}{|\tilde{\alpha}|^2} = \sum_{\substack{\beta\in\Delta \\ \langle\beta,\alpha\rangle>0}} \frac{2\langle\beta,\alpha\rangle}{|\alpha|^2}. \qquad (2.12)$$

We shall prove that

$$\sum_{\substack{\beta\in\Delta \\ \langle\beta,\alpha\rangle>0}} \frac{2\langle\beta,\alpha\rangle}{|\alpha|^2} = 2\#\ \{\beta\in\Delta_n \mid \beta-\alpha\in\Delta\}+2. \qquad (2.13)$$

To do so, we consider all α strings of roots in Δ. The only singleton string that contributes to either side of (2.13) is $\{\alpha\}$, which contributes 2 to the left side and the constant term 2 to the right side. We consider a doubleton string $\{\gamma, \gamma-\alpha\}$ together with its negative $\{-\gamma+\alpha,\ -\gamma\}$. The roots γ and $-\gamma+\alpha$ contribute a total of 2 to the left side of (2.13); exactly one of γ and $-\gamma+\alpha$ is noncompact, and it contributes a matching 2 to the right side of (2.13). Consider a tripleton string $\{\gamma, \gamma-\alpha, \gamma-2\alpha\}$ together with its negative $\{-\gamma+2\alpha,\ -\gamma+\alpha,\ -\gamma\}$. The roots γ and $-\gamma+2\alpha$ contribute a total of 4 to the left side of (2.13), and one of the pairs $(\gamma,\ -\gamma+2\alpha)$ or $(\gamma-\alpha,\ -\gamma+\alpha)$ contributes a matching 4 to the right side of (2.13). This proves (2.13), and also we have

$$2\#\ \{\beta\in\Delta_n \mid \beta-\alpha\in\Delta\} = 2\#\ \{\beta\in\Delta_n \mid \beta+\alpha\in\Delta\} \qquad (2.14)$$

by replacing β by $-\beta$. Combining (2.11), (2.12), (2.13), and (2.14), we obtain conclusion (a) of the theorem.

Proof of (b). We give the argument for the top choice of signs. Write $(\mu'+\alpha)^\vee=\mu'+\beta_0$, and suppose I + II is nonzero. Choose β in Δ_n with $\beta-\alpha\in\Delta$, $|\beta|^2>|\alpha|^2$, and $\mu'\perp\beta-\alpha$. Since α is short, Lemma 1.2 allows us to write $\beta_0 = w\alpha$ for some w in $W_{K,\mu'}$. Put $\gamma = w(\beta - \alpha)$. Then γ is in $\Delta_{K,\mu'}$, $\gamma \perp \beta_0$, and $\gamma \perp\!\!\!\!/\!\!\perp \beta_0$. Possibly replacing γ by $-\gamma$, we may assume γ is positive.

Thus $\Delta^+_{K,\mu'}$ has a member γ with $\gamma \perp \beta_0$ and $\gamma \perp\!\!\!\!/\!\!\perp \beta_0$. Changing notation, let γ be a minimal element in $\Delta^+_{K,\mu'}$ with $\gamma \perp \beta_0$ and $\gamma \perp\!\!\!\!/\!\!\perp \beta_0$. We shall prove that γ is Δ^+_K simple.

Assuming the contrary, write $\gamma = \gamma_1 + \gamma_2$ with γ_1 and γ_2 in Δ^+_K. Since μ' is assumed Δ^+_K dominant, γ_1 and γ_2 are in $\Delta^+_{K,\mu'}$. But β_0 is $\Delta^+_{K,\mu'}$ dominant, and thus $\langle \beta_0, \gamma_1 \rangle \geqslant 0$ and $\langle \beta_0, \gamma_2 \rangle \geqslant 0$. Since $\langle \beta_0, \gamma_1 + \gamma_2 \rangle = 0$, we conclude

$$\langle \beta_0, \gamma_1 \rangle = \langle \beta_0, \gamma_2 \rangle = 0. \tag{2.15}$$

Now $0 < |\gamma|^2 = \langle \gamma, \gamma_1 \rangle + \langle \gamma, \gamma_2 \rangle$ implies $\langle \gamma, \gamma_1 \rangle > 0$ or $\langle \gamma, \gamma_2 \rangle > 0$. Without loss of generality, let us say $\langle \gamma, \gamma_1 \rangle > 0$. We know that $\gamma \pm \beta_0$ are roots, and (2.15) gives

$$\langle \gamma_1, \gamma \pm \beta_0 \rangle = \langle \gamma_1, \gamma \rangle \pm \langle \gamma_1, \beta_0 \rangle = \langle \gamma_1, \gamma \rangle > 0.$$

Hence $\gamma_2 \pm \beta_0 = (\gamma \pm \beta_0) - \gamma_1$ are roots. Thus γ_2 is in $\Delta^+_{K,\mu'}$ with $\gamma_2 \perp \beta_0$ and $\gamma_2 \perp\!\!\!\!/\!\!\perp \beta_0$. So γ_2 contradicts the minimality of γ, and we conclude γ is Δ^+_K simple.

Taking the existence of γ into account in Theorem 1.3, we see that $\tau_{\mu' + \beta_0}$ does not occur in $\tau_{\mu'} \otimes \mathfrak{p}^{\mathbb{C}}$, in contradiction to our hypothesis. This proves (b).

Proof of (c). We give the argument for the top choice of signs. In view of (b), we have only to show that $\mathrm{I} = \nu_0^+$. Substituting from (2.12) and (2.13) for $\rho(X_\alpha + X_{-\alpha})$ and from (2.8) for $\Lambda_\alpha = 2\langle \Lambda, \alpha \rangle / |\alpha|^2$, we have

$$\begin{aligned}
\mathrm{I} = {} & \# \{\beta \in \Delta_n \mid \beta - \alpha \in \Delta\} + 1 - 2(2\delta_K)_\alpha + \mu_\alpha \\
& - \# \{\beta \in \Delta_n \mid \beta - \alpha \in \Delta,\ |\beta|^2 < |\alpha|^2,\ \Lambda_{\beta-\alpha} = 1\} \\
& - 2\# \{\beta \in \Delta_n \mid \beta - \alpha \in \Delta,\ |\beta|^2 \geqslant |\alpha|^2,\ \Lambda_{\beta-\alpha} \geqslant 1\} \\
& - 2\# \{\beta \in \Delta_n \mid \beta - \alpha \in \Delta,\ |\beta|^2 < |\alpha|^2,\ \Lambda_{\beta-\alpha} \geqslant 2\} \\
= {} & 1 + \mu_\alpha + \# \{\beta \in \Delta_n \mid \beta - \alpha \in \Delta\} - 2(2\delta_K)_\alpha \\
& - 2\# \{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta,\ \Lambda_{\beta-\alpha} \geqslant 1\} \\
& + \# \{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta,\ |\beta|^2 < |\alpha|^2,\ \Lambda_{\beta-\alpha} = 1\} \\
= {} & 1 + \mu_\alpha + \# \{\beta \in \Delta_n \mid \beta - \alpha \in \Delta\} - 2(2\delta_K)_\alpha \\
& - 2\# \{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta\} + 2\# \{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta,\ \Lambda_{\beta-\alpha} = 0\} \\
& + \# \{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta,\ |\beta|^2 < |\alpha|^2,\ \Lambda_{\beta-\alpha} = 1\}.
\end{aligned}$$

Comparing this expression with v_0^+, we see that it is enough to prove

$$\# \{\beta \in \Delta_n \mid \beta - \alpha \in \Delta\} - 2\# \{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta\} = \frac{2\langle 2\delta_K, \alpha\rangle}{|\alpha|^2}. \tag{2.16}$$

To do so, we consider all α strings of roots in Δ. The singleton strings do not contribute to (2.16). We consider a doubleton string together with its negative:

$$\{\gamma, \gamma - \alpha\} \quad \text{and} \quad \{-\gamma + \alpha, -\gamma\},$$

where $\gamma > 0$. If γ is noncompact, the contribution to (2.16) is $1 - 2 = 2\langle \gamma - \alpha, \alpha\rangle / |\alpha|^2$, and equality is preserved. If γ is compact, the contribution to (2.16) is $1 - 0 = 2\langle \gamma, \alpha\rangle / |\alpha|^2$, and equality is preserved. This handles doubleton strings. We consider a tripleton string together with its negative:

$$\{\gamma, \gamma - \alpha, \gamma - 2\alpha\} \quad \text{and} \quad \{-\gamma + 2\alpha, -\gamma + \alpha, -\gamma\},$$

where $\gamma > 0$. If γ is noncompact, the contribution to (2.16) is $2 - 2 = 2\langle \gamma - \alpha, \alpha\rangle / |\alpha|^2$, and equality is preserved. If γ is compact, the contribution to (2.16) is $2 - 2 = 2\langle (\gamma) + (\gamma - 2\alpha), \alpha\rangle / |\alpha|^2$, and equality is preserved. This handles tripleton strings. Thus (2.16) is proved, and the proof of Theorem 2.1 is complete.

Remarks. In [2] we used a version of Theorem 2.1 applicable to a superorthogonal set $\{\alpha_1, ..., \alpha_l\}$ in place of $\{\alpha\}$. For the precise statement, we fix j and replace α by α_j in (2.10) and the statement of Theorem 2.1. Also we add to each of the sets of conditions $\{\beta \in \Delta_n \mid \cdots\}$ the condition $\beta \perp\perp \alpha_1, ..., \alpha_{j-1}$. If we maintain these conventions throughout the proof, the proof goes through without further change. Let us define $v_{0,j}^+$ and $v_{0,j}^-$ as the versions of v_0^+ and v_0^- that are modified in this way.

3. Multiplicity Questions for K Types

We continue with the notation of Sections 1 and 2, but we take $l = 1$ through Proposition 3.4, so that $\{\alpha_1, ..., \alpha_l\}$ reduces to $\{\alpha\}$. In this section we shall compute the multiplicities of certain K types and weights.

Since $\sigma = \operatorname{ind}_{M^\#}^M \sigma^\#$, the double induction formula shows that

$$U(\nu) = \operatorname{ind}_{MAN}^G(\sigma \otimes e^\nu \otimes 1) = \operatorname{ind}_{M^\# AN}^G(\sigma^\# \otimes e^\nu \otimes 1). \tag{3.1a}$$

Thus

$$U(\nu)|_K = \operatorname{ind}_{K \cap M^\#}^K (\sigma^\# |_{K \cap M^\#}). \tag{3.1b}$$

Let us write

$$\sigma^{\#}|_{K\cap M^{\#}} = \sum_{\lambda'} n_{\lambda'}\tau_{\lambda'} \tag{3.2}$$

as the decomposition into $(K\cap M^{\#})$ types. Since $\sigma^{\#}$ is determined by $\sigma^{\#}|_{M_0}$ and the scalar $\sigma^{\#}(\gamma_\alpha)$, we can regard the λ's in (3.2) as distinct integral forms on $\mathfrak{b}_-$. Frobenius reciprocity gives us the multiplicity formula

$$[U(v)|_K : \tau_{\Lambda'}] = \sum_{\lambda'} n_{\lambda'}[\tau_{\Lambda'}|_{K\cap M^{\#}} : \tau_{\lambda'}], \tag{3.3}$$

a formula that motivates a number of the results of this section.

Let δ^+ be the result of making α dominant relative to $\Delta_{K,\Lambda}^+$ (by means of $W_{K,\Lambda}$), and let δ^- be the result of making $-\alpha$ dominant relative to $\Delta_{K,\Lambda}^+$. By the δ^+ *subgroup of* G, we mean the reductive subgroup built from $\mathfrak{b}$, α, and all the simple roots of Δ^+ needed for the expansion of δ^+ in terms of Δ^+ simple roots. The δ^- *subgroup of* G is defined analogously in terms of δ^- and $s_\alpha\Delta^+$.

LEMMA 3.1. *If G has no factor split F_4 and if σ is nondegenerate in the sense of* [16], *then the following are equivalent*:

(a) α *and* $-\alpha$ *are conjugate via* $W_{K,\Lambda}$

(b) $\Lambda+\alpha$ *and* $\Lambda-\alpha$ *are conjugate via* $W_{K,\Lambda}$

(c) $\Lambda+\alpha$ *and* $\Lambda-\alpha$ *are conjugate via* W_K

(d) $\Lambda+\delta^+ = (\Lambda+\alpha)^\vee = (\Lambda-\alpha)^\vee = \Lambda+\delta^-$

(e) $\langle\Lambda,\alpha\rangle = 0$ *and* $\Lambda+\alpha$ *is a weight of* $\tau_{(\Lambda-\alpha)^\vee}$

(f) $\langle\Lambda,\alpha\rangle = 0$ *and* $\Lambda-\alpha$ *is a weight of* $\tau_{(\Lambda+\alpha)^\vee}$.

Moreover, these conditions can hold only if there exist roots of two different lengths and α is short.

Proof. It is clear that (a) and (b) are equivalent. To see that (c), (e), and (f) are equivalent, first assume (c) holds. Then $\Lambda+\alpha$ is certainly a weight of $\tau_{(\Lambda-\alpha)^\vee}$, and $\Lambda+\alpha$ has the same length as $\Lambda-\alpha$, so that $\langle\Lambda,\alpha\rangle=0$. Thus (e) holds, and similarly (f) holds. Conversely if (e) holds, then $\langle\Lambda,\alpha\rangle=0$ implies $|\Lambda+\alpha|=|\Lambda-\alpha|$. Since $\Lambda+\alpha$ is a weight of $\tau_{(\Lambda-\alpha)^\vee}$, it must be an extreme weight, and we conclude $\Lambda+\alpha$ is a W_K transform of $\Lambda-\alpha$. Thus (e) implies (c), and similarly (f) implies (c).

Next let us observe that (d) implies (a). In fact, (d) gives $\delta^+=\delta^-$. Thus α is conjugate by $W_{K,\Lambda}$ to $\delta^+=\delta^-$, which is conjugate by $W_{K,\Lambda}$ to $-\alpha$. Hence (d) implies (a). Clearly (a) and (b) imply (c).

To complete the equivalences, we show (c) implies (d). It is enough to prove (c) implies $(\Lambda+\alpha)^\vee=\Lambda+\delta^+$ and $(\Lambda-\alpha)^\vee=\Lambda+\delta^-$. If all roots have the same length or if α is short, this follows from Lemma 1.2. Thus we may assume that G is simple, that there are roots of two different lengths, and that α is long. Then G is locally $Sp(n, \mathbb{R})$ or $SO(\text{odd, even})$, since split F_4 is excluded by hypothesis. Lemma 7.1 of [5] says that either $\Lambda+\alpha$ is not a weight of $\tau_{(\Lambda-\alpha)^\vee}$ or else $\Lambda-\alpha$ is not a weight of $\tau_{(\Lambda+\alpha)^\vee}$. Hence (e) or (f) fails, and (c) cannot hold when α is long and there are roots of two lengths. In particular, the six statements are always equivalent.

Finally we show that (a) is impossible if all roots have the same length. We may assume G is simple. By (d), we have $\delta^+=\delta^-$. In the terminology of [5], the roots of the δ^+ and δ^- subgroups, as well as the members of $\Delta_{K,\Lambda}$, lie in the special basic case. Thus we may assume G is a special basic case. By Lemma 4.1 of [5], either the δ^+ subgroup or the δ^- subgroup has real rank one and is thus of the form $SU(n, 1)$ locally. Since $\delta^+=\delta^-$, α and $-\alpha$ are conjugate via the W_K of a subgroup of the form $SU(n, 1)$. But this is a contradiction, since 2α is not the sum of compact roots in $SU(n, 1)$.

LEMMA 3.2. *If β_1 and β_2 are noncompact short roots that are conjugate via $W_{K,\Lambda}$, then there exists p in $W_{K,\Lambda}$ with $p^2=1$ such that $\beta_2=p\beta_1$.*

Proof. Since β_1 and β_2 are conjugate via $W_{K,\Lambda}$, we can write

$$\beta_1=\beta_2+\sum_{\gamma\in\Delta^+_{K,\Lambda}} n_\gamma\gamma \qquad \text{with} \quad n_\gamma\in\mathbb{Z}. \tag{3.4}$$

Among all such expressions (3.4), choose one with $\sum|n_\gamma|$ as small as possible. In this case we shall prove that the γ's with $n_\gamma\neq 0$ are strongly orthogonal.

In fact, we write

$$0=|\beta_1|^2-|\beta_2|^2=\sum_\gamma (n_\gamma 2\langle\beta_2,\gamma\rangle+n_\gamma^2|\gamma|^2)+\sum_{\substack{\gamma,\gamma'\\ \gamma\neq\gamma'}} 2n_\gamma n_{\gamma'}\langle\gamma,\gamma'\rangle.$$

The first sum on the right side is $\geqslant 0$ term by term since

$$n_\gamma 2\langle\beta_2,\gamma\rangle+n_\gamma^2|\gamma|^2=|n_\gamma|\left(\frac{2\langle\beta_2,\gamma\rangle}{|\gamma|^2}\operatorname{sgn} n_\gamma+1\right)|\gamma|^2$$
$$+|\gamma|^2(n_\gamma^2-|n_\gamma|);$$

here both terms on the right side are $\geqslant 0$, the first one since γ compact and β_2 short noncompact force $|2\langle\beta_2,\gamma\rangle/|\gamma|^2|\leqslant 1$. Thus

$$0\geqslant\sum_{\gamma\neq\gamma'} n_\gamma n_{\gamma'}\langle\gamma,\gamma'\rangle.$$

We claim that each term on the right is 0. Assuming the contrary, choose γ and γ' with $n_\gamma n_{\gamma'}\langle\gamma, \gamma'\rangle < 0$.

If $n_\gamma n_{\gamma'} > 0$, then $\gamma+\gamma'$ is a root. Without loss of generality, let $|n_{\gamma'}| \geqslant |n_\gamma|$. We can rewrite

$$n_\gamma \gamma + n_{\gamma'}\gamma' \qquad \text{as} \qquad n_\gamma(\gamma+\gamma') + (n_{\gamma'} - n_\gamma)\,\gamma' \tag{3.5a}$$

with $|n_{\gamma'} - n_\gamma| < |n_{\gamma'}|$ and get a contradiction to the minimality of $\sum |n_\gamma|$. If $n_\gamma n_{\gamma'} < 0$, then $\gamma-\gamma'$ is a root. Without loss of generality, let $|n_{\gamma'}| \geqslant |n_\gamma|$. We can rewrite

$$n_\gamma \gamma + n_{\gamma'}\gamma' \qquad \text{as} \qquad n_\gamma(\gamma-\gamma') + (n_\gamma + n_{\gamma'})\,\gamma' \tag{3.5b}$$

with $|n_\gamma + n_{\gamma'}| < |n_{\gamma'}|$ and get a contradiction to the minimality of $\sum |n_\gamma|$.

Thus the roots γ in our minimal expression (3.4) are orthogonal. Let us prove they are strongly orthogonal. Suppose γ and γ' occur with $\gamma \perp \gamma'$ and $\gamma \perp\!\!\!\!/\!\!\!\perp \gamma'$. If $n_\gamma n_{\gamma'} > 0$, then we argue as in (3.5a), using that $\gamma+\gamma'$ is a root, to get a contradiction. If $n_\gamma n_{\gamma'} < 0$, then we argue as in (3.5b), using that $\gamma-\gamma'$ is a root, to get a contradiction. We conclude that the γ's in our minimal expression (3.4) are strongly orthogonal.

Put $p = \prod_{n_\gamma \neq 0} s_\gamma$. Then p is in $W_{K,A}$ with $p^2 = 1$. We shall prove that $p\beta_1 = \beta_2$. If s_γ occurs in p, then

$$\frac{2\langle \beta_1 - \beta_2, \gamma\rangle}{|\gamma|^2} = \frac{2\langle \sum n_{\gamma'}\gamma', \gamma\rangle}{|\gamma|^2} = 2n_\gamma. \tag{3.6}$$

Here $2\langle\beta_1, \gamma\rangle/|\gamma|^2$ is 0, $+1$, or -1 since $|\beta_1| \leqslant |\gamma|$, and $2\langle\beta_2, \gamma\rangle/|\gamma|^2$ similarly is 0, $+1$, or -1. Since $n_\gamma \neq 0$, the only possibilities to achieve (3.6) are

$$\frac{2\langle\beta_1, \gamma\rangle}{|\gamma|^2} = +1 \qquad \text{and} \qquad \frac{2\langle\beta_2, \gamma\rangle}{|\gamma|^2} = -1 \tag{3.7a}$$

or

$$\frac{2\langle\beta_1, \gamma\rangle}{|\gamma|^2} = -1 \qquad \text{and} \qquad \frac{2\langle\beta_2, \gamma\rangle}{|\gamma|^2} = +1. \tag{3.7b}$$

In either case, (3.7a) or (3.7b), we see that $\langle\beta_1, -\gamma\rangle = \langle\beta_2, \gamma\rangle$. Hence for each γ,

$$\langle p\beta_1, \gamma\rangle = \langle\beta_1, p\gamma\rangle = \langle\beta_1, -\gamma\rangle = \langle\beta_2, \gamma\rangle. \tag{3.8}$$

Now (3.4) gives

$$p\beta_1 = p\beta_2 - \sum n_\gamma \gamma,$$

and it is clear that $p\beta_2 \in \beta_2 + \sum_\gamma \mathbb{R}\gamma$. Hence

$$p\beta_1 = \beta_2 + \sum c_\gamma \gamma \tag{3.9}$$

for suitable $c_\gamma \in \mathbb{R}$. Taking the inner product of both sides of (3.9) with each γ and comparing with (3.8), we see that each c_γ is 0. Hence $p\beta_1 = \beta_2$ as required.

THEOREM 3.3. *Suppose that G has no factor split F_4, that σ is nondegenerate in the sense of* [16], *and that α and $-\alpha$ are conjugate via $W_{K,\Lambda}$. Let $\Lambda_1 = (\Lambda+\alpha)^\vee = (\Lambda-\alpha)^\vee$. Then*

(a) $[\tau_{\Lambda_1} : \Lambda+\alpha] = [\tau_{\Lambda_1} : \Lambda-\alpha] = 1$ (*for the multiplicities of weights*)

(b) $[\tau_{\Lambda_1}|_{K\cap M^\#} : \tau_\lambda] = 1$.

Proof. Conclusion (a) is trivial: Since $\Lambda+\alpha$ and $\Lambda-\alpha$ are extreme weights of τ_{Λ_1}, they have multiplicity one. Toward conclusion (b), first let us prove that any weight μ' of τ_{Λ_1} with $\mu'|_{\mathfrak{b}_-} = \lambda$ has the form

$$\mu' = \Lambda + (2n+1)\,\alpha \tag{3.10}$$

for some integer n. In fact, $\mu'|_{\mathfrak{b}_-} = \Lambda|_{\mathfrak{b}_-} = \lambda$ implies $\mu' - \Lambda = c\alpha$ for some constant c. Since μ' is a weight of τ_{Λ_1}, Lemma 3.1d gives

$$\mu' = \Lambda_1 - \sum_{\gamma \in \Delta_K^+} n_\gamma \gamma = \Lambda + \delta^+ - \sum_{\gamma \in \Delta_K^+} n_\gamma \gamma.$$

Hence

$$c\alpha = \delta^+ - \sum_{\gamma \in \Delta_K^+} n_\gamma \gamma.$$

Taking the inner product of both sides with the half sum δ shows that c is an integer. Since δ^+ and α are noncompact, c is odd. (See Problems 12 and 13 on p. 478 of [12].) This proves (3.10).

Parts (e) and (f) of Lemma 3.1 say that $\langle \Lambda, \alpha \rangle = 0$. Therefore $|\mu'|$ in (3.10) is too large for μ' to be a weight unless $\mu' = \Lambda \pm \alpha$. Taking into account (a) of the theorem, we see that the span of the weight vectors whose weight restricts to λ has dimension 2, coming from weights $\Lambda + \alpha$ and $\Lambda - \alpha$. Consequently $[\tau_{\Lambda_1}|_{K\cap M^\#} : \tau_\lambda] \leqslant 2$.

To see that $1 \leqslant [\tau_{\Lambda_1}|_{K\cap M^\#} : \tau_\lambda]$, let $v_{\Lambda+\alpha}$ and $v_{\Lambda-\alpha}$ be nonzero weight vectors for τ_{Λ_1} of weights $\Lambda+\alpha$ and $\Lambda-\alpha$, respectively. We show for suitable normalization of these vectors that $v_{\Lambda+\alpha} + v_{\Lambda-\alpha}$ is a highest weight vector for a $K \cap M^\#$ representation of type τ_λ. We have

$$\tau_{\Lambda_1}(\gamma_\alpha)\, v_{\Lambda+\alpha} = \exp \frac{2\pi i \langle \Lambda+\alpha, \alpha \rangle}{|\alpha|^2}\, v_{\Lambda+\alpha} = \exp \frac{2\pi i \langle \Lambda, \alpha \rangle}{|\alpha|^2}\, v_{\Lambda+\alpha},$$

and the scalar on the right is the same one by which $\tau_\Lambda(\gamma_\alpha)$ acts on a highest weight vector of τ_Λ. Thus

$$\tau_{\Lambda_1}(\gamma_\alpha)\, v_{\Lambda+\alpha} = \sigma^{\#}(\gamma_\alpha)\, v_{\Lambda+\alpha}.$$

Applying the same argument to $v_{\Lambda-\alpha}$, we obtain

$$\tau_{\Lambda_1}(\gamma_\alpha)(v_{\Lambda+\alpha} + v_{\Lambda-\alpha}) = \sigma^{\#}(\gamma_\alpha)(v_{\Lambda+\alpha} + v_{\Lambda-\alpha}).$$

Thus it is enough to exhibit a normalization such that the root vectors for the positive roots of $(m^{\mathbb{C}}, b_-^{\mathbb{C}})$ annihilate $v_{\Lambda+\alpha} + v_{\Lambda-\alpha}$. Let us set aside temporarily the question of which normalization to use and work with an arbitrary normalization.

Let γ be an M-compact positive root for $(m^{\mathbb{C}}, b_-^{\mathbb{C}})$ with $\tilde{X}_\gamma$ as root vector. Then $\gamma \perp \alpha$. Suppose $\gamma \perp\!\!\perp \alpha$. Then $\tilde{X}_\gamma$ can be taken to be X_γ. We have $\tau_{\Lambda_1}(\tilde{X}_\gamma)\, v_{\Lambda+\alpha} = 0$ since

$$|\Lambda+\alpha+\gamma|^2 - |\Lambda+\alpha|^2 = 2\langle \Lambda+\alpha, \gamma\rangle + |\gamma|^2 = 2\langle \Lambda, \gamma\rangle + |\gamma|^2 > 0$$

shows $\Lambda+\alpha+\gamma$ cannot be a weight of τ_{Λ_1}. Similarly $\tau_{\Lambda_1}(\tilde{X}_\gamma)\, v_{\Lambda-\alpha} = 0$. Thus

$$\tau_{\Lambda_1}(\tilde{X}_\gamma)(v_{\Lambda+\alpha} + v_{\Lambda-\alpha}) = 0.$$

Next suppose that $\gamma \perp \alpha$ but $\gamma \not\perp\!\!\perp \alpha$. By Lemma 5.4 of [15], we can take $\tilde{X}_\gamma = [X_\gamma, X_\alpha] - [X_\gamma, X_{-\alpha}]$. Then $\tau_{\Lambda_1}[X_\gamma, X_\alpha]\, v_{\Lambda+\alpha} = 0$ since

$$\begin{aligned} |(\Lambda+\alpha) + (\gamma+\alpha)|^2 - |\Lambda+\alpha|^2 &= 2\langle \Lambda+\alpha, \gamma+\alpha\rangle + |\gamma+\alpha|^2 \\ &= 2\langle \Lambda, \gamma\rangle + 2\,|\alpha|^2 + |\gamma+\alpha|^2 > 0 \end{aligned}$$

shows $(\Lambda + \alpha) + (\gamma + \alpha)$ cannot be a weight of τ_{Λ_1}. Similarly $\tau_{\Lambda_1}[X_\gamma, X_{-\alpha}]\, v_{\Lambda-\alpha} = 0$. Consequently

$$\tau_{\Lambda_1}(\tilde{X}_\gamma)(v_{\Lambda+\alpha} + v_{\Lambda-\alpha}) = \tau_{\Lambda_1}[X_\gamma, X_\alpha]\, v_{\Lambda-\alpha} - \tau_{\Lambda_1}[X_\gamma, X_{-\alpha}]\, v_{\Lambda+\alpha}. \tag{3.11}$$

Actually each term on the right side of (3.11) is 0 unless $\langle \Lambda, \gamma\rangle = 0$. In fact, each term is a weight vector of weight $\Lambda+\gamma$, and the equation

$$|\Lambda+\gamma|^2 - |\Lambda+\alpha|^2 = 2\langle \Lambda, \gamma\rangle - 2\langle \Lambda, \alpha\rangle = 2\langle \Lambda, \gamma\rangle$$

shows $\Lambda+\gamma$ is a weight only if $\langle \Lambda, \gamma\rangle = 0$. In this case, $\Lambda+\gamma$ is an extreme weight.

For use below, let us observe conversely that

$$\tau_{\Lambda_1}[X_\gamma, X_\alpha]\, v_{\Lambda-\alpha} \neq 0 \tag{3.12}$$

if $\gamma \perp \alpha$, $\gamma \not\perp\!\!\perp \alpha$, and $\langle \Lambda, \gamma\rangle = 0$. Hence $\Lambda+\gamma$ is a weight, necessarily extreme. In fact, all we have to observe is that $\langle \Lambda-\alpha, \gamma+\alpha \rangle < 0$. Then (3.12) follows from standard facts about weight strings.

Returning to (3.11), we see that every normalization of $v_{\Lambda+\alpha}+v_{\Lambda-\alpha}$ gives a highest weight vector unless there is some γ_0 with $\gamma_0 \perp \alpha$, $\gamma_0 \perp\!\!\!\!/\!\!\perp \alpha$, and $\langle \Lambda, \gamma_0 \rangle = 0$. Thus, in proving $1 \leqslant [\tau_{\Lambda_1}|_{K \cap M^\#} : \tau_\lambda]$, we may assume there is such a root γ_0. In this case, let $v_{\Lambda+\gamma_0}$ denote a nonzero weight vector of weight $\Lambda+\gamma_0$; this is unique up to a scalar since $\Lambda+\gamma_0$ is extreme. Then we can write

$$\tau_{\Lambda_1}[X_{\gamma_0}, X_\alpha]\, v_{\Lambda-\alpha} = a_{\gamma_0} v_{\Lambda+\gamma_0}$$
$$\tau_{\Lambda_1}[X_{\gamma_0}, X_{-\alpha}]\, v_{\Lambda+\alpha} = b_{\gamma_0} v_{\Lambda+\gamma_0},$$

so that

$$\tau_{\Lambda_1}(\tilde{X}_{\gamma_0})(v_{\Lambda+\alpha}+v_{\Lambda-\alpha}) = (a_{\gamma_0}-b_{\gamma_0})\, v_{\Lambda+\gamma_0}. \tag{3.13}$$

If γ is another root with $\gamma \perp \alpha$, $\gamma \perp\!\!\!\!/\!\!\perp \alpha$, and $\langle \Lambda, \gamma \rangle = 0$, then we have similarly

$$\tau_{\Lambda_1}(\tilde{X}_\gamma)(v_{\Lambda+\alpha}+v_{\Lambda-\alpha}) = (a_\gamma-b_\gamma)\, v_{\Lambda+\gamma}. \tag{3.14}$$

The proof that $1 \leqslant [\tau_{\Lambda_1}|_{K\cap M^\#} : \tau_\lambda]$ will be complete if we show that $a_\gamma - b_\gamma$ is a multiple of $a_{\gamma_0}-b_{\gamma_0}$ (since then we can adjust $v_{\Lambda-\alpha}$ by a scalar to make $a_{\gamma_0}-b_{\gamma_0}$ be 0, with the result that the adjusted $v_{\Lambda+\alpha}+v_{\Lambda-\alpha}$ is $K \cap M^\#$ highest).

Now $\tau_{\Lambda_1}(\tilde{X}_{\gamma_0})\, v_{\Lambda+\gamma_0} = 0$ since

$$\begin{aligned} &|(\Lambda+\gamma_0)+(\gamma_0 \pm \alpha)|^2 - |\Lambda \pm \alpha|^2 \\ &\quad = 4\langle \Lambda \pm \alpha, \gamma_0 \rangle + |2\gamma_0|^2 = 4\,|\gamma_0|^2 > 0 \end{aligned}$$

shows that neither of $(\Lambda+\gamma_0)+(\gamma_0 \pm \alpha)$ is a weight. From (3.13) it follows that there is a nonzero constant c such that

$$\begin{aligned} c(v_{\Lambda+\alpha}+v_{\Lambda-\alpha}) &= \tau_{\Lambda_1}(\tilde{X}_{-\gamma_0})\, \tau_{\Lambda_1}(\tilde{X}_{\gamma_0})(v_{\Lambda+\alpha}+v_{\Lambda-\alpha}) \\ &= (a_{\gamma_0}-b_{\gamma_0})\, \tau_{\Lambda_1}(\tilde{X}_{-\gamma_0})\, v_{\Lambda+\gamma_0}. \end{aligned}$$

Substituting into (3.14), we obtain

$$(a_\gamma - b_\gamma)\, v_{\Lambda+\gamma} = c^{-1}(a_{\gamma_0}-b_{\gamma_0})\, \tau_{\Lambda_1}(\tilde{X}_\gamma)\, \tau_{\Lambda_1}(\tilde{X}_{-\gamma_0})\, v_{\Lambda+\gamma_0},$$

which is the desired relation. This proves that $1 \leqslant [\tau_{\Lambda_1}|_{K\cap M^\#} : \tau_\lambda]$.

To complete the proof, we show that $[\tau_{\Lambda_1}|_{K\cap M^\#} : \tau_\lambda]$ is not 2. It is enough to prove that $v_{\Lambda-\alpha}$ is not $K \cap M^\#$ highest. In view of (3.12), it is enough to produce an M-compact positive root γ_0 with $\gamma_0 \perp \alpha$, $\gamma_0 \perp\!\!\!\!/\!\!\perp \alpha$, and $\langle \Lambda, \gamma_0 \rangle = 0$.

Applying Lemma 3.2 with $\beta_1 = \alpha$ and $\beta_2 = -\alpha$, we can find strongly orthogonal roots γ_i in $\Delta_{K,\Lambda}$ with $(\prod s_{\gamma_i})\,\alpha = -\alpha$. Then

$$\alpha - \sum \frac{2\langle \alpha, \gamma_i \rangle}{|\gamma_i|^2}\gamma_i = \left(\prod s_{\gamma_i}\right)\alpha = -\alpha$$

says that

$$\alpha = c_1\gamma_1 + \cdots + c_r\gamma_r.$$

We may assume all the c_i are nonzero. Taking into account the result of Lemma 3.1 that α is short, we find the following possibilities, all having all $|c_i| = \frac{1}{2}$:

(i) $r = 2$, both γ_i's are long

(ii) $r = 3$, one γ_i is long, two γ_i's are short

(iii) $r = 4$, all γ_i's are short.

Meanwhile Lemma 3.1 says that there do exist roots of two different lengths. We may assume G is simple, and then Δ is of type B_n, C_n, or F_4. In B_n and F_4, any two orthogonal short roots fail to be strongly orthogonal. Thus (ii) and (iii) are ruled out above, and $\alpha = c_1\gamma_1 + c_2\gamma_2$ with $|c_1| = |c_2| = \frac{1}{2}$. Put $\gamma_0 = s_{\gamma_2}\alpha = c_1\gamma_1 - c_2\gamma_2$. Then $\gamma_0 \perp \alpha$, $\gamma_0 \perp\!\!\!\!/\!\!\perp \alpha$, and $\langle \Lambda, \gamma_0 \rangle = 0$. The equation $\alpha + \gamma_0 = \pm\gamma_1$ shows that γ_0 is G-noncompact. Thus γ_0 is M-compact, as required.

We are left with Δ of type C_n. In $Sp(n, \mathbb{R})$, no noncompact root is W_K conjugate to its negative. So we may assume G is locally $Sp(p, q)$ with $p + q = n$ and with K locally $Sp(p) \times Sp(q)$. In standard notation we can write $\alpha = e_j - e_{j+1}$, and we may assume $2e_j$ is a root of $Sp(p)$ and $2e_{j+1}$ is a root of $Sp(q)$. For α to be $W_{K,\Lambda}$ conjugate to $-\alpha$, $2e_j$ must be $W_{K,\Lambda}$ conjugate to $-2e_j$, and $2e_{j+1}$ must be $W_{K,\Lambda}$ conjugate to $-2e_{j+1}$. Consequently $\langle \Lambda, e_j \rangle = \langle \Lambda, e_{j+1} \rangle = 0$. Then $\gamma_0 = e_j + e_{j+1}$ is a root with $\gamma_0 \perp \alpha$, $\gamma_0 \perp\!\!\!\!/\!\!\perp \alpha$, and $\langle \Lambda, \gamma_0 \rangle = 0$. The equation $\alpha + \gamma_0 = 2e_j$ with $2e_j$ compact shows that γ_0 is G-noncompact. Thus γ_0 is M-compact, as required.

PROPOSITION 3.4. *Suppose that α and $-\alpha$ are not conjugate via $W_{K,\Lambda}$. Suppose $\Lambda_1^+ = (\Lambda + \alpha)^\vee$* equals *$\Lambda + \delta^+$ and $\Lambda_1^- = (\Lambda - \alpha)^\vee$ equals $\Lambda + \delta^-$. With the signs taken consistently throughout,*

(a) $[\tau_{\Lambda_1^\pm} : \Lambda \pm \alpha] = 1$.

(b) $[\tau_{\Lambda_1^\pm}|_{K \cap M^\#} : \tau_\lambda] \leqslant 1 + \sum_{n \geqslant 0} [\tau_{\Lambda_1^\pm} : \Lambda \mp (2n+1)\,\alpha]$.

(c) $1 \leqslant [\tau_{\Lambda_1^\pm}|_{K \cap M^\#} : \tau_\lambda]$ *if $\Lambda \mp \alpha$ is not a weight of $\tau_{\Lambda_1^\pm}$.*

A sufficient condition for $\Lambda \mp \alpha$ not to be a weight of $\tau_{\Lambda_1^\pm}$ is that $\langle \Lambda, \pm\alpha \rangle \leqslant 0$.

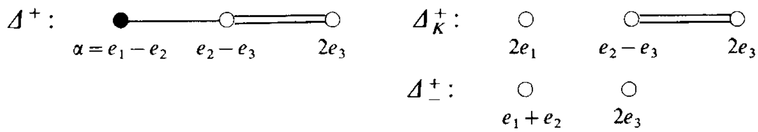
(d) *The* nth *term on the right side of* (b) *is* 0 *for* $n \geqslant 0$ *if* $\langle \Lambda, \pm\alpha \rangle \leqslant 0$.

(e) *The* nth *term on the right side of* (b) *is* 0 *for* $n > 0$ *if the* $\delta^\pm$ *group is classical. If, in addition,* $2\delta^\pm$ *is not a sum of members of* Δ_K^+, *then the* 0*th term is* 0.

Remarks. (1) The results for Λ_1^+ depend on having $(\Lambda+\alpha)^\vee = \Lambda + \delta^+$, and those for Λ_1^- depend on having $(\Lambda-\alpha)^\vee = \Lambda + \delta^-$. Both these conditions are satisfied if all roots have the same length or if α is short, by Lemma 1.2.

(2) In [1] we announced that $1 \leqslant [\tau_{\Lambda_1^\pm}|_{K\cap M^\#} : \tau_\lambda]$ without the hypothesis on weights in (c). We are not able to prove the version in the announcement. But see Remark 3 with Corollary 5.2 for further information.

(3) The condition on weights in (c) is not a necessary condition. For example, take $G = Sp(2, 1)$ with



If $\Lambda = (1, 0, 0)$, then $(\Lambda+\alpha)^\vee = (2, 1, 0)$ and $\Lambda - \alpha = (0, 1, 0)$. So $\Lambda - \alpha$ is a weight of $\tau_{(\Lambda+\alpha)^\vee}$. A little computation shows nevertheless that $[\tau_{(\Lambda+\alpha)^\vee}|_{K\cap M^\#} : \tau_\lambda] = 1$.

Proof. We give the argument for Λ_1^+. The results for Λ_1^- then follow by replacing Δ^+ by $s_\alpha \Delta^+$ and α by $-\alpha$.

(a) $\Lambda + \alpha$ is an extreme weight in $\tau_{\Lambda_1^+}$ and therefore has multiplicity one.

(b) The same argument as for (3.10), but with a reference to the hypothesis rather than to Lemma 3.1d, shows that any weight μ' of $\tau_{\Lambda_1^\pm}$ with $\mu'|_{\mathfrak{b}_-} = \lambda$ has the form

$$\mu' = \Lambda + (2n+1)\alpha \tag{3.15}$$

for some integer n. Choose $w \in W_{K,\Lambda}$ with $w\delta^+ = \alpha$. Then $\Lambda + \alpha$ is the highest weight of $\tau_{\Lambda_1^+}$ relative to $w\Delta_K^+$, and thus

$$\begin{aligned} \mu' &= \Lambda + \alpha - \sum_{\gamma' \in w\Delta_K^+} n_{\gamma'} \gamma' \\ &= \Lambda + \alpha - \sum_{\gamma \in \Delta_K^+} n_{w\gamma} w\gamma, \quad n_{w\gamma} \geqslant 0. \end{aligned}$$

Substitution from (3.15) gives

$$(2n+1)\,\alpha = \alpha - \sum_{\gamma \in \Delta_K^+} n_{w\gamma} w\gamma.$$

Applying w^{-1}, we have

$$2n\delta^+ = -\sum_{\gamma \in \Delta_K^+} n_{w\gamma}\gamma. \tag{3.16}$$

Since δ^+ and each γ are positive, we conclude $n \leqslant 0$.

The subspace of highest weight vectors for occurrences of τ_λ in $\tau_{\Lambda_1^+}|_{K \cap M^\#}$ must be contained in the span of the μ' weight spaces, with μ' as in (3.15) (and also $n \leqslant 0$, as we have just seen). Conclusion (b) of the proposition therefore follows when we take (a) into account.

(c) Let $v_{\Lambda+\alpha}$ be a nonzero weight vector of weight $\Lambda+\alpha$ in $\tau_{\Lambda_1^+}$. First we prove that $v_{\Lambda+\alpha}$ is a highest weight vector under $K \cap M^\#$ for an occurrence of τ_λ if and only if there exists no M-compact positive root γ with $\gamma \perp \alpha$, $\gamma \perp\!\!\!\not\perp \alpha$, and $\langle \Lambda, \gamma - \alpha \rangle = 0$. The behavior under $\tau_{\Lambda_1^+}(\gamma_\alpha)$ is no problem and is handled as in Theorem 3.3.

Let γ be an M-compact positive root for $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}_-^{\mathbb{C}})$ with $\tilde{X}_\gamma$ as root vector. Then $\gamma \perp \alpha$. Suppose $\gamma \perp\!\!\perp \alpha$. Then $\tilde{X}_\gamma$ can be taken to be X_γ. We have $\tau_{\Lambda_1^+}(\tilde{X}_\gamma)\, v_{\Lambda+\alpha} = 0$ since

$$|\Lambda+\alpha+\gamma|^2 - |\Lambda+\alpha|^2 = 2\langle \Lambda+\alpha, \gamma\rangle + |\gamma|^2 = 2\langle \Lambda, \gamma\rangle + |\gamma|^2 > 0$$

shows $\Lambda+\alpha+\gamma$ cannot be a weight of $\tau_{\Lambda_1^+}$.

Next suppose that $\gamma \perp \alpha$ but $\gamma \perp\!\!\!\not\perp \alpha$. By Lemma 5.4 of [15], we can take $\tilde{X}_\gamma = [X_\gamma, X_\alpha] - [X_\gamma, X_{-\alpha}]$. Then $\tau_{\Lambda_1^+}[X_\gamma, X_\alpha]\, v_{\Lambda+\alpha} = 0$ since

$$\begin{aligned}
|(\Lambda+\alpha)+(\gamma+\alpha)|^2 - |\Lambda+\alpha|^2 &= 2\langle \Lambda+\alpha, \gamma+\alpha\rangle + |\gamma+\alpha|^2 \\
&= 2\langle \Lambda, \gamma+\alpha\rangle + 2\,|\alpha|^2 + |\gamma+\alpha|^2 \\
&> 0
\end{aligned}$$

shows $(\Lambda+\alpha)+(\gamma+\alpha)$ cannot be a weight of $\tau_{\Lambda_1^+}$. Consequently

$$\tau_{\Lambda_1^+}(\tilde{X}_\gamma)\, v_{\Lambda+\alpha} = -\tau_{\Lambda_1^+}[X_\gamma, X_{-\alpha}]\, v_{\Lambda+\alpha}. \tag{3.17}$$

To complete the proof of our assertion, we show that the right side of (3.17) is 0 if and only if $\langle \Lambda, \gamma-\alpha\rangle > 0$. (Note $\gamma-\alpha$ is positive since γ is positive and α is simple.) If $\langle \Lambda, \gamma-\alpha\rangle = 0$, then $s_{\gamma-\alpha}(\Lambda+\alpha) = \Lambda+\gamma$ shows that $\Lambda+\gamma$ is a weight, and the right side of (3.17) is nonzero as a result of standard facts about weight strings. If $\langle \Lambda, \gamma-\alpha\rangle > 0$, then

$$|\Lambda+\gamma|^2 - |\Lambda+\alpha|^2 = 2\langle \Lambda, \gamma-\alpha\rangle > 0$$

shows $(\Lambda+\alpha)+(\gamma-\alpha)=\Lambda+\gamma$ is not a weight of Λ_1^+, and thus the right side of (3.16) is 0. This proves our assertion.

To prove (c), it is therefore enough to prove that there exists no M-compact positive root γ with $\gamma \perp \alpha$, $\gamma \perp\!\!\!\!/\!\!\perp \alpha$, and $\langle \Lambda, \gamma-\alpha\rangle=0$. Assume the contrary. Since $\gamma+\alpha$ is in Δ_K^+, we have

$$0 \leqslant \langle \Lambda, \gamma+\alpha\rangle=\langle \Lambda, \gamma-\alpha\rangle+2\langle \Lambda, \alpha\rangle=2\langle \Lambda, \alpha\rangle.$$

Thus $\langle \Lambda, \alpha\rangle \geqslant 0$. Meanwhile, we have seen that $\Lambda+\gamma$ is a weight of $\tau_{\Lambda_1^+}$. Also $\gamma+\alpha$ is a member of Δ_K^+ with

$$\langle \Lambda+\gamma, \gamma+\alpha\rangle=\langle \Lambda, \gamma-\alpha\rangle+2\langle \Lambda, \alpha\rangle+|\gamma|^2>2\langle \Lambda, \alpha\rangle,$$

and we know the right side is $\geqslant 0$. Thus $(\Lambda+\gamma)-(\gamma+\alpha)=\Lambda-\alpha$ is a weight, in contradiction to hypothesis. We conclude that τ_λ occurs in $\tau_{\Lambda_1^+}$ if $\Lambda-\alpha$ is not a weight of $\tau_{\Lambda_1^+}$.

Suppose $\langle \Lambda, \alpha\rangle<0$. Then

$$|\Lambda-\alpha|^2-|\Lambda+\alpha|^2=-2\langle \Lambda, \alpha\rangle>0$$

shows $\Lambda-\alpha$ cannot be a weight of $\tau_{\Lambda_1^+}$. If $\langle \Lambda, \alpha\rangle=0$ and $\Lambda-\alpha$ is a weight, then the implication (f) implies (d) implies (a) in Lemma 3.1 (which does not need the hypotheses about F_4 or nondegeneracy) says α and $-\alpha$ are indeed conjugate via $W_{K,\Lambda}$, in contradiction to hypothesis. Thus $\langle \Lambda, \alpha\rangle \leqslant 0$ implies $\Lambda-\alpha$ is not a weight of $\tau_{\Lambda_1^+}$.

(d) If $\langle \Lambda, \alpha\rangle \leqslant 0$, we have just seen that $\Lambda-\alpha$ is not a weight of $\tau_{\Lambda_1^+}$. For $n>0$, we have

$$|\Lambda-(2n+1)\,\alpha|^2-|\Lambda+\alpha|^2=-2\langle \Lambda, (2n+2)\,\alpha\rangle+(4n^2+4n)|\alpha|^2>0,$$

and thus $\Lambda-(2n+1)\,\alpha$ cannot be a weight.

(e) If we write n for $-n$ in (3.15), then the derivation of (3.10) shows that the weight

$$\mu'=\Lambda-(2n+1)\,\alpha$$

arises in tandem with an identity

$$-(2n+1)\,\alpha=\delta^+-\sum_{\gamma\in\Delta_K^+} k_\gamma\gamma, \qquad k_\gamma \geqslant 0.$$

Applying s_α to this identity, we have

$$s_\alpha\delta^+=(2n+1)\,\alpha+\sum_{\gamma\in\Delta_K^+} k_\gamma(s_\alpha\gamma), \qquad k_\gamma \geqslant 0.$$

Here $s_\alpha\delta^+$ and $s_\alpha\gamma$ are positive roots (unless $\delta^+=\alpha$, in which case there is no difficulty). If we refine this expansion into an expansion of $s_\alpha\delta^+$ in terms of simple roots, we see that the simple root α occurs at least 3 times if $n>0$. But a simple root does not occur more than twice in a root expansion within a classical group. This proves the first assertion in (e).

For the second assertion, suppose $\Lambda-\alpha$ is a weight of $\tau_{\Lambda_1^+}$. Then we have

$$\Lambda-\alpha\in\Lambda+\delta^+-\sum\Delta_K^+,$$

as well as

$$\Lambda+\alpha\in\Lambda+\delta^+-\sum\Delta_K^+.$$

Adding these relations, we see that $2\delta^+$ is in $\sum\Delta_K^+$. This completes the proof of the proposition.

The point of Theorem 3.3 and Proposition 3.4 is to address the first term $[\tau_{(\Lambda+\alpha)^\vee}|_{K\cap M^\#}:\tau_\lambda]$ on the right side of (3.3) when $\tau_{\Lambda'}=\tau_{(\Lambda+\alpha)^\vee}$. The following considerations allow us sometimes to show that the remaining terms are 0.

We return to a general superorthogonal set $\{\alpha_1,\ldots,\alpha_l\}$ in place of $\{\alpha\}$. Suppose that $\tau_{\Lambda'}$ occurs in $U(v)|_K$ because of some $\tau_{\lambda'}\subseteq\tau_{\Lambda'}|_{K\cap M^\#}$ with $\lambda'\neq\lambda$. Then

$$\lambda'=\lambda+\sum_{\beta\in\Delta^+_{-,n}}k_\beta\beta,$$

and the sum is nonzero. For suitable real numbers s_j, it must be true that $\lambda'+\sum s_j\alpha_j$ is a weight of $\tau_{\Lambda'}$, hence is of the form

$$\lambda'+\sum s_j\alpha_j=\Lambda'-\sum_{\gamma\in\Delta_K^+}n_\gamma\gamma.$$

Hence

$$\Lambda'=\lambda+\sum_{\beta\in\Delta^+_{-,n}}k_\beta\beta+\sum_{\gamma\in\Delta_K^+}n_\gamma\gamma+\sum s_j\alpha_j.$$

If we write $\lambda=\Lambda+\sum t_j\alpha_j$, then we find

$$\Lambda'=\Lambda+\sum_{\beta\in\Delta^+_{-,n}}k_\beta\beta+\sum_{\gamma\in\Delta_K^+}n_\gamma\gamma+\sum(s_j+t_j)\,\alpha_j.$$

Any two weights of $U(v)$ must differ by a member of the root lattice (relative to Δ), and it follows that each s_j+t_j is an integer. Thus we must have

$$\boxed{\Lambda' = \Lambda + \sum_{\beta \in \Delta^+_{-,n}} k_\beta \beta + \sum_{\gamma \in \Delta_K^+} n_\gamma \gamma + \sum_{j=1}^{l} m_j \alpha_j} \tag{3.18}$$

with integer coefficients, $k_\beta \geqslant 0$, $n_\gamma \geqslant 0$, and some $k_\beta > 0$.

PROPOSITION 3.5. *If Λ' is integral and Δ_K^+ dominant and if $\Lambda' - \Lambda$ lies in the linear span of a real-rank l subsystem $\Delta_L \subseteq \Delta$ that is generated by Δ^+ simple roots and contains $\alpha_1, \ldots, \alpha_l$, then*

$$[U(v)|_K : \tau_{\Lambda'}] = [\tau_{\Lambda'}|_{K \cap M^\#} : \tau_\lambda].$$

Proof. By (3.18), $\sum k_\beta \beta + \sum n_\gamma \gamma$ lies in the span of Δ_L. Since each term is positive and since Δ_L is spanned by simple roots, each such β lies in Δ_L. Since Δ_L has real rank l and since $\alpha_1, \ldots, \alpha_l$ are in Δ_L, the M corresponding to Δ_L is compact. Thus $\Delta^+_{-,n} \cap \Delta_L$ is empty, and there are no nontrivial solutions to (3.18). The proposition follows since $n_\lambda = 1$ in (3.3).

COROLLARY 3.6. *Suppose $l = 1$ and $\{\alpha_1, \ldots, \alpha_l\} = \{\alpha\}$. If $\Lambda_1^+ = (\Lambda + \alpha)^\vee$ equals $\Lambda + \delta^+$ and if the δ^+ subgroup has real rank one, then*

$$[U(v)|_K : \tau_{\Lambda_1^+}] = [\tau_{\Lambda_1^+}|_{K \cap M^\#} : \tau_\lambda].$$

If $\Lambda_1^- = (\Lambda - \alpha)^\vee$ equals $\Lambda + \delta^-$ and if the δ^- subgroup has real rank one, then

$$[U(v)|_K : \tau_{\Lambda_1^-}] = [\tau_{\Lambda_1^-}|_{K \cap M^\#} : \tau_\lambda].$$

4. NECESSARY CONDITIONS FOR UNITARITY

Continuing with notation as in Section 2, we recall the techniques of [1, 2] for proving nonunitarity. Since G is connected and rank G = rank K, we can deduce from [13] and from [12, p. 478] that there exists a unique family of intertwining operators $T(v)$ with the following properties:

(1) $T(v)$ is defined for Re v in a neighborhood of the closed positive Weyl chamber of the dual $\mathfrak{a}'$ of $\mathfrak{a}$.

(2) For each Λ', $T(v)$ carries the $\tau_{\Lambda'}$ K type for $U(v)$ into the $\tau_{\Lambda'}$ K type for $U(-v)$, varies holomorphically in v, and satisfies

$$U(-v, X)\, T(v) = T(v)\, U(v, x)$$

for all X in $\mathfrak{g}^{\mathbb{C}}$.

(3) $T(v)$ is the identity on the τ_Λ K type.

For Re ν in the open positive Weyl chamber, $U(\nu)$ has a unique irreducible quotient $J(\nu)$, and $J(\nu)$ contains the τ_Λ K type with multiplicity one. For Re ν on the edge of the positive Weyl chamber, the uniqueness of $J(\nu)$ may break down; but there exists a unique irreducible quotient of $U(\nu)$ containing the τ_Λ K type, and this we take to be $J(\nu)$. If ν is real-valued, $J(\nu)$ admits an invariant Hermitian form, unique up to a real scalar; this form lifts to $U(\nu)$, where it is given by

$$\langle f, g\rangle = (T(\nu) f, g)_{L^2(K)}. \tag{4.1}$$

Since the normalization (3) makes $T(\nu)$ positive definite on the τ_Λ K type, (4.1) shows that $J(\nu)$ will fail to be infinitesimally unitary for some real ν in the closed positive Weyl chamber where $J(\nu)$ is the unique irreducible quotient if we can produce a K type $\tau_{\Lambda'}$ such that $T(\nu)$ fails to be positive semidefinite on that K type.

The papers [1, 2] introduce two techniques for finding such a Λ'. Both use the following definitions. If τ_{Λ_1} is an irreducible representation of K, we let P_{Λ_1} be the projection of the induced space to the τ_{Λ_1} subspace given by

$$P_{\Lambda_1} f(k_0) = d_{\Lambda_1} \int_K \overline{\chi_{\Lambda_1}(k)}\, f(k^{-1}k_0)\, dk. \tag{4.2}$$

Here d_{Λ_1} is the degree of τ_{Λ_1}, and χ_{Λ_1} is the character. Next if h is any scalar-valued function on K and ω is an integral form on $\mathfrak{b}$, we let h_ω be the $-\omega$ Fourier component of h under the action of B on the right:

$$h(k)_\omega = \int_B h(kb)\, \xi_\omega(b)\, db, \tag{4.3}$$

where ξ_ω is the character of B corresponding to ω.

Fix f_0 in the induced space to be a nonzero highest weight vector for the minimal K type τ_Λ. If v_0 denotes a nonzero highest weight vector in an abstract representation space V^Λ of K of type τ_Λ, then f_0 is necessarily of the form

$$f_0(k) = A\tau_\Lambda(k)^{-1}\, v_0 \tag{4.4}$$

for a unique operator A in $\mathrm{Hom}_{K\cap M^\#}(V^\Lambda, V^{\sigma^\#})$. It follows from the remarks after (2.9) that there exists a unique element u_0 in $V^{\sigma^\#}$ of weight λ in the τ_λ subspace such that

$$A^* u_0 = v_0. \tag{4.5}$$

We fix this element u_0.

Fix ω integral on $\mathfrak{b}$, let $\tau_{\Lambda_1}, \ldots, \tau_{\Lambda_n}$ be irreducible representations of K, and let $X_1, \ldots, X_n$ be in $\mathfrak{g}^{\mathbb{C}}$. Define

$$a(\nu, k) = \langle P_{\Lambda_n} U(\nu, X_n)\, P_{\Lambda_{n-1}} \cdots P_{\Lambda_1} U(\nu, X_1) f_0(k), u_0 \rangle_\omega, \qquad (4.6)$$

the inner product being taken in V^{σ^*}. Let $b(\nu, k)$ be the special case

$$\begin{aligned} b(\nu, k) = \langle & P_\Lambda U(\nu, \bar{X}_1)\, P_{\Lambda_1} \cdots P_{\Lambda_{n-1}} U(\nu, \bar{X}_n) \\ & \times P_{\Lambda_n} U(\nu, X_n)\, P_{\Lambda_{n-1}} \cdots P_{\Lambda_1} U(\nu, X_1) f_0(k), u_0 \rangle_\Lambda. \end{aligned} \qquad (4.7)$$

THEOREM 4.1. *Let ν be real-valued and be in the closed positive Weyl chamber.*

(a) *Suppose τ_{Λ_n} has multiplicity one in $U(\nu)$ and $a(\nu, k)$ is not identically* 0 *as a function of k in K. Then the quotient*

$$c(\nu) = a(-\nu, k)/a(\nu, k)$$

is independent of k. Also $c(\nu) < 0$ implies that $T(\nu)$ is not positive semidefinite on the τ_{Λ_n} K type.

(b) *Regardless of whether τ_{Λ_n} has multiplicity one in $U(\nu)$, $(-1)^n b(-\nu, 1) < 0$ implies that $T(\nu)$ is not positive semidefinite on the τ_{Λ_n} K type.*

Part (a) is proved in [1] in the special case $n = 1$. In Section 1 of [2], both parts of this theorem are proved in full generality.

5. GENERAL ONE-STEP FORMULA

The main result of this section, Theorem 5.1, will give information about the quantities $a(\nu, k)$ and $b(\nu, k)$ in Theorem 4.1 in the case that one advances by one step in the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$ (for example, from $a(\nu, k)$ with $n-1$ steps to $a(\nu, k)$ with n steps). Many of the ideas are already present in the proof of this theorem, and the proof will serve as a model for a number of later results. We continue with notation as in Section 2, and we introduce K-invariant inner products on finite-dimensional representation spaces of K as needed.

THEOREM 5.1. *Fix an index r with $1 \leqslant r \leqslant l$, an integral form μ' on* $\mathfrak{b}$, *and a choice of a sign $\pm$. Let $\Lambda' = (\mu')^\vee$ and $\Lambda'' = (\mu' \pm \alpha_r)^\vee$. Fix a nonzero vector v' of weight μ' in $\tau_{\Lambda'}$, and for each ν, let $B(\nu)$ be a member of* $\mathrm{Hom}_{K \cap M^*}(V^{\Lambda'}, V^{\sigma^*})$. *Let f_1 be the member of the induced space given by*

$$f_1(k) = B(\nu)\, \tau_{\Lambda'}(k)^{-1} v'.$$

Suppose that

(a) *the only weight in* $\tau_{\Lambda'}$ *of the form* $\mu' \pm \alpha_r + \alpha_j$ *or* $\mu' \pm \alpha_r - \alpha_j$ *is* μ' *itself,*

(b) $\tau_{\Lambda''}$ *occurs in* $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$, *so that* $v'' = E_{\Lambda''}(v' \otimes X_{\pm\alpha_r})$ *is not zero,*

(c) *Eq.* (3.18) *for* $\Lambda' - \Lambda$ *has no solutions with* $\sum_{\beta \in \Delta^+_{-,n}} k_\beta \beta$ *nonzero.*

Then

$$\frac{\langle P_{\Lambda''} U(v, X_{\pm\alpha_r}) f_1(k), u_0 \rangle_{\mu' \pm \alpha_r}}{\langle \tau_{\Lambda''}(k)^{-1} v'', v'' \rangle} = \frac{|\alpha_r|^2}{4} d(v) \frac{\langle f_1(k), u_0 \rangle_{\mu'}}{\langle \tau_{\Lambda'}(k)^{-1} v', v' \rangle},$$

where the subscripts $\mu' \pm \alpha_r$ *and* μ' *are as in* (4.3), u_0 *is as in* (4.5), *and* $d(v)$ *is given by*

$$\begin{aligned} d(v) = {} & (v+\rho)(X_{\alpha_r} + X_{-\alpha_r}) + \frac{2\langle \mu', \pm\alpha_r \rangle}{|\alpha_r|^2} \\ & - 2\# \Big\{ \beta \in \Delta_n \mid \beta \perp\!\!\perp \alpha_1, ..., \alpha_{r-1}; \\ & \qquad \beta \mp \alpha_r \in \Delta, |\beta|^2 \geqslant |\alpha_r|^2, \frac{2\langle \mu', \beta \mp \alpha_r \rangle}{|\beta \mp \alpha_r|^2} \geqslant 1 \Big\} \\ & - 2\# \Big\{ \beta \in \Delta_n \mid \beta \perp\!\!\perp \alpha_1, ..., \alpha_{r-1}; \\ & \qquad \beta \mp \alpha_r \in \Delta, |\beta|^2 < |\alpha_r|^2, \frac{2\langle \mu', \beta \mp \alpha_r \rangle}{|\beta \mp \alpha_r|^2} \geqslant 2 \Big\} \\ & - \# \Big\{ \beta \in \Delta_n \mid \beta \perp\!\!\perp \alpha_1, ..., \alpha_{r-1}; \\ & \qquad \beta \mp \alpha_r \in \Delta, |\beta|^2 < |\alpha_r|^2, \frac{2\langle \mu', \beta \mp \alpha_r \rangle}{|\beta \mp \alpha_r|^2} = 1 \Big\}. \end{aligned}$$

Remarks. (1) In (b) the fact that $v'' \neq 0$ follows from Theorem 1.5.

(2) If all noncompact roots have the same length, then (b) is automatically satisfied, according to Corollary 1.4a.

(3) Condition (c) is implied by the condition

(c′) there exists a system $\Delta_{L'} \subseteq \Delta$ generated by Δ^+ simple roots such that

(i) $\alpha_1, ..., \alpha_l$ are in $\Delta_{L'}$, and $\Delta_{L'}$ has real rank exactly l

(ii) $\Lambda' - \Lambda$ is an integral linear combination of roots in $\Delta_{L'}$.

To see the implication, suppose (3.18) has a nontrivial solution. Putting $\sum m_j\alpha_j$ on the correct side of the equation, we see from (c′) that $\sum_{\beta\in\Delta^+_{-,n}} k_\beta\beta + \sum_{\gamma+\Delta^+_K} n_\gamma\gamma$ is in $\Delta_{L'}$. Since $\Delta_{L'}$ is generated by Δ^+ simple roots, each β with $k_\beta > 0$ is in $\Delta_{L'}$. But $\Delta^+_{-,n} \cap \Delta_{L'}$ is empty since $\Delta_{L'}$ has real rank l and contains $\alpha_1, ..., \alpha_l$.

Preliminaries for the proof. In the proof we shall use formulas relating the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ (relative to $\mathfrak{b}^{\mathbb{C}}$) and the Iwasawa-like decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus (\mathfrak{m} \cap \mathfrak{p})^{\mathbb{C}} \oplus \mathfrak{a}^{\mathbb{C}} \oplus \mathfrak{n}^{\mathbb{C}}. \tag{5.1}$$

Some of these formulas are taken from Section 5 of [15], and the rest are from the preprint version of Blank [6]. Let $P_{\mathfrak{k}}$, $P_{\mathfrak{m}}$, $P_{\mathfrak{m}}$, $P_{\mathfrak{a}}$, and $P_{\mathfrak{n}}$ be the projections to the respective summands in (5.1). Here $P_{\mathfrak{m}}$ and $P_{\mathfrak{a}}$ are orthogonal projections, but $P_{\mathfrak{k}}$ and $P_{\mathfrak{n}}$ are not. For $1 \leqslant j \leqslant l$, we have

$$\begin{aligned} &P_{\mathfrak{k}}(X_{\alpha_j}) = \tfrac{1}{2}H_{\alpha_j} \qquad \text{and} \qquad P_{\mathfrak{k}}(X_{-\alpha_j}) = -\tfrac{1}{2}H_{\alpha_j} \\ &P_{\mathfrak{a}}(X_{\alpha_j}) = P_{\mathfrak{a}}(X_{-\alpha_j}) = \tfrac{1}{2}(X_{\alpha_j} + X_{-\alpha_j}) \\ &P_{\mathfrak{m}}(X_{\alpha_j}) = P_{\mathfrak{m}}(X_{-\alpha_j}) = 0. \end{aligned} \tag{5.2}$$

If β is a noncompact root with $\beta \perp\!\!\!\perp \alpha_1, ,..., \alpha_l$, then

$$P_{\mathfrak{m}}(X_\beta) = X_\beta \qquad \text{and} \qquad P_{\mathfrak{k}}(X_\beta) = P_{\mathfrak{a}}(X_\beta) = 0. \tag{5.3}$$

Suppose β is a noncompact root with $\beta \perp\!\!\!\perp \alpha_1, ..., \alpha_{j-1}$ and $\beta \not\perp\!\!\!\perp \alpha_j$ and $\beta \neq \pm\alpha_j$. If the α_j root string through β is

$$\beta - p\alpha_j, ..., \beta + q\alpha_j, \tag{5.4a}$$

then

$$P_{\mathfrak{k}}(X_\beta) = -\frac{1}{p+q}([X_{-\alpha_j}, X_\beta] + [X_{\alpha_j}, X_\beta]) \tag{5.4b}$$

$$P_{\mathfrak{a}}(X_\beta) = 0. \tag{5.4c}$$

In this situation

$$P_{\mathfrak{m}}(X_\beta) = \begin{cases} \tfrac{1}{2}X_\beta - \tfrac{1}{4}[X_{-\alpha_j}, [X_{-\alpha_j}, X_\beta]] & \text{when} \quad p = 0, q = 2 \\ \tfrac{1}{2}X_\beta - \tfrac{1}{4}[X_{\alpha_j}, [X_{\alpha_j}, X_\beta]] & \text{when} \quad p = 2, q = 0 \\ 0 & \text{when} \quad p \leqslant 1, q \leqslant 1. \end{cases} \tag{5.4d}$$

The next ingredient in the proof is a formula for ad $X_{-\alpha_j}$ ad X_{α_j} and for ad X_{α_j} ad $X_{-\alpha_j}$. Let p and q be as in (5.4a). Then

$$[X_{-\alpha_j}, [X_{\alpha_j}, X_\gamma]] = q(p+1)\, X_\gamma \tag{5.5a}$$

$$[X_{\alpha_j}, [X_{-\alpha_j}, X_\gamma]] = p(q+1)\, X_\gamma \tag{5.5b}$$

by [8, p. 143].

The third ingredient in the proof is a start at a computation of $U(\nu, X) f_1(k)$, where f_1 is in the space of the induced representation and is given on K by the formula

$$f_1(k) = B(\nu)\, \tau_{\Lambda'}(k)^{-1}\, v' \tag{5.6}$$

with $B(\nu)$ in $\mathrm{Hom}_{K \cap M^\#}(V^{\Lambda'}, V^{\sigma^\#})$. If $c(t)$ denotes any curve in G with $c(0) = 1$ and with tangent vector $X \in \mathfrak{g}$ at $t = 0$, then

$$U(\nu, X) f_1(k) = \frac{d}{dt} f_1(c(t)^{-1} k)|_{t=0} = \frac{d}{dt} f_1(k \cdot k^{-1} c(t)^{-1} k)|_{t=0}.$$

For $c(t)$, let us put $Y = \mathrm{Ad}(k)^{-1} X$ and take

$$c(t) = k \exp(tP_{\mathfrak{k}}\, Y) \exp(tP_{\mathfrak{m}}\, Y) \exp(tP_{\mathfrak{a}}\, Y) \exp(tP_{\mathfrak{n}}\, Y)\, k^{-1}.$$

Because of the transformation law of f_1 under $M^\# AN$ on the right, we obtain

$$\begin{aligned} U(\nu, X) f_1(k) &= \frac{d}{dt} \{ e^{(\nu+\rho)}(\exp(tP_{\mathfrak{a}}\, Y))\, \sigma^\#(\exp tP_{\mathfrak{m}}\, Y) \\ &\qquad \times f_1(k(\exp tP_{\mathfrak{k}}\, Y)^{-1})\}_{t=0} \\ &= (\nu+\rho)(P_{\mathfrak{a}}\, Y) f_1(k) + \sigma^\#(P_{\mathfrak{m}}\, Y) f_1(k) \\ &\qquad + \frac{d}{dt} f_1(k(\exp tP_{\mathfrak{k}}\, Y)^{-1})|_{t=0}. \end{aligned}$$

Now we use (5.6) to write

$$f_1(k(\exp tP_{\mathfrak{k}}\, Y)^{-1}) = B(\nu)\, \tau_{\Lambda'}(\exp tP_{\mathfrak{k}}\, Y)\, \tau_{\Lambda'}(k)^{-1}\, v'.$$

Differentiating and again substituting from (5.6), we conclude

$$\begin{aligned} U(\nu, X) f_1(k) &= [(\nu+\rho)(P_{\mathfrak{a}}\, Y)]\, B(\nu)\, \tau_{\Lambda'}(k)^{-1}\, v' \\ &\qquad + [\sigma^\#(P_{\mathfrak{m}}\, Y)]\, B(\nu)\, \tau_{\Lambda'}(k)^{-1}\, v' \\ &\qquad + B(\nu)\, \tau_{\Lambda'}(P_{\mathfrak{k}}\, Y)\, \tau_{\Lambda'}(k)^{-1}\, v', \end{aligned} \tag{5.7}$$

and this relation extends by linearity to $X \in \mathfrak{g}^{\mathbb{C}}$. We shall refer to the three terms on the right side of (5.7) as the $\mathfrak{a}$ term, the $\mathfrak{m}$ term, and the $\mathfrak{k}$ term.

Proof of Theorem 5.1. From (2.1) and (2.4), we have $|X_\beta|^2 = 2/|\beta|^2$ for every root β. Therefore the vectors $(\frac{1}{2}|\beta|^2) X_\beta$, $\beta \in \Delta_n$, form an orthonormal basis of $\mathfrak{p}^{\mathbb{C}}$. If X is in $\mathfrak{p}^{\mathbb{C}}$, we therefore have

$$Y = \mathrm{Ad}(k)^{-1} X = \sum_{\beta \in \Delta_n} \tfrac{1}{2}|\beta|^2 \langle \mathrm{Ad}(k)^{-1} X, X_\beta \rangle X_\beta. \tag{5.8}$$

Let us compute the $\mathfrak{a}$ term of (5.7). We have $P_{\mathfrak{a}} X_\beta = 0$ unless $\beta = \pm\alpha_j$ for some j, and then (5.2) applies. Hence

$$P_{\mathfrak{a}} Y = \sum_{j=1}^{l} \tfrac{1}{4}|\alpha|^2 \langle \mathrm{Ad}(k)^{-1} X, X_{\alpha_j} + X_{-\alpha_j} \rangle (X_{\alpha_j} + X_{-\alpha_j})$$

and

$$\begin{aligned} \langle \mathfrak{a}\text{ term}, u_0 \rangle &= \sum_{j=1}^{l} \tfrac{1}{4}|\alpha_j|^2 [(\nu+\rho)(X_{\alpha_j} + X_{-\alpha_j})] \\ &\qquad \times \langle B(\nu)\, \tau_{\Lambda'}(k)^{-1} v', u_0 \rangle \langle \mathrm{Ad}(k)^{-1} X, X_{\alpha_j} + X_{-\alpha_j} \rangle \\ &= \sum_{j=1}^{l} \tfrac{1}{4}|\alpha_j|^2 [(\nu+\rho)(X_{\alpha_j} + X_{-\alpha_j})] \\ &\qquad \times \langle \pi(k)^{-1} (v' \otimes X), B(\nu)^* u_0 \otimes (X_{\alpha_j} + X_{-\alpha_j}) \rangle, \end{aligned}$$

where $\pi = \tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$. To this function of k, we apply the projection operator $P_{\Lambda''}$ of (4.2), and we see that the effect is to project $\pi(k)^{-1}(v' \otimes X)$ according to $\tau_{\Lambda''}$. Hence

$$\begin{aligned} \langle P_{\Lambda''}(\mathfrak{a}\text{ term})(k_0), u_0 \rangle &= \sum_{j=1}^{l} \tfrac{1}{4}|\alpha_j|^2 [(\nu+\rho)(X_{\alpha_j} + X_{-\alpha_j})] \\ &\qquad \times \langle E_{\Lambda''}(v' \otimes X), \pi(k_0)(B(\nu)^* u_0 \otimes (X_{\alpha_j} + X_{-\alpha_j})) \rangle. \end{aligned}$$

We put $X = X_{\pm\alpha_r}$ and apply the subscript $\mu' \pm \alpha_r$ as in (4.3). Only terms in $B(\nu)^* u_0 \otimes (X_{\alpha_j} + X_{-\alpha_j})$ of weight $\mu' \pm \alpha_r$ survive this operation. Thus the surviving terms have

$$(\text{weight of } B(\nu)^* u_0) + \alpha_j = \mu' \pm \alpha_r$$

or

$$(\text{weight of } B(\nu)^* u_0) - \alpha_j = \mu' \pm \alpha_r,$$

and assumption (a) allows us to conclude that the surviving terms have $j = r$, $\pm\alpha_j = \pm\alpha_r$, and

$$(\text{weight of } B(\nu)^* u_0) = \mu'.$$

Now μ' is an extreme weight of $\tau_{\Lambda'}$, and v' is a nonzero weight vector for this weight. Thus we can write

$$B(v)^* u_0 = |v'|^{-2} \langle B(v)^* u_0, v' \rangle v' + \text{irrelevant terms}, \tag{5.9a}$$

and we abbreviate the first term as

$$\overline{a(v)}\, v'. \tag{5.9b}$$

Thus

$$\begin{aligned} &\langle P_{\Lambda''}(\mathfrak{a}\text{ term})(k_0), u_0 \rangle_{\mu' \pm \alpha_r} \\ &\quad = a(v)\, \tfrac{1}{4}\, |\alpha_r|^2 \langle E_{\Lambda''}(v' \otimes X_{\pm\alpha_r}), \pi(k_0)(v' \otimes X_{\pm\alpha_r}) \rangle \\ &\quad\quad \times \{(v+\rho)(X_{\alpha_r} + X_{-\alpha_r})\}. \end{aligned} \tag{5.10}$$

Next let us compute the $\mathfrak{m}$ term of (5.7), showing that it makes no contribution. Using (5.8), we have

$$\begin{aligned} \langle \mathfrak{m}\text{ term}, u_0 \rangle &= \sum_{\beta \in \Delta_n} \tfrac{1}{2} |\beta|^2 \langle \tau_{\Lambda'}(k)^{-1} v', B(v)^* \sigma^{\#}(P_{\mathfrak{m}} X_\beta)^* u_0 \rangle \\ &\quad \times \langle \mathrm{Ad}(k)^{-1} X, X_\beta \rangle \\ &= \sum_{\beta \in \Delta_n} \tfrac{1}{2} |\beta|^2 \langle \pi(k)^{-1} (v' \otimes X), B(v)^* \sigma^{\#}(P_{\mathfrak{m}} X_\beta)^* u_0 \otimes X_\beta \rangle. \end{aligned}$$

To see that the $\mathfrak{m}$ term does not contribute, we shall show that

$$B(v)^* \sigma^{\#}(P_{\mathfrak{m}} X_\beta)^* u_0 = 0 \tag{5.11}$$

for all $\beta \in \Delta_n$. We may assume that $B(v) \neq 0$. In this case Frobenius reciprocity says that $\tau_{\Lambda'}$ occurs in $U(v)$. On the other hand, assumption (c) and Eq. (3.18) say that no $\tau_{\lambda'} \subseteq \sigma|_{K \cap M^{\#}}$ occurs in $\tau_{\Lambda'}|_{K \cap M^{\#}}$ other than for $\lambda' = \lambda$. Consequently $\sigma^{\#}(P_{\mathfrak{m}} X_\beta)^* u_0$ has a nonzero component in τ_λ. The roots $\beta \in \Delta_n$ for which $P_{\mathfrak{m}} X_\beta$ is not zero are of two types, given by (5.3) and (5.4d). The first kind has $\beta \perp\!\!\!\perp \alpha_1, ..., \alpha_l$. Then β is M-noncompact, and $\sigma^{\#}(P_{\mathfrak{m}} X_\beta)^* u_0$ has $\mathfrak{b}_-$ weight $\lambda - \beta$ (cf. (2.5)). Thus τ_λ has $\lambda - \beta$ as a weight. But this conclusion is contradictory since β cannot be the sum of M-compact roots (Problems 12 and 13 on p. 478 of [12]); so the first kind of β cannot contribute. The other kind of β has $\beta \pm \alpha_j$ orthogonal to α_j for some j and for some choice of sign. Then $\beta \pm \alpha_j$ is M-noncompact, and $\sigma^{\#}(P_{\mathfrak{m}} X_\beta)^* u_0$ has $\mathfrak{b}_-$ weight $\lambda - (\beta \pm \alpha_j)$. Thus τ_λ has $\lambda - (\beta \pm \alpha_j)$ as a weight, and again we have a contradiction; so the second kind of β cannot contribute. Thus (5.11) holds, and we obtain

$$\langle P_{A''}(\mathfrak{m}\text{ term})(k_0), u_0\rangle_{\mu'\pm\alpha_r}=0. \tag{5.12}$$

Now let us compute the $\mathfrak{k}$ term of (5.7). Arguing as with the $\mathfrak{a}$ term, we obtain

$$\langle P_{A''}(\mathfrak{k}\text{ term})(k_0), u_0\rangle$$

$$=\sum_{\beta\in\Delta_n}\tfrac{1}{2}|\beta|^2\langle E_{A''}(v'\otimes X), \pi(k_0)(\tau_{A'}(P_{\mathfrak{k}}X_\beta)^*B(v)^*u_0\otimes X_\beta)\rangle. \tag{5.13}$$

We put $X=X_{\pm\alpha_r}$ and apply the subscript $\mu'\pm\alpha_r$ as in (4.3). Only terms in $\tau_{A'}(P_{\mathfrak{k}}X_\beta)^*B(v)^*u_0\otimes X_\beta$ of weight $\mu'\pm\alpha_r$ survive this operation. Formulas (2.5) and (5.2)–(5.4) show that $\tau_{A'}(P_{\mathfrak{k}}X_\beta)^*$ reduces weights by $\beta+\alpha_j$ or $\beta-\alpha_j$ for some j. Thus the surviving terms have

$$-(\beta+\alpha_j)+(\text{weight of } B(v)^*u_0)+\beta=\mu'\pm\alpha_r$$

or

$$-(\beta-\alpha_j)+(\text{weight of } B(v)^*u_0)+\beta=\mu'\pm\alpha_r.$$

Assumption (a) allows us to conclude that the surviving terms have $j=r$, a particular choice of sign on α_j, and

$$(\text{weight of } B(v)^*u_0)=\mu'.$$

As with our argument for the $\mathfrak{a}$ term, we can therefore replace $B(v)^*u_0$ in (5.13) by $\overline{a(v)}v'$. Hence

$$\langle P_{A''}(\mathfrak{k}\text{ term})(k_0), u_0\rangle_{\mu'\pm\alpha_r}$$

$$=a(v)\sum_{\substack{\beta\in\Delta_n\\ \beta\perp\perp\alpha_1,\ldots,\alpha_{r-1}\\ \beta\perp\!\!/\!\!\perp\alpha_r}}\tfrac{1}{2}|\beta|^2$$

$$\times\langle E_{A''}(v'\otimes X_{\pm\alpha_r}), \pi(k_0)(\tau_{A'}(P_{\mathfrak{k}}X_\beta)^*v'\otimes X_\beta)\rangle_{\mu'\pm\alpha_r}, \tag{5.14}$$

and the only contribution from $P_{\mathfrak{k}}X_\beta$ comes from a term with $[X_{\mp\alpha_r}, X_\beta]$.

If $\beta=\mp\alpha_r$, we get 0. If $\beta=\pm\alpha_r$, (5.2) gives $P_{\mathfrak{k}}X_{\pm\alpha_r}=\pm\frac{1}{2}H_{\alpha_r}$. From (2.5),

$$\tau_{A'}(P_{\mathfrak{k}}X_{\pm\alpha_r})^*v'=\pm\frac{1}{2}\frac{2\langle\mu',\alpha_r\rangle}{|\pm\alpha_r|^2}.$$

So $\beta = \pm\alpha_r$ contributes to (5.14) an amount

$$a(v)\frac{1}{4}|\alpha_r|^2 \langle E_{\Lambda'}(v' \otimes X_{\pm\alpha_r}), \pi(k_0)(v' \otimes X_{\pm\alpha_r})\rangle \left\{\frac{2\langle \mu', \pm\alpha_r\rangle}{|\alpha_r|^2}\right\}. \tag{5.15}$$

For the remaining β's, the relevant term of $P_{\mathfrak{k}}X_\beta$ is $-(p+q)^{-1}[X_{\mp\alpha_r}, X_\beta]$, by (5.4b); here $\beta - p(\pm\alpha_r), ..., \beta + q(\pm\alpha_r)$ is the $\pm\alpha_r$ root string through β. Thus (2.5c) shows that the relevant term of $\tau_{\Lambda'}(P_{\mathfrak{k}}X_\beta)^* v' \otimes X_\beta$ is

$$+(p+q)^{-1}\tau_{\Lambda'}[X_{\pm\alpha_r}, X_{-\beta}]\, v' \otimes X_\beta. \tag{5.16}$$

Let us show that (5.16) is 0 if

$$\frac{2\langle \mu', \beta \mp \alpha_r\rangle}{|\beta \mp \alpha_r|^2} < 1. \tag{5.17}$$

In fact, under this condition, the computation

$$\begin{aligned} |\mu' - \beta \pm \alpha_r|^2 - |\mu'|^2 &= -2\langle \mu', \beta \mp \alpha_r\rangle + |\beta \mp \alpha_r|^2 \\ &= |\beta \mp \alpha_r|^2 \left(1 - \frac{2\langle \mu', \beta \mp \alpha_r\rangle}{|\beta \mp \alpha_r|^2}\right) > 0 \end{aligned}$$

shows that $\mu' - \beta \pm \alpha_r$ is too long to be a weight of $\tau_{\Lambda'}$. Thus (5.16) is 0 when β satisfies (5.17).

For any of the remaining β's, (5.5a) gives

$$\begin{aligned} &\tau_{\Lambda'}[X_{\pm\alpha_r}, X_{-\beta}]\, v' \otimes X_\beta \\ &\quad = \pi[X_{\pm\alpha_r}, X_{-\beta}](v' \otimes X_\beta) - v' \otimes (\mathrm{ad}[X_{\pm\alpha_r}, X_{-\beta}])\, X_\beta \\ &\quad = \pi[X_{\pm\alpha_r}, X_{-\beta}](v' \otimes X_\beta) - q'(p'+1)\, v' \otimes X_{\pm\alpha_r}, \end{aligned} \tag{5.18}$$

where p' and q' determine the β root string through $\mp\alpha_r$ as $\mp\alpha_r - p'\beta, ..., \mp\alpha_r + q'\beta$. The remaining β's are in one of the three sets

$$\begin{aligned} S_1 = \Big\{&\beta \in \Delta_n \,|\, \beta \perp\!\!\!\perp \alpha_1, ..., \alpha_{r-1}; \\ &\beta \mp \alpha_r \in \Delta,\ |\beta|^2 \geqslant |\alpha_r|^2, \frac{2\langle \mu', \beta \mp \alpha_r\rangle}{|\beta \mp \alpha_r|^2} \geqslant 1\Big\} \end{aligned}$$

$$\begin{aligned} S_2 = \Big\{&\beta \in \Delta_n \,|\, \beta \perp\!\!\!\perp \alpha_1, ..., \alpha_{r-1}; \\ &\beta \mp \alpha_r \in \Delta,\ |\beta|^2 < |\alpha_r|^2, \frac{2\langle \mu', \beta \mp \alpha_r\rangle}{|\beta \mp \alpha_r|^2} \geqslant 2\Big\} \end{aligned}$$

or

$$S_3=\Big\{\beta\in\Delta_n\,|\,\beta\perp\!\!\perp\alpha_1,\dots,\alpha_{r-1};$$

$$\beta\mp\alpha_r\in\Delta,\ |\beta|^2<|\alpha_r|^2,\ \frac{2\langle\mu',\beta\mp\alpha_r\rangle}{|\beta\mp\alpha_r|^2}=1\Big\}.$$

We shall show that

$$E_{\Lambda''}(\pi[X_{\pm\alpha_r},X_{-\beta}](v'\otimes X_\beta))=CE_{\Lambda''}(v'\otimes X_{\pm\alpha_r}),\tag{5.19a}$$

where

$$C=\begin{cases}0 & \text{if}\quad \beta\in S_1 \text{ or } \beta\in S_2\\ 1 & \text{if}\quad \beta\in S_3.\end{cases}\tag{5.19b}$$

Accepting this relationship for the moment, we see from (5.16) that we can replace the βth term of (5.14) in such cases by

$$a(v)\frac{1}{4}|\alpha_r|^2\langle E_{\Lambda''}(v'\otimes X_{\pm\alpha_r}),\pi(k_0)(v'\otimes X_{\pm\alpha_r})\rangle$$
$$\times\left\{\frac{2|\beta|^2}{|\alpha_r|^2}(p+q)^{-1}(C-q'(p'+1))\right\}.\tag{5.20}$$

To verify (5.19) for $\beta\in S_1\cup S_2$, it is enough to check that $\mu'+\beta$ is not a weight of $\tau_{\Lambda''}$. In fact, we have

$$|\mu'+\beta|^2-|\mu'\pm\alpha_r|^2=2\langle\mu',\beta\mp\alpha_r\rangle+|\beta|^2-|\alpha_r|^2$$
$$=|\beta\mp\alpha_r|^2\left(\frac{2\langle\mu',\beta\mp\alpha_r\rangle}{|\beta\mp\alpha_r|^2}+\frac{|\beta|^2-|\alpha_r|^2}{|\beta\mp\alpha_r|^2}\right),\tag{5.21}$$

and this is >0 if β is in S_1 or S_2. Thus (5.19) holds for β in $S_1\cup S_2$. Putting $C=0$ in (5.20), we calculate that the expression in braces in (5.20) is -2 for every possible configuration of root lengths. Thus the roots β in S_1 or S_2 contribute

$$a(v)\tfrac{1}{4}|\alpha_r|^2\langle E_{\Lambda''}(v'\otimes X_{\pm\alpha_r}),\pi(k_0)(v'\otimes X_{\pm\alpha_r})\rangle$$
$$\times\{-2\#\{\beta\in S_1\}-2\#\{\beta\in S_2\}\}.\tag{5.22}$$

To verify (5.19) for $\beta\in S_3$, we make the following preliminary calculation:

$$\begin{aligned}
&[[X_{\mp\alpha_r}, X_\beta], [X_{\pm\alpha_r}, X_{-\beta}]] \\
&\quad = (\operatorname{ad} X_{\mp\alpha_r})[X_\beta, [X_{\pm\alpha_r}, X_{-\beta}]] - (\operatorname{ad} X_\beta)[X_{\mp\alpha_r}, [X_{\pm\alpha_r}, X_{-\beta}]] \\
&\quad = -q'(p'+1)[X_{\mp\alpha_r}, X_{\pm\alpha_r}] - p(q+1)[X_\beta, X_{-\beta}]. \qquad (5.23\text{a})
\end{aligned}$$

Since β is in S_3, we have $p=1$, $q=0$, $p'=0$, $q'=2$. Thus (5.23a) is

$$= 2H_{\pm\alpha_r} - H_\beta = -H_{\beta\mp\alpha_r}, \qquad (5.23\text{b})$$

the last step following from easy computation. Now under our assumption that β is in S_3, we have

$$\begin{aligned}
\frac{2\langle \mu' \pm \alpha_r, \beta \mp \alpha_r\rangle}{|\beta \mp \alpha_r|^2} &= \frac{2\langle \mu', \beta \mp \alpha_r\rangle}{|\beta \mp \alpha_r|^2} + \frac{2\langle \pm\alpha_r, \beta \mp \alpha_r\rangle}{|\beta \mp \alpha_r|^2} \\
&= 1 - 2 = -1 < 0, \qquad (5.24)
\end{aligned}$$

and thus $\mu' + \beta = (\mu' \pm \alpha_r) + (\beta \mp \alpha_r)$ is a weight of $\tau_{\Lambda''}$. The fact that the right side of (5.21) is 0 shows that $\mu' + \beta$ is an extreme weight. Thus $\mu' + \beta$ and $\mu' \pm \alpha_r$ both have multiplicity one in $\tau_{\Lambda''}$. Assumption (b) and Theorem 1.5 show as a consequence that $E_{\Lambda''}(v' \otimes X_\beta)$ and $E_{\Lambda''}(v' \otimes X_{\pm\alpha_r})$ are both nonzero. Taking (5.24) into account, we obtain (5.19a) for some as yet unknown constant C. Now we apply $\pi[X_{\mp\alpha_r}, X_\beta]$ to both sides of (5.19a). Since

$$\begin{aligned}
|(\mu'+\beta) + (\beta \mp \alpha_r)|^2 - |\mu'+\beta|^2 &= 2\langle \mu'+\beta, \beta \mp \alpha_r\rangle + |\beta \mp \alpha_r|^2 \\
&= 2\langle \mu', \beta \mp \alpha_r\rangle + |\beta \mp \alpha_r|^2 > 0,
\end{aligned}$$

$\pi[X_{\mp\alpha_r}, X_\beta]\, E_{\Lambda''}(v' \otimes X_\beta)$ equals 0. So (5.23a) gives

$$\begin{aligned}
&\pi[X_{\mp\alpha_r}, X_\beta]\, \pi[X_{\pm\alpha_r}, X_{-\beta}]\, E_{\Lambda''}(v' \otimes X_\beta) \\
&\quad = \pi[[X_{\mp\alpha_r}, X_\beta], [X_{\pm\alpha_r}, X_{-\beta}]]\, E_{\Lambda''}(v' \otimes X_\beta) \\
&\quad = -\pi(H_{\beta\mp\alpha_r})\, E_{\Lambda''}(v' \otimes X_\beta) \\
&\quad = -\frac{2\langle \mu'+\beta, \beta \mp \alpha_r\rangle}{|\beta \mp \alpha_r|^2}\, E_{\Lambda''}(v' \otimes X_\beta) \\
&\quad = -E_{\Lambda''}(v' \otimes X_\beta). \qquad (5.25)
\end{aligned}$$

Meanwhile (5.19a) gives

$$\begin{aligned}
&\pi[X_{\mp\alpha_r}, X_\beta]\, \pi[X_{\pm\alpha_r}, X_{-\beta}]\, E_{\Lambda''}(v' \otimes X_\beta) \\
&\quad = C E_{\Lambda''}(\pi[X_{\mp\alpha_r}, X_\beta](v' \otimes X_{\pm\alpha_r})). \qquad (5.26\text{a})
\end{aligned}$$

When we expand the right side, the $\tau_{\Lambda'}$ term gives 0 since

$$|\mu'+\beta\mp\alpha_r|^2-|\mu'|^2=2\langle\mu',\beta\mp\alpha_r\rangle+|\beta\mp\alpha_r|^2>0.$$

Thus (5.26a) is

$$\begin{aligned}
&=CE_{\Lambda''}(v'\otimes \operatorname{ad}[X_{\mp\alpha_r},X_\beta]\,X_{\pm\alpha_r})\\
&=-Cp(q+1)\,E_{\Lambda''}(v'\otimes X_\beta)\\
&=-CE_{\Lambda''}(v'\otimes X_\beta).
\end{aligned}\tag{5.26b}$$

Comparing (5.25) and (5.26), we obtain $C=1$, as asserted in (5.19b). Putting $C=1$ in (5.20) and using our known values for p, q, p', and q', we see that the expression in braces in (5.20) is -1. Thus the roots β in S_3 contribute

$$a(v)\tfrac{1}{4}|\alpha_r|^2\,\langle E_{\Lambda''}(v'\otimes X_{\pm\alpha_r}),\pi(k_0)(v'\otimes X_{\pm\alpha_r})\rangle\{-\#\{\beta\in S_3\}\}.\tag{5.27}$$

Finally we add the results of (5.10), (5.12), (5.15), (5.22), and (5.27) to get

$$\begin{aligned}
&\langle P_{\Lambda''}U(v,X_{\pm\alpha_r})f_1(k_0),u_0\rangle\\
&\quad=a(v)\frac{1}{4}|\alpha_r|^2\,\langle E_{\Lambda''}(v'\otimes X_{\pm\alpha_r}),\pi(k_0)(v'\otimes X_{\pm\alpha_r})\rangle\\
&\qquad\times\Big\{(v+\rho)(X_{\alpha_r}+X_{-\alpha_r})+\frac{2\langle\mu',\pm\alpha_r\rangle}{|\alpha_r|^2}-2\#\{\beta\in S_1\}\\
&\qquad-2\#\{\beta\in S_2\}-\#\{\beta\in S_3\}\Big\}\\
&\quad=\frac{|\alpha_r|^2}{4}\,d(v)\,a(v)\langle E_{\Lambda''}(v'\otimes X_{\pm\alpha_r}),\pi(k_0)(v'\otimes X_{\pm\alpha_r})\rangle\\
&\quad=\frac{|\alpha_r|^2}{4}\,d(v)\,a(v)\langle\tau_{\Lambda''}(k_0)^{-1}v'',v''\rangle,
\end{aligned}$$

with $d(v)$ and v'' as in the statement of the theorem. Assumption (b) in the theorem allows us to divide by $\langle\tau_{\Lambda''}(k_0)^{-1}v'',v''\rangle$, and thus we are to show that

$$a(v)=\langle f_1(k_0),u_0\rangle_{\mu'}/\langle\tau_{\Lambda'}(k_0)^{-1}v',v'\rangle.\tag{5.28}$$

We have

$$\begin{aligned}
\langle f_1(k_0),u_0\rangle_{\mu'}&=\langle B(v)\,\tau_{\Lambda'}(k_0)^{-1}v',u_0\rangle_{\mu'}\\
&=\langle\tau_{\Lambda'}(k_0)^{-1}v',B(v)^*u_0\rangle_{\mu'}.
\end{aligned}\tag{5.29a}$$

Since μ' has multiplicity one in $\tau_{\Lambda'}$, the projection to the μ' weight space of $B(v)^* u_0$ is $|v'|^{-2} \langle B(v)^* u_0, v' \rangle v' = \overline{a(v)}\, v'$, by (5.9). Thus (5.29a) is

$$= \langle \tau_{\Lambda'}(k_0)^{-1} v', \overline{a(v)}\, v' \rangle = a(v) \langle \tau_{\Lambda'}(k_0)^{-1} v', v' \rangle, \qquad (5.29b)$$

and (5.28) follows from (5.29). This completes the proof of Theorem 5.1.

COROLLARY 5.2. *Suppose* $l=1$ *and* $\{\alpha_1, ..., \alpha_l\} = \{\alpha\}$. *Let* $\Lambda' = (\Lambda + \alpha)^\vee$, *and suppose that either* (a), (b), *and* (c) *or* (a'), (b), *and* (c) *hold*:

(a) $\Lambda - \alpha$ *is not a weight of* $\tau_{\Lambda'}$.

(a') $\Lambda - \alpha$ *is conjugate to* $\Lambda + \alpha$ *by the Weyl group* W_K, G *has no factor split* F_4, *and* σ *is nondegenerate in the sense of* [16].

(b) $\tau_{\Lambda'}$ *occurs in* $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$.

(c) Eq. (3.18) *for* $\Lambda' - \Lambda$ *has no solutions with* $\sum_{\beta \in \Delta^+_{-,n}} k_\beta \beta$ *nonzero.*

Then $\tau_{\Lambda'}$ *occurs in* $U(v)|_K$, *and the pair of* K *types* $\{\Lambda, (\Lambda + \alpha)^\vee\}$ *exhibits* $J(\frac{1}{2}c\tilde{\alpha})$ *as not infinitesimally unitary for* $c > v_0^+$.

Remarks. (1) Condition (c) is implied by condition (c') in Remark 3 after Theorem 5.1.

(2) Corollary 5.2 has a dual result obtained by reflection in α. In it we use $\Lambda' = (\Lambda - \alpha)^\vee$. Hypotheses (b), (a'), and (c) are unchanged, while (a) is to say that $\Lambda + \alpha$ is not a weight of $\tau_{\Lambda'}$. The conclusion is that $\{\Lambda, (\Lambda - \alpha)^\vee\}$ detects nonunitarity for $c > v_0^-$.

(3) If $\tau_{\Lambda'}$ is known to have multiplicity at most one in $U(v)$, then the proof will show that (b) and (c) are sufficient without (a) or (a').

Proof. First suppose that (a), (b), and (c) hold. We shall prove non-unitarity by applying Theorem 4.1b. Taking (2.1b) into account, we see that it suffices to show that

$$b(v, k) = \langle P_\Lambda U(v, X_{-\alpha}) P_{\Lambda'} U(v, X_\alpha) f_0(k), u_0 \rangle_\Lambda$$

has $b(-v, 1) > 0$ for the indicated values of v.

We put

$$a(v, k) = \langle P_{\Lambda'} U(v, X_\alpha) f_0(k), u_0 \rangle_{\Lambda + \alpha} \qquad (5.30a)$$

and prepare to apply Theorem 5.1 to $+\alpha$, $\mu' = \Lambda$, v_0, and f_0. Assumption (a) of Theorem 5.1 is that $\Lambda + 2\alpha$ is not a weight of τ_Λ; it holds since $\Lambda - (\Lambda + 2\alpha)$ is not a sum of positive compact roots. Assumption (b) of Theorem 5.1 is assumption (b) of Corollary 5.2, and assumption (c) of Theorem 5.1 is trivial. Thus Theorem 5.1 gives

$$a(v, k) = \frac{|\alpha|^2}{4} d(v) \frac{\langle f_0(k), u_0 \rangle_\Lambda}{\langle \tau_\Lambda(k)^{-1} v_0, v_0 \rangle} \langle \tau_{\Lambda'}(k)^{-1} v', v' \rangle,$$

where $v' = E_{\Lambda'}(v_0 \otimes X_\alpha)$ and $d(v)$ is the expression in Theorem 5.1 with adjusted notation. By (4.4) and (4.5),

$$\begin{aligned}\langle f_0(k), u_0 \rangle &= \langle A\tau_\Lambda(k)^{-1} v_0, u_0 \rangle = \langle \tau_\Lambda(k)^{-1} v_0, A^* u_0 \rangle \\ &= \langle \tau_\Lambda(k)^{-1} v_0, v_0 \rangle.\end{aligned}$$

Also Theorem 2.1c says that $d(v) = v(X_\alpha + X_{-\alpha}) + v_0^+$. Hence

$$a\left(\frac{1}{2} c\tilde{\alpha}, k\right) = \frac{|\alpha|^2}{4} \langle \tau_{\Lambda'}(k)^{-1} v', v' \rangle \{c + v_0^+\}. \tag{5.30b}$$

By Frobenius reciprocity the map $V^{\Lambda'} \otimes_{\mathbb{C}} \operatorname{Hom}_{K \cap M^\#}(V^{\Lambda'}, V^{\sigma^\#})$ into the induced space, given by

$$v \otimes B \to B\tau_{\Lambda'}(k)^{-1} v, \tag{5.31}$$

is one–one onto the $\tau_{\Lambda'}$ K type of the induced space. Put

$$f_1 = P_{\Lambda'} U(v, X_\alpha) f_0.$$

This is a member of the $\tau_{\Lambda'}$ K type, and it has weight $\Lambda + \alpha$, which is extreme for $\tau_{\Lambda'}$. Since $\Lambda + \alpha$ is extreme, it has multiplicity one, and multiples of v' are the only v's that can contribute to the realization of f_1 via (5.31). Thus

$$f_1(k) = B(v)\, \tau_{\Lambda'}(k)^{-1} v' \tag{5.32}$$

for unique members $B(v)$ of $\operatorname{Hom}_{K \cap M^\#}(V^{\Lambda'}, V^{\sigma^\#})$.

We prepare to apply Theorem 5.1 to $-\alpha$, $\mu' = \Lambda + \alpha$, v_1, and f_1. Assumption (a) of Theorem 5.1 is that $\Lambda - \alpha$ is not a weight of $\tau_{\Lambda'}$; it holds by assumption (a) of Corollary 5.2. Applying Theorem 1.3 twice, we see that $\tau_\Lambda \subseteq \tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$ if and only if $\tau_{\Lambda'} \subseteq \tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$; thus assumption (b) of Corollary 5.2 implies assumption (b) of Theorem 5.1. And assumption (c) of Theorem 5.1 is given as assumption (c) of Corollary 5.2. Thus Theorem 5.1 gives

$$b(v, k) = \frac{|\alpha|^2}{4} d(v) \frac{\langle f_1(k), u_0 \rangle_{\Lambda + \alpha}}{\langle \tau_{\Lambda'}(k)^{-1} v', v' \rangle} \langle \tau_{\Lambda''}(k)^{-1} v'', v'' \rangle,$$

where $v'' = E_{\Lambda''}(v' \otimes X_{-\alpha})$ and $d(v)$ takes on the new meaning attached to it by Theorem 5.1. Also $\langle f_1(k), u_0 \rangle_{\Lambda + \alpha} = a(v, k)$ by (5.30a), and $d(v) = v(X_\alpha + X_{-\alpha}) - v_0^+$ by Theorem 2.1c. Substituting from (5.30b), we obtain

$$b\left(\frac{1}{2} c\tilde{\alpha}, k\right) = \left(\frac{|\alpha|^2}{4}\right)^2 \langle \tau_{\Lambda''}(k)^{-1} v'', v'' \rangle \{(c + v_0^+)(c - v_0^+)\}.$$

Then $b(-\frac{1}{2}c\tilde{\alpha}, 1) > 0$ for $c > v_0^+$, as asserted.

Finally suppose that (a′), (b), and (c) hold. We shall prove nonunitarity by applying Theorem 4.1a. Assumption (c) and the discussion with Eq. (3.18) say that

$$[U(\nu)|_K : \tau_{\Lambda'}] = [\tau_{\Lambda'}|_{K \cap M^*} : \tau_\lambda].$$

Assumption (a′) and Theorem 3.3b therefore allow us to conclude that $\tau_{\Lambda'}$ has multiplicity one in $U(\nu)|_K$. Thus Theorem 4.1a is applicable. The derivation of (5.30b) did not use assumption (a) of Corollary 5.2, and thus

$$\frac{a(-\frac{1}{2}c\tilde{\alpha})}{a(\frac{1}{2}c\tilde{\alpha})} = \frac{-c+\nu_0^+}{c+\nu_0^+}.$$

This expression is <0 for $c > \nu_0^+$, and the asserted nonunitarity follows.

6. Two-Step Formula with Some Conjugacy

We now take up a number of results that either relax the hypotheses of Theorem 5.1 or else impose stronger hypotheses and get stronger conclusions. Each of them will involve the ideas that enter the proof of Theorem 5.1, as well as a few new ideas, and we shall emphasize only the new ideas in each.

Theorem 5.1 says that the ν dependence in $a(\nu, k)$ is under suitable circumstances a product of linear factors $d(\nu)$, each coming from a single step of the action of $\mathfrak{g}^{\mathbb{C}}$ on the representation space, times a function of k. A simple way in which these circumstances can fail is when the theorem is to be applied twice, first to pass from $(\mu')^\vee$ to $(\mu'+\alpha_r)^\vee$ and then to pass from $(\mu'+\alpha_r)^\vee$ to $(\mu'+\alpha_r+\alpha_s)^\vee$; assumption (a) will fail at the second step if $\mu'+\alpha_r$ is conjugate to $\mu'+\alpha_s$ by the Weyl group of Δ_K. The main result of this section, Theorem 6.4, addresses this situation, giving a formula for the combined effect of the two steps. We continue with notation as in Section 2.

Let γ be in Δ_K, and let s_γ be the corresponding reflection in W_K. A *standard representative* of s_γ is a representative w_γ of s_γ in K that lies in the three-dimensional subgroup of K corresponding to γ and has square lying in the center of this three-dimensional subgroup (e.g., w_γ could be the image of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ or of $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$).

Lemma 6.1. *Let β_1 and β_2 be strongly orthogonal noncompact roots such that $\beta_2 = p\beta_1$ with p in W_K and $p^2 = 1$. Let $p = \prod s_\gamma$ a nonredundant decomposition of p into the commuting product of strongly orthogonal reflections,* *and let w be a representative of p in K obtained by multiplying standard representatives of the s_γ's. Then*

$$\mathrm{Ad}(w)(X_{\beta_1} \otimes X_{\beta_2}) = X_{\beta_2} \otimes X_{\beta_1}. \tag{6.1}$$

Moreover if c is the number such that

$$\mathrm{Ad}(w) X_{\beta_1} = c X_{\beta_2}, \tag{6.2a}$$

then $|c| = 1$ and

$$\mathrm{Ad}(w) X_{\beta_2} = \bar{c} X_{\beta_1}, \qquad \mathrm{Ad}(w) X_{-\beta_1} = \bar{c} X_{-\beta_2}, \qquad \mathrm{Ad}(w) X_{-\beta_2} = c X_{-\beta_1}. \tag{6.2b}$$

Proof. Define c by (6.2a) and d by

$$\mathrm{Ad}(w) X_{\beta_2} = d X_{\beta_1}.$$

Since $\mathrm{Ad}(w)$ is unitary on $\mathfrak{p}^{\mathbb{C}}$, (2.1a) gives $|c| = |d| = 1$. Also it is clear that

$$\mathrm{Ad}(w)(X_{\beta_1} \otimes X_{\beta_2}) = cd(X_{\beta_2} \otimes X_{\beta_1}),$$

and (2.1b) gives

$$\mathrm{Ad}(w) X_{-\beta_1} = -\mathrm{Ad}(w) \overline{\theta X_{\beta_1}} = -\theta\, \overline{\mathrm{Ad}(w) X_{\beta_1}} = -\bar{c} \theta \bar{X}_{\beta_2} = \bar{c} X_{-\beta_2}$$
$$\mathrm{Ad}(w) X_{-\beta_2} = -\mathrm{Ad}(w) \overline{\theta X_{\beta_2}} = -\theta\, \overline{\mathrm{Ad}(w) X_{\beta_2}} = -\bar{d} \theta \bar{X}_{\beta_1} = \bar{d} X_{-\beta_1}.$$

Thus it is enough to prove $cd = 1$.

We can get a handle on the number cd from the formula

$$\mathrm{Ad}(w^2) X_{\beta_1} = \mathrm{Ad}(w)(c X_{\beta_2}) = cd X_{\beta_1}. \tag{6.3}$$

The standard representative of each s_γ is in $\exp(\mathbb{C}E_\gamma + \mathbb{C}E_{-\gamma})$, and the strong orthogonality of the γ's forces these representatives to commute. Thus

$$w^2 = \prod_\gamma \exp \pi i H_\gamma = \exp \pi i \sum_\gamma H_\gamma,$$

and

$$\begin{aligned}
\mathrm{Ad}(w^2) X_{\beta_1} &= \left\{ \exp \pi i \operatorname{ad} \sum H_\gamma \right\} X_{\beta_1} \\
&= \left\{ \prod \exp 2\pi i \langle \beta_1, \gamma \rangle / |\gamma|^2 \right\} X_{\beta_1} \\
&= (-1)^{\sum 2\langle \beta_1, \gamma \rangle / |\gamma|^2} X_{\beta_1}.
\end{aligned} \tag{6.4}$$

We have

$$\beta_2 = \left(\prod s_\gamma\right)\beta_1 = \beta_1 - \sum \frac{2\langle \beta_1, \gamma\rangle}{|\gamma|^2}\gamma.$$

Taking the inner product with β_1 gives

$$|\beta_1|^2 = \sum \frac{2\langle \beta_1, \gamma\rangle^2}{|\gamma|^2}.$$

Thus

$$\sum_\gamma \frac{4\langle \beta_1, \gamma\rangle^2}{|\gamma|^2\,|\beta_1|^2} = 2. \tag{6.5}$$

Consequently the number of γ's in (6.5) with $\langle \beta_1, \gamma\rangle \neq 0$ is at most 2 and is 2 if and only if β_1 and the two γ's have the same length; in this case the exponent of (-1) in (6.4) is even and $\mathrm{Ad}(w^2)X_{\beta_1} = X_{\beta_1}$. Suppose the number of γ's is 1. If $|\gamma| < |\beta_1|$, then again (6.4) shows that $\mathrm{Ad}(w^2)X_{\beta_1} = X_{\beta_1}$. If $|\gamma| > |\beta_1|$, then

$$\beta_2 = p\beta_1 = s_\gamma\beta_1 = \beta_1 \pm \gamma$$

shows that $\beta_2 - \beta_1 = \pm\gamma$, in contradiction to the assumed strong orthogonality of β_1 and β_2. Thus $\mathrm{Ad}(w^2)X_{\beta_1} = X_{\beta_1}$ in every case, and (6.3) shows $cd = 1$, as required.

LEMMA 6.2. *Let μ' be an integral form on $\mathfrak{b}$, let $\Lambda' = (\mu')^\vee$, let v' be a weight vector for $\tau_{\Lambda'}$ with weight μ', let γ be in $\Delta_{K,\mu'}$, and suppose w is a standard representative of s_γ. Then $\tau_{\Lambda'}(w)v' = v'$.*

Proof. Under our assumption, X_γ and $X_{-\gamma}$ act as 0 on v'. Using the series for the exponential, we see that w acts as 1.

LEMMA 6.3. *Let β_1 and β_2 be strongly orthogonal noncompact roots, and let μ' be an integral form on $\mathfrak{b}$. Suppose β_1 and β_2 are conjugate by a member p of W_K such that $p^2 = 1$ and $p\mu' = \mu'$. Let $p = \prod s_\gamma$ be a nonredundant decomposition of p into the commuting product of strongly orthogonal reflections that fix μ' and $\beta_1 + \beta_2$, and let w be a representative of p in K obtained by multiplying standard reflections of the s_γ's. Let $\Lambda' = (\mu')^\vee$, $\Lambda'' = (\mu' + \beta_1)^\vee = (\mu' + \beta_2)^\vee$, and $\Lambda''' = (\mu' + \beta_1 + \beta_2)^\vee$. Fix a nonzero vector v' of weight μ' in $\tau_{\Lambda'}$, and define*

$$v_1'' = E_{\Lambda''}(v' \otimes X_{\beta_1}) \qquad \textit{and} \qquad v_2'' = \tau_{\Lambda''}(w)v_1''.$$

Then

$$E_{\Lambda'''}(v_1''\otimes X_{\beta_2})=\bar{c}E_{\Lambda'''}(v_2''\otimes X_{\beta_1}),$$

where c *is the number such that* $\mathrm{Ad}(w)X_{\beta_1}=cX_{\beta_2}$.

Remark. The existence of the decomposition $p=\prod s_\gamma$ follows from Chevalley's Lemma [12, p. 81].

Proof. By Lemma 6.1 we have $\bar{c}X_{\beta_1}=\mathrm{Ad}(w)X_{\beta_2}$. Thus

$$\bar{c}E_{\Lambda'''}(v_2''\otimes X_{\beta_1})=\tau_{\Lambda'''}(w)\,E_{\Lambda'''}(v_1''\otimes X_{\beta_2}). \tag{6.6}$$

By assumption each s_γ fixes μ' and $\beta_1+\beta_2$, hence is in $\Delta_{K,\mu'+\beta_1+\beta_2}$. Lemma 6.2 thus says we can drop $\tau_{\Lambda'''}(w)$ from the right side of (6.6), and Lemma 6.3 follows.

THEOREM 6.4. *Fix roots* $\pm\alpha_r$ *and* $\pm\alpha_s$ *with* $r\neq s$ *and with the two choices of sign not necessarily the same, and fix an integral form* μ' *on* $\mathfrak{b}$. *Suppose that* $\pm\alpha_r$ *and* $\pm\alpha_s$ *are conjugate by an element* p *of order* 2 *in the Weyl group of* $\Delta_{K,\mu'}$. *Let* $\Lambda'=(\mu')^\vee$, $\Lambda''=(\mu'\pm\alpha_r)^\vee=(\mu'\pm\alpha_s)^\vee$, *and* $\Lambda'''=(\mu'\pm\alpha_r\pm\alpha_s)^\vee$. *Fix a nonzero vector* v' *of weight* μ' *in* $\tau_{\Lambda'}$, *and, for each* v, *let* $B(v)$ *be a member of* $\mathrm{Hom}_{K\cap M^\#}(V^{\Lambda'},V^{\sigma\#})$. *Let* f_1 *be the member of the induced space given by*

$$f_1(k)=B(v)\,\tau_{\Lambda'}(k)^{-1}v'.$$

Suppose that

(a1) *the only weight in* $\tau_{\Lambda'}$ *obtainable by adding or subtracting some* α_j *from* $\mu'\pm\alpha_r$ *or* $\mu'\pm\alpha_s$ *is* μ' *itself*;

(a2) *the only weights in* $\tau_{\Delta''}$ *obtainable from* $\mu'\pm\alpha_r\pm\alpha_s$ *by adding or subtracting some* α_j *are the two weights* $\mu'\pm\alpha_r$ *and* $\mu'\pm\alpha_s$;

(b1) $\tau_{\Lambda''}$ *occurs in* $\tau_{\Lambda'}\otimes\mathfrak{p}^{\mathbb{C}}$, *so that* $v_r''=E_{\Lambda''}(v'\otimes X_{\pm\alpha_r})$ *is nonzero;*

(b2) $\tau_{\Lambda'''}$ *occurs in* $\tau_{\Lambda''}\otimes\mathfrak{p}^{\mathbb{C}}$, *so that* $v'''=E_{\Lambda'''}(v_r''\otimes X_{\pm\alpha_s})$ *is nonzero;*

(c1) *Eq.* (3.18) *for* $\Lambda'-\Lambda$ *has no solutions with* $\sum_{\beta\in\Delta^+_{-,n}}k_\beta\beta$ *nonzero;*

(c2) *Eq.* (3.18) *rewritten for* $\Lambda''-\Lambda$ *has no solutions with* $\sum_{\beta\in\Delta^+_{-,n}}k_\beta\beta$ *nonzero;*

(d1) *no* $\beta\in\Delta_n$ *with* $\beta\perp\perp\alpha_1,\ldots,\alpha_{s-1}$, $\beta-(\pm\alpha_s)\in\Delta$, *and* $|\beta|^2<|\alpha_s|^2$ *has either*

$$\frac{2\langle\mu',\beta-(\pm\alpha_s)\rangle}{|\beta-(\pm\alpha_s)|^2}=1\qquad\textit{or}\qquad\frac{2\langle\mu'\pm\alpha_r,\beta-(\pm\alpha_s)\rangle}{|\beta-(\pm\alpha_s)|^2}=1;$$

(d2) *no* $\beta\in\Delta_n$ *with* $\beta\perp\!\!\!\perp\alpha_1,\ldots,\alpha_{r-1}$, $\beta-(\pm\alpha_r)\in\Delta$, *and* $|\beta|^2<|\alpha_r|^2$ *has either*

$$\frac{2\langle\mu',\beta-(\pm\alpha_r)\rangle}{|\beta-(\pm\alpha_r)|^2}=1 \qquad or \qquad \frac{2\langle\mu'\pm\alpha_s,\beta-(\pm\alpha_r)\rangle}{|\beta-(\pm\alpha_r)|^2}=1.$$

Then $v'''\neq 0$ *and*

$$\frac{\langle P_{\Lambda'''}U(v,X_{\pm\alpha_s})\,P_{\Lambda''}U(v,X_{\pm\alpha_r})\,f_1(k),u_0\rangle_{\mu'\pm\alpha_r\pm\alpha_s}}{\langle\tau_{\Lambda'''}(k)^{-1}v''',v'''\rangle}$$
$$=\frac{|\alpha_r|^2\,|\alpha_s|^2}{16}\,(d_1(v)\,d_3(v)+d_2(v)\,d_4(v))\,\frac{\langle f_1(k),u_0\rangle_{\mu'}}{\langle\tau_{\Lambda'}(k)^{-1}v',v'\rangle},$$

where

$$\begin{aligned} d_1(v)={}&(v+\rho)(X_{\alpha_s}+X_{-\alpha_s})+\frac{2\langle\mu',\pm\alpha_s\rangle}{|\alpha_s|^2}\\ &-2\#\{\beta\in\Delta_n\mid\beta\perp\!\!\!\perp\alpha_1,\ldots,\alpha_{s-1};\beta-(\pm\alpha_s)\in\Delta;\\ &\qquad\langle\mu'\pm\alpha_r,\beta-(\pm\alpha_s)\rangle>0\};\end{aligned}$$

$d_2(v)$ *is* $d_1(v)$ *with* r *and* s *interchanged*;

$$\begin{aligned} d_3(v)={}&(v+\rho)(X_{\alpha_r}+X_{-\alpha_r})+\frac{2\langle\mu',\pm\alpha_r\rangle}{|\alpha_r|^2}\\ &-2\#\{\beta\in\Delta_n\mid\beta\perp\!\!\!\perp\alpha_1,\ldots,\alpha_{r-1};\beta-(\pm\alpha_r)\in\Delta);\\ &\qquad\langle\mu',\beta-(\pm\alpha_r)\rangle>0\};\end{aligned}$$

$d_4(v)$ *is* $d_3(v)$ *with* r *and* s *interchanged.*

Remarks. (1) The vectors v_r'', v_s'', and v''' are nonzero by Theorem 1.5.

(2) If all noncompact roots have the same length, then (b1) and (b2) are automatically satisfied, according to Corollary 1.4a, and (d1) and (d2) are trivially satisfied.

(3) Conditions (c1) and (c2) are implied by the condition

(c′) there exists a system $\Delta_{L'}\subseteq\Delta$ generated by Δ^+ simple roots such that

(i) $\alpha_1,\ldots,\alpha_l$ are in $\Delta_{L'}$, and $\Delta_{L'}$ has real rank exactly l

(ii) $\Lambda'-\Lambda$ is an integral linear combination of roots in $\Delta_{L'}$

(iii) $\Lambda''-\Lambda$ is an integral linear combination of roots in $\Delta_{L'}$.

The argument that (c′) implies (c1) and (c2) is the same as the corresponding implication in Theorem 5.1.

Proof. In Lemma 6.3, we take $\beta_1 = \pm\alpha_r$ and $\beta_2 = \pm\alpha_s$, we choose w as in the statement of that lemma, and we put $v_s'' = \tau_{\Lambda''}(w)v_r''$. Then Lemmas 6.1 and 6.3 give

$$\mathrm{Ad}(w)\, X_{\pm\alpha_r} = cX_{\pm\alpha_s}, \qquad \mathrm{Ad}(w)\, X_{\pm\alpha_s} = \bar{c}X_{\pm\alpha_r} \tag{6.7a}$$

$$\mathrm{Ad}(w)\, X_{-(\pm\alpha_r)} = \bar{c}X_{-(\pm\alpha_s)}, \qquad \mathrm{Ad}(w)\, X_{-(\pm\alpha_s)} = cX_{-(\pm\alpha_r)} \tag{6.7b}$$

$$E_{\Lambda'''}(v_r'' \otimes X_{\pm\alpha_s}) = \bar{c}E_{\Lambda'''}(v_s'' \otimes X_{\pm\alpha_r}). \tag{6.7c}$$

Set

$$f_r(k) = P_{\Lambda''}\, U(v, X_{\pm\alpha_r})\, f_1(k)$$
$$f_s(k) = P_{\Lambda''}\, U(v, X_{\pm\alpha_s})\, f_1(k).$$

The same argument as with (5.31) shows that we can write

$$f_r(k) = C(v)\, \tau_{\Lambda''}(k)^{-1} v_r'' \tag{6.8}$$

for a unique $C(v)$ in $\mathrm{Hom}_{K\cap M^\#}(V^{\Lambda''}, V^{\sigma^\#})$. Notice from Lemma 6.2 that

$$\begin{aligned} U(v, w^{-1})\, f_1(k) = f_1(wk) &= B(v)\, \tau_{\Lambda'}(k)^{-1} \tau_{\Lambda'}(w)^{-1} v' \\ &= B(v)\, \tau_{\Lambda'}(k)^{-1} v' = f_1(k). \end{aligned} \tag{6.9}$$

Thus (6.7a) gives

$$\begin{aligned} cf_s(k) &= P_{\Lambda''}\, U(v, \mathrm{Ad}(w)\, X_{\pm\alpha_r})\, f_1(k) \\ &= U(v, w)\, P_{\Lambda''}\, U(v, X_{\pm\alpha_r})\, U(v, w^{-1})\, f_1(k) \\ &= P_{\Lambda''}\, U(v, X_{\pm\alpha_r})\, U(v, w^{-1})\, f_1(w^{-1}k) \\ &= P_{\Lambda''}\, U(v, X_{\pm\alpha_r})\, f_1(w^{-1}k) \qquad \text{by (6.9)} \\ &= f_r(w^{-1}k), \end{aligned}$$

and (6.8) gives

$$f_s(k) = \bar{c}f_r(w^{-1}k) = \bar{c}C(v)\, \tau_{\Lambda''}(k)^{-1} v_s''. \tag{6.10}$$

Hypotheses (a1), (b1), and (c1) allow us to use Theorem 5.1 to compute $\langle f_r(k), u_0\rangle_{\mu'\pm\alpha_r}$ and $\langle f_s(k), u_0\rangle_{\mu'\pm\alpha_s}$. Because of (d1) and (d2), the result is simply

$$\frac{\langle f_r(k), u_0 \rangle_{\mu' \pm \alpha_r}}{\langle \tau_{\Lambda''}(k)^{-1} v_r'', v_r'' \rangle} = \frac{|\alpha_r|^2}{4} d_3(v) \frac{\langle f_1(k), u_0 \rangle_{\mu'}}{\langle \tau_{\Lambda'}(k)^{-1} v', v' \rangle} \tag{6.11a}$$

$$\frac{\langle f_s(k), u_0 \rangle_{\mu' \pm \alpha_s}}{\langle \tau_{\Lambda''}(k)^{-1} v_s'', v_s'' \rangle} = \frac{|\alpha_s|^2}{4} d_4(v) \frac{\langle f_1(k), u_0 \rangle_{\mu'}}{\langle \tau_{\Lambda'}(k)^{-1} v', v' \rangle}. \tag{6.11b}$$

This is all the information we need about the first step taken by $U(v)$. For the second step, we cannot apply Theorem 5.1 directly to $f_r(k)$, since the appropriate assumption (a) is not satisfied. But we run through as much of the proof of Theorem 5.1 as we can, obtaining

$$\begin{aligned}
&\langle P_{\Lambda'''} U(v, X_{\pm\alpha_s}) f_r(k), u_0 \rangle_{\mu' \pm \alpha_r \pm \alpha_s} \\
&\quad = \sum_{j=1}^{l} \tfrac{1}{4} |\alpha_j|^2 [(v+\rho)(X_{\alpha_j} + X_{-\alpha_j})] \\
&\qquad \times \langle E_{\Lambda'''}(v_r'' \otimes X_{\pm\alpha_s}), \pi(k)(C(v)^* u_0 \otimes (X_{\alpha_j} + X_{-\alpha_j})) \rangle_{\mu' \pm \alpha_r \pm \alpha_s} \\
&\qquad + \sum_{\beta \in \Delta_n} \tfrac{1}{2} |\beta|^2 \\
&\qquad \times \langle E_{\Lambda'''}(v_r'' \otimes X_{\pm\alpha_s}), \pi(k)(\tau_{\Lambda''}(P_{\mathfrak{k}} X_\beta)^* C(v)^* u_0 \otimes X_\beta) \rangle_{\mu' \pm \alpha_r \pm \alpha_s}.
\end{aligned} \tag{6.12}$$

(The $\mathfrak{m}$ term gives 0 because of assumption (c2).) In the first sum, only terms in $C(v)^* u_0 \otimes (X_{\alpha_j} + X_{-\alpha_j})$ of weight $\mu' \pm \alpha_r \pm \alpha_s$ survive, and in the second sum, only terms in $\tau_{\Lambda''}(P_{\mathfrak{k}} X_\beta)^* C(v)^* u_0 \otimes X_\beta$ of weight $\mu' \pm \alpha_r \pm \alpha_s$ survive. Using assumption (a2), we see that

$$\begin{aligned}
C(v)^* u_0 &= \frac{\langle C(v)^* u_0, v_r'' \rangle}{|v_r''|^2} v_r'' + \frac{\langle C(v)^* u_0, v_s'' \rangle}{|v_s''|^2} v_s'' + \text{irrelevant terms} \\
&= \overline{a_r(v)} v_r'' + \overline{a_s(v)} v_s'' + \text{irrelevant terms}.
\end{aligned} \tag{6.12}$$

Substituting into (6.12), we have

$$\begin{aligned}
&\langle P_{\Lambda'''} U(v, X_{\pm\alpha_s}) f_r(k), u_0 \rangle_{\mu' \pm \alpha_r \pm \alpha_s} \\
&\quad = a_r(v) \Bigg\{ \tfrac{1}{4} |\alpha_s|^2 [(v+\rho)(X_{\alpha_s} + X_{-\alpha_s})] \\
&\qquad \times \langle E_{\Lambda'''}(v_r'' \otimes X_{\pm\alpha_s}), \pi(k)(v_r'' \otimes X_{\pm\alpha_s}) \rangle_{\mu' \pm \alpha_r \pm \alpha_s} \\
&\qquad + \sum_{\substack{\beta \in \Delta_n \\ \beta \perp\!\!\perp \alpha_1, \ldots, \alpha_{s-1} \\ \beta \perp\!\!\!/\!\!\!\perp \alpha_s}} \tfrac{1}{2} |\beta|^2 \\
&\qquad \times \langle E_{\Lambda'''}(v_r'' \otimes X_{\pm\alpha_s}), \pi(k)(\tau_{\Lambda''}(P_{\mathfrak{k}} X_\beta)^* v_r'' \otimes X_\beta) \rangle_{\mu' \pm \alpha_r \pm \alpha_s} \Bigg\}
\end{aligned}$$

$$+a_s(v)\left\{\tfrac{1}{4}|\alpha_r|^2[(v+\rho)(X_{\alpha_r}+X_{-\alpha_r})]\right.$$

$$\times\langle E_{\Lambda'''}(v_r''\otimes X_{\pm\alpha_s}),\pi(k)(v_s''\otimes X_{\pm\alpha_r})\rangle_{\mu'\pm\alpha_r\pm\alpha_s}$$

$$+\sum_{\substack{\beta\in\Delta_n\\ \beta\perp\perp\alpha_1,\dots,\alpha_{r-1}\\ \beta\perp\!\!/\!\!\perp\alpha_r}}\tfrac{1}{2}|\beta|^2$$

$$\left.\times\langle E_{\Lambda'''}(v_r''\otimes X_{\pm\alpha_s}),\pi(k)(\tau_{\Lambda''}(P_{\mathfrak{k}}X_\beta)^*v_s''\otimes X_\beta)\rangle_{\mu'\pm\alpha_r\pm\alpha_s}\right\}.$$

In the second expression in braces, we can substitute from (6.7c), and then each term in braces is computed by the same argument as in Theorem 5.1. Assumptions (d1) and (d2) make the result a little tidier:

$$\begin{aligned}
&\langle P_{\Lambda'''}U(v,X_{\pm\alpha_s})f_r(k),u_0\rangle_{\mu'\pm\alpha_r\pm\alpha_s}\\
&\quad=\{\tfrac{1}{4}|\alpha_s|^2a_r(v)\,d_1(v)\}\langle E_{\Lambda'''}(v_r''\otimes X_{\pm\alpha_s}),\pi(k)(v_r''\otimes X_{\pm\alpha_s})\rangle\\
&\qquad+\{\tfrac{1}{4}|\alpha_r|^2\,\bar{c}a_s(v)\,d_2(v)\}\langle E_{\Lambda'''}(v_s''\otimes X_{\pm\alpha_r}),\pi(k)(v_s''\otimes X_{\pm\alpha_r})\rangle\\
&\quad=\{\tfrac{1}{4}|\alpha_s|^2\,a_r(v)\,d_1(v)+\tfrac{1}{4}|\alpha_r|^2\,\bar{c}a_s(v)\,d_2(v)\}\langle\tau_{\Lambda'''}(k)^{-1}v''',v'''\rangle,
\end{aligned}\qquad(6.13)$$

the last equality following by a second application of (6.7c).

Now (6.12) gives

$$\begin{aligned}
\langle f_r(k),u_0\rangle_{\mu'\pm\alpha_r}&=\langle C(v)\,\tau_{\Lambda''}(k)^{-1}v_r'',u_0\rangle_{\mu'\pm\alpha_r}\\
&=\langle\tau_{\Lambda''}(k)^{-1}v_r'',C(v)^*u_0\rangle_{\mu'\pm\alpha_r}\\
&=a_r(v)\langle\tau_{\Lambda''}(k)^{-1}v_r'',v_r''\rangle,
\end{aligned}$$

and so (6.11a) gives

$$a_r(v)=\frac{|\alpha_r|^2}{4}d_3(v)\frac{\langle f_1(k),u_0\rangle_{\mu'}}{\langle\tau_{\Lambda'}(k)^{-1}v',v'\rangle}.\qquad(6.14a)$$

Moreover (6.12) gives

$$\begin{aligned}
\langle f_s(k),u_0\rangle_{\mu'\pm\alpha_s}&=\bar{c}\langle C(v)\,\tau_{\Lambda''}(k)^{-1}v_s'',u_0\rangle_{\mu'\pm\alpha_s}\\
&=\bar{c}\langle\tau_{\Lambda''}(k)^{-1}v_s'',C(v)^*u_0\rangle_{\mu'\pm\alpha_s}\\
&=\bar{c}a_s(v)\langle\tau_{\Lambda''}(k)^{-1}v_s'',v_s''\rangle,
\end{aligned}$$

and so (6.11b) gives

$$\bar{c}a_s(v)=\frac{|\alpha_s|^2}{4}d_4(v)\frac{\langle f_1(k),u_0\rangle_{\mu'}}{\langle\tau_{\Lambda'}(k)^{-1}v',v'\rangle}.\qquad(6.14b)$$

Substituting (6.14) into (6.13), we obtain the conclusion of the theorem.

7. Two-Step Formula Applicable to Gaps in $Sp(n, 1)$

In Sections 7–10 we give further results that are in the spirit of Sections 5 and 6 but are more specialized. The ones in Sections 7–9 are all needed in the classification [5] of unitary Langlands quotients obtained from maximal parabolic subgroups, and the one in Section 10 is needed for the determination of the unitary dual of $SU(N, 2)$ in [2]. For each of the results the proof has three ingredients, partly in common with Theorem 5.1 and partly new:

(1) Calculation of some expression (4.6) or (4.7) corresponding to application of successive operators $U(\nu, X)$. In Sections 7–10 the expression will involve two steps by operators $U(\nu, X)$. The methods of Theorems 5.1 and 6.4 will handle much of the first step and some of the second. The remainder of the calculation will require new ideas.

(2) Verification that some projection is nonzero. In easy cases this follows from Theorem 1.5. In harder cases a direct argument is needed.

(3) Derivation of a multiplicity result.

With the three ingredients in place, we apply Theorem 4.1, and the conclusion is a result about unitary representations.

The first result we shall state in some generality. In real-rank-one cases it leads to the nonunitarity of the gap when there is an isolated representation. A corollary will make this implication precise for $Sp(n, 1)$.

Lemma 7.1. *If γ is in $\Delta_{K,\Lambda}$, then*

$$(\operatorname{ad} X_\gamma)\Bigg(\sum_{\substack{\beta\in\Delta_n\\ \beta\perp\Lambda}} |\beta|^2 X_{-\beta}\otimes X_\beta\Bigg)=0.$$

Proof. The product in question is

$$\sum_{\substack{\beta\in\Delta_n\\ \beta\perp\Lambda}} |\beta|^2 [X_\gamma, X_{-\beta}]\otimes X_\beta + \sum_{\substack{\beta'\in\Delta_n\\ \beta'\perp\Lambda}} |\beta'|^2 X_{-\beta'}\otimes [X_\gamma, X_{\beta'}].$$

If $\beta=\gamma+\beta'$, it is enough to show

$$|\beta|^2 [X_\gamma, X_{-\beta}]\otimes X_\beta + |\beta'|^2 X_{-\beta'}\otimes [X_\gamma, X_{\beta'}]=0. \tag{7.1}$$

Put

$$[X_\gamma, X_{\beta'}]=aX_\beta \qquad \text{and} \qquad [X_\gamma, X_{-\beta}]=a'X_{-\beta'}.$$

Then

$$a[X_{-\beta'}, X_\beta] = -[X_{-\beta'}, [X_{\beta'}, X_\gamma]] = -(p'+1)\,q'X_\gamma,$$

where $\gamma - p'\beta', ..., \gamma + q'\beta'$ is the β' string through γ. Also

$$a'[X_{-\beta'}, X_\beta] = [[X_\gamma, X_{-\beta}], X_\beta] = p(q+1)X_\gamma,$$

where $\gamma - p\beta, ..., \gamma + q\beta$ is the β string through γ. So the left side of (7.1) is

$$\begin{aligned}(a'\,|\beta|^2 + a\,|\beta'|^2)&(X_{-\beta'} \otimes X_\beta)\\ &= c[p(q+1)\,|\beta|^2 - (p'+1)\,q'\,|\beta'|^2](X_{-\beta'} \otimes X_\beta),\end{aligned}$$

and we readily check that the expression in brackets is 0. Thus (7.1) holds, and the lemma follows.

THEOREM 7.2. *With $l=1$ and $\{\alpha_1, ..., \alpha_l\} = \{\alpha\}$, suppose that G has no factor split F_4, that σ is nondegenerate in the sense of* [16], *and that α and $-\alpha$ are conjugate via $W_{K,\Lambda}$. Let $\Lambda_1 = (\Lambda+\alpha)^\vee = (\Lambda-\alpha)^\vee = \Lambda + \delta$, and suppose that $\langle \delta, \alpha \rangle < 0$. Let $\Lambda_2 = (\Lambda_1 + \alpha)^\vee = \Lambda + \delta_1$. Suppose that*

(a) *τ_{Λ_1} occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$,*

(b) *τ_{Λ_2} occurs in $\tau_{\Lambda_1} \otimes \mathfrak{p}^{\mathbb{C}}$, and*

(c) *the $\delta + \delta_1$ subgroup (i.e., the semisimple subgroup built from all simple roots of Δ^+ needed for the expansion of $\delta + \delta_1$) has real rank one.*

Then

(1) $\langle P_{\Lambda_2} U(v, X_{-\alpha})\, P_{\Lambda_1} U(v, X_\alpha) f_0(k), u_0 \rangle_\Lambda = (|\alpha|^4/8)\, c(v) \langle \tau_{\Lambda_2}(k)^{-1} v_2, v_2 \rangle$,

where

$$c(v) = v(X_\alpha + X_{-\alpha}) + v_0^+ - 2\#\{\beta \in \Delta_n^+ \mid \langle \Lambda, \beta \rangle = 0, \beta \perp \alpha, \beta \perp\!\!\!\!/\!\!\perp \alpha\}$$

and

$$v_2 = E_{\Lambda_2}(E_{\Lambda_1}(v_0 \otimes X_\alpha) \otimes X_{-\alpha}) \tag{7.2}$$

(2) *the vector v_2 in (7.2) is not zero*

(3) *τ_{Λ_2} has multiplicity one in $U(v)|_K$*

(4) *the pair of K types $\{\Lambda, \Lambda_2\}$ exhibits $J(\frac{1}{2}c\tilde{\alpha})$ as not infinitesimally unitary for*

$$v_0^+ - 2\#\{\beta \in \Delta_n^+ \mid \langle \Lambda, \beta \rangle = 0, \beta \perp \alpha, \beta \perp\!\!\!\!/\!\!\perp \alpha\} < c < v_0^+.$$

Proof of conclusion (1). By Lemma 3.1, there exist roots of two different lengths, and α is short. Moreover,

$$\langle \Lambda, \alpha \rangle = 0, \tag{7.3}$$

and $\delta=\delta^+=\delta^-$ has the property that $\Lambda+\delta$ is Δ_K^+ dominant. If w_1 is a member of $W_{K,\Lambda}$ such that $\delta=w_1\alpha$, then (7.3) implies

$$\langle \Lambda, \delta\rangle = \langle \Lambda, w_1\alpha\rangle = \langle w_1^{-1}\Lambda, \alpha\rangle = \langle \Lambda, \alpha\rangle = 0. \tag{7.4}$$

Choose w_2 in $W_{K,\Lambda}$ such that $w_2(\Lambda_1+\alpha)$ is $\Delta_{K,\Lambda}^+$ dominant. We claim that $w_2(\Lambda_1+\alpha)$ is actually Δ_K^+ dominant. In fact, if γ is in Δ_K^+ but not $\Delta_{K,\Lambda}^+$, then

$$\frac{2\langle w_2(\Lambda_1+\alpha), \gamma\rangle}{|\gamma|^2} = \frac{2\langle \Lambda+w_2(\delta+\alpha), \gamma\rangle}{|\gamma|^2} = \frac{2\langle \Lambda, \gamma\rangle}{|\gamma|^2} + \frac{2\langle w_2(\delta+\alpha), \gamma\rangle}{|\gamma|^2}$$

can be negative only if $2\langle \Lambda, \gamma\rangle/|\gamma|^2=1$ and $w_2(\delta+\alpha)=-\gamma$, since $\delta+\alpha$ is a short root. But (7.3) and (7.4) show that $\delta+\alpha$ is in $\Delta_{K,\Lambda}$, and thus $\gamma=-w_2(\delta+\alpha)$ is in $\Delta_{K,\Lambda}$, contradiction. We conclude that $w_2(\Lambda_1+\alpha)$ is Δ_K^+ dominant.

Consequently

$$\begin{aligned}\Lambda_2=(\Lambda_1+\alpha)^\vee &= w_2(\Lambda_1+\alpha)=\Lambda+w_2\delta+w_2\alpha\\ &=\Lambda_1+(w_2\delta+w_2\alpha-\delta),\end{aligned}$$

and $\delta_1=w_2\delta+w_2\alpha-\delta$. Therefore

$$\langle \Lambda, \delta_1\rangle = \langle \Lambda, w_2\delta+w_2\alpha-\delta\rangle = 0+0-0=0 \tag{7.5}$$

and

$$\langle \delta, \delta_1\rangle = \langle \Lambda+\delta, \delta_1\rangle = \langle \Lambda_1, \delta_1\rangle = \langle \Lambda_1, \alpha\rangle = \langle \Lambda+\delta, \alpha\rangle = \langle \delta, \alpha\rangle <0. \tag{7.6}$$

The argument in the last three paragraphs of the proof of Theorem 3.3 shows that there exist long orthogonal roots γ_1 and γ_2 in $\Delta_{K,\Lambda}$ such that the involution $p=s_{\gamma_1}s_{\gamma_2}$ in $W_{K,\Lambda}$ has $p\alpha=-\alpha$. Let w be a representative of p in K obtained by multiplying standard representatives of s_{γ_1} and s_{γ_2}, and define a number c with $|c|=1$ by

$$\operatorname{Ad}(w)X_\alpha = cX_{-\alpha}. \tag{7.7}$$

Let us prove that

$$\operatorname{Ad}(w)X_{-\alpha} = \bar{c}X_{\alpha}. \tag{7.8}$$

In fact, α has to be in the span of γ_1 and γ_2 and must therefore be given by $\alpha=c_1\gamma_1+c_2\gamma_2$ with $|c_1|=|c_2|=\frac{1}{2}$. Hence

$$\frac{2\langle \alpha, \gamma_1\rangle}{|\gamma_1|^2} + \frac{2\langle \alpha, \gamma_2\rangle}{|\gamma_2|^2} = 2(c_1+c_2) \qquad \text{is in } 2\mathbb{Z}. \tag{7.9}$$

If d is such that $\operatorname{Ad}(w)X_{-\alpha} = dX_\alpha$, then (7.7) gives

$$c\,dX_\alpha = \operatorname{Ad}(w)(cX_{-\alpha}) = \operatorname{Ad}(w^2)X_\alpha = \operatorname{Ad}(\exp \pi i(H_{\gamma_1} + H_{\gamma_2}))X_\alpha$$
$$(-1)^{2\langle\alpha,\gamma_1\rangle/|\gamma_1|^2 + 2\langle\alpha,\gamma_2\rangle/|\gamma_2|^2} X_\alpha = X_\alpha.$$

This proves (7.8).

Now we imitate a certain amount of the proof of Theorem 6.4. Let

$$v_+ = E_{\Lambda_1}(v_0 \otimes X_\alpha) \tag{7.10a}$$

and

$$v_- = \tau_{\Lambda_1}(w)v_+ = cE_{\Lambda_1}(v_0 \otimes X_{-\alpha}). \tag{7.10b}$$

(Here the second formula for v_- uses (7.7) and Lemma 6.2.) By assumption (a), τ_{Λ_1} occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$. Thus by Theorem 1.5, v_+ and v_- are nonzero. Let

$$f_+(k) = P_{\Lambda_1} U(v, X_\alpha) f_0(k)$$
$$f_-(k) = P_{\Lambda_1} U(v, X_{-\alpha}) f_0(k).$$

The same argument as with (5.31) shows that we can write

$$f_+(k) = B(v)\,\tau_{\Lambda_1}(k)^{-1} v_+ \tag{7.11}$$

for a unique $B(v)$ in $\operatorname{Hom}_{K\cap M^\#}(V^{\Lambda_1}, V^{\sigma^\#})$. Then the proof of (6.10) shows that

$$f_-(k) = \bar{c}B(v)\,\tau_{\Lambda_1}(k)^{-1} v_-. \tag{7.12}$$

We shall use Theorem 5.1 to compute $\langle f_+(k), u_0\rangle_{\Lambda+\alpha}$ and $\langle f_-(k), u_0\rangle_{\Lambda-\alpha}$. Condition (a) is satisfied since $\Lambda - (\Lambda \pm 2\alpha) = \mp 2\alpha$ is not the sum of positive compact roots. Condition (b) holds by hypothesis. And condition (c) is satisfied trivially. Since $\langle f_0(k), u_0\rangle_\Lambda = \langle \tau_\Lambda(k)^{-1}v_0, v_0\rangle$ by (4.4) and (4.5), Theorems 5.1 and 2.1 give

$$\langle f_+(k), u_0\rangle_{\Lambda+\alpha} = \frac{|\alpha|^2}{4}(v_0^+ + v(X_\alpha + X_{-\alpha}))\langle \tau_{\Lambda_1}(k)^{-1}v_+, v_+\rangle \tag{7.13}$$

$$\langle f_-(k), u_0\rangle_{\Lambda-\alpha} = \frac{|\alpha|^2}{4}(v_0^- + v(X_\alpha + X_{-\alpha}))\langle \tau_{\Lambda_1}(k)^{-1}v_-, v_-\rangle. \tag{7.14}$$

Moreover the respective expressions I for α and $-\alpha$ in Theorem 2.1 are equal when $\mu' = \Lambda$, since $\langle \Lambda, \alpha\rangle = 0$, and thus

$$v_0^+ = v_0^-. \tag{7.15}$$

This is the information that we need about the first step taken by $U(v)$.

The new ingredient is the nature of the second step. We run through as much of the proof of Theorem 5.1 as we can, obtaining

$$\begin{aligned}
&\langle P_{\Lambda_2} U(\nu, X_{-\alpha}) f_+(k), u_0 \rangle_\Lambda \\
&\quad = \tfrac{1}{4} |\alpha|^2 [(\nu+\rho)(X_\alpha + X_{-\alpha})] \\
&\qquad \times \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(B(\nu)^* u_0 \otimes (X_\alpha + X_{-\alpha})) \rangle_\Lambda \\
&\qquad + \sum_{\beta \in \Delta_n} \tfrac{1}{2} |\beta|^2 \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(\tau_{\Lambda_1}(P_{\mathfrak{k}} X_\beta)^* B(\nu)^* u_0 \otimes X_\beta) \rangle_\Lambda.
\end{aligned} \tag{7.16}$$

(The $\mathfrak{m}$ term gives 0 because of assumption (c).) In the first sum, only terms in $B(\nu)^* u_0 \otimes (X_\alpha + X_{-\alpha})$ of weight Λ survive, and in the second sum, only terms in $\tau_{\Lambda_1}(P_{\mathfrak{k}} X_\beta)^* B(\nu)^* u_0 \otimes X_\beta$ of weight Λ survive. Since $\Lambda + 2\alpha$ and $\Lambda - 2\alpha$ are too long to be weights of τ_{Λ_2}, we see that

$$\begin{aligned}
B(\nu)^* u_0 &= \frac{\langle B(\nu)^* u_0, v_+ \rangle}{|v_+|^2} v_+ + \frac{\langle B(\nu)^* u_0, v_- \rangle}{|v_-|^2} v_- + \text{irrelevant terms} \\
&= \overline{a_+(\nu)} v_+ + \overline{a_-(\nu)} v_- + \text{irrelevant terms}.
\end{aligned} \tag{7.17}$$

Substituting into (7.16), we have

$$\begin{aligned}
&\langle P_{\Lambda_2} U(\nu, X_{-\alpha}) f_+(k), u_0 \rangle_\Lambda \\
&\quad = a_+(\nu) \Big\{ \tfrac{1}{4} |\alpha|^2 [(\nu+\rho)(X_\alpha + X_{-\alpha})] \\
&\qquad \times \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(v_+ \otimes X_{-\alpha}) \rangle_\Lambda \\
&\qquad + \sum_{\substack{\beta \in \Delta_n \\ \beta \perp\!/\!\!\perp \alpha}} \tfrac{1}{2} |\beta|^2 \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(\tau_{\Lambda_1}(P_{\mathfrak{k}} X_\beta)^* v_+ \otimes X_\beta) \rangle_\Lambda \Big\} \\
&\qquad + a_-(\nu) \Big\{ \tfrac{1}{4} |\alpha|^2 [(\nu+\rho)(X_\alpha + X_{-\alpha})] \\
&\qquad \times \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(v_- \otimes X_\alpha) \rangle_\Lambda \\
&\qquad + \sum_{\substack{\beta \in \Delta_n \\ \beta \perp\!/\!\!\perp \alpha}} \tfrac{1}{2} |\beta|^2 \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(\tau_{\Lambda_1}(P_{\mathfrak{k}} X_\beta)^* v_- \otimes X_\beta) \rangle_\Lambda \Big\}.
\end{aligned} \tag{7.18}$$

Here we can compute $a_+(\nu)$ and $a_-(\nu)$ as follows: From (7.11), we have

$$\begin{aligned}
\langle f_+(k), u_0 \rangle_{\Lambda+\alpha} &= \langle B(\nu)\, \tau_{\Lambda_1}(k)^{-1} v_+, u_0 \rangle_{\Lambda+\alpha} \\
&= \langle \tau_{\Lambda_1}(k)^{-1} v_+, B(\nu)^* u_0 \rangle_{\Lambda+\alpha} \\
&= a_+(\nu) \langle \tau_{\Lambda_1}(k)^{-1} v_+, v_+ \rangle,
\end{aligned}$$

so that (7.13) gives

$$a_+(v) = \frac{|\alpha|^2}{4}(v_0^+ + v(X_\alpha + X_{-\alpha})). \tag{7.19a}$$

From (7.12), we have

$$\begin{aligned}\langle f_-(k), u_0\rangle_{\Lambda-\alpha} &= \bar{c}\langle B(v)\,\tau_{\Lambda_1}(k)^{-1}v_-, u_0\rangle_{\Lambda-\alpha}\\ &= \bar{c}\langle \tau_{\Lambda_1}(k)^{-1}v_-, B(v)^* u_0\rangle_{\Lambda-\alpha}\\ &= \bar{c}a_-(v)\langle \tau_{\Lambda_1}(k)^{-1}v_-, v_-\rangle,\end{aligned}$$

so that (7.14) and the equality $v_0^+ = v_0^-$ give

$$\bar{c}a_-(v) = \frac{|\alpha|^2}{4}(v_0^+ + v(X_\alpha + X_{-\alpha})). \tag{7.19b}$$

Let us concentrate on the $\mathfrak{k}$ terms in the first set of braces in (7.18). The only contribution from $P_{\mathfrak{k}}X_\beta$ comes from a term with $[X_\alpha, X_\beta]$. We imitate a certain amount of the proof of Theorem 5.1, starting after (5.14). The term $\beta = \alpha$ gives 0, and the term $\beta = -\alpha$ has $P_{\mathfrak{k}}X_{-\alpha} = -\frac{1}{2}H_\alpha$ and gives

$$\tfrac{1}{4}|\alpha|^2\langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(v_+ \otimes X_{-\alpha})\rangle\{-2\}.$$

For the remaining β's, the relevant term of $P_{\mathfrak{k}}X_\beta$ is $-(p+q)^{-1}[X_\alpha, X_\beta]$ by (5.4b); here $\beta - p\alpha, ..., \beta + q\alpha$ is the α root string through β. Thus the relevant term of $\tau_{\Lambda_1}(P_{\mathfrak{k}}X_\beta)^* v_+ \otimes X_\beta$ is

$$(p+q)^{-1}\tau_{\Lambda_1}[X_{-\alpha}, X_{-\beta}]\,v_+ \otimes X_\beta. \tag{7.20}$$

If $\langle \Lambda+\alpha, \beta+\alpha\rangle \leqslant 0$ or if $\beta+\alpha$ is not a root, then the first factor of (7.20) gives 0. The remaining β's are those in the set

$$T_1 = \{\beta \in \Delta_n \mid \beta + \alpha \in \Delta \text{ and } \langle \Lambda+\alpha, \beta+\alpha\rangle > 0\}.$$

For these β's, (5.5a) gives

$$\begin{aligned}&\tau_{\Lambda_1}[X_{-\alpha}, X_{-\beta}]v_+ \otimes X_\beta\\ &\quad = \pi[X_{-\alpha}, X_{-\beta}](v_+ \otimes X_\beta) - v_+ \otimes (\mathrm{ad}[X_{-\alpha}, X_{-\beta}])X_\beta\\ &\quad = \pi[X_{-\alpha}, X_{-\beta}](v_+ \otimes X_\beta) - q'(p'+1)\,v_+ \otimes X_{-\alpha},\end{aligned} \tag{7.21}$$

where p' and q' determine the β root string through α as $\alpha - p'\beta, ..., \alpha + q'\beta$.

For β in T_1 the weight of $v_+ \otimes X_\beta$ is too long to contribute to τ_{Λ_2} if

$$\begin{aligned}0 &< |\Lambda+\alpha+\beta|^2 - |\Lambda+\delta+\alpha|^2\\ &= 2\langle \Lambda, \beta-\delta\rangle + |\alpha+\beta|^2 - |\delta+\alpha|^2 = 2\langle \Lambda, \beta\rangle + |\alpha+\beta|^2 - |\delta+\alpha|^2.\end{aligned} \tag{7.22}$$

(Here we have used (7.4).) On the other hand, the β's in T_1 all have $\langle \Lambda+\alpha, \beta+\alpha\rangle > 0$. Since $\langle \Lambda, \alpha\rangle = 0$ and since $\beta+\alpha$ is in Δ, this latter condition means

$$1 \leqslant \frac{2\langle \Lambda+\alpha, \beta+\alpha\rangle}{|\beta+\alpha|^2} = \frac{2\langle \Lambda, \beta\rangle}{|\beta+\alpha|^2} + \frac{2\langle \alpha, \beta+\alpha\rangle}{|\beta+\alpha|^2} = \frac{2\langle \Lambda, \beta\rangle}{|\beta+\alpha|^2} + (1 \text{ or } 0).$$

Comparing this condition with (7.22), we see that the weight of $v_+ \otimes X_\beta$ is too long to contribute to τ_{Λ_2} unless β is in

$$T_2 = \{\beta \in T_1 \mid \langle \beta, \alpha\rangle < 0, |\beta| = |\alpha|, \beta \neq -\alpha, \text{and } \langle \Lambda, \beta\rangle = 0\}.$$

If β is in T_2, we make a different calculation of the left side of (7.21), obtaining

$$\begin{aligned}
&\tau_{\Lambda_1}[X_{-\alpha}, X_{-\beta}] v_+ \otimes X_\beta \\
&\quad = \tau_{\Lambda_1}[X_{-\alpha}, X_{-\beta}]\, E_{\Lambda_1}(v_0 \otimes X_\alpha) \otimes X_\beta \\
&\quad = E_{\Lambda_1}(v_0 \otimes (\operatorname{ad}[X_{-\alpha}, E_{-\beta}]) X_\alpha) \otimes X_\beta \qquad \text{since } \langle \Lambda, \beta+\alpha\rangle = 0 \\
&\quad = -q(p+1)\, E_{\Lambda_1}(v_0 \otimes X_{-\beta}) \otimes X_\beta. \qquad (7.23)
\end{aligned}$$

Now we combine (7.21) and (7.23), taking into account the identity

$$\frac{|\beta|^2}{|\alpha|^2} \frac{q'(p'+1)}{p+q} = 1,$$

and we see that the $\mathfrak{k}$ terms in the first set of braces in (7.18) contribute

$$\begin{aligned}
&\frac{1}{4} |\alpha|^2 \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(v_+ \otimes X_{-\alpha})\rangle \{-2\#\{\beta \in T_1 - T_2\} - 2\} \\
&\quad - \frac{1}{2} \sum_{\beta \in T_2} |\beta|^2 \frac{q(p+1)}{p+q} \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(E_{\Lambda_1}(v_0 \otimes X_{-\beta}) \otimes X_\beta)\rangle.
\end{aligned} \qquad (7.24\text{a})$$

Similarly the $\mathfrak{k}$ terms in the second set of braces in (7.18) contribute

$$\begin{aligned}
&\frac{1}{4} |\alpha|^2 \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(v_- \otimes X_\alpha)\rangle \{-2\#\{\beta \in T_3 - T_4\} - 2\} \\
&\quad - \bar{c} \frac{1}{2} \sum_{\beta \in T_4} |\beta|^2 \frac{p(q+1)}{p+q} \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(E_{\Lambda_1}(v_0 \otimes X_{-\beta}) \otimes X_\beta)\rangle,
\end{aligned} \qquad (7.24\text{b})$$

where

$$T_3 = \{\beta \in \Delta_n \mid \beta - \alpha \in \Delta \text{ and } \langle \Lambda - \alpha, \beta - \alpha \rangle > 0\}$$
$$T_4 = \{\beta \in T_3 \mid \langle \beta, \alpha \rangle > 0, |\beta| = |\alpha|, \beta \neq \alpha, \text{ and } \langle \Lambda, \beta \rangle = 0\},$$

and where the factor of $\bar{c}$ enters the second term of (7.24b) because of (7.10b). If β is in T_2, then $p = 0$ and $q = 1$. So $q(p+1)(p+q)^{-1} = 1$. Similarly if β is in T_4, then $p(q+1)(p+q)^{-1} = 1$. Thus we may drop these factors from (7.24a) and (7.24b).

Now we substitute into (7.18) from (7.24), obtaining

$$\begin{aligned}
&\langle P_{\Lambda_2} U(v, X_{-\alpha}) f_+(k), u_0 \rangle_\Lambda \\
&\quad = \frac{|\alpha|^4}{16} \{v_0^+ + v(X_\alpha + X_{-\alpha})\} \\
&\qquad \times [\langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(v_+ \otimes X_{-\alpha}) \rangle \\
&\qquad \times \{(v+\rho)(X_\alpha + X_{-\alpha}) - 2 - 2\#\{\beta \in T_1 - T_2\}\} \\
&\qquad + c\langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(v_- \otimes X_\alpha) \rangle \\
&\qquad \times \{(v+\rho)(X_\alpha + X_{-\alpha}) - 2 - 2\#\{\beta \in T_3 - T_4\}\}] \\
&\qquad - \frac{|\alpha|^2}{8} \{v_0^+ + v(X_\alpha + X_{-\alpha})\} \\
&\qquad \times \left[\sum_{\beta \in T_2 \cup T_4} |\beta|^2 \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(E_{\Lambda_1}(v_0 \otimes X_{-\beta}) \otimes X_\beta) \rangle \right].
\end{aligned} \tag{7.25}$$

Let $T = \{\beta \in \Delta_n \mid \beta \perp \Lambda\}$. We shall prove below that

$$\sum_{\beta \in T} |\beta|^2 E_{\Lambda_2}(\tau_{\Lambda_1}(v_0 \otimes X_{-\beta}) \otimes X_\beta) = 0. \tag{7.26}$$

Assuming this result for the moment, we write T as a disjoint union

$$T = T_2 \cup T_4 \cup T_5 \cup T_6 \cup T_7 \cup \{\alpha, -\alpha\},$$

where

$$T_5 = \{\beta \in T \mid \beta \perp \alpha \text{ and } \beta \perp\!\!\!\!/\!\!\perp \alpha\}$$
$$T_6 = \{\beta \in T \mid |\beta| > |\alpha|\}$$
$$T_7 = \{\beta \in T \mid |\beta| = |\alpha| \text{ and } \beta \perp\!\!\perp \alpha\}.$$

The set T_6 makes no contribution to (7.26) because the weight of $v_0 \otimes X_{-\beta}$

has length $>|\Lambda_1|$. Let us see that T_7 makes no contribution. If $\beta \in T_7$ makes a contribution, then the weight of $v_0 \otimes X_{-\beta}$, namely $\Lambda - \beta$, has the same length as $\Lambda + \alpha$ in τ_{Λ_1} and hence is conjugate to $\Lambda + \alpha$ via W_K. Thus it is conjugate to $\Lambda + \delta$. Since β is short, $\Lambda - \beta$ can be made Δ_K^+ dominant by a member of $W_{K,\Lambda}$. Thus β is conjugate to δ via $W_{K,\Lambda}$. Bringing in the conjugacy of α and $-\alpha$ via $W_{K,\Lambda}$, we see that $-\beta$ is conjugate to δ, too. Thus

$$\delta - \beta = \sum_{\gamma \in \Delta_{K,\Lambda}^+} m_\gamma \gamma \qquad \text{and} \qquad \delta + \beta = \sum_{\gamma \in \Delta_{K,\Lambda}^+} n_\gamma \gamma,$$

and one of these relations implies that β is in the δ subgroup. But the δ subgroup has real rank one by assumption (c), and no noncompact root can be strongly orthogonal to α within this subgroup. We conclude that T_7 makes no contribution.

Next let us observe that T_5 is nonempty. In fact, in our earlier notation we wrote $\alpha = c_1\gamma_1 + c_2\gamma_2$, and $\beta = c_1\gamma_1 - c_2\gamma_2$ is in T_5. Let β be any member of T_5. Then we have

$$\begin{aligned}
&E_{\Lambda_2}(E_{\Lambda_1}(v_0 \otimes X_{-\beta}) \otimes X_\beta) \\
&\quad = \tfrac{1}{2} E_{\Lambda_2}(E_{\Lambda_1}(v_0 \otimes X_{-\beta}) \otimes [[X_\beta, X_\alpha], X_{-\alpha}]) \\
&\quad = -\tfrac{1}{2} E_{\Lambda_2}(\tau_{\Lambda_1}[X_\beta, X_\alpha]\, E_{\Lambda_1}(v_0 \otimes X_{-\beta}) \otimes X_{-\alpha}) \\
&\qquad\qquad \text{since } \Lambda - \beta - \alpha \text{ is not a weight of } \tau_{\Lambda_2} \\
&\quad = -\tfrac{1}{2} E_{\Lambda_2}(E_{\Lambda_1}(v_0 \otimes [[X_\beta, X_\alpha], X_{-\beta}]) \otimes X_{-\alpha}) \\
&\qquad\qquad \text{since } \Lambda \text{ is orthogonal to } \beta + \alpha \\
&\quad = E_{\Lambda_2}(E_{\Lambda_1}(v_0 \otimes X_\alpha) \otimes X_{-\alpha}).
\end{aligned} \tag{7.27}$$

Hence all the members of T_5 make the same contribution to (7.26), namely $|\alpha|^2 E_{\Lambda_2}(v_+ \otimes X_{-\alpha})$. Moreover we could have replaced α by $-\alpha$ in the derivation of (7.27), and we conclude from the fact that T_5 is nonempty that

$$E_{\Lambda_2}(E_{\Lambda_1}(v_0 \otimes X_\alpha) \otimes X_{-\alpha}) = E_{\Lambda_2}(E_{\Lambda_1}(v_0 \otimes X_{-\alpha}) \otimes X_\alpha), \tag{7.28a}$$

i.e., that

$$E_{\Lambda_2}(v_- \otimes X_\alpha) = c E_{\Lambda_2}(v_+ \otimes X_{-\alpha}). \tag{7.28b}$$

Using (7.26), (7.27), and (7.28a), we see that the last two lines of (7.25) are

$$\begin{aligned}
&= +\frac{|\alpha|^4}{8} \{v_0^+ + v(X_\alpha + X_{-\alpha})\} \\
&\qquad \times \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(v_+ \otimes X_{-\alpha})\rangle \{\#\{\beta \in T_5\} + 2\}.
\end{aligned} \tag{7.29}$$

Now recall our element $p \in W_{K,\Lambda}$ with $p^2 = 1$ and $p\alpha = -\alpha$. If β is in T_1, then $p\beta$ has

$$
\begin{aligned}
&p\beta - \alpha \in \Delta && \text{since} \quad p\beta - \alpha = p(\beta + \alpha) \text{ and } \beta + \alpha \in \Delta \\
&\langle \Lambda - \alpha, p\beta - \alpha \rangle > 0 && \text{since} \quad \langle \Lambda - \alpha, p\beta - \alpha \rangle = \langle \Lambda - \alpha, p(\beta + \alpha) \rangle \\
& && \qquad = (p(\Lambda - \alpha), \beta + \alpha \rangle \\
& && \qquad = \langle \Lambda + \alpha, \beta + \alpha \rangle > 0.
\end{aligned}
$$

Thus p carries T_1 to T_3, and clearly p carries T_3 to T_1. In the process, we can check that p carries T_2 to T_4 and vice versa. Thus

$$\#\{\beta \in T_1 - T_2\} = \#\{\beta \in T_3 - T_4\}. \tag{7.30}$$

Using (7.28b), (7.29), and (7.30), we can simplify (7.25) to

$$
\begin{aligned}
&\langle P_{\Lambda_2} U(v, X_{-\alpha}) f_+(k), u_0 \rangle_\Lambda \\
&\quad = \frac{|\alpha|^4}{8} \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(v_+ \otimes X_{-\alpha}) \rangle [v_0^+ + v(X_\alpha + X_{-\alpha})] \\
&\qquad \times [(v + \rho)(X_\alpha + X_{-\alpha}) - 2 - 2\#\{\beta \in T_1 - T_2\} + \#\{\beta \in T_5\} + 2].
\end{aligned}
$$

We readily check that T_1 is the disjoint union

$$T_1 = T_2 \cup T_5 \cup \{\beta \in \Delta_n \mid \beta + \alpha \in \Delta \text{ and } \langle \Lambda, \beta + \alpha \rangle > 0\}.$$

Meanwhile the equality $I = v_0^-$ in Theorem 2.1 says

$$\rho(X_\alpha + X_{-\alpha}) + 0 - 2\#\{\beta \in \Delta_n \mid \beta + \alpha \in \Delta \text{ and } \langle \Lambda, \beta + \alpha \rangle > 0\} = v_0^+.$$

Hence

$$
\begin{aligned}
&\langle P_{\Lambda_2} U(v, X_{-\alpha}) f_+(k), u_0 \rangle_\Lambda \\
&\quad = \frac{|\alpha|^4}{8} \langle E_{\Lambda_2}(v_+ \otimes X_{-\alpha}), \pi(k)(v_+ \otimes X_{-\alpha}) \rangle [v_0^+ + v(X_\alpha + X_{-\alpha})] \\
&\qquad \times [v_0^+ + v(X_\alpha + X_{-\alpha}) - \#\{\beta \in T_5\}].
\end{aligned}
$$

This is conclusion (1) of the theorem, except that we have not yet proved (7.26).

To prove (7.26), we shall show that the vector on the left side is a highest weight vector (of weight Λ). Since $\Lambda \neq \Lambda_2$, (7.26) will follow. First suppose γ is in $\Delta_{K,\Lambda}$. Then

$$\tau_{\Lambda_2}(X_\gamma) \sum_{\beta\in T} |\beta|^2 E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes X_{-\beta})\otimes X_\beta)$$

$$= E_{\Lambda_2}(E_{\Lambda_1}\otimes I)\,\pi(X_\gamma)\left\{v_0\otimes\left(\sum_{\beta\in T}|\beta|^2 X_{-\beta}\otimes X_\beta\right)\right\}$$

$$= E_{\Lambda_2}(E_{\Lambda_1}\otimes I)\left\{v_0\otimes \operatorname{ad}(X_\gamma)\left(\sum_{\beta\in T}|\beta|^2 X_{-\beta}\otimes X_\beta\right)\right\} \quad \text{since } \langle \Lambda,\gamma\rangle = 0$$

$$= 0 \qquad \text{by Lemma 7.1.} \tag{7.31}$$

Since τ_{Λ_2} by assumption (b) occurs in $\tau_{\Lambda_1}\otimes \mathfrak{p}^{\mathbb{C}}$, it must occur with multiplicity one, by Proposition 1.1b. Moreover a nonzero highest weight vector is $E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes X_\delta)\otimes X_{\delta_1})$, by Theorem 1.5. Thus we can choose u in the universal enveloping algebra of $\sum_{\gamma\in\Delta_K^+}\mathbb{C}X_{-\gamma}$ such that

$$\sum_{\beta\in T}|\beta|^2 E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes X_{-\beta})\otimes X_\beta)$$
$$= \tau_{\Lambda_2}(u)\, E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes X_\delta)\otimes X_{\delta_1}). \tag{7.32}$$

We can discard from u any monomial not of total weight $-\delta-\delta_1$. Any remaining monomial is of the form $X_{-\gamma_1}\cdots X_{-\gamma_n}$ with $\gamma_j\in\Delta_K^+$ and $\sum\gamma_j = \delta+\delta_1$. Taking the inner product with Λ and invoking (7.4), (7.5), and the Δ_K^+ dominance of Λ, we see that γ_j is in $\Delta_{K,\Lambda}^+$.

Let $\gamma\in\Delta_K^+$ be simple, and apply $\tau_{\Lambda_2}(X_\gamma)$ to the left side of (7.32). If γ is in $\Delta_{K,\Lambda}^+$, then (7.31) shows we get 0. Otherwise X_γ commutes with each factor $X_{-\gamma_j}$ in each monomial of u, and (7.32) shows we get 0, since $E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes X_\delta)\otimes X_{\delta_1})$ is a highest weight vector. Consequently the left side of (7.32) is indeed a highest weight vector, necessarily of weight Λ, and (7.26) is proved. This completes the proof of conclusion (1).

Proof of conclusion (2). We start from the fact observed above that

$$E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes X_\delta)\otimes X_{\delta_1})\neq 0.$$

Applying the element w_3 of W_{K,Λ_1} such that $w_3\delta_1=\alpha$, we see that

$$E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes X_\delta)\otimes X_\alpha)\neq 0.$$

The weight of this vector satisfies $\langle \Lambda+\delta+\alpha, \delta+\alpha\rangle > 0$ by (7.3) and (7.4), and hence

$$0\neq \tau_{\Lambda_2}[X_{-\delta}, X_{-\alpha}]\, E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes X_\delta)\otimes X_\alpha)$$
$$= E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes [[X_{-\delta}, X_{-\alpha}], X_\delta])\otimes X_\alpha)$$
$$+ E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes X_\delta)\otimes [[X_{-\delta}, X_{-\alpha}], X_\alpha])$$
$$= -E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes X_{-\alpha})\otimes X_\alpha) + E_{\Lambda_2}(E_{\Lambda_1}(v_0\otimes X_\delta)\otimes X_{-\delta}).$$

One of the two terms on the right side must be nonzero. If it is the first term, then (7.28a) finishes the proof of conclusion (2). Otherwise the second term is nonzero and remains nonzero when we apply the element $w_1^{-1} \in W_{K,\Lambda}$ such that $w_1^{-1}\delta = \alpha$; again we obtain conclusion (2).

Proof of conclusion (3). It is immediate from conclusions (1) and (2) that τ_{Λ_2} has multiplicity at least one in $U(\nu)|_K$. Proposition 3.5 and assumption (c) therefore show that it is enough to prove $[\tau_{\Lambda_2}|_{K\cap M^\#} : \tau_\lambda] \leqslant 1$.

First we show that any $K \cap M^\#$ highest weight vector for τ_λ within τ_{Λ_2} has weight Λ relative to $\mathfrak{b}$. In fact, the vector must be a linear combination of weight vectors whose weights restrict to λ on $\mathfrak{b}_-$, thus are of the form $\Lambda + n\alpha$ with $n \in \mathbb{Z}$. For $|n| \geqslant 2$, we have

$$\begin{aligned} |\Lambda + n\alpha|^2 &\geqslant |\Lambda|^2 + 4\,|\alpha|^2 > |\Lambda|^2 + |\alpha|^2 \\ &= |\Lambda|^2 + |\delta + \alpha|^2 = |\Lambda + \delta + \alpha|^2 = |\Lambda_2|^2, \end{aligned}$$

and no such $\Lambda + n\alpha$ can be a weight. For $|n| = 1$, $\Lambda + n\alpha$ differs from Λ_2 by the sum of a noncompact root and a compact root, and this is not the sum of compact roots (Problems 12 and 13 on p. 478 of [12]). So $n = 0$ is the only possibility.

Second let K_1 be the analytic subgroup of K containing B and built from the root system

$$\Delta_{K_1} = \{\gamma \in \Delta_K \mid 2\langle \gamma, \alpha\rangle/|\alpha|^2 \text{ is even}\};$$

we shall show that any $K \cap M^\#$ highest weight vector of weight Λ in τ_{Λ_2} is highest for K_1. In fact, let $\gamma \in \Delta_{K_1}^+$ be given. If $|\gamma|^2 = 2\,|\alpha|^2$, then $\Lambda + \gamma$ is too long to be a weight of τ_{Λ_2} since

$$\begin{aligned} |\Lambda + \gamma|^2 &= |\Lambda|^2 + 2\langle \Lambda, \gamma\rangle + |\gamma|^2 \\ &\geqslant |\Lambda|^2 + 2\,|\alpha|^2 > |\Lambda|^2 + |\delta + \alpha|^2 = |\Lambda + \delta + \alpha|^2 = |\Lambda_2|^2. \end{aligned}$$

Since γ is in Δ_{K_1}, we may thus assume γ is short and $\gamma \perp \alpha$. If $\langle \Lambda, \gamma\rangle > 0$, then $\Lambda + \gamma$ is too long to be a weight of τ_{Λ_2} since

$$|\Lambda + \gamma|^2 = |\Lambda|^2 + 2\langle \Lambda, \gamma\rangle + |\gamma|^2 > |\Lambda|^2 + |\alpha|^2 = |\Delta_2|^2.$$

So we may assume also that $\gamma \perp \Lambda$. Let us see that $\Lambda + \gamma$ cannot be a weight if $\gamma \perp \alpha$ but $\gamma \perp/\perp \alpha$. In fact, otherwise $|\Lambda + \gamma|^2 = |\Lambda + \delta + \delta_1|^2$, and $\Lambda + \gamma$ is conjugate to $\Lambda + \delta + \delta_1$ by W_K and even by $W_{K,\Lambda}$. So γ is conjugate to $\delta + \delta_1$ by $W_{K,\Lambda}$ and

$$\delta + \delta_1 = \gamma + \sum_{\gamma_j \in \Delta_K^+} n_j \gamma_j, \qquad n_j \geqslant 0.$$

Thus γ is in the $\delta+\delta_1$ subgroup. Since γ is M-noncompact, this conclusion contradicts assumption (c). The only remaining possibility is that $\gamma \perp\perp \alpha$. In this case X_γ is in $(\mathfrak{k}\cap\mathfrak{m})^{\mathbb{C}}$ and acts as 0 on any $K\cap M^\#$ highest weight vector.

Third let $K^\perp$ be the analytic subgroup of K containing B and built from the root system $\Delta_{K,\Lambda}$, and let $K_1^\perp$ be the analytic subgroup of K containing B and built from the root system

$$\Delta_{K_1,\Lambda}=\Delta_{K,\Lambda}\cap\Delta_{K_1}.$$

If v_{Λ_2} denotes a nonzero highest weight vector for τ_{Λ_2} and if $U(\mathfrak{k}^\perp)$ denotes the universal enveloping algebra of $(\mathfrak{k}^\perp)^{\mathbb{C}}$, we shall show that $V=U(\mathfrak{k}^\perp)v_{\Lambda_2}$ is irreducible under $K^\perp$ and contains the full Λ weight space of τ_{Λ_2}. In fact, the irreducibility follows from the Theorem of the Highest Weight. Also V contains the full Λ weight space by the same argument used in connection with (7.32).

Fourth we observe that $(K^\perp, K_1^\perp)$ is a symmetric pair. The involution is $+1$ on $\mathfrak{k}_1^\perp$ and is -1 on

$$\mathfrak{k}^\perp\cap\Big\{\sum_{2\langle\gamma,\alpha\rangle/|\gamma|^2\ \mathrm{odd}}\mathbb{C}X_\gamma\Big\}.$$

Let $K_{ss}^\perp$ be the semisimple subgroup of $K^\perp$.

Finally we can complete the proof. Since $(K^\perp, K_1^\perp)$ is a symmetric pair and V is irreducible under $K^\perp$, the subspace of $(K_{ss}^\perp\cap K_1^\perp)$-fixed vectors in V is at most one-dimensional. (This is a well known implication due to Gelfand and Naimark; see [8, p. 416].) Since V contains the full Λ weight space, the subspace of $(K_{ss}^\perp\cap K_1^\perp)$-fixed vectors in the Λ weight space is at most one-dimensional. This subspace is the same as the subspace of K_1 highest weight vectors in the Λ weight space. We saw that every $K\cap M^\#$ highest weight vector of type τ_λ lies in this subspace, and hence the subspace of $K\cap M^\#$ highest weight vectors of type τ_λ is at most one-dimensional.

Proof of conclusion (4). In the presence of the first three conclusions, conclusion (4) follows from Theorem 4.1a.

COROLLARY 7.3. *Suppose $n\geqslant 2$ and $\mathfrak{g}=\mathfrak{sp}(n,1)$, possibly with abelian and compact factors, and suppose in the terminology of* [5] *that the special basic case for the infinitesimal character λ_0 is all of Δ. Suppose that $\mu=0$, that α is adjacent to the long simple root, and that α is the only noncompact simple root. Put $\Lambda_1=(\Lambda+\alpha)^\vee$ and $\Lambda_2=(\Lambda_1+\alpha)^\vee$. Then τ_{Λ_1} and τ_{Λ_2} have multiplicity one in $U(\frac{1}{2}c\tilde{\alpha})$, the signature of the standard form on τ_{Λ_1} is* $\mathrm{sgn}(v_0^+-c)=\mathrm{sgn}(v_0^--c)$, *and the signature of the standard form on τ_{Λ_2} is* $\mathrm{sgn}(v_0^+-c)(v_0^+-c-2)$.

Proof. We may as well assume $\mathfrak{g} = \mathfrak{sp}(n, 1)$. With standard notation for Δ^+, the long simple root is $2e_{n+1}$ and α is $e_n - e_{n+1}$. The other short simple roots are compact. The assumptions force $\Lambda = 0$. Then $\delta = e_1 + e_{n+1}$ and $\delta_1 = e_2 - e_{n+1}$. The hypotheses of Corollary 5.2 are satisfied (with (b) valid by Corollary 1.4a), and the conclusions about τ_{Λ_1} follow from Theorem 3.3, Proposition 3.5, and Corollary 5.2. The hypotheses of Theorem 7.2 are satisfied (with (b) valid by Corollary 1.4a), and the conclusions about τ_{Λ_2} follow.

8. Two-Step Formulas Applicable to Sharper Estimates in $SO(N, 2)$

We continue with the more specialized results that we began to discuss in Section 7. The main result of this section we shall state in some generality as Theorem 8.1. Our interest in this result is in the two corollaries, which specialize the theorem essentially to $SO(\text{even}, 2)$ and to $SO(\text{odd}, 2)$. In reading Theorem 8.1, one should keep these groups in mind and think of α as $e_j - e_{j+1}$ (in standard notation), β_0 as $e_j + e_{j+1}$, and the $SO(2)$ part of K as corresponding to weights e_j.

THEOREM 8.1. *With $l = 1$ and $\{\alpha_1, ..., \alpha_l\} = \{\alpha\}$, suppose that*

(a) *α is long (if there exist roots of two lengths),*

(b) *there exists a unique positive noncompact root β_0 that is orthogonal to α,*

(c) *α is conjugate to β_0 via an element p of order 2 in $W_{K,A}$,*

(d) *if $\Lambda_1 = (\Lambda + \alpha)^\vee$ and $\Lambda_2 = (\Lambda + \alpha + \beta_0)^\vee$, then τ_{Λ_1} occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$ and τ_{Λ_2} occurs in $\tau_{\Lambda_1} \otimes \mathfrak{p}^{\mathbb{C}}$,*

(e) *if $\lambda' = \lambda + \beta_0$, then τ_{Λ_1} occurs in $U(v)$ exactly twice, once because $[\tau_{\Lambda_1}|_{K \cap M^\#} : \tau_\lambda] = 1$ and once because $[\tau_{\Lambda_1}|_{K \cap M^\#} : \tau_{\lambda'}] = 1$ and $[\sigma^\# |_{K \cap M^\#} : \tau_{\lambda'}] = 1$.*

Then

(1) *the vector $v_2 = E_{\Lambda_2}(E_{\Lambda_1}(v_0 \otimes X_\alpha) \otimes X_{\beta_0})$ is not zero, and*

(2) *$\langle P_{\Lambda_2} U(v, X_{\beta_0})\ P_{\Lambda_1} U(v, X_\alpha)\ f_0(k), u' \rangle_{\Lambda + \alpha + \beta_0} = \frac{1}{4}\ |\alpha|^4\ d(v)$ $\langle \tau_{\Lambda_2}(k)^{-1} v_2, v_2 \rangle$ for a suitable normalization (independent of v) of a highest weight vector u' of $V^{\lambda'} \subseteq V^{\sigma^\#}$, where*

$$d(v) = v(X_\alpha + X_{-\alpha}) + v_0^+$$
$$- \# \left\{ \beta \in \Delta_n \mid \beta - \alpha \in \Delta, \frac{2\langle \Lambda + \beta_0, \beta - \alpha \rangle}{|\beta - \alpha|^2} \geqslant 1 \right\}$$

$$+\ \#\left\{\beta\in\Delta_n\,|\,\beta-\alpha\in\Delta,\frac{2\langle\Lambda,\beta-\alpha\rangle}{|\beta-\alpha|^2}\geqslant 1\right\}$$

$$+\frac{1}{2}\ \#\left\{\beta\in\Delta_n\,|\,\beta-\alpha\in\Delta,\ |\beta|^2<|\alpha|^2,\frac{2\langle\Lambda+\beta_0,\beta-\alpha\rangle}{|\beta-\alpha|^2}=1\right\}$$

$$-\frac{1}{2}\ \#\left\{\beta\in\Delta_n\,|\,\beta-\alpha\in\Delta,\ |\beta|^2<|\alpha|^2,\frac{2\langle\Lambda,\beta-\alpha\rangle}{|\beta-\alpha|^2}=1\right\}.$$

Moreover if τ_{Λ_2} *has multiplicity at most one in* $U(v)$, *then*

(3) *the pair of K types* $\{\Lambda,\Lambda_2\}$ *exhibits* $J(\frac{1}{2}c\tilde{\alpha})$ *as not infinitesimally unitary for* $c>d(0)$.

Remark. Theorem 8.1 has a dual result obtained by reflection in α. In it we use $\Lambda_1=(\Lambda-\alpha)^\vee$, β_0 conjugate to $-\alpha$, and $\Lambda_2=(\Lambda-\alpha+\beta_0)^\vee$. The formula for $d(v)$ is changed by changing v_0^+ to v_0^- and by replacing α everywhere by $-\alpha$.

Proof of conclusion (1). We begin by defining u'. If γ is in $\Delta_K\cap\Delta_-$, then $\langle\beta_0,\gamma\rangle=0$ since otherwise one of $\pm\beta_0\pm\gamma$ would contradict the uniqueness in assumption (b). Since α is long, it follows that

$$\mathbb{C}X_{-\beta_0}+\mathbb{C}H_{\beta_0}+\mathbb{C}X_{\beta_0}$$

is an ideal in $\mathfrak{m}^{\mathbb{C}}$, and $\mathfrak{m}$ is isomorphic to the sum of $\mathfrak{sl}(2,\mathbb{R})$ and the Lie algebra of a compact group. The root β_0 is the positive root of the $\mathfrak{sl}(2,\mathbb{R})$ factor of $\mathfrak{m}$.

Meanwhile λ is the Blattner parameter for the representation $\sigma^\#$ of $M^\#$, and the theory of $\mathfrak{sl}(2,\mathbb{R})$ tells us that the other $K\cap M^\#$ types of $\sigma^\#$ have highest weights $\lambda+\beta_0$, $\lambda+2\beta_0$, ... The element $\sigma^\#(X_{\beta_0})^*$ carries $\tau_{\lambda+n\beta_0}$ onto $\tau_{\lambda+(n-1)\beta_0}$ if $n>0$, again by the theory for $\mathfrak{sl}(2,\mathbb{R})$. Thus we define u' to be the highest weight vector of $\tau_{\lambda+\beta_0}$ such that

$$\sigma^\#(X_{\beta_0})^*u'=u_0. \tag{8.1}$$

This will be the normalization of u' that we use in conclusion (1).

In Lemmas 6.1 and 6.3, we take $\beta_1=\alpha$ and $\beta_2=\beta_0$, using the element p supplied by assumption (3). The result is an element w satisfying

$$\mathrm{Ad}(w)X_\alpha=cX_{\beta_0}\qquad\text{and}\qquad\mathrm{Ad}(w)X_{\beta_0}=\bar{c}X_\alpha\text{ with }|c|=1.$$

Moreover we can take

$$v_1=E_{\Lambda_1}(v_0\otimes X_\alpha)$$

$$v_1'=\tau_{\Lambda_1}(w)\,v_1=cE_{\Lambda_1}(v_0\otimes X_{\beta_0}),$$

the last equality following by Lemma 6.2, and then

$$E_{\Lambda_2}(E_{\Lambda_1}(v_0 \otimes X_\alpha) \otimes X_{\beta_0}) = E_{\Lambda_2}(E_{\Lambda_1}(v_0 \otimes X_{\beta_0}) \otimes X_\alpha) \tag{8.2a}$$

$$E_{\Lambda_2}(v_1 \otimes X_{\beta_0}) = \bar{c} E_{\Lambda_2}(v_1' \otimes X_\alpha). \tag{8.2b}$$

Theorem 1.5 and assumption (d) imply that the vectors v_1, v_1', and (8.2a) are all nonzero. In particular, this proves conclusion (1).

Proof of conclusion (2). Define

$$f_1(k) = P_{\Lambda_1} U(v, X_\alpha) f_0(k)$$
$$f_1'(k) = P_{\Lambda_1} U(v, X_{\beta_0}) f_0(k).$$

The same argument as with (5.31) shows that we can write

$$f_1(k) = B(v)\, \tau_{\Lambda_1}(k)^{-1} v_1 \tag{8.3}$$

for a unique $B(v)$ in $\mathrm{Hom}_{K \cap M^\#}(V^{\Lambda_1}, V^{\sigma^\#})$. Then the proof of (6.10) shows that

$$f_1'(k) = \bar{c} B(v)\, \tau_{\Lambda_1}(k)^{-1} v_1'. \tag{8.4}$$

Assumption (e) implies that we can write

$$B(v) = b_1(v)\, B_1 + b_1'(v)\, B_1', \tag{8.5}$$

where B_1 carries V^{Λ_1} to the V^λ subspace of $V^{\sigma^\#}$ and B_1' carries V^{Λ_1} to the $V^{\lambda'}$ subspace. Moreover the decomposition into the two terms in (8.5) is unique. We shall fix normalizations of B_1 and B_1' so that the whole decomposition (8.5) is unique. First we write

$$B(v)^* = \overline{b_1(v)}\, B_1^* + \overline{b_1'(v)}\, B_1'^* \tag{8.6a}$$

with B_1^* mapping the V^λ subspace of $V^{\sigma^\#}$ into V^{Λ_1} and with $B_1'^*$ mapping the $V^{\lambda'}$ subspace of $V^{\sigma^\#}$ into V^{Λ_1}. The vectors $B_1^* u_0$ and $B_1'^* u'$ are nonzero $K \cap M^\#$ highest weight vectors within V^{Λ_1} of respective weights λ and λ', and assumption (e) says these highest weight vectors are determined up to scalars. Since α is long, it is a simple matter to check that v_1 (of weight $\Lambda + \alpha$) is $K \cap M^\#$ highest of weight λ and v_1' (of weight $\Lambda + \beta_0$) is $K \cap M^\#$ highest of weight λ'. We normalize B_1^* and $B_1'^*$ by the conditions

$$B_1^* u_0 = v_1 \qquad \text{and} \qquad B_1'^* u' = v_1'. \tag{8.6b}$$

We shall apply Theorem 5.1 to compute $\langle f_1(k), u_0 \rangle_{\Lambda + \alpha}$, and we shall compute $\langle f_1'(k), u' \rangle_{\Lambda + \beta_0}$ directly. In Theorem 5.1 with $\mu' = \Lambda$, condition (a) is satisfied since $\Lambda - (\Lambda + 2\alpha) = -2\alpha$ is not the sum of positive compact roots, condition (b) holds by assumption (d), and condition (c) is satisfied trivially. Since

$$\langle f_0(k), u_0\rangle_\Lambda = \langle \tau_\Lambda(k)^{-1} v_0, v_0\rangle$$

by (4.4) and (4.5), Theorems 5.1 and 2.1 give

$$\langle f_1(k), u_0\rangle_{\Lambda+\alpha} = \frac{|\alpha|^2}{4}\,(v_0^+ + v(X_\alpha + X_{-\alpha}))\langle \tau_{\Lambda_1}(k)^{-1} v_1, v_1\rangle. \tag{8.7}$$

To compute $\langle f_1'(k), u'\rangle_{\Lambda+\beta_0}$, we review the beginning of the proof of Theorem 5.1 and see that

$$\begin{aligned}
&\langle f_1'(k), u'\rangle_{\Lambda+\beta_0}\\
&\quad= \tfrac{1}{4}\,|\alpha|^2 [(v+\rho)(X_\alpha + X_{-\alpha})]\langle v_1', \pi(k)(A^*u' \otimes (X_\alpha + X_{-\alpha}))\rangle_{\Lambda+\beta_0}\\
&\qquad+ \sum_{\beta\in\Delta_n} \tfrac{1}{2}\,|\beta|^2 \langle v_1', \pi(k)(A^*\sigma^\#(P_{\mathfrak{m}} X_\beta)^* u' \otimes X_\beta)\rangle_{\Lambda+\beta_0}\\
&\qquad+ \sum_{\beta\in\Delta_n} \tfrac{1}{2}\,|\beta|^2 \langle v_1', \pi(k)(\tau_{\Lambda_1}(P_{\mathfrak{k}} X_\beta)^* A^* u' \otimes X_\beta)\rangle_{\Lambda+\beta_0},
\end{aligned} \tag{8.8}$$

with A as in (4.4). Let us refer to the terms as the $\mathfrak{a}$ term, the $\mathfrak{m}$ terms, and the $\mathfrak{k}$ terms, respectively. Here A carries V^Λ to the V^λ subspace of $V^{\sigma^\#}$, and $A^*: V^\lambda \to V^\Lambda$ vanishes on the $V^{\lambda'}$ subspace. Consequently the $\mathfrak{a}$ term and the $\mathfrak{k}$ terms have $A^*u' = 0$. Thus only the $\mathfrak{m}$ terms of (8.8) survive. Since α is long, the only β's in Δ_n that have $P_{\mathfrak{m}} X_\beta \neq 0$ are roots orthogonal to α. By assumption (b), only β_0 and $-\beta_0$ can contribute. For $\beta = -\beta_0$, $\sigma^\#(P_{\mathfrak{m}} X_{-\beta_0})^* u'$ has weight $\lambda + 2\beta_0$ relative to $\mathfrak{b}_-$, and so A^* must annihilate this vector. Hence

$$\begin{aligned}
\langle f_1'(k), u'\rangle_{\Lambda+\beta_0} &= \tfrac{1}{2}\,|\beta_0|^2 \langle v_1', \pi(k)(A^*\sigma^\#(P_{\mathfrak{m}} X_{\beta_0})^* u' \otimes X_{\beta_0})\rangle_{\Lambda+\beta_0}\\
&= \tfrac{1}{2}\,|\alpha|^2 \langle v_1', \pi(k)(A^* u_0 \otimes X_{\beta_0})\rangle_{\Lambda+\beta_0}\\
&= \tfrac{1}{2}\,|\alpha|^2 \langle v_1', \pi(k)(v_0 \otimes X_{\beta_0})\rangle
\end{aligned}$$

and

$$\langle f_1'(k), u'\rangle_{\Lambda+\beta_0} = \tfrac{1}{2}\,|\alpha|^2 \langle \tau_{\Lambda_1}(k)^{-1} v_1', v_1'\rangle. \tag{8.9}$$

Now we can compute $b_1(v)$ and $b_2(v)$. From (8.3), we have

$$\begin{aligned}
\langle f_1(k), u_0\rangle_{\Lambda+\alpha} &= \langle B(v)\,\tau_{\Lambda_1}(k)^{-1}\, v_1, u_0\rangle_{\Lambda+\alpha}\\
&= \langle \tau_{\Lambda_1}(k)^{-1} v_1, B(v)^* u_0\rangle_{\Lambda+\alpha}\\
&= b_1(v)\langle \tau_{\Lambda_1}(k)^{-1} v_1, v_1\rangle,
\end{aligned}$$

so that (8.7) gives

$$b_1(v)=\frac{|\alpha|^2}{4}(v_0^+ + v(X_\alpha + X_{-\alpha})). \tag{8.10a}$$

From (8.4), we have

$$\begin{aligned}\langle f_1'(k), u'\rangle_{\Lambda+\beta_0} &= \bar{c}\langle B(v)\,\tau_{\Lambda_1}(k)^{-1}v_1', u'\rangle_{\Lambda+\beta_0}\\ &= \bar{c}\langle \tau_{\Lambda_1}(k)^{-1}v_1', B(v)^* u'\rangle_{\Lambda+\beta_0}\\ &= \bar{c}b_1'(v)\langle \tau_{\Lambda_1}(k)^{-1}v_1', v_1'\rangle,\end{aligned}$$

so that (8.9) gives

$$\bar{c}b_1'(v)=\tfrac{1}{2}|\alpha|^2. \tag{8.10b}$$

Turning attention to the second step taken by $U(v)$, we run through as much of the proof of Theorem 5.1 as we can, obtaining

$$\begin{aligned}&\langle P_{\Lambda_2}U(v, X_{\beta_0})\, f_1(k), u'\rangle_{\Lambda+\alpha+\beta_0}\\ &\quad=\tfrac{1}{4}|\alpha|^2[(v+\rho)(X_\alpha+X_{-\alpha})]\\ &\qquad\times\langle E_{\Lambda_2}(v_1\otimes X_{\beta_0}), \pi(k)(B(v)^*u'\otimes(X_\alpha - X_{-\alpha}))\rangle_{\Lambda+\alpha+\beta_0}\\ &\qquad+\sum_{\beta\in\Delta_n}\tfrac{1}{2}|\beta|^2\langle E_{\Lambda_2}(v_1\otimes X_{\beta_0}), \pi(k)(B(v)^*\sigma^{\#}(P_{\mathfrak{m}}X_\beta)^*u'\otimes X_\beta)\rangle_{\Lambda+\alpha+\beta_0}\\ &\qquad+\sum_{\beta\in\Delta_n}\tfrac{1}{2}|\beta|^2\langle E_{\Lambda_2}(v_1\otimes X_{\beta_0}), \pi(k)(\tau_{\Lambda_1}(P_{\mathfrak{k}}X_\beta)^*u'\otimes X_\beta)\rangle_{\Lambda+\alpha+\beta_0}.\end{aligned}$$

In the $\mathfrak{a}$ term and the $\mathfrak{k}$ terms, we have $B(v^*)u'=\overline{b_1'(v)}\,v_1'$. If we make this substitution, we see that the $\mathfrak{a}$ and $\mathfrak{k}$ terms are just what Theorem 5.1 calculates when $\mu'=\Lambda+\beta_0$, apart from the common factor $b_1'(v)$. Meanwhile the $\mathfrak{m}$ terms are all 0 except possibly for $\beta=\beta_0$ and $\beta=-\beta_0$, by assumption (b). The term with $\beta=-\beta_0$ involves $\sigma^{\#}(X_{-\beta_0})^*u'$, which has $\mathfrak{b}_-$ weight $\lambda+2\beta_0$ and is annihilated by $B(v)^*$. The term with $\beta=\beta_0$ involves $B(v)^*\sigma^{\#}(P_{\mathfrak{m}}X_{\beta_0})^*u'=\overline{b_1(v)}\,v_1$. Hence

$$\begin{aligned}&\langle P_{\Lambda_2}U(v, X_{\beta_0})\, f_1(k), u'\rangle_{\Lambda+\alpha+\beta_0}\\ &\quad=b_1'(v)\frac{|\alpha|^2}{4}\langle E_{\Lambda_2}(v_1\otimes X_{\beta_0}), \pi(k)(v_1'\otimes X_\alpha)\rangle\\ &\qquad\times\Bigg[(v+\rho)(X_\alpha+X_{-\alpha})+\frac{2\langle\Lambda+\beta_0,\alpha\rangle}{|\alpha|^2}\\ &\qquad-2\#\left\{\beta\in\Delta_n\,|\,\beta-\alpha\in\Delta,\ |\beta|^2=|\alpha|^2, \frac{2\langle\Lambda+\beta_0,\beta-\alpha\rangle}{|\beta-\alpha|^2}\geqslant 1\right\}\end{aligned}$$

$$-2\# \left\{\beta \in \Delta_n \mid \beta-\alpha \in \Delta,\ |\beta|^2<|\alpha|^2, \frac{2\langle \Lambda+\beta_0, \beta-\alpha\rangle}{|\beta-\alpha|^2} \geqslant 2\right\}$$

$$\left. - \# \left\{\beta \in \Delta_n \mid \beta-\alpha \in \Delta,\ |\beta|^2<|\alpha|^2, \frac{2\langle \Lambda+\beta_0, \beta-\alpha\rangle}{|\beta-\alpha|^2} = 1\right\}\right]$$

$$+ b_1(v) \frac{|\alpha|^2}{2} \langle E_{\Lambda_2}(v_1 \otimes X_{\beta_0}), \pi(k)(v_1 \otimes X_{\beta_0})\rangle.$$

Let us write $d_1(v)$ for the expression in brackets. Substitution from (8.2b) and (8.10) gives

$$\langle P_{\Lambda_2} U(v, X_{\beta_0}) f_1(k), u'\rangle_{\Lambda+\alpha+\beta_0}$$
$$= \frac{|\alpha|^4}{8} \langle E_{\Lambda_2}(v_1 \otimes X_{\beta_0}), \pi(k)(v_1 \otimes X_{\beta_0})\rangle [d_1(v) + v_0^+ + v(X_\alpha + X_{-\alpha})]. \tag{8.11}$$

In the expression for $d_1(v)$, we substitute for

$$\rho(X_\alpha + X_{-\alpha}) + 2\langle \Lambda, \alpha\rangle/|\alpha|^2$$

from Theorem 2.1c (taking into account assumption (d)), and we obtain

$$d_1(v) + v_0^+ + v(X_\alpha + X_{-\alpha})$$
$$= 2v(X_\alpha + X_{-\alpha}) + 2v_0^+$$
$$-2\# \left\{\beta \in \Delta_n \mid \beta-\alpha \in \Delta, \frac{2\langle \Lambda+\beta_0, \beta-\alpha\rangle}{|\beta-\alpha|^2} \geqslant 1\right\}$$
$$+2\# \left\{\beta \in \Delta_n \mid \beta-\alpha \in \Delta, \frac{2\langle \Lambda, \beta-\alpha\rangle}{|\beta-\alpha|^2} \geqslant 1\right\}$$
$$+ \# \left\{\beta \in \Delta_n \mid \beta-\alpha \in \Delta,\ |\beta|^2<|\alpha|^2, \frac{2\langle \Lambda+\beta_0, \beta-\alpha\rangle}{|\beta-\alpha|^2} = 1\right\}$$
$$- \# \left\{\beta \in \Delta_n \mid \beta-\alpha \in \Delta,\ |\beta|^2<|\alpha|^2, \frac{2\langle \Lambda, \beta-\alpha\rangle}{|\beta-\alpha|^2} = 1\right\}.$$

Substituting into (8.11), we obtain conclusion (2).

Proof of conclusion (3). If we refer to [1], we see that Theorem 4.1a remains valid if u_0 is replaced by u' in (4.6). We are assuming τ_{Λ_2} has multiplicity $\leqslant 1$ in $U(v)$, and conclusions (1) and (2) show it has multiplicity $\geqslant 1$. The right side of conclusion (2) is of the form $I(k)\, d(v)$, and $I(k)$ is not identically 0. Then the modified Theorem 4.1a says we have nonunitarity when $d(-v)/d(v)$ is negative, and this is conclusion (3).

COROLLARY 8.2. *Suppose $n \geqslant 2$ and $\mathfrak{g} = \mathfrak{so}(2n, 2)$, possibly with abelian and compact factors, and suppose in the terminology of* [16, 5] *that σ is nondegenerate and that the special basic case for λ_0 is all of Δ. Then there is a choice $\pm$ of sign to that $\pm\alpha$ is conjugate by W_K to the unique positive noncompact root β_0 orthogonal to α; fix this choice of sign. Put $\Lambda_2 = (\Lambda \pm \alpha + \beta_0)^\vee = \Lambda \pm \alpha + \beta_0$. Then τ_{Λ_2} has multiplicity one in $U(\frac{1}{2}c\tilde{\alpha})$, and the signature of the standard form on τ_{Λ_2} is $\operatorname{sgn}(v_{0,L}^{\pm} - c)$, where $v_{0,L}^{+}$ and $v_{0,L}^{-}$ are the quantities v_0^+ and v_0^- computed in an $\mathfrak{su}(n, 1)$ subdiagram containing α and generated by simple roots of Δ^+.*

Proof. In standard notation let

$$\Delta = \{\pm e_i \pm e_m \mid 1 \leqslant i \leqslant n+1,\ 1 \leqslant m \leqslant n+1,\ m \neq i\}$$

and let the noncompact roots be given in terms of an index j by

$$\Delta_n = \{\pm e_i \pm e_j \mid 1 \leqslant i \leqslant n+1,\ i \neq j\}.$$

We may assume that the simple roots are $e_1 - e_2, ..., e_n - e_{n+1}, e_n + e_{n+1}$ as usual. If $\alpha = e_j - e_{j+1}$, then $\beta_0 = e_j + e_{j+1}$ and β_0 is conjugate to α by W_K. If $\alpha = e_{j-1} - e_j$, then $\beta_0 = e_{j-1} + e_j$ is conjugate to $-\alpha$ by W_K. Because of the possibility of reflecting in α, we may assume that we are in the first case.

Thus $\alpha = e_j - e_{j+1}$ and $\beta_0 = e_j + e_{j+1}$ for some j with $1 \leqslant j \leqslant n$. Since we are in a special basic case, Lemma 2.2 of [5] and a little computation show that $\Lambda = ae_j$ with $a \in \mathbb{Z}$. Actually one can show $a > 0$ from Table 2.1 and (1.3) of [5]. Hence $W_{K,\Lambda} = W_K$. So conditions (a), (b), and (c) in Theorem 8.1 are obviously satisfied. Condition (d) is satisfied because of Corollary 1.4a.

Let us consider (e), taking $\Lambda_1 = (\Lambda + \alpha)^\vee$. The Blattner parameter λ is the projection of Λ orthogonal to α, hence is $\lambda = \frac{1}{2}a(e_j + e_{j+1}) = \frac{1}{2}a\beta_0$. As we showed at the beginning of the proof of Theorem 8.1, $\mathfrak{m}$ is the sum of $\mathfrak{sl}(2, \mathbb{R})$ and a compact subalgebra. By the theory for $\mathfrak{sl}(2, \mathbb{R})$, the $K \cap M^\#$ types of $\sigma^\#$ have highest weights $\lambda + m\beta_0$, $m \geqslant 0$, relative to $\mathfrak{b}_-$, and they have multiplicity one in $\sigma^\# |_{K \cap M^\#}$. For $\lambda + m\beta_0$ to extend to a weight $\lambda + m\beta_0 + c\alpha$ of τ_{Λ_1} relative to $\mathfrak{b}$, we must have

$$\lambda + m\beta_0 + c\alpha = \Lambda + \alpha + \left(\sum \text{compact roots}\right).$$

Extracting the coefficient of e_j, we obtain $\frac{1}{2}a + m + c = a + 1$. Thus $c = \frac{1}{2}a - m + 1$ and the weight is

$$\tfrac{1}{2}(a + 2m)\,\beta_0 + \tfrac{1}{2}(a - 2m + 2)\alpha. \tag{8.12}$$

Now $\Lambda+\alpha=(a+1)e_j-e_{j+1}$ is an extreme weight of τ_{Λ_1} and the e_j term is the same for all weights of τ_{Λ_1}. Thus the coefficient of e_{j+1} in any weight must be $\leqslant 1$ in absolute value. Applying this fact to the weight (8.12), we see that

$$\tfrac{1}{2}(a+2m)-\tfrac{1}{2}(a-2m+2)=2m-1 \qquad \text{is } \pm 1.$$

Thus $m=0$ or $m=-1$. In these cases, the weight (8.12) is $\Lambda+\alpha$ or $\Lambda+\beta_0$, and it is extreme, hence of multiplicity one. Thus τ_{Λ_1} has multiplicity at most two in $U(v)|_K$. Since α is long, it is a simple matter to check that the weight vectors for weights $\Lambda+\alpha$ and $\Lambda+\beta_0$ are indeed $K\cap M^{\#}$ highest. Then (e) follows.

Moreover $\Lambda+\alpha+\beta_0=\Lambda+2e_j=(a+2)e_j$ is Δ_K^+ dominant and is orthogonal to all members of Δ_K. Thus τ_{Λ_2} is one-dimensional. Consequently τ_{Λ_2} has multiplicity at most one in $U(v)$.

Therefore conclusion (3) of Theorem 8.1 is applicable. Since all roots have the same length and since Λ is orthogonal to all members of Δ_K, we have

$$\begin{aligned} d(0) &= v_0^+ - \#\left\{\beta\in\Delta_n \mid \beta-\alpha\in\Delta, \frac{2\langle\beta_0,\beta-\alpha\rangle}{|\beta-\alpha|^2}\geqslant 1\right\}\\ &= v_0^+ - \#\{\beta \mid \beta=e_j\pm e_i \text{ with } i\neq j \text{ and } i\neq j+1\}\\ &= v_0^+ - 2\#\{\beta \mid \beta=e_j+e_i \text{ with } i\neq j \text{ and } i\neq j+1\}\\ &= 1+\frac{2\langle\mu,\alpha\rangle}{|\alpha|^2}+2\#\{\beta\in\Delta_n^+ \mid \beta-\alpha\in\Delta\}\\ &\qquad -2\#\{\beta \mid \beta=e_j+e_i \text{ with } i\neq j \text{ and } i\neq j+1\}\\ &= v_{0,L}^+. \end{aligned}$$

The $\mathfrak{su}(n,1)$ diagram can be taken as the one with roots $\pm(e_i-e_m)$, and the result follows.

COROLLARY 8.3. *Suppose $n\geqslant 2$ and $\mathfrak{g}=\mathfrak{so}(2n+1,2)$, possibly with abelian and compact factors, suppose that α is long, and suppose in the terminology of* [16, 5] *that σ is nondegenerate and that the special basic case for λ_0 is all of Δ. Then there is a choice $\pm$ of sign so that $\pm\alpha$ is conjugate by W_K to the unique positive noncompact root β_0 orthogonal to α; fix this choice of sign. Put $\Lambda_2=(\Lambda\pm\alpha+\beta_0)^\vee=\Lambda\pm\alpha+\beta_0$. Then τ_{Λ_2} has multiplicity one in $U(\frac{1}{2}c\tilde{\alpha})$, and the signature of the standard form on τ_{Λ_2} is $\operatorname{sgn}(v_{0,L}^{\pm}+1-c)$, where $v_{0,L}^+$ and $v_{0,L}^-$ are the quantities v_0^+ and v_0^- computed in an $\mathfrak{su}(n,1)$ subdiagram containing α and generated by simple roots of Δ^+.*

Proof. In standard notation let

$$\begin{aligned}\Delta &= \{\pm e_i \pm e_m \mid 1 \leqslant i \leqslant n+1 \text{ and } 1 \leqslant m \leqslant n+1, m \neq i\} \\ &\cup \{\pm e_i \mid 1 \leqslant i \leqslant n+1\},\end{aligned}$$

and let the noncompact roots be given in terms of an index j by

$$\Delta_n = \{\pm e_j\} \cup \{\pm e_i \pm e_j \mid 1 \leqslant i \leqslant n+1, i \neq j\}.$$

We may assume that the simple roots are $e_1 - e_2, ..., e_n - e_{n+1}, e_{n+1}$ as usual. Normalizing matters as in Corollary 8.2, we may take $\alpha = e_j - e_{j+1}$ and $\beta_0 = e_j + e_{j+1}$, so that β_0 is conjugate to α by W_K.

As in Corollary 8.2, we have $\Lambda = ae_j$ with $a \in \mathbb{Z}$ and $a > 0$. Hence $W_{K,\Lambda} = W_K$. Then (a), (b), and (c) of Theorem 8.1 are certainly satisfied, but (d) needs to be checked. Let $\Lambda_1 = (\Lambda + a)^\vee$. We have

$$\Lambda_1 = (ae_j + (e_j - e_{j+1}))^\vee = \begin{cases} e_1 + (a+1)e_j & \text{if } j > 1 \\ e_2 + (a+1)e_1 & \text{if } j = 1, \end{cases}$$

and in either case, $(\Lambda + \alpha)^\vee = \Lambda + \beta$ with β long. By Theorem 1.3, τ_{Λ_1} occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$. Also Λ_2 has

$$\Lambda_2 = (\Lambda + \alpha + \beta_0)^\vee = ((a+2)e_j)^\vee = (a+2)e_j, \tag{8.13}$$

and this is $\Lambda_1 + \beta$ with β long. Again by Theorem 1.3, τ_{Λ_2} occurs in $\tau_{\Lambda_1} \otimes \mathfrak{p}^{\mathbb{C}}$. This proves (d).

The proof of (e) is the same as in Corollary 8.2. Moreover τ_{Λ_2} is one-dimensional, by (8.13), and hence has multiplicity at most one in $U(v)|_K$.

Therefore conclusion (3) of Theorem 8.1 is applicable. Since Λ is orthogonal to all members of Δ_K and since $2\langle \beta_0, \gamma \rangle / |\gamma|^2$ is even when $\gamma \in \Delta_K$ is short, we have

$$\begin{aligned} d(0) &= v_0^+ - \# \left\{ \beta \in \Delta_n \mid \beta - \alpha \in \Delta, \frac{2\langle \beta_0, \beta - \alpha \rangle}{|\beta - \alpha|^2} \geqslant 1 \right\} \\ &= v_0^+ - 2\# \{\beta \mid \beta = e_j + e_i \text{ with } i \neq j \text{ and } i \neq j+1\} - \# \{\beta \mid \beta = e_j\} \\ &= 1 + \frac{2\langle \mu, \alpha \rangle}{|\alpha|^2} + 2\# \{\beta \in \Delta_n^+ \mid \beta - \alpha \in \Delta\} \\ &\quad - 2\# \{\beta \mid \beta = e_j + e_i \text{ with } i \neq j \text{ and } i \neq j+1\} - \# \{\beta \mid \beta = e_j\} \\ &= v_{0,L}^+ + \# \{\beta \mid \beta = e_j\} \\ &= v_{0,L}^+ + 1. \end{aligned}$$

The $\mathfrak{su}(n, 1)$ diagram is the one with roots $\pm(e_i - e_m)$, and the result follows.

9. Two-Step Formula Applicable to Gap in $SO(2n, 3)$

The third kind of specialized result is applicable in certain situations in $SO(2n, 3)$ and establishes a gap of nonunitarity that is only half the width of the gap that occurs in Section 7 for $Sp(n, 1)$. We shall state the result in the same generality as in [5].

THEOREM 9.1. *With $l=1$ and $\{\alpha_1, ..., \alpha_l\} = \{\alpha\}$, suppose $n \geqslant 2$ and $\mathfrak{g} = \mathfrak{so}(2n, 3)$. Suppose further that α is long, that σ is nondegenerate in the sense of [16], that the short Δ^+ simple root ε is basic (in the terminology of [5]), and that the special basic case for λ_0 is the maximal $\mathfrak{su}(n, 1)$ diagram containing α that is generated by simple roots of Δ^+. Let ζ be the sum of the simple roots strictly between α and ε in the Dynkin diagram, and suppose ζ is (nonzero and) noncompact. Put $\Lambda_1^- = (\Lambda - \alpha)^\vee$ and $\Lambda_2 = (\Lambda + (\zeta + \varepsilon))^\vee = \Lambda + \zeta + \varepsilon$. Then*

(1) *τ_{Λ_2} has multiplicity one in $U(\nu)|_K$*

(2) $$\langle P_{\Lambda_2} U(\nu, X_{e_j}) P_{\Lambda_1^-} U(\nu, X_{-\alpha}) f_0(k),\ u_0 \rangle_\Lambda = \frac{|\alpha|^4}{16} [(\nu(X_\alpha + X_{-\alpha}) + \nu_0^-)(\nu(X_\alpha + X_{-\alpha}) + \nu_0^- - 1] \times \langle \tau_{\Lambda_2}(k)^{-1} v_2, v_2 \rangle,$$

where

$$v_2 = E_{\Lambda_2}(E_{\Lambda_1^-}(v_0 \otimes X_{-\alpha}) \otimes X_\alpha) \tag{9.1}$$

(3) *the vector v_2 in (9.1) is not zero*

(4) *the signature of the standard form for $U(\frac{1}{2}c\tilde{\alpha})|_K$ on τ_{Λ_2} is* $\operatorname{sgn}(\nu_0^- - c)(\nu_0^- - 1 - c)$.

Preliminaries. In standard notation let

$$\Delta = \{\pm e_i \pm e_m \mid 1 \leqslant i \leqslant n+1,\ 1 \leqslant m \leqslant n+1,\ m \neq i\} \cup \{\pm e_i \mid 1 \leqslant i \leqslant n+1\},$$

and let the simple roots be $e_1 - e_2, ..., e_n - e_{n+1}, e_{n+1}$ as usual. Then we have $\varepsilon = e_{n+1}$, and we define j by the condition $\alpha = e_j - e_{j+1}$, $1 \leqslant j \leqslant n$. The root $\zeta = e_{j+1} - e_{n+1}$ is assumed noncompact. Therefore j is $< n$, and the noncompact simple roots are exactly

$$e_j - e_{j+1},\ e_{j+1} - e_{j+2},\ e_{n+1},$$

in order that $\mathfrak{g} \cong \mathfrak{so}(2n, 3)$. So

$$\Delta_K = \{\pm e_i \pm e_m \mid i \neq j+1, m \neq j+1, i \neq m\} \cup \{\pm e_{j+1}\}.$$

Notice that $\{\pm e_{j+1}\}$ corresponds to an $\mathfrak{su}(2)$ factor of $\mathfrak{k}$. By assumption the special basic case is to correspond to all roots $\pm(e_i - e_m)$.

Computing with the formulas of [5] (namely (1.3), Table 2.1, and Lemma 2.2), we find

$$\Lambda = ae_{j+1} + \sum_{i=1}^{n+1} \tfrac{1}{2} e_i \tag{9.2}$$

with $a \in \mathbb{Z}$ and $a \geqslant 0$. Then

$$\Lambda - \alpha = (a+1) e_{j+1} + \sum_{i \neq j} \tfrac{1}{2} e_i - \tfrac{1}{2} e_j$$

$$(\Lambda - \alpha)^\vee = \Lambda - \alpha + (e_j - e_{n+1}) = \Lambda + (e_{j+1} - e_{n+1}).$$

This is of the form $\Lambda + \delta^-$ with δ^- long, and thus Theorem 1.3 shows that

$$\tau_{\Lambda_1^-} \text{ occurs in } \tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}. \tag{9.3}$$

Since

$$\Lambda + \alpha = (a-1) e_{j+1} + \tfrac{3}{2} e_j + \sum_{i \neq j} \tfrac{1}{2} e_i,$$

we see from the presence of the 3/2 that

$$\Lambda + \alpha \text{ is not a weight of } \tau_{\Lambda_1^-}. \tag{9.4}$$

Meanwhile

$$\Lambda_2 = (\Lambda + e_{j+1})^\vee = \Lambda + e_{j+1} = (a+1) e_{j+1} + \sum_{i=1}^{n+1} \tfrac{1}{2} e_i. \tag{9.5a}$$

Notice that another formula for Λ_2 is

$$\Lambda_2 = \Lambda - \alpha + e_j. \tag{9.5b}$$

We have

$$\Lambda_2 = \Lambda + e_{j+1} = \Lambda_1^- - \delta^- + e_{j+1} = \Lambda_1^- + e_{n+1}. \tag{9.6}$$

The only short Δ_K^+ simple root is e_{j+1}, and this is not in $\Delta_{K,\Lambda}$ since by assumption e_{j+1} is not in the special basic case; hence Theorem 1.3 and the formula (9.6) show that

$$\tau_{\Lambda_2} \text{ occurs in } \tau_{\Lambda_1^-} \otimes \mathfrak{p}^{\mathbb{C}}. \tag{9.7}$$

Proof of conclusion (1). First let us show that

$$[U(v)|_K : \tau_{\Lambda_2}] = [\tau_{\Lambda_2}|_{K \cap M^\#} : \tau_\lambda] \tag{9.8}$$

by showing that (3.18) has no solutions. If there is a solution, we have

$$\Lambda_2 = \Lambda + \sum_{\beta \in \Delta^+_{-,n}} k_\beta \beta + \sum_{\gamma \in \Delta^+_K} n_\gamma \gamma + m\alpha \tag{9.9}$$

with integer coefficients, $k_\beta \geqslant 0$, $n_\gamma \geqslant 0$, and some $k_\beta > 0$. All the β's and γ's in (9.9) have zero coefficient for e_i if $i < j$, as we see inductively by taking the inner product of (9.9) with the dominant form $e_1 + \cdots + e_i$. Consequently no γ in (9.9) is of the form $-e_j + e_i$. The projection of (9.9) to the span of e_j and e_{j+1} is therefore

$$e_{j+1} = k(e_j + e_{j+1}) + ne_{j+1} + m(e_j - e_{j+1}) \qquad \text{with} \quad k \geqslant 0, n \geqslant 0.$$

The coefficients of e_j give $0 = k + m$. So $m = -k$, and the coefficients of e_{j+1} therefore give $1 = k + n - (-k)$. So $2k + n = 1$, and we conclude $k = 0$ and $n = 1$. The remaining contribution to (9.9) satisfies

$$0 = \sum (\text{positive roots})$$

and hence is all zero. Thus $\sum k_\beta \beta = 0$, and (3.18) has no solutions.

Next let us see that λ extends to a weight of τ_{Λ_2} in only one way, with the weight of multiplicity one. Thus suppose $\Lambda + c\alpha$ is a weight of τ_{Λ_2} with $c \in \mathbb{Z}$. From (9.2), we obtain

$$\Lambda + c\alpha = (a - c)e_{j+1} + (\tfrac{1}{2} + c)e_j + \sum_{i \neq j} \tfrac{1}{2} e_i.$$

From (9.5a), we see that any weight of τ_{Λ_2} must have e_j component between $-\frac{1}{2}$ and $+\frac{1}{2}$. Thus $c = 0$ or $c = -1$. If $c = -1$, the candidate for a weight is $\Lambda - \alpha$. Since (9.5b) gives

$$\Lambda_2 - (\Lambda - \alpha) = e_j,$$

which is a noncompact root, $\Lambda - \alpha$ does not differ from Λ_2 by the sum of compact roots. (See Problems 12 and 13 on p. 478 of [12].) Thus $\Lambda - \alpha$ is not a weight. Thus $c = 0$ is the only possibility, and the candidate for a weight is Λ. Since Λ differs from the highest weight Λ_2 only on the $\mathfrak{su}(2)$ factor of $\mathfrak{k}$, and since irreducible representations of $\mathfrak{su}(2)$ have all weights of multiplicity one, Λ has multiplicity one in τ_{Λ_2}. Hence λ extends to a weight of τ_{Λ_2} only as Λ, and Λ has multiplicity one. Consequently τ_λ occurs in $\tau_{\Lambda_2}|_{K \cap M^\#}$ at most once.

Hence τ_{Λ_2} occurs in $U(\nu)|_K$ at most once. The fact that it occurs at least once will follow from conclusions (2) and (3).

Proof of conclusion (2). Let

$$v_1 = E_{\Lambda_1^-}(v_0 \otimes X_{-\alpha}).$$

This is nonzero by (9.3) and Theorem 1.5. Define

$$f_1(k) = P_{\Lambda_1^-} U(\nu, X_{-\alpha}) f_0(k).$$

The same argument as with (5.31) shows that we can write

$$f_1(k) = B(\nu)\, \tau_{\Lambda_1^-}(k)^{-1} v_1 \tag{9.10}$$

for a unique $B(\nu)$ in $\mathrm{Hom}_{K \cap M^{\#}}(V^{\Lambda_1^-}, V^{\sigma^{\#}})$.

We can apply Theorem 5.1 with $\Lambda' = \Lambda$ to evaluate $\langle f_1(k), u_0 \rangle_{\Lambda - \alpha}$. The only nontrivial hypothesis is (b), which was verified above in (9.3). Since $\langle f_0(k), u_0 \rangle_\Lambda = \langle \tau_\Lambda(k)^{-1} v_0, v_0 \rangle$ by (4.4) and (4.5), Theorems 5.1 and 2.1 give

$$\langle f_1(k), u_0 \rangle_{\Lambda - \alpha} = \frac{|\alpha|^2}{4}(v_0^- + \nu(X_\alpha + X_{-\alpha})) \langle \tau_{\Lambda_1^-}(k)^{-1} v_1, v_1 \rangle. \tag{9.11}$$

Turning attention to the second step taken by $U(\nu)$, we run through as much of the proof of Theorem 5.1 as we can, obtaining

$$\begin{aligned}
&\langle P_{\Lambda_2} U(\nu, X_{e_j}) f_1(k), u_0 \rangle_\Lambda \\
&\quad = \tfrac{1}{4}|\alpha|^2 [(\nu + \rho)(X_\alpha + X_{-\alpha})] \\
&\qquad \times \langle E_{\Lambda_2}(v_1 \otimes X_{e_j}), \pi(k)(B(\nu)^* u_0 \otimes (X_\alpha + X_{-\alpha})) \rangle_\Lambda \\
&\qquad + \sum_{\beta \in \Delta_n} \tfrac{1}{2}|\beta|^2 \langle E_{\Lambda_2}(v_1 \otimes X_{e_j}), \pi(k)(\tau_{\Lambda_1^-}(P_{\mathfrak{k}} X_\beta)^* B(\nu)^* u_0 \otimes X_\beta) \rangle_\Lambda .
\end{aligned} \tag{9.12}$$

(The $\mathfrak{m}$ term gives 0 because of the calculation that proves (9.8).) In the first sum, only the terms in $B(\nu)^* u_0 \otimes (X_\alpha + X_{-\alpha})$ of weight Λ survive, and in the second sum, only terms in $\tau_{\Lambda_1^-}(P_{\mathfrak{k}} X_\beta)^* B(\nu)^* u_0 \otimes X_\beta$ of weight Λ survive. Since (9.4) shows that $\Lambda + \alpha$ is not a weight of $\tau_{\Lambda_1^-}$, we see that

$$\begin{aligned}
B(\nu)^* u_0 &= \frac{\langle B(\nu)^* u_0, v_1 \rangle}{|v_1|^2} v_1 + \text{irrelevant terms} \\
&= \overline{b(\nu)}\, v_1 + \text{irrelevant terms}.
\end{aligned}$$

Substituting into (9.12), we obtain

$$\begin{aligned}
&\langle P_{\Lambda_2} U(\nu, X_{e_j}) f_1(k), u_0 \rangle_\Lambda \\
&\quad = b(\nu) \Big\{ \tfrac{1}{4} |\alpha|^2 [(\nu+\rho)(X_\alpha + X_{-\alpha})] \\
&\qquad \times \langle E_{\Lambda_2}(v_1 \otimes X_{e_j}), \pi(k)(v_1 \otimes X_\alpha) \rangle_\Lambda \\
&\qquad + \sum_{\substack{\beta \in \Delta_n \\ \beta \perp \alpha}} \tfrac{1}{2} |\beta|^2 \langle E_{\Lambda_2}(v_1 \otimes X_{e_j}), \pi(k)(\tau_{\Lambda_1^-}(P_{\mathfrak{k}} X_\beta)^* v_1 \otimes X_\beta) \rangle_\Lambda \Big\}. \qquad (9.13)
\end{aligned}$$

Here we can compute $b(\nu)$ in the usual way: From (9.10) we have

$$\begin{aligned}
\langle f_1(k), u_0 \rangle_{\Lambda-\alpha} &= \langle B(\nu)\, \tau_{\Lambda_1^-}(k)^{-1} v_1, u_0 \rangle_{\Lambda-\alpha} \\
&= \langle \tau_{\Lambda_1^-}(k)^{-1} v_1, B(\nu)^* u_0 \rangle_{\Lambda-\alpha} \\
&= b(\nu) \langle \tau_{\Lambda_1^-}(k)^{-1} v_1, v_1 \rangle,
\end{aligned}$$

so that (9.11) gives

$$b(\nu) = \frac{|\alpha|^2}{4} (\nu_0^- + \nu(X_\alpha + X_{-\alpha})). \qquad (9.14)$$

Let us concentrate on the $\mathfrak{k}$ terms in (9.13). The only contribution from $P_{\mathfrak{k}} X_\beta$ comes from a term with $[X_{-\alpha}, X_\beta]$. We imitate a certain amount of the proof of Theorem 5.1, starting after (5.14). The term $\beta = -\alpha$ gives 0, and the term $\beta = \alpha$ gives

$$\frac{1}{4} |\alpha|^2 \langle E_{\Lambda_2}(v_1 \otimes X_{e_j}), \pi(k)(v_1 \otimes X_\alpha) \rangle \left\{ \frac{2\langle \Lambda - \alpha, \alpha \rangle}{|\alpha|^2} \right\}. \qquad (9.15)$$

For the remaining β's, the relevant term of $P_{\mathfrak{k}} X_\beta$ is $-(p+q)^{-1} [X_{-\alpha}, X_\beta]$ by (5.4b); here $\beta - p\alpha, ..., \beta + q\alpha$ is the α root string through β. Thus the relevant term of $\tau_{\Lambda_1^-}(P_{\mathfrak{k}} X_\beta)^* v_1 \otimes X_\beta$ is

$$(p+q)^{-1} \tau_{\Lambda_1^-} [X_\alpha, X_{-\beta}]\, v_1 \otimes X_\beta. \qquad (9.16)$$

If $\langle \Lambda - \alpha, \beta - \alpha \rangle \leqslant 0$ or if $\beta - \alpha$ is not a root, then the first factor of (9.16) gives 0. The remaining β's are those in the set

$$T = \{ \beta \in \Delta_n \mid \beta - \alpha \in \Delta \text{ and } \langle \Delta - \alpha, \beta - \alpha \rangle > 0 \}.$$

The long roots $\beta \in \Delta_n$ with $\beta - \alpha \in \Delta$ are $\beta = \pm e_i - e_{j+1}$ with $i \neq j$ and $i \neq j+1$. Then $\beta - \alpha = \pm e_i - e_j$. For such a root β, $-2\langle \alpha, \beta - \alpha \rangle / |\beta - \alpha|^2 = 1$, so that the condition to be in T is $\langle \Lambda, \beta - \alpha \rangle \geqslant 0$. From (9.2), we see that $\langle \Lambda, \beta-\alpha\rangle = 0$ for $\beta = e_i - e_{j+1}$ and $\langle \Lambda, \beta-\alpha\rangle < 0$ for $\beta = -e_i - e_{j+1}$. Thus the long roots in T are those in

$$T_1 = \{\beta \in \Delta_n \mid \beta = e_i - e_{j+1} \text{ with } i \neq j \text{ and } i \neq j+1\}.$$

The only short root $\beta \in \Delta_n$ with $\beta - \alpha \in \Delta$ is $\beta = e_j$, which has $\langle \Lambda - \alpha, \beta - \alpha\rangle = \langle (a+\frac{3}{2})\, e_{j+1}, e_{j+1}\rangle > 0$. Thus

$$T = T_1 \cup \{e_j\}.$$

The roots β in T_1 we can handle in the usual way: $\Lambda - \alpha + \beta$ is not a weight of τ_{Λ_2} since

$$\Lambda - \alpha + \beta = \Lambda + e_i - e_j = (a+\tfrac{1}{2})\, e_{j+1} + \tfrac{3}{2} e_i - \tfrac{1}{2} e_j + \tfrac{1}{2} \sum_{\text{other } k} e_k$$

and since 3/2 is too large to occur as a coefficient of a weight (except on e_{j+1}). Thus (9.16), when projected by E_{Λ_2}, is

$$\begin{aligned} &E_{\Lambda_2}((p+q)^{-1} \tau_{\Lambda_1^-} [X_\alpha, X_{-\beta}]\, v_1 \otimes X_\beta) \\ &\quad = -(p+q)^{-1} E_{\Lambda_2}(v_1 \otimes (\operatorname{ad}[X_\alpha, X_{-\beta}]) X_\beta) = -E_{\Lambda_2}(v_1 \otimes X_\alpha). \qquad (9.17) \end{aligned}$$

For $\beta = e_j$, which is the remaining root in T, we have $p+q=2$, and we shall prove below that

$$E_{\Lambda_2}(\tau_{\Lambda_1^-}[X_\alpha, X_{-e_j}]\, v_1 \otimes X_{e_j}) = \frac{2\langle \Lambda, e_{j+1}\rangle}{|e_{j+1}|^2} E_{\Lambda_2}(v_1 \otimes X_\alpha). \qquad (9.18)$$

Putting together formulas (9.13) through (9.18), we thus obtain

$$\begin{aligned} &\langle P_{\Lambda_2} U(\nu, X_{e_j}) f_1(k), u_0\rangle_\Lambda \\ &\quad = \frac{|\alpha|^4}{16} \langle E_{\Lambda_2}(v_1 \otimes X_{e_j}), \pi(k)(v_1 \otimes X_\alpha)\rangle [v_0^- + \nu(X_\alpha + X_{-\alpha})] \\ &\qquad + \left[(\nu+\rho)(X_\alpha + X_{-\alpha}) + \frac{2\langle \Lambda - \alpha, \alpha\rangle}{|\alpha|^2} - 2\#\{\beta \in T_1\} + \frac{2\langle \Lambda, e_{j+1}\rangle}{|e_{j+1}|^2} \right]. \end{aligned} \qquad (9.19)$$

Let us consider roots $\beta \in \Delta_n$ with $\beta + \alpha \in \Delta$. The only short such root is $\beta = -e_j$, and it has $\langle \Lambda, \beta + \alpha\rangle = \langle \Lambda, -e_{j+1}\rangle < 0$. The long such roots are $\beta = \pm e_i + e_{j+1}$, and the subset

$$T_2 = \{\beta \in \Delta_n \mid \beta + \alpha \in \Delta,\ |\beta| = |\alpha|,\ \langle \Lambda, \beta + \alpha\rangle > 0\}$$

works out as

$$= \{\beta \in \Delta_n \mid \beta = e_i + e_{j+1} \text{ with } i \neq j \text{ and } i \neq j+1\}.$$

Since T_2 and T_1 have the same number of elements, (9.3) and the equality $\mathrm{I} = v_0^-$ in Theorem 2.1 give

$$\rho(X_\alpha + X_{-\alpha}) - \frac{2\langle \Lambda, \alpha\rangle}{|\alpha|^2} - 2\#\{\beta \in T_1\} = v_0^-.$$

The last line of (9.19) is thus

$$\begin{aligned} &= v(X_\alpha + X_{-\alpha}) + v_0^- + \frac{2\langle \Lambda, \alpha\rangle}{|\alpha|^2} + \frac{2\langle \Lambda - \alpha, \alpha\rangle}{|\alpha|^2} + \frac{2\langle \Lambda, e_{j+1}\rangle}{|e_{j+1}|^2} \\ &= v(X_\alpha + X_{-\alpha}) + v_0^- + \frac{2\langle \Lambda, e_j\rangle}{|e_j|^2} - 2 \\ &= v(X_\alpha + X_{-\alpha}) + v_0^- - 1. \end{aligned}$$

Substituting into (9.19), we obtain the desired formula for conclusion (2). Thus the proof of conclusion (2) will be complete once we prove (9.18).

To prove (9.18), we recall that $\Lambda_2 - \Lambda = e_{j+1}$ shows that Λ differs from the highest weight of τ_{Λ_2} only in the $\mathfrak{su}(2)$ part of $\mathfrak{k}$; thus Λ has multiplicity one in τ_{Λ_2}. Since $v = E_{\Lambda_2}(v_1 \otimes X_{e_j})$ is a nonzero highest weight vector (Theorem 1.5) and since $\langle \Lambda_2, e_{j+1}\rangle > 0$, $\tau_{\Lambda_2}[X_\alpha, X_{-e_j}]v$ is a nonzero vector of weight Λ. Thus

$$E_{\Lambda_2}(v_1 \otimes X_\alpha) = c\tau_{\Lambda_2}[X_\alpha, X_{-e_j}]\, E_{\Lambda_2}(v_1 \otimes X_{e_j}) \tag{9.20}$$

for some constant c.

Before determining c, let us write

$$[X_{-\alpha}, X_{e_j}] = aX_{e_{j+1}} \qquad \text{and} \qquad [X_\alpha, X_{-e_j}] = bX_{-e_{j+1}}.$$

Our bilinear form B_0 has

$$\begin{aligned} abB_0(X_{e_{j+1}}, X_{-e_{j+1}}) &= B_0([X_{-\alpha}, X_{e_j}], [X_\alpha, X_{-e_j}]) \\ &= -B_0(X_{e_j}, [X_{-\alpha}, [X_\alpha, X_{-e_j}]]) \\ &= -B_0(X_{e_j}, X_{-e_j}), \end{aligned}$$

and thus (2.1a) shows $ab = -1$. Hence

$$[[X_{-\alpha}, X_{e_j}], [X_\alpha, X_{-e_j}]] = -[X_{e_{j+1}}, X_{-e_{j+1}}] = -H_{e_{j+1}}. \tag{9.21}$$

Returning to (9.20), we apply $\tau_{\Lambda_2}[X_{-\alpha}, X_{e_j}]$ to both sides. Then we find

$$\begin{aligned}
&-c\frac{2\langle \Lambda_2, e_{j+1}\rangle}{|e_{j+1}|^2} E_{\Lambda_2}(v_1\otimes X_{e_j})\\
&\quad = -c\tau_{\Lambda_2}(H_{e_{j+1}})\, E_{\Lambda_2}(v_1\otimes X_{e_j})\\
&\quad = c[\tau_{\Lambda_2}[X_{-\alpha}, X_{e_j}], \tau_{\Lambda_2}[X_\alpha, X_{-e_j}]]\, E_{\Lambda_2}(v_1\otimes X_{e_j}) \qquad \text{by (9.21)}\\
&\quad = c\tau_{\Lambda_2}[X_{-\alpha}, X_{e_j}]\,\tau_{\Lambda_2}[X_\alpha, X_{-e_j}]\, E_{\Lambda_2}(v_1\otimes X_{e_j})\\
&\quad = \tau_{\Lambda_2}[X_{-\alpha}, X_{e_j}]\, E_{\Lambda_2}(v_1\otimes X_\alpha) \qquad \text{by (9.20)}\\
&\quad = E_{\Lambda_2}(v_1\otimes [[X_{-\alpha}, X_{e_j}], X_\alpha])\\
&\quad = -E_{\Lambda_2}(v_1\otimes X_{e_j}).
\end{aligned}$$

Thus c is nonzero and is given by $c^{-1} = 2\langle \Lambda_2, e_{j+1}\rangle/|e_{j+1}|^2$, and (9.20) gives

$$\begin{aligned}
&\frac{2\langle \Lambda_2, e_{j+1}\rangle}{|e_{j+1}|^2} E_{\Lambda_2}(v_1\otimes X_\alpha)\\
&\quad = E_{\Lambda_2}(\tau_{\Lambda_1^-}[X_\alpha, X_{-e_j}]\, v_1\otimes X_{e_j}) + E_{\Lambda_2}(v_1\otimes [[X_\alpha, X_{-e_j}], X_{e_j}])\\
&\quad = E_{\Lambda_2}(\tau_{\Lambda_1^-}[X_\alpha, X_{-e_j}]\, v_1\otimes X_{e_j}) + 2E_{\Lambda_2}(v_1\otimes X_\alpha).
\end{aligned}$$

Formula (9.18) follows immediately from this equation.

Proof of conclusion (3). We have seen that the vector on the right side of (9.20) is nonzero and that c is nonzero. The left side is the vector v_2, and hence v_2 is nonzero. Combining this result with conclusion (2), we see that τ_{Λ_2} must occur in $U(\nu)|_K$. This proves the remaining part of conclusion (1).

Proof of conclusion (4). By conclusion (1), τ_{Λ_2} occurs in $U(\nu)|_K$ with multiplicity one. Thus Theorem 4.1a is applicable, and the result follows from conclusions (2) and (3).

10. Two-Step Formula Giving Elliptical Cut-Offs

For the final specialized result, we return to a general set $\{\alpha_1, ..., \alpha_l\}$. The interest is in passing from Λ to some $(\Lambda \pm \alpha_r)^\vee$ and then back to Λ when the argument in Corollary 5.2 breaks down. Theorem 6.4 dealt with one situation where this argument breaks down, namely when there is W_K conjugacy between $\Lambda \pm \alpha_r$ and some $\Lambda \pm \alpha_s$. Theorem 10.1 will give a different estimate in this situation and will treat also the case where $\Lambda \pm \alpha_s$ is a non-extreme weight in $\tau_{(\Lambda \pm \alpha_r)^\vee}$, provided the weight has multiplicity one. We used this result in [2] in the determination of the unitary dual of $SU(N, 2)$, and the result is especially helpful also with $SO(N, 2)$ and $Sp(N, 2)$.

THEOREM 10.1. *Fix roots $\pm\alpha_r$ and $\pm\alpha_s$ with $r \neq s$ and with the two choices of sign not necessarily the same. Put $\Lambda' = (\Lambda \pm \alpha_r)^\vee$, and assume that $\Lambda \pm \alpha_s$ is a weight of multiplicity one in $\tau_{\Lambda'}$. Suppose that*

(a) *the only weights in $\tau_{\Lambda'}$ of the form $\Lambda + \alpha_j$ or $\Lambda - \alpha_j$ are $\Lambda \pm \alpha_r$ and $\Lambda \pm \alpha_s$,*

(b1) *$\tau_{\Lambda'}$ occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$,*

(b2) *no $\beta \in \Delta_n$ has $\beta \perp\!\!\perp \alpha_1, ..., \alpha_{s-1}$, $\beta - (\pm\alpha_s) \in \Delta$, $|\beta|^2 > |\alpha_s|^2$, and $\langle \Lambda, \beta - (\pm\alpha_s)\rangle = 0$,*

(c) *there exists $C > 0$ such that the nonzero vector*

$$v'' = E_\Lambda(E_{\Lambda'}(v_0 \otimes X_{\pm\alpha_r}) \otimes X_{-(\pm\alpha_r)}) \tag{10.1}$$

satisfies

$$v'' = CE_\Lambda(E_{\Lambda'}(v_0 \otimes X_{\pm\alpha_s}) \otimes X_{-(\pm\alpha_s)}), \tag{10.2}$$

(d) *whenever β in Δ_n is such that $\beta \perp\!\!\perp \alpha_1, ..., \alpha_{s-1}$, $\beta + (\pm\alpha_s) \in \Delta$, and $\Lambda - \beta$ is a weight of $\tau_{\Lambda'}$, then $\langle \Lambda, \beta + (\pm\alpha_s)\rangle \geq 0$,*

(e) *Eq. (3.18) for $\Lambda' - \Lambda$ has no solutions with $\sum_{\beta \in \Delta^+_{-,n}} k_\beta \beta$ nonzero.*

Then

$$\begin{aligned}
&\langle P_\Lambda U(v, X_{-(\pm\alpha_r)})\, P_{\Lambda'} U(v, X_{\pm\alpha_r}) f_0(k), u_0\rangle_\Lambda \\
&= \tfrac{1}{16}\langle \tau_\Lambda(k)^{-1} v'', v''\rangle \\
&\quad \times [|\alpha_r|^2\{v(X_{\alpha_r} + X_{-\alpha_r})^2 - (v^\pm_{0,r})^2\} \\
&\quad + C^{-1}\, |\alpha_s|^2 \{v(X_{\alpha_s} + X_{-\alpha_s})^2 - (v^\pm_{0,s})^2\}],
\end{aligned}$$

where $v^\pm_{0,r}$ and $v^\pm_{0,s}$ are the versions of $v^\pm_0$ defined at the end of Section 2 for α_r and α_s.

Remarks. (1) If all noncompact roots are short, then (b1) is satisfied automatically, according to Corollary 1.4, and (b2) is satisfied trivially.

(2) When (b1) holds, then also τ_Λ occurs in $\tau_{\Lambda'} \otimes \mathfrak{p}^{\mathbb{C}}$. This is a formal consequence of Theorem 1.3, and also it follows immediately by using characters.

(3) Theorem 2.1, as amplified in the remarks at the end of Section 2, applies to $v^\pm_{0,r}$ because of assumption (b1). Therefore we can write, in obvious notation,

$$\mathrm{I}(\pm\alpha_r) = -\mathrm{II}(\pm\alpha_r) = v^\pm_{0,r}.$$

(4) A similar formula

$$\mathrm{I}(\pm\alpha_s) = -\mathrm{II}(\pm\alpha_s) = v_{0,s}^{\pm}$$

is applicable to $\pm\alpha_s$, even though we have not assumed that $\tau_{(\Lambda\pm\alpha_s)^\vee}$ occurs in $\tau_\Lambda \otimes \mathfrak{p}^{\mathbb{C}}$. What we have assumed is (b2) above, and the proof of Theorem 2.1c shows that no further assumption is needed.

(5) The expressions for $\mathrm{I}(\pm\alpha_s)$ and $\mathrm{II}(\pm\alpha_s)$ simplify under our assumptions. Let us see that

$$\Big\{\beta \in \Delta_n \mid \beta \perp\!\!\perp \alpha_1, ..., \alpha_{s-1};$$
$$\beta - (\pm\alpha_s) \in \Delta;\ |\beta|^2 < |\alpha_s|^2; \frac{2\langle \Lambda, \beta - (\pm\alpha_s)\rangle}{|\beta - (\pm\alpha_s)|^2} = 1\Big\}$$

is empty. In fact, if β is in the set, then $-\beta$ is strongly orthogonal to $\alpha_1, ..., \alpha_{s-1}$ and has $(-\beta) + (\pm\alpha_s) \in \Delta$, $|-\beta|^2 < |\alpha_s|^2$, and

$$\frac{2\langle \Lambda, (-\beta) + (\pm\alpha_s)\rangle}{|(-\beta) + (\pm\alpha_s)|^2} < 0.$$

By the contrapositive of assumption (d), applied to $-\beta$, we see that $\Lambda + \beta$ is not a weight of $\tau_{\Lambda'}$. On the other hand, $\Lambda \pm \alpha_s$ is a weight of $\tau_{\Lambda'}$ with $\langle \Lambda \pm \alpha_s, \beta - (\pm\alpha_s)\rangle < 0$, and thus the sum $\Lambda + \beta$ is a weight, contradiction. As a result, $\mathrm{I}(\pm\alpha_s)$ simplifies to

$$\mathrm{I}(\pm\alpha_s) = \rho(X_{\alpha_s} + X_{-\alpha_s}) + \frac{2\langle \Lambda, \pm\alpha_s\rangle}{|\alpha_s|^2}$$
$$-2\#\{\beta \in \Delta_n \mid \beta \perp\!\!\perp \alpha_1, ..., \alpha_{s-1}; \beta - (\pm\alpha_s) \in \Delta;$$
$$\langle \Lambda, \beta - (\pm\alpha_s)\rangle > 0\}.$$

Similarly

$$\Big\{\beta \in \Delta_n \mid \beta \perp\!\!\perp \alpha_1, ..., \alpha_{s-1};$$
$$\beta + (\pm\alpha_s) \in \Delta,\ |\beta|^2 < |\alpha_s|^2, \frac{2\langle \Lambda \pm \alpha_s, \beta \pm \alpha_s\rangle}{|\beta \pm \alpha_s|^2} = 1\Big\}$$

is empty as a consequence of (d), and $\mathrm{II}(\pm\alpha_s)$ simplifies to

$$\mathrm{II}(\pm\alpha_s) = \rho(X_{\alpha_s} + X_{-\alpha_s}) - \frac{2\langle \Lambda \pm \alpha_s, \pm\alpha_s\rangle}{|\alpha_s|^2}$$
$$-2\#\{\beta \in \Delta_n \mid \beta \perp\!\!\perp \alpha_1, ..., \alpha_{s-1}; \beta + (\pm\alpha_s) \in \Delta;$$
$$\langle \Lambda \pm \alpha_s, \beta \pm \alpha_s\rangle > 0\}.$$

Proof. Let

$$v_r = E_{\Lambda'}(v_0 \otimes X_{\pm\alpha_r}).$$

This is nonzero by assumption (b1) and Theorem 1.5. Then v'' in (10.1) is nonzero by remark (2) and Theorem 1.5. Define

$$f_r(k) = P_{\Lambda'} U(\nu, X_{\pm\alpha_r}) f_0(k).$$

The same argument as with (5.31) shows that we can write

$$f_r(k) = B(\nu)\tau_{\Lambda'}(k)^{-1} v_r \tag{10.3}$$

for a unique $B(\nu)$ in $\mathrm{Hom}_{K\cap M^\#}(V^{\Lambda'}, V^{\sigma^\#})$.
Let

$$v_s = E_{\Lambda'}(v_0 \otimes X_{\pm\alpha_s}).$$

Since $v'' \neq 0$, it follows from assumption (c) that $v_s \neq 0$. Therefore

$$v_s \text{ is a nonzero weight vector in } \tau_{\Lambda'} \text{ of weight } \Lambda \pm \alpha_s. \tag{10.4}$$

We can apply Theorem 5.1 with $\Lambda' = \Lambda$ to evaluate $\langle f_r(k), u_0\rangle_{\Lambda\pm\alpha_r}$. Hypothesis (a) holds because $\Lambda - (\Lambda \pm \alpha_r \pm \alpha_j)$ is not a sum of positive compact roots, and hypothesis (c) is satisfied trivially. The only nontrivial hypothesis is (b), which is given here as assumption (b1). Since $\langle f_0(k), u_0\rangle_\Lambda = \langle \tau_\Lambda(k)^{-1} v_0, v_0\rangle$ by (4.4) and (4.5), Theorem 5.1 and remark (3) give

$$\langle f_r(k), u_0\rangle_{\Lambda\pm\alpha_r} = \frac{|\alpha_r|^2}{4} (v_{0,r}^{\pm} + \nu(X_{\alpha_r} + X_{-\alpha_r}))\langle \tau_{\Lambda'}(k)^{-1} v_r, v_r\rangle. \tag{10.5}$$

We can try to use the argument of Theorem 5.1 to evaluate $\langle f_r(k), u_0\rangle_{\Lambda\pm\alpha_s}$. When we treat the $\mathfrak{k}$ term corresponding to $\beta \in \Delta_n$, we get 0 as usual if β is not strongly orthogonal to $\alpha_1, ..., \alpha_{s-1}$, if $\beta - (\pm\alpha_s)$ is not a root, or if $\langle \Lambda, \beta - (\pm\alpha_s)\rangle \leqslant 0$. Applying the contrapositive of assumption (d) to $-\beta$, we see in the remaining cases that $\Lambda + \beta$ is not a weight of $\tau_{\Lambda'}$. Therefore $\tau_{\Lambda'}[X_{\pm\alpha_s}, X_{-\beta}]$ moves over in the usual way to act on X_β by $-\mathrm{ad}[X_{\pm\alpha_s}, X_{-\beta}]$. We arrive at

$$\langle f_r(k), u_0\rangle_{\Lambda\pm\alpha_s} = \frac{|\alpha_s|^2}{4} (\mathrm{I}(\pm\alpha_s) + \nu(X_{\alpha_s} + X_{-\alpha_s}))\langle \tau_{\Lambda'}(k)^{-1} v_s, v_s\rangle,$$

with $\mathrm{I}(\pm\alpha_s)$ given as in remark (5). By remark (4), we can rewrite this as

$$\langle f_r(k), u_0\rangle_{\Lambda\pm\alpha_s} = \frac{|\alpha_s|^2}{4}(v_{0,s}^{\pm} + v(X_{\alpha_s} + X_{-\alpha_s}))\langle \tau_{\Lambda'}(k)^{-1} v_s, v_s\rangle. \tag{10.6}$$

Turning attention to the second step taken by $U(v)$, we run through the first part of the proof of Theorem 5.1, obtaining

$$\begin{aligned}
&\langle P_\Lambda U(v, X_{-(\pm\alpha_r)}) f_r(k), u_0\rangle_\Lambda \\
&\quad = \sum_{j=1}^{l} \tfrac{1}{4}|\alpha_j|^2[(v+\rho)(X_{\alpha_j} + X_{-\alpha_j})] \\
&\qquad \times \langle E_\Lambda(v_r \otimes X_{-(\pm\alpha_r)}), \pi(k)(B(v)^* u_0 \otimes (X_{\alpha_j} + X_{-\alpha_j}))\rangle_\Lambda \\
&\qquad + \sum_{\beta\in\Delta_n} \tfrac{1}{2}|\beta|^2 \langle E_\Lambda(v_r \otimes X_{-(\pm\alpha_r)}), \pi(k)(\tau_{\Lambda'}(P_{\mathfrak{k}} X_\beta)^* B(v)^* u_0 \otimes X_\beta)\rangle_\Lambda,
\end{aligned} \tag{10.7}$$

with the m terms giving 0 by assumption (e).

We go through the usual argument with weights to determine as much of $B(v)^* u_0$ as possible. Since $\Lambda \pm \alpha_s$ has multiplicity one in $\tau_{\Lambda'}$, (10.4) and assumption (a) lead us in the usual way to write

$$\begin{aligned}
B(v)^* u_0 &= \frac{\langle B(v)^* u_0, v_r\rangle}{|v_r|^2} v_r \\
&\quad + \frac{\langle B(v)^* u_0, v_s\rangle}{|v_s|^2} v_s + \text{irrelevant terms} \\
&= \overline{b_r(v)} v_r + \overline{b_s(v)} v_s + \text{irrelevant terms}.
\end{aligned} \tag{10.8}$$

Here we can compute $b_r(v)$ and $b_s(v)$ as follows: From (10.3) we have

$$\begin{aligned}
\langle f_r(k), u_0\rangle &= \langle B(v)\,\tau_{\Lambda'}(k)^{-1} v_r, u_0\rangle = \langle \tau_{\Lambda'}(k)^{-1} v_r, B(v)^* u_0\rangle \\
&= b_r(v)\langle \tau_{\Lambda'}(k)^{-1} v_r, v_r\rangle \\
&\quad + b_s(v)\langle \tau_{\Lambda'}(k)^{-1} v_s, v_s\rangle + \text{irrelevant terms}.
\end{aligned}$$

Taking Fourier coefficients and using (10.5) and (10.6) gives us

$$b_r(v) = \frac{|\alpha_r|^2}{4}(v_{0,r}^{\pm} + v(X_{\alpha_r} + X_{-\alpha_r})) \tag{10.9a}$$

$$b_s(v) = \frac{|\alpha_s|^2}{4}(v_{0,s}^{\pm} + v(X_{\alpha_s} + X_{-\alpha_s})). \tag{10.9b}$$

To evaluate (10.7), we substitute from (10.8) and obtain

$$
\begin{aligned}
&\langle P_\Lambda U(\nu, X_{-(\pm\alpha_r)}) f_r(k), u_0\rangle_\Lambda \\
&= b_r(\nu)\Big\{\tfrac{1}{4}|\alpha_r|^2[(\nu+\rho)(X_{\alpha_r}+X_{-\alpha_r})] \\
&\quad\times\langle E_\Lambda(v_r\otimes X_{-(\pm\alpha_r)}), \pi(k)(v_r\otimes X_{-(\pm\alpha_r)})\rangle \\
&\quad+\sum_{\substack{\beta\perp\perp\alpha_1,\ldots,\alpha_{r-1}\\ \beta\perp\!\!/\!\!\perp\alpha_r}}\tfrac{1}{2}|\beta|^2 \\
&\quad\times\langle E_\Lambda(v_r\otimes X_{-(\pm\alpha_r)}), \pi(k)(\tau_{\Lambda'}(P_{\mathfrak{k}}X_\beta)^* v_r\otimes X_\beta)\rangle_\Lambda\Big\} \\
&\quad+b_s(\nu)\Big\{\tfrac{1}{4}|\alpha_s|^2[(\nu+\rho)(X_{\alpha_s}+X_{-\alpha_s})] \\
&\quad\times\langle E_\Lambda(v_r\otimes X_{-(\pm\alpha_r)}), \pi(k)(v_s\otimes X_{-(\pm\alpha_s)})\rangle \\
&\quad+\sum_{\substack{\beta\perp\perp\alpha_1,\ldots,\alpha_{s-1}\\ \beta\perp\!\!/\!\!\perp\alpha_s}}\tfrac{1}{2}|\beta|^2 \\
&\quad\times\langle E_\Lambda(v_r\otimes X_{(\pm\alpha_r)}), \pi(k)(\tau_{\Lambda'}(P_{\mathfrak{k}}X_\beta)^* v_s\otimes X_\beta)\rangle_\Lambda\Big\}. \qquad (10.10)
\end{aligned}
$$

Here the only contribution from the first occurrence of $\tau_{\Lambda'}(P_{\mathfrak{k}}X_\beta)^*$ is from the term involving $\tau_{\Lambda'}[X_{-(\pm\alpha_r)}, X_{-\beta}]$, and the only contribution from the second occurrence of $\tau_{\Lambda'}(P_{\mathfrak{k}}X_\beta)^*$ is from the term involving $\tau_{\Lambda'}[X_{-(\pm\alpha_s)}, X_{-\beta}]$.

For the first expression in braces in (10.10), we argue just as in the proof of Theorem 5.1. If we take into account the equality $\mathrm{II}(\pm\alpha_r)=-\nu_{0,r}^{\pm}$ in remark (3), we see that the expression in braces is

$$
\begin{aligned}
&\tfrac{1}{4}|\alpha_r|^2\langle E_\Lambda(v_r\otimes X_{-(\pm\alpha_r)}), \pi(k)(v_r\otimes X_{-(\pm\alpha_r)})\rangle \\
&\quad\times[-\nu_{0,r}^{\pm}+\nu(X_{\alpha_r}+X_{-\alpha_r})]. \qquad (10.11\text{a})
\end{aligned}
$$

For the second expression in braces in (10.10), we begin to process the $\mathfrak{k}$ term from β just as in Theorem 5.1. Suppose $\beta\neq-(\pm\alpha_s)$ and $\tau_{\Lambda'}[X_{-(\pm\alpha_s)}, X_{-\beta}]v_s\neq 0$. Then $\beta+(\pm\alpha_s)$ is in Δ and $\Lambda-\beta$ is a weight of $\tau_{\Lambda'}$. Assumption (d) gives $\langle\Lambda, \beta+(\pm\alpha_s)\rangle\geqslant 0$. Hence

$$|(\Lambda\pm\alpha_s)+\beta|^2-|\Lambda|^2=2\langle\Lambda, \beta+(\pm\alpha_s)\rangle+|\beta+(\pm\alpha_s)|^2>0,$$

and $(\Lambda \pm \alpha_s) + \beta$ is not a weight of τ_Λ. This means that the $\tau_{\Lambda'}[X_{-(\pm\alpha_s)}, X_{-\beta}]$ moves over as $-\mathrm{ad}[X_{-(\pm\alpha_s)}, X_{-\beta}]$ to act on X_β, and the argument of Theorem 5.1 goes through. We are led to the simplified expression for $\mathrm{II}(\pm\alpha_s)$ in remark (5), and we see that the second expression in braces in (10.10) is

$$\tfrac{1}{4}|\alpha_s|^2 \langle E_\Lambda(v_r \otimes X_{-(\pm\alpha_r)}), \pi(k)(v_s \otimes X_{-(\pm\alpha_s)})\rangle$$
$$\times [-v_{0,s}^{\pm} + v(X_{\alpha_s} + X_{-\alpha_s})]. \qquad (10.11\mathrm{b})$$

Now we substitute (10.9) and (10.11) into (10.10), and then we substitute (10.2) into the result. The theorem follows.

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Printed in Belgium