# UNITARY REPRESENTATIONS AND BASIC CASES

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In attempting to classify the irreducible unitary representations of linear semisimple Lie groups, one knows that it is enough to decide which of certain standard representations in Hilbert space admit new inner products with respect to which they are unitary. In this context B. Speh and the author [3] introduced a notion of basic case and gave a conjecture that would if true reduce the classification problem to a study of finitely many basic cases in each group. The paper [3] did not, however, tell how to calculate what the basic cases are. The present paper will address this question, giving some theorems that usually make it a simple matter to identify the basic cases.

The paper is organized as follows: In §1 we review the setting of the classification problem and restate the existence-uniqueness theorem for basic cases. In §2 we give two reduction theorems for calculating basic cases and show how to apply them. The proof of the second reduction theorem is in §3.

The development of the theory of basic cases has been influenced extensively by conversations with David Vogan. Vogan's paper [5] may be viewed as a related but different attempt to isolate the phenomena that lead to unitary representations.

## 1. Definition of basic cases

Let $G$ be a connected linear semisimple group with maximal compact subgroup $K$. We assume as in [3] that $\operatorname{rank} G = \operatorname{rank} K$.

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Let $P = MAN$ be a parabolic subgroup of $G$ with $\operatorname{rank} M = \operatorname{rank}(K \cap M)$, let $\sigma$ be a discrete series or limit of discrete series representation of $M$, and let $e^{\nu}$ be a homomorphism of $A$ into $\mathbb{C}^{\times}$. We denote by $U(P,\sigma,\nu)$ the unitarily induced representation

$$U(P,\sigma,\nu) = \operatorname{ind}_P^G(\sigma \otimes e^{\nu} \otimes 1) .$$

This representation may be regarded as acting in a closed subspace of $L^2$ functions on $K$ with values in the space on which $\sigma$ operates. When $\operatorname{Re} \nu$ is in the closed positive Weyl chamber relative to $N$ and when a certain computable finite group (known as an "R group") is trivial, this representation has a unique irreducible quotient called the Langlands quotient and denoted $J(P,\sigma,\nu)$. We shall assume these conditions on $\nu$ are satisfied; they are always satisfied when $\operatorname{Re} \nu$ is in the open positive Weyl chamber.

The representations $J(P,\sigma,\nu)$ act in quotient Hilbert spaces, and the classification question for the unitary dual comes down to deciding which of the $J$'s admit new inner products that make them unitary. In fact, by an observation of Vogan's recited in [2], it is enough to handle $\nu$ real-valued. For $\nu$ real-valued (and $\operatorname{rank} G = \operatorname{rank} K$), the representation $U(P,\sigma,\nu)$ (and hence also $J(P,\sigma,\nu)$) always admits a nonzero invariant Hermitian form, and the question is whether the known operator that relates this form to the $L^2$ inner product is semidefinite.

We think of $\sigma$ as fixed and $\nu$ as varying, and we look for those real $\nu$ in the closed positive Weyl chamber for which $J(P,\sigma,\nu)$ can be made unitary. Then it appears from examples that there are only finitely many distinct pictures of unitary points for a given $G$ and that most of these pictures are associated to subgroups of $G$. The idea behind "basic cases" is to pick out finitely many $\sigma$'s whose pictures ought to include all the pictures that are new for $G$. Then we want to associate to a general $(G,\sigma)$ a pair $(L,\sigma^L)$ with $L \subseteq G$ and $\sigma^L$ basic such that the pictures of unitary points for $\sigma$ and $\sigma^L$ ought to match exactly.

To define basic cases $\sigma$, we restrict attention to a class of $\sigma$'s for which some minimal K-type of $U(P,\sigma,\nu)$ depends coherently upon $\sigma$, and then the basic case is the $\sigma$ of smallest parameter in the class.

In more detail, let $\mathfrak{b} \subseteq \mathfrak{k}$ be a compact Cartan subalgebra of $\mathfrak{g}$, let $\Delta$ be the roots of $(\mathfrak{g}^{\mathbb{C}},\mathfrak{b}^{\mathbb{C}})$, and let $\Delta_K$ be the subset of compact roots. We may assume that the Lie algebra $\mathfrak{a}$ of $A$ is built by Cayley transform from strongly orthogonal noncompact roots $\{\alpha_1,\ldots,\alpha_\ell\}$. We decompose $\mathfrak{b} = \mathfrak{b}_- \oplus \mathfrak{b}_r$, where $\mathfrak{b}_-$ is the common kernel of the $\alpha_j$, and we let

$$\Delta_r = \Delta \cap \sum \mathbb{R}\alpha_j .$$

From $\mathfrak{b}_r$ and $\Delta_r$ we can construct a split semisimple subalgebra $\mathfrak{g}_r$ of $\mathfrak{g}$. Let $G_r$ be the corresponding analytic subgroup, and choose $K_r = K \cap G_r$ as maximal compact subgroup. Since each $\pm\alpha_j$ is in $\Delta_r$, we have $\mathfrak{a} \subseteq \mathfrak{g}_r$. In fact, $\mathfrak{a}$ can be taken as the Iwasawa $\mathfrak{a}$ of $\mathfrak{g}_r$. The $M$ of a corresponding minimal parabolic subgroup of $G_r$ is then $M_r = Z_{K_r}(\mathfrak{a})$; $M_r$ is a direct sum of two-element groups, and it is a subgroup of the center $Z_M$ of $M$.

The roots of $M$ can be naturally identified with the subset $\Delta_-$ of $\Delta$ orthogonal to all $\alpha_j$. If $\sigma$ is a discrete series or limit of discrete series representation of $M$, then we know that $\sigma$ is induced from some $\sigma^\#$ on

$$M^\# = M_0 Z_M = M_0 M_r .$$

We let $\chi$ denote the scalar value of $\sigma$ on the subgroup $M_r$ of $Z_M$, and we let $(\lambda_0,C)$ be a Harish-Chandra parameter of $\sigma^\#$. Here $\lambda_0$

is dominant for the Weyl chamber $C$ of $i\mathfrak{b}_-$, and we let $(\Delta_-)^+$ be the corresponding positive system in $\Delta_-$.

The paper [1] shows how to obtain a positive system $\Delta^+$ such that

(i) $\lambda_0$ is $\Delta^+$ dominant

(ii) $\Delta^+ \supseteq (\Delta_-)^+$

(iii) $\Delta_r$ is generated by the $\Delta^+$ simple roots that it contains

(iv) some other properties hold.

The theorems of [1] then identify the (highest weights of the) minimal K-types of $U(P,\sigma,\nu)$ as all $\Delta_K^+$ dominant expressions of the form

$$\Lambda = \lambda - E(2\rho_K) + 2\rho_{K_r} + \mu . \tag{1.1}$$

Here $\rho$ refers to a half-sum of positive roots, $\lambda$ is the Blattner parameter of $\sigma^\#$ given by

$$\lambda = \lambda_0 - \rho_{-,c} + \rho_{-,n} , \tag{1.2}$$

and $E$ is the orthogonal projection on $\sum \mathbb{R}\alpha_j$. The linear functional $\mu$ is any minimal (= fine) $K_r$-type for the principal series representations of $G_r$ with $M_r$ parameter the translate of $\chi$ given by

$$\omega = \chi \cdot \exp(E(2\rho_K) - 2\rho_{K_r})|_{M_r} .$$

We say $\sigma$ has $(\{\alpha_j\},\Delta^+,\chi,\mu)$ as a <u>format</u> if $\Lambda$ in (1.1) is a minimal K-type of $U(P,\sigma,\nu)$, i.e., if $\Lambda$ is $\Delta_K^+$ dominant. We consider simultaneously all $\sigma$'s with a common format and pick out a smallest one. Theorem 3.1 of [3], reproduced below, gives the sense in which there exists a unique smallest one.

<u>Theorem 1.1</u>. Suppose $G^{\mathbb{C}}$ is simply connected. Among all infinitesimal characters $\lambda_0$ of discrete series or limits of discrete series of $M$ with a particular format for $G$, there exists a unique one $\lambda_{0,b}$ such that any other $\lambda_0$ for that format has $\lambda_0 - \lambda_{0,b}$ dominant for $\Delta^+$ and G-integral.

We call $\lambda_{0,b}$ or its associated $\sigma_b$ the basic case for the format. When $G^{\mathbb{C}}$ is not simply connected, we pass to the appropriate cover of $G$ in order to use Theorem 1.1 to define "basic case"; back in the original $G$, the parameter $\lambda_{0,b}$ continues to make sense, but $\sigma_b$ may no longer be single-valued. In any event there are only finitely many basic cases for each $G$.

Some detailed examples appear in [3], all attached to minimal parabolic subgroups. For the double cover of $SO(2n,1)$, $\sigma_b$ is the trivial representation or the spin representation. For $SU(n,1)$ and $SU(N,2)$ the basic cases are finitely many one-dimensional representations of $M$ close to the trivial representation. For $Sp(n,1)$ with $n \geq 2$, $M$ is $SU(2)\times Sp(n-1)$. The basic cases $(k \times \text{fundamental}) \otimes 1$, with $0 \leq k \leq 2n-1$, were listed in [3]; there is one other basic case——given by $\sigma_b = 1 \otimes \sigma_0$, where $\sigma_0$ is the fundamental representation attached to the long simple root of $Sp(n-1)$.[1]

Returning to the general $(G,\sigma)$, we recall how [3] associates to $(G,\sigma)$ a basic case $(L,\sigma^L)$ for a certain subgroup $L$ of $G$. Let $\lambda_0$ be the infinitesimal character of $\sigma$, let $(\{\alpha_j\},\Delta^+,\chi,\mu)$ be a compatible format, and let $\lambda_{0,b}$ be the basic case for this format. Let $\mathfrak{q} = \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u}$ be the parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ defined by the $\Delta^+$ dominant form $\lambda_0 - \lambda_{0,b}$:

$\mathfrak{q}$ is built from $\mathfrak{b}^{\mathbb{C}}$ and all $\beta \in \Delta$ with $\langle \lambda_0 - \lambda_{0,b}, \beta\rangle \geq 0$,
$\mathfrak{l}^{\mathbb{C}}$ is built from $\mathfrak{b}^{\mathbb{C}}$ and all $\beta \in \Delta$ with $\langle \lambda_0 - \lambda_{0,b}, \beta\rangle = 0$,
$\mathfrak{u}$ is built from all $\beta \in \Delta$ with $\langle \lambda_0 - \lambda_{0,b}, \beta\rangle > 0$.

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[1] This additional basic case for $Sp(n,1)$ was inadvertently omitted from the list in [3]. For it the induced representation has two minimal K-types, and $J(P,\sigma_b,t\rho_A)$ is not infinitesimally unitary for any $t > 0$.

Set $\mathfrak{l} = \mathfrak{l}^{\mathbb{C}} \cap \mathfrak{g}$, and let $L$ be the corresponding analytic subgroup of $G$. The root system of $(\mathfrak{l}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, namely

$$\Delta^L = \{\beta \in \Delta \mid \langle \lambda_0 - \lambda_{0,\mathfrak{b}}, \beta\rangle = 0\},$$

contains all $\pm\alpha_j$, and thus $\mathfrak{l}^{\mathbb{C}}$ contains $\mathfrak{a}^{\mathbb{C}}$. Then it follows that

$$P^L = (M \cap L)A(N \cap L)$$

is a parabolic subgroup of $L$. We define $\sigma^L$ by

$$\lambda_0^L = \lambda_0 - \rho(\mathfrak{u}) \tag{1.3a}$$

$$\chi^L = \chi \cdot [\exp E(2\rho(\mathfrak{u} \cap \mathfrak{l}^{\mathbb{C}}))]|_{M_r}. \tag{1.3b}$$

The propositions in §4 of [3] establish the following.

<u>Theorem 1.2</u>. The definitions (1.3) consistently define $\sigma^L$, and $(\{\alpha_j\}, \Delta^+ \cap \Delta^L, \chi^L, \mu)$ is a compatible format for $\sigma^L$. Moreover, $\sigma^L$ is the basic case for this format.

Remark. The group $L$ is reductive, not necessarily semisimple, and the statement that $\sigma^L$ is a basic case is more precisely a statement about the restriction of $\sigma^L$ within the derived group of $L$.

Conjecture 5.1 of [3] expects that $J(P,\sigma,\nu)$ is infinitesimally unitary for $G$ if and only if $J(P^L,\sigma^L,\nu)$ is infinitesimally unitary for $L$.

## 2. Reduction theorems

Even in ostensibly easy examples, it is a bit subtle to determine the basic cases without a guess as to what they are.[2] In this section we give two reduction theorems to make this determination easier. We apply the theorems to give formulas for the basic cases attached to maximal and minimal parabolic subgroups.

Throughout this section we work with a fixed format $(\{\alpha_j\},\Delta^+,\chi,\mu)$. Following [1], we say that

$$\text{a root in } \Delta \text{ is } \begin{cases} \underline{\text{real}} & \text{if in } \sum \mathbb{R}\alpha_j \\ \underline{\text{imaginary}} & \text{if orthogonal to } \sum \mathbb{R}\alpha_j \\ \underline{\text{complex}} & \text{otherwise.} \end{cases}$$

The first theorem is a kind of localization theorem for the calculation of basic cases. Fix a complex or imaginary $\Delta^+$ simple root $\beta$, and let

$$\begin{aligned} \Delta^H &= \text{root system generated by } \beta \text{ and } \Delta_r \\ \mathfrak{b}^H &= \mathbb{R}iH_\beta + \mathfrak{b}_r \\ \mathfrak{h}^{\mathbb{C}} &= (\mathfrak{b}^H)^{\mathbb{C}} + \sum_{\gamma\in\Delta^H} \mathbb{C}X_\gamma \\ \mathfrak{h} &= \mathfrak{g} \cap \mathfrak{h}^{\mathbb{C}} \\ H &= \text{(semisimple) connected subgroup of } G \text{ corresponding} \\ &\quad\ \text{to } \mathfrak{h}\,. \end{aligned} \qquad (2.1)$$

We use a superscript $H$ to denote the usual subgroups, subalgebras, etc., associated with $H$. Note that $\Delta_r^H = \Delta_r$ and thus $G_r^H = G_r$. Let

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[2] Cf. Footnotes 1 and 3 elsewhere in this paper.

$$(\Delta^H)^+ = \Delta^+ \cap \Delta^H$$

$$\mathfrak{v} = \sum_{\substack{\gamma\in\Delta^+ \\ \gamma\notin\Delta^H}} \mathbb{C} X_\gamma . \qquad (2.2)$$

<u>Theorem 2.1</u>. Fix a format $(\{\alpha_j\}, \Delta^+, \chi, \mu)$ and a complex or imaginary $\Delta^+$ simple root $\beta$, and make the corresponding definitions (2.1) and (2.2). Let $\lambda_0 = \lambda_{0,b}$ be the basic case for the format $(\{\alpha_j\}, \Delta^+, \chi, \mu)$ for $G$, and define

$$\lambda_0^H = \lambda_0 - \rho(\mathfrak{v}) \qquad (\text{restricted to } \mathfrak{b}^H)$$

$$\chi^H = \chi \cdot [\exp E(2\rho(\mathfrak{v} \cap \mathfrak{k}^{\mathbb{C}}))]|_{M_r} .$$

Then $(\{\alpha_j\}, (\Delta^H)^+, \chi^H, \mu)$ is a format for $H$, and the basic case for this format is exactly $\lambda_0^H$.

Proof. This is proved in the same way as Propositions 4.1 and 4.2 of [3] but with $H$ in place of $L$.

<u>Corollary 2.2</u>. Let $\lambda_{0,b}$ be the basic case for the format $(\{\alpha_j\}, \Delta^+, \chi, \mu)$ of $G$, and let $\beta$ be an imaginary $\Delta^+$ simple root. Then

$$\frac{2\langle \lambda_{0,b}, \beta\rangle}{|\beta|^2} = \begin{cases} 1 & \text{if } \beta \text{ is in } \Delta_K^+ \\ 0 & \text{if not}. \end{cases}$$

Proof. We apply Theorem 2.1. Since $\pm\beta$ are orthogonal to the other members of $\Delta^H$, we may think of $\Delta^H$ as being just $\{\pm\beta\}$. Then $H$ is locally $SU(2)$ or $SL(2,\mathbb{R})$, and the corollary results from direct calculation.

Corollary 2.3. Let $\lambda_{0,b}$ be the basic case for a format $(\{\alpha\},\Delta^+,\chi,\mu)$ of $G$ that corresponds to a maximal parabolic subgroup other than in a real form of $G_2^{\mathbb{C}}$, and let $\beta$ be a $\Delta^+$ simple root. Then

$$\frac{2\langle\lambda_{0,b},\beta\rangle}{|\beta|^2} = \begin{cases} 1 & \text{if } \beta \text{ is compact imaginary} \\ 0 & \text{if } \beta \text{ is real or noncompact imaginary} \\ \text{correction}(\beta) & \text{if } \beta \text{ is complex}. \end{cases}$$

Here correction($\beta$) is always $0$, $\frac{1}{2}$, or $1$, depending on the form of $\beta$. With $\varepsilon$ denoting a member of $(i\mathfrak{b}_-)'$, the formula for correction($\beta$) is

$$\text{correction}(\beta) = \begin{cases} \frac{1}{2} & \text{if } \beta=\varepsilon-\frac{1}{2}\alpha,\ |\beta|=|\alpha|, \\ & \text{and } \mu=0 \\ \frac{1}{2}(1-\operatorname{sgn}\langle\mu,\gamma\rangle) & \text{if } \beta=\varepsilon-\frac{1}{2}\alpha,\ |\beta|=|\alpha|,\ \mu=\pm\frac{1}{2}\alpha, \\ & \text{and a sign} \pm \text{ is fixed so that} \\ & \gamma=\varepsilon\pm\frac{1}{2}\alpha \text{ is compact} \\ \frac{1}{2}\pm\left|\frac{2\langle\mu,\beta\rangle}{|\beta|^2}\right| & \text{if } \beta=\varepsilon-\alpha,\ |\beta|^2=2|\alpha|^2, \text{ and} \\ & \text{the sign} \pm \text{ is fixed to be} + \text{ if} \\ & \beta \text{ is compact for } G \text{ and} - \text{ if } \beta \\ & \text{is noncompact for } G \\ \max\{0,-\frac{2\langle\mu,\gamma\rangle}{|\gamma|^2}\} & \text{if } \beta=\varepsilon-\frac{1}{2}\alpha,\ |\alpha|^2=2|\beta|^2, \text{ and} \\ & \text{a sign} \pm \text{ is fixed so that} \\ & \gamma=\varepsilon\pm\frac{1}{2}\alpha \text{ is compact}. \end{cases}$$

Proof. For $\beta$ real the formula is trivial, and for $\beta$ imaginary the formula comes from Corollary 2.2. For $\beta$ complex we apply Theorem 2.1 and are led to a group $H$ of rank 2, where we make an explicit computation. The group $H$ is locally $SU(2,1)$ for the first two forms of $\beta$, $SO(4,1)$ for $\beta$ compact of the third form, and $Sp(2,\mathbb{R})$ in the remaining cases.

If we try to use Theorem 2.1 to handle the general case, we find that $H$ can still be fairly complicated. Specifically the subgroup $G_r$ of $H$ is split with rank $G_r$ = rank $K_r$, and its simple components are of type $A_1$, $B_n$, $C_n$, $D_{2n}$, $E_7$, $E_8$, $F_4$, or $G_2$. However, not all of $G_r$ is needed to handle the projection $E(\beta)$ of the simple root $\beta$, and we seek a second reduction theorem that allows us to discard the unnecessary part of $G_r$.

The success of such a reduction depends upon the nature of $\beta$. The full list of possibilities is enumerated in the following lemma.

Lemma 2.4. Apart from indexing and signs, the following expressions $\beta = \varepsilon + \sum c_j\alpha_j$ (with $\varepsilon$ in $(i\mathfrak{b}_-)'$) are the only possibilities for a complex root in $\Delta$ other than in a factor of type $G_2^{\mathbb{R}}$. Each such possibility has

$$c_j = \frac{\langle \beta , \alpha_j \rangle}{|\alpha_j|^2} \quad \text{and} \quad \sum \frac{4\langle \beta , \alpha_j \rangle^2}{|\beta|^2|\alpha_j|^2} = n < 4 .$$

(1) $n = 1$. $\beta$ and $\alpha_1$ of equal length; $\beta = \varepsilon + \frac{1}{2}\alpha_1$.

(2) $n = 2$.

a) $\beta, \alpha_1, \alpha_2$ of same length; $\beta = \varepsilon + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$.

b) $\beta$ long relative to $\alpha_1$; $\beta = \varepsilon + \alpha_1$.

c) $\beta$ short relative to $\alpha_1$; $\beta = \varepsilon + \frac{1}{2}\alpha_1$.

(3) $n = 3$.

a) $\beta, \alpha_1, \alpha_2, \alpha_3$ of same length; $\beta = \varepsilon + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3$.

b) $\beta$ and $\alpha_2$ long, $\alpha_1$ short; $\beta = \varepsilon + \alpha_1 + \frac{1}{2}\alpha_2$.

c) $\beta$ and $\alpha_2$ short, $\alpha_1$ long; $\beta = \varepsilon + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$.

Proof. We apply Parseval's equality to the expansion of $\beta$ in terms of the orthogonal elements $\epsilon, \alpha_1, \ldots,$ and we are led to the list of possibilities in the statement of the lemma.

Taking advantage of the reduction in Theorem 2.1, let us now suppose that $G$ and $\Delta^+$ are such that there are no imaginary simple roots and there is exactly one complex simple root, which we call $\beta$. Fix a subset $S$ of the indices $1, \ldots, \ell$ such that

$$\{\alpha_j \mid j \in S\} \supseteq \{\alpha_j \mid \langle \alpha_j, \beta \rangle \neq 0\},$$

and let

$$\begin{aligned}
\Delta^H &= \Delta \cap [\mathbb{R}\beta + \sum_{j\in S} \mathbb{R}\alpha_j] \\
\mathfrak{b}^H &= \mathbb{R}iH_\beta + \sum_{j\in S} \mathbb{R}iH_{\alpha_j} \\
\mathfrak{h}^{\mathbb{C}} &= (\mathfrak{b}^H)^{\mathbb{C}} + \sum_{\gamma\in\Delta^H} \mathbb{C}X_\gamma \\
\mathfrak{h} &= \mathfrak{g} \cap \mathfrak{h}^{\mathbb{C}} \\
H &= \text{(semisimple) connected subgroup of } G \text{ corresponding to } \mathfrak{h}.
\end{aligned} \tag{2.3}$$

Again we use a superscript $H$ to denote the usual subgroups, subalgebras, etc., associated with $H$. In terms of the given format for $G$, we define

$$\begin{aligned}
(\Delta^H)^+ &= \Delta^+ \cap \Delta^H \\
\mu^H &= \mu|_{\mathfrak{b}^H} \\
\omega_0^H &= \omega|_{M_r^H \cap \exp \mathfrak{b}_-} \\
\chi_0^H &= \omega_0^H \cdot \exp(E^H(2\rho_K^H) - 2\rho_{K_r^H})|_{M_r^H \cap \exp \mathfrak{b}_-}.
\end{aligned} \tag{2.4}$$

Theorem 2.5. Fix a format $(\{\alpha_j\},\Delta^+,\chi,\mu)$ for which the only nonreal $\Delta^+$ simple root is the complex root $\beta$, and make the corresponding definitions (2.3) and (2.4). Assume further that

(i) if $\epsilon$ is in $\Delta_-$ and $\epsilon \pm \alpha_{j_0}$ are roots, then $j_0$ is in $S$

(ii) in the notation of Lemma 2.4, $\beta$ is of type (1), (2a), or (3a), or else $\beta$ is compact and is of type (2b).

Then there exists an extension $\omega^H$ of $\omega_0^H$ to a character of $M_r^H$ such that if $\mu$ is a fine $K_r$-type for $\omega$ then $\mu^H$ is a fine $K_r^H$-type for $\omega^H$. For any such extension $\omega^H$ and corresponding $\chi^H$, if $\lambda_0 = \lambda_{0,b}$ is the basic case for the format $(\{\alpha_j\},\Delta^+,\chi,\mu)$, then $(\{\alpha_j\}_{j\in S}, (\Delta^H)^+,\chi^H,\mu^H)$ is a format for $H$ and the basic case $\lambda_{0,b}^H$ for this format is exactly $\lambda_0^H = \lambda_0|_{b^H}$.

Remarks. We may ignore real forms of $G_2^{\mathbb{C}}$, since otherwise $\Delta^H = \Delta$. Then we see that Assumptions (i) and (ii) are satisfied if all roots in $\Delta$ have the same length. In particular the theorem reduces calculations of basic cases for $\Delta$ of type $E_6$, $E_7$, or $E_8$ to a classical root system $\Delta^H$ of rank at most four. Moreover, the corollary below shows that the theorem handles formats associated to minimal parabolic subgroups. With a little additional work, one can weaken Assumption (ii) in the general case, but we shall not do so here.

Theorem 2.5 will be proved in §3. The proof uses the following lemma, which we need also when we apply Theorem 2.5 to obtain Corollary 2.7.

Lemma 2.6. In the notation of Theorem 2.5, if the component of $\beta$ orthogonal to all $\alpha_j$ is $\epsilon$, then every member of $\Delta^+$ is of the form $\gamma = n\epsilon + \sum c_j\alpha_j$ with $n = 0, 1,$ or $2$.

Proof. Given $\gamma$ in $\Delta^+$, expand $\gamma$ in terms of the simple roots and then regroup to see that $\gamma = n\varepsilon + \Sigma c_j \alpha_j$ with $n$ an integer $\geq 0$. Then

$$2|\beta|^2 \geq |\gamma|^2 \geq |n\varepsilon|^2 = n^2|\varepsilon|^2 = n^2 \times \left(\tfrac{1}{4} \text{ or } \tfrac{1}{2} \text{ or } \tfrac{3}{4}\right)|\beta|^2 \qquad (2.5)$$

by Lemma 2.4. So $2 \geq \frac{1}{4}n^2$ and $n \leq 2$.

Corollary 2.7.[3] Let $\lambda_{0,b}$ be the basic case for a format $(\{\alpha_j\}, \Delta^+, \chi, \mu)$ of $G$ that corresponds to a minimal parabolic subgroup, and let $\beta$ be a $\Delta^+$ simple root. Then

$$\frac{2\langle \lambda_{0,b}, \beta\rangle}{|\beta|^2} = \begin{cases} 1 & \text{if } \beta \text{ imaginary (and compact)} \\ 0 & \text{if } \beta \text{ real} \\ \text{correction}(\beta) & \text{if } \beta \text{ complex}. \end{cases}$$

Here correction$(\beta)$ is always $0$, $\frac{1}{2}$, or $1$, depending on the form of $\beta$. With $\varepsilon$ denoting a member of $(i\mathfrak{b}_-)'$, the formula for correction$(\beta)$ is

$$\text{correction}(\beta) = \begin{cases} \frac{1}{2} & \text{if } \beta = \varepsilon - \frac{1}{2}\alpha_j,\ |\beta| = |\alpha_j|,\ \mu \perp \alpha_j \\ \frac{1}{2}(1 - \operatorname{sgn}\langle \mu, \gamma\rangle) & \text{if } \beta = \varepsilon - \frac{1}{2}\alpha_j,\ |\beta| = |\alpha_j|,\ \mu \not\perp \alpha_j, \\ & \text{and a sign} \pm \text{ is fixed so that} \\ & \gamma = \varepsilon \pm \frac{1}{2}\alpha_j \text{ is compact} \\ \frac{1}{2} + \left|\frac{2\langle \mu, \beta\rangle}{|\beta|^2}\right| & \text{if } \beta = \varepsilon - \alpha_j \text{ with } |\beta|^2 = 2|\alpha_j|^2 \\ c + \left|\frac{2\langle \mu, \gamma\rangle}{|\gamma|^2}\right| & \text{if } \beta = \varepsilon - \frac{1}{2}\alpha_i - \frac{1}{2}\alpha_j,\ |\beta| = |\alpha_i| \\ & = |\alpha_j|,\ c = \frac{1}{2} \text{ or } 0 \text{ according as} \\ & \frac{1}{2}(\alpha_i + \alpha_j) \text{ is or is not a root, and} \\ & \text{a sign} \pm \text{ is fixed so that} \\ & \gamma = \varepsilon - \frac{1}{2}\alpha_i \pm \frac{1}{2}\alpha_j \text{ is compact}. \end{cases}$$

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[3] A result of this sort was announced in [3]. However, the formula for correction$(\beta)$ in [3] contains some misprints and an omission.

Proof. For $\beta$ real the formula is trivial, and for $\beta$ imaginary the formula comes from Corollary 2.2. For $\beta$ complex we apply Lemma 2.4. Our assumption about a minimal parabolic subgroup means that $\Delta_{-,n} = \emptyset$. If $\beta$ were of type (2c) or (3a) or (3b), then $2\varepsilon$ would be in $\Delta_{-,n}$; if $\beta$ were of type (2b) with $\beta$ noncompact, then $\varepsilon$ would be in $\Delta_{-,n}$. So $\beta$ is of none of these types.

We apply Theorem 2.1 and prepare to apply Theorem 2.5. Normally we let $S = \{j \mid \langle \beta, \alpha_j \rangle \neq 0\}$. But if there is an index $j_0$ such that $\varepsilon' + \alpha_{j_0}$ is in $\Delta$ with $\varepsilon'$ orthogonal to all $\alpha_j$, then we adjoin $j_0$ to $S$. (Such an index $j_0$ is necessarily unique.) Then Assumption (i) is certainly satisfied.

In view of what we have already proved, Assumption (ii) will be satisfied if we show that $\beta$ cannot be of type (3c). If, on the contrary, $\beta$ is of type (3c), then the component $\Delta_\beta$ of $\Delta$ to which $\beta$ belongs is of type $B_n$, $C_n$, or $F_4$. For the case of $B_n$, we note that $\beta$ and $\alpha_2$ are short and nonorthogonal, contradiction. For the case of $C_n$ or $F_4$, $\alpha_1$ shows the existence of long noncompact roots, and it follows that the group corresponding to $\Delta_\beta$ is split over $\mathbb{R}$. Since the format is assumed attached to a minimal parabolic subgroup, $\Delta_\beta$ can contain no complex roots, in contradiction to the assumed form of $\beta$.

Thus we can apply Theorem 2.5. We consider the possibilities for $S$. First suppose $\Delta^+$ contains some member $\gamma = \varepsilon' - \alpha_{j_0}$ with $\varepsilon' \neq 0$ orthogonal to all $\alpha_j$, and let $\varepsilon$ be the projection of $\beta$ orthogonal to all $\alpha_j$. Lemma 2.6 shows that $\varepsilon' = 2\varepsilon$ or $\varepsilon' = \varepsilon$. If $\varepsilon' = 2\varepsilon$, examination of (2.5) shows that $|\varepsilon|^2 = \frac{1}{4}|\beta|^2$, i.e., $\beta$ is of type (3), in contradiction with what we already know about $\beta$.

So $\gamma$ must be of the form $\varepsilon - \alpha_{j_0}$. For $\varepsilon$ to be a root, $\beta$ must be of type (2a) or (2b). If $\beta$ is of type (2a), say

$\beta = \varepsilon - \frac{1}{2}\alpha_i - \frac{1}{2}\alpha_j$, then $\frac{1}{2}\alpha_i + \frac{1}{2}\alpha_j$ is in $\Delta$. If $\beta$ is of type (2b), then $\beta$ must be $\gamma$. In any event, the only way $S$ can have more than one element is if $\beta$ is of type (2a).

When $\Delta^+$ does not contain a member $\varepsilon' - \alpha_{j_0}$, then the only way $S$ can have more than one element is if $\beta$ is of type (2a), since $\beta$ is not of type (3).

Thus if $\beta$ is not of type (2a), we can apply Corollary 2.3 to $H$ to handle matters. If $\beta$ is of type (2a), say with $\beta = \varepsilon - \frac{1}{2}\alpha_i - \frac{1}{2}\alpha_j$, then we make an explicit calculation. If $\frac{1}{2}(\alpha_i + \alpha_j)$ is a root, the calculation is in a group locally isomorphic to $SO(5,2)$ or $SO(6,3)$; if $\frac{1}{2}(\alpha_i + \alpha_j)$ is not a root, the calculation is in a group locally isomorphic to $SU(2,2)$. There are no other possibilities, and the corollary follows.

## 3. Proof of Theorem 2.5

Throughout this section the notation and assumptions in Theorem 2.5 will be in force. We write $\beta = \varepsilon - \Sigma b_i \alpha_i$ with $\varepsilon$ orthogonal to all $\alpha_j$.

Lemma 3.1. $\Delta_-^H = \Delta_-$ and $\Delta_{-,c}^H = \Delta_{-,c}$.

Proof. Let $\gamma$ be in $\Delta_-$ and write $\gamma = c\beta + \Sigma c_i \alpha_i$. Here $c \neq 0$ to make $\gamma$ orthogonal to all $\alpha_j$. Taking the inner product with $\alpha_j$, we have

$$0 = \langle \gamma, \alpha_j \rangle = c\langle \beta, \alpha_j \rangle + c_j |\alpha_j|^2.$$

Thus $c_j \neq 0$ implies $\langle \beta, \alpha_j \rangle \neq 0$, which implies $j$ is in $S$. Thus $\gamma$ is in $\Delta^H$, and we obtain $\Delta_-^H = \Delta_-$. The equality $\Delta_{-,c}^H = \Delta_{-,c}$ then follows from Assumption (i).

Lemma 3.2. $\rho - \rho^H - \rho_r + \rho_r^H$ is analytically integral on $\mathfrak{b}^H$.

Remark. Assumption (ii) is used only in this lemma and in Lemma 3.3.

Proof. The expression in question is half the sum of the members of $\Delta^+$ that are in neither $\Delta^H$ nor $\Delta_r$, and it is enough to show that the expression is actually a sum of members of $\Delta^+$. The roots contributing to the expression are all complex, and we write them out as in Lemma 2.4. Then we consider together roots differing only in the signs attached to each $\alpha$. If there are at least two $\alpha$'s, then half the sum of the members of a class is an integer multiple of the sum of the root with all plus signs and the root with all minus signs. So the only problem is with roots of types (1), (2b), and (2c). We show the only ones that contribute are of type (2b).

For type (1) let $\gamma = \varepsilon' \pm \frac{1}{2}\alpha_k$. Consideration of lengths shows that $\beta = \varepsilon' - \frac{1}{2}\alpha_j$, and then $j = k$ to force $2\langle\gamma,\beta\rangle/|\beta|^2$ to be an integer. Thus $\gamma$ is already in $\Delta^H$. For type (2c) let $\gamma = \varepsilon' \pm \frac{1}{2}\alpha_k$. Then $2\varepsilon'$ is a root, and Lemma 2.6 implies $\varepsilon' = \varepsilon$. Assumption (ii) forces $\beta$ to be of type (2a), (2b), or (3a), and (2b) is ruled out since $2\varepsilon$ is a root. If $\beta = \varepsilon - \frac{1}{2}\alpha_i - \frac{1}{2}\alpha_j$ is of type (2a), then $\frac{1}{2}(\alpha_i + \alpha_j) \pm \frac{1}{2}\alpha_k$ is a pair of orthogonal roots whose difference is a root. Then the sum $\alpha_i + \alpha_j$ must be a root, in contradiction to strong orthogonality. Finally if $\beta$ is of type (3a), then $\langle\beta,\alpha_k\rangle \neq 0$ in order to avoid $2\langle\beta,\gamma\rangle/|\beta|^2 = 1/2$. In short, no root $\gamma$ of type (1) or (2c) makes a contribution.

For type (2b) let $\gamma = \varepsilon' \pm \alpha_k$. Half the sum of these two roots is $\varepsilon'$, which is a root. This completes the proof.

Lemma 3.3. Every member of $\Delta_K^+$ is the sum of members of $(\Delta_K^H)^+$ and $\Delta_r^+$.

Proof. Let $\gamma$ in $\Delta_K^+$ be given. In view of Lemma 2.6, we can write $\gamma = c\epsilon + \Sigma c_i\alpha_i$ with $c = 1$ or $2$. Lemma 3.1 shows we may assume $\Sigma c_i\alpha_i \neq 0$.

First suppose $c = 2$. Then $|\gamma|^2 > 4|\epsilon|^2$, and (2.5) shows that $|\epsilon|^2 = \frac{1}{4}|\beta|^2$ and $2|\beta|^2 = |\gamma|^2$. That is, $\beta$ is of type (3) in Lemma 2.4, hence of type (3a) by Assumption (ii). Also $\gamma$ is of type (2); being long, it must be of type (2a) or (2b). If $\gamma$ is of type (2a), then the $\alpha$'s in it are all long, while the $\alpha$'s in $\beta$ are short; thus we obtain $2\langle\gamma,\beta\rangle/|\gamma|^2 = 1/2$, contradiction. So $\gamma$ is of type (2b), and then Assumption (i) and Lemma 3.1 show that $\gamma$ is already in $(\Delta_K^H)^+$.

Thus $c = 1$. Suppose now that $|\beta|^2 = 2|\gamma|^2$. Then $\beta$ is of type (3), and $\gamma$ is of type (2). Hence $\beta$ is of type (3a) by Assumption (ii), and $\gamma$ (being short) must be of type (2a) or (2c). If $\gamma$ is of type (2a), then the $\alpha$'s in it are all short, while the $\alpha$'s in $\beta$ are all long; thus we obtain $2\langle\gamma,\beta\rangle/|\beta|^2 = 1/2$, contradiction. So $\gamma$ is of type (2c): $\gamma = \epsilon \pm \frac{1}{2}\alpha_i$. The requirement $2\langle\gamma,\beta\rangle/|\beta|^2 \neq 1/2$ forces $\alpha_i$ to occur in $\beta$, and we conclude that $\gamma$ is already in $(\Delta_K^H)^+$.

With $c = 1$, suppose next that $|\gamma|^2 = 2|\beta|^2$. Then $\beta$ is of type (2) and $\gamma$ is of type (3). Since $\beta$ is short and $\gamma$ is long, $\beta$ is of type (2a) or (2c), and $\gamma$ is of type (3a) or (3b). Assumption (ii) says $\beta$ is not of type (2c). So $\beta$ is of type (2a), say $\beta = \epsilon - \frac{1}{2}\alpha_i - \frac{1}{2}\alpha_j$. Then $2\langle\beta,\gamma\rangle/|\gamma|^2 \neq 1/2$ implies $\gamma$ is of type (3b) of the form $\gamma = \epsilon \pm \alpha_i \pm \frac{1}{2}\alpha_k$. It follows that $\frac{1}{2}(\alpha_i + \alpha_j) \pm \frac{1}{2}\alpha_k$ are orthogonal roots whose difference is a root; hence the sum $\alpha_i + \alpha_j$ is a root, in contradiction with strong orthogonality.

Consequently we may assume that $c = 1$ and $|\beta| = |\gamma|$. Then $\beta$ and $\gamma$ are both of type (n) for $n = 1, 2$, or $3$. If $n = 1$, write

$\beta = \epsilon - \frac{1}{2}\alpha_i$. Then $2\langle\beta,\gamma\rangle/|\gamma|^2 \neq 3/2$ implies $\gamma = \epsilon \pm \frac{1}{2}\alpha_i$; hence $\gamma$ is in $(\Delta_K^H)^+$.

Suppose $n = 2$. If $\beta$ is of type (2b) (and is thus compact, by assumption), say $\beta = \epsilon - \alpha_i$, then $\gamma = \epsilon \pm \alpha_i$ or else $\langle\gamma,\alpha_i\rangle = 0$, by consideration of the lengths of the $\alpha$'s in $\gamma$. In the first case $\gamma$ is in $(\Delta_K^H)^+$, and in the second case $\langle\gamma,\beta\rangle > 0$ and $\beta$ simple imply that $\gamma = \beta + (\gamma - \beta)$ is the required decomposition of $\gamma$.

If $\beta$ is of type (2a), write $\beta = \epsilon - \frac{1}{2}\alpha_i - \frac{1}{2}\alpha_j$. If $\gamma$ is of type (2b), Assumption (i) shows $\gamma$ is in $(\Delta_K^H)^+$. If $\gamma$ is of type (2c), say $\gamma = \epsilon \pm \frac{1}{2}\alpha_k$, then $\frac{1}{2}(\alpha_i+\alpha_j) \pm \frac{1}{2}\alpha_k$ are orthogonal roots, and we are led to conclude that $\alpha_i + \alpha_j$ is a root, contradiction. So we may assume $\gamma$ is of type (2a). We may assume that $\gamma$ is not $\epsilon \pm \frac{1}{2}\alpha_i \pm \frac{1}{2}\alpha_j$, and then the fact that $2\langle\gamma,\beta\rangle/|\beta|^2$ is an integer means that $\gamma = \epsilon \pm \frac{1}{2}\alpha_k \pm \frac{1}{2}\alpha_m$ with $\{i,j\} \cap \{k,m\}$ empty. Let us assume that $i$ precedes $j$; we choose a sign $\pm$ so that $\tilde{\beta} = \epsilon - \frac{1}{2}\alpha_i \pm \frac{1}{2}\alpha_j$ is compact. Then $\gamma = \tilde{\beta} + (\gamma - \tilde{\beta})$ is the required decomposition; $\gamma - \tilde{\beta}$ is positive because $\gamma - \beta$ is positive and because $i$ precedes $j$.

Finally suppose $n = 3$. By Assumption (ii) $\beta$ is of type (3a), say $\beta = \epsilon - \frac{1}{2}\alpha_i - \frac{1}{2}\alpha_j - \frac{1}{2}\alpha_k$. Then $2\epsilon$ is noncompact, and so $\gamma$ is of type (3a) or (3b). If $\gamma$ is of type (3b), then $2\langle\beta,\gamma\rangle/|\gamma|^2 \neq 1/2$ implies $\gamma$ is of the form $\epsilon \pm \frac{1}{2}\alpha_i \pm \alpha_m$. Then $\alpha_m - \frac{1}{2}(\alpha_j + \alpha_k)$ is a root, necessarily negative for $\beta$ to be simple. Hence $j$ or $k$ precedes $m$. Let us say that $j$ precedes $k$, for definiteness. Then the roots $\frac{1}{2}\alpha_j \pm \frac{1}{2}\alpha_k \pm \alpha_m$ are all positive. Choose the coefficient of $\alpha_m$ to match that in $\gamma$, choose the coefficient of $\frac{1}{2}\alpha_k$ to make the whole root compact, and call the result $\delta$. Then $\gamma = \delta + (\gamma - \delta)$ is the required decomposition of $\gamma$.

For $\gamma$ of type (3a), we argue in the same way. Since

$2\langle\beta,\gamma\rangle/|\gamma|^2 \neq 1/2$, we may assume $\gamma = \varepsilon \pm \frac{1}{2}\alpha_i \pm \frac{1}{2}\alpha_{j'} \pm \frac{1}{2}\alpha_{k'}$. Say $j$ precedes $k$ and $j'$ precedes $k'$. Since $\beta$ is simple, $j$ precedes $j'$. Then the roots

$$\tfrac{1}{2}\alpha_j \pm \tfrac{1}{2}\alpha_k \pm \tfrac{1}{2}\alpha_{j'} \pm \tfrac{1}{2}\alpha_{k'}$$

are all positive. We choose the signs for $\alpha_{j'}$ and $\alpha_{k'}$ to match those in $\gamma$ and the sign for $\alpha_k$ to make the whole root compact. If we call the result $\delta$, then $\gamma = \delta + (\gamma - \delta)$ is the required decomposition of $\gamma$. This proves the lemma.

<u>Proof of Theorem 2.5</u>. We may assume $G^{\mathbb{C}}$ is simply connected. It is clear that $\lambda_0^H$ is dominant for $(\Delta_-^H)^+$. From (1.2) and Lemma 3.1 it follows that $\lambda^H = \lambda$ on $\mathfrak{b}^H$ and hence that $\lambda^H$ is analytically integral on $\mathfrak{b}_-^H$ and is dominant for $(\Delta_{-,c}^H)^+$. Then it follows that $\lambda_0^H$ is the infinitesimal character of a discrete series or limit of discrete series of $(M \cap H)_0$.

Next we show that $\exp \lambda^H$ and any extension $\chi^H$ of $\chi_0^H$ agree on $(\exp \mathfrak{b}_-^H) \cap (\exp \mathfrak{b}_r^H)$, so that we obtain a well defined representation of $(M \cap H)^\#$, then of $M \cap H$. By Lemma 3.2, the restriction to $\mathfrak{b}^H$ of the linear functional

$$2\rho_K - 2\rho_K^H - (\rho - \rho^H - \rho_r + \rho_r^H) \tag{3.1}$$

is analytically integral on $\mathfrak{b}^H$. By Lemma 3 of [1], we have

$$2\rho_K - 2\rho_{-,c} - \rho + \rho_- + \rho_r - E(2\rho_K) = 0 \tag{3.2}$$

and a similar identity on $H$. Restricting (3.2) to $\mathfrak{b}^H$, subtracting the corresponding identity on $H$, and taking into account Lemma 3.1, we obtain

$$2\rho_K - 2\rho_K^H - (\rho - \rho^H - \rho_r + \rho_r^H) = E(2\rho_K) - E^H(2\rho_K^H) \tag{3.3}$$

on $\mathfrak{b}^H$. The integrality of (3.1) means that the right side of (3.3) is analytically integral on $\mathfrak{b}^H$. Therefore

$$\exp(E^H(2\rho_K^H) - 2\rho_{K_r}^H) = \exp(E(2\rho_K) - 2\rho_{K_r}) \tag{3.4}$$

on $\exp \mathfrak{b}_-^H$. The required consistency

$$\chi_0^H = \exp \lambda^H \quad \text{on} \quad (\exp \mathfrak{b}_-^H) \cap (\exp \mathfrak{b}_r^H)$$

follows immediately by combining (3.4) with the identities

$$\begin{aligned}
\omega_0^H &= \omega && \text{on } M_r^H \cap \exp \mathfrak{b}_- \\
\omega_0^H &= \chi_0^H \cdot \exp(E^H(2\rho_K^H) - 2\rho_{K_r}^H) && \text{on } M_r^H \cap \exp \mathfrak{b}_-^H \\
\omega &= \chi \cdot \exp(E(2\rho_K) - 2\rho_{K_r}) && \text{on } M_r \\
\chi &= \exp \lambda && \text{on } (\exp \mathfrak{b}_-) \cap (\exp \mathfrak{b}_r) \\
\lambda^H &= \lambda && \text{on } \mathfrak{b}^H .
\end{aligned}$$

It follows from the definition of "fine" in [4] that every irreducible constituent of $\tau_\mu |_{K_r^H}$ is fine. Since $\tau_{\mu^H}$ is such a constituent, $\mu^H$ is fine. Fix $\omega^H$ as some constituent of $\tau_{\mu^H}|_{M_r^H}$. Since $M_r^H \cap \exp \mathfrak{b}_-^H$ centralizes $G_r$, $\tau_\mu$ is scalar on $M_r^H \cap \exp \mathfrak{b}_-^H$, and thus $\omega^H$ has to agree with $\omega$ on $M_r^H \cap \exp \mathfrak{b}_-^H$. Thus $\omega^H$ is an extension of $\omega_0^H$, and $\mu^H$ is a fine $K_r^H$-type for $\omega^H$.

Now we check that $\Lambda^H$ is $(\Delta_K^H)^+$ dominant. The formula (1.1) for $\Lambda$ is

$$\Lambda = \lambda - E(2\rho_K) + 2\rho_{K_r} + \mu ,$$

and there is a similar formula for $\Lambda^H$. Restricting the two formulas to $\mathfrak{b}^H$ and subtracting, we obtain

$$\Lambda^H - \Lambda = [E(2\rho_K) - 2\rho_{K_r}] + [E^H(2\rho_K^H) - 2\rho_{K_r}^H] . \tag{3.5}$$

Both bracketed terms on the right are orthogonal to $\Delta_K^H$, and the $(\Delta_K^H)^+$ dominance of $\Lambda^H$ therefore follows from the $\Delta_K^+$ dominance of $\Lambda$.

This establishes that $(\{\alpha_j\}_{j\in S}, (\Delta^H)^+, \chi^H, \mu^H)$ is a format for H and that $\lambda_0^H$ is compatible with it. We now want to see that $\lambda_0^H$ is the basic case for this format. Assume the contrary. Then there exists a $\Delta_H^+$ dominant H-algebraically integral form $\xi^H$ on $\mathfrak{b}^H$ not identically $0$ such that $\lambda_0^H - \xi^H$ corresponds to a nonzero representation (corresponding to a cover of H) compatible with our format for $H$. One of the conditions on $\xi^H$ is that $\langle \xi^H, \alpha_j\rangle = 0$ for $j$ in $S$. We extend $\xi^H$ to $\xi$ on $\mathfrak{b}$ by requiring $\langle \xi, \alpha_j\rangle = 0$ for all $j$ (and also $\langle \xi, \beta\rangle = \langle \xi^H, \beta\rangle$). Then $\xi$ is a multiple $n\Lambda_\beta$ of the fundamental form $\Lambda_\beta$ for $G$ corresponding to $\beta$. Since $G^{\mathbb{C}}$ is simply connected, $\xi$ is analytically integral on $\mathfrak{b}$. From the fact that $\lambda_0^H - n\Lambda_\beta|_{\mathfrak{b}^H}$ corresponds to a nonzero representation compatible with the format of $H$, we shall prove that $\lambda_0' = \lambda_0 - n\Lambda_\beta$ corresponds to a nonzero representation compatible with the format of $G$, in contradiction with the fact that $\lambda_0$ is a basic case. This contradiction will prove that $\lambda_0^H$ is a basic case and will complete the proof of the theorem.

The integrality conditions are no problem. We need to see that $\lambda_0'$ is dominant for $(\Delta_-)^+$, that its $\lambda'$ is dominant for $\Delta_{-,c}^+$, and that its $\Lambda'$ is dominant for $\Delta_K^+$. The required dominance for $\lambda_0'$ and $\lambda'$ follows from Lemma 3.1 and the corresponding properties in $H$ of $\lambda_0^H - n\Lambda_\beta|_{\mathfrak{b}^H}$.

For $\Lambda'$, formula (3.5) shows that $\langle \Lambda, \gamma\rangle \geq 0$ for $\gamma$ in $(\Delta_K^H)^+$, and each $\gamma$ in $\Delta_{K_r}^+$ satisfies

$$\langle \Lambda', \gamma\rangle = \langle \Lambda, \gamma\rangle - n\langle \Lambda_\beta, \gamma\rangle = \langle \Lambda, \gamma\rangle \geq 0 .$$

Thus $\Lambda'$ is $\Delta_K^+$ dominant by Lemma 3.3. This completes the proof of the theorem.

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