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Introduction to Representations in Analytic Cohomology

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ABSTRACT. This is a survey of background and old results concerning representations in cohomology sections of vector bundles. The base space is a homogeneous space G/L, where G is a connected reductive Lie group and L is the centralizer of a torus. When G is compact, the representations in question are the subject of the Bott-Borel-Weil Theorem. When G is noncompact and L is compact, the representations are identified by the Langlands Conjecture, which was proved by Schmid. For noncompact L, difficult analytic problems blocked progress initially. To avoid these difficulties, Zuckerman and Vogan developed an algebraic analog, cohomological induction, that gave a construction of identifiable representations that were often irreducible unitary. Recent progress has related the analytic representations and their algebraic analogs in various ways.

1. Sections of homogeneous vector bundles

This paper gives some background from representation theory for understanding the connection between the Penrose transform and analytic realizations of group representations. It is assumed that the reader is acquainted with elementary facts about holomorphic vector bundles and the elementary structure theory of semisimple groups. Discussions of these two topics may be found in Wells [**19**, Chapter I] and Knapp [**8**, Chapter V], respectively. The results in this paper largely are not new, and, for the most part, references will be given in place of proofs.

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1.1. Setting.

Throughout the paper we work with the following situation, sometimes limiting ourselves to special cases: G is a connected linear reductive Lie group with complexification $G^{\mathbb{C}}$, T is a torus subgroup, and $L = Z_G(T)$ is the centralizer of T in G. It is known that L is connected; a proof may be constructed by combining [**8**, Corollary 4.22] with the style of argument at the top of p. 126 of that book. Therefore the complexification $L^{\mathbb{C}}$ is meaningful. Let Q be a parabolic subgroup of $G^{\mathbb{C}}$ with Levi factor $L^{\mathbb{C}}$.

We denote Lie algebras of Lie groups A, B, etc., by $\mathfrak{a}_0$, $\mathfrak{b}_0$, etc., and we denote their complexifications by $\mathfrak{a}$, $\mathfrak{b}$, etc. The complex Lie algebras of complex Lie groups $G^{\mathbb{C}}$, $L^{\mathbb{C}}$, Q are denoted $\mathfrak{g}$, $\mathfrak{l}$, $\mathfrak{q}$. We use an overbar to denote the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

We can decompose the Lie algebra $\mathfrak{q}$ of Q as a vector space direct sum $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{u}$ is the nilradical. Then $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are both nilpotent complex Lie algebras, and we have $[\mathfrak{l}, \mathfrak{u}] \subseteq \mathfrak{u}$ and $[\mathfrak{l}, \bar{\mathfrak{u}}] \subseteq \bar{\mathfrak{u}}$.

We assume that $\mathfrak{q}$ is a **θ-stable parabolic**; this condition means that

$$\mathfrak{g}_0 \cap \mathfrak{q} = \mathfrak{l}_0. \tag{1.1a}$$

It is equivalent to assume a vector space sum decomposition

$$\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u}. \tag{1.1b}$$

Under the condition (1.1), the natural mapping $G/L \to G^{\mathbb{C}}/Q$ is an inclusion, and the image is an open set. Thus the choice of Q has made G/L into a complex manifold with G operating holomorphically.

An example to keep in mind is the group $G = U(m,n)$ of complex matrices that preserve an indefinite Hermitian form. Here $G^{\mathbb{C}} = GL(m+n, \mathbb{C})$. If we take T to be any closed connected subgroup of the diagonal, then L will be a block diagonal subgroup within G, necessarily connected. We can choose $\mathfrak{u}$ to be the complex Lie algebra of corresponding block upper triangular matrices and $\bar{\mathfrak{u}}$ to consist of the corresponding block lower triangular matrices.

1.2. Associated bundles.

It is well known (see [**16**]) that

$$p : G \to G/L \tag{1.2}$$

is a C^∞ principal fiber bundle with structure group L. Let V be a finite-dimensional real or complex vector space, let $GL(V)$ be its general linear group, and let $\rho : L \to GL(V)$ be a C^∞ homomorphism. The **associated vector bundle**

$$p_V : G \times_L V \to G/L \tag{1.3a}$$

is a vector bundle with structure group $GL(V)$ whose bundle space is given by

$$G \times_L V = \{(g,v)/\sim\} \qquad \text{with} \qquad (gl, v) \sim (g, \rho(l)v) \tag{1.3b}$$

for $g \in G$, $l \in L$, and $v \in V$; the bundle structure will now be described.

The bundle (1.2) can be given in terms of transition functions. Namely for a suitably fine open cover $\{U\}$ of G/L, there are fiber preserving C^∞ maps $h_U : p^{-1}(U) \to U \times L$ that specify the local product structure on G, and the assumption is that, for $x \in U \cap V$, $h_{VU}(x) = h_V \circ h_U^{-1}|_{p^{-1}(x)}$ is a member of L and depends in C^∞ fashion on x. The functions $\{\rho(h_{VU}(x))\}$ are the transition functions for (1.3). These determine a vector bundle structure for (1.3) by Steenrod [**16**, Theorem I.3.2].

The space of C^∞ sections of (1.3) is denoted $\mathcal{E}(G \times_L V)$. The group G acts on $G \times_L V$ by left translation: $g_0(g, v) = (g_0 g, v)$ in the notation of (1.3b), and this action induces an action of G on $\mathcal{E}(G \times_L V)$ by $(g_0\gamma)(g, v) = \gamma(g_0^{-1}g, v)$ for $\gamma \in \mathcal{E}(G \times_L V)$. When V is complex, this construction yields a representation of G (understood to be on a complex vector space) with a natural continuity property: $(g_0, \gamma) \mapsto g_0\gamma$ is continuous from $G \times \mathcal{E}(G \times_L V)$ to $\mathcal{E}(G \times_L V)$ if $\mathcal{E}(G \times_L V)$ is given its usual C^∞ topology.

Similarly

$$p : G^{\mathbb{C}} \to G^{\mathbb{C}}/Q \tag{1.4}$$

is a holomorphic principal fiber bundle with structure group Q. In the above situation if V is complex and if ρ extends to a holomorphic homomorphism $\rho : Q \to GL(V)$, then we can construct an associated vector bundle

$$p_V : G^{\mathbb{C}} \times_{\mathbb{C}} V \to G^{\mathbb{C}}/Q \tag{1.5a}$$

with bundle space given by

$$G^{\mathbb{C}} \times_Q V = \{(g^{\mathbb{C}}, v)/\sim\} \qquad \text{with} \qquad (g^{\mathbb{C}}q, v) \sim (g^{\mathbb{C}}, \rho(q)v). \tag{1.5b}$$

The bundle (1.5) is a holomorphic vector bundle.

The inclusion $G/L \hookrightarrow G^{\mathbb{C}}/Q$ induces via pullback from (1.5a) a bundle map

$$G \times_L V \hookrightarrow G^{\mathbb{C}} \times_Q V. \tag{1.6}$$

In terms of (1.3b) and (1.5b), this map is given simply by $(g, v) \mapsto (g, v)$. The result is that the C^∞ complex vector bundle $G \times_L V$ acquires the structure of a holomorphic vector bundle. We can regard the space of holomorphic sections $\mathcal{O}(G \times_L V)$ of $G \times_L V$ as a vector subspace of $\mathcal{E}(G \times_L V)$.

In applications it is important to be able to relax the assumptions on the original $\rho : L \to GL(V)$ and still be able to use Q to impose the structure of a holomorphic vector bundle on $G \times_L V$. See Tirao-Wolf [**17**] for this generalization.

To any section γ of $G \times_L V$ we can associate a function $\varphi_\gamma : G \to V$ by the definition

$$\gamma(gL) = (g, \varphi_\gamma(g)) \in G \times_L V. \tag{1.7a}$$

Under this correspondence, C^∞ sections γ go to C^∞ functions φ_γ, and we obtain an isomorphism

(1.7b)
$$\mathcal{E}(G \times_L V) \cong \left\{ \varphi : G \to V \;\middle|\; \begin{array}{l} \varphi \text{ of class } C^\infty, \\ \varphi(gl) = \rho(l)^{-1}\varphi(g) \text{ for } l \in L,\, g \in G \end{array} \right\}.$$

The usual C^∞ topology on $\mathcal{E}(G \times_L V)$ corresponds to the C^∞ topology on the space of φ's.

The correspondence $\gamma \leftrightarrow \varphi_\gamma$ works locally as well, with sections over an open set $U \subseteq G/L$ corresponding to functions φ on the open subset $p^{-1}(U)$ of G transforming as in (1.7b). Again γ of class C^∞ corresponds to φ_γ of class C^∞. Let $\mathcal{E}(U)$ be the space of C^∞ sections over U.

In the special case that $G \times_L V$ admits the structure of a holomorphic vector bundle because of (1.6) and (1.5), we can speak of the space of holomorphic sections $\mathcal{O}(U)$ over an open set $U \subseteq G/L$. The first proposition tells how to use φ_γ to decide whether γ is holomorphic.

PROPOSITION 1. *Suppose that ρ extends to a holomorphic homomorphism $\rho : Q \to GL(V)$ and thereby makes $G \times_L V$ into a holomorphic vector bundle. Let $U \subseteq G/L$ be open, let γ be in $\mathcal{E}(U)$, and let φ_γ be the corresponding function from G to V given by (1.7). Then γ is holomorphic if and only if*

$$(Z\varphi_\gamma)(g) = -\rho(Z)(\varphi_\gamma(g)) \tag{1.8}$$

for all $g \in p^{-1}(U)$ and $Z \in \mathfrak{q}$, with Z acting on φ_γ as a complex left-invariant vector field.

PROOF. Suppose γ is in $\mathcal{E}(U)$. We can regard U as open in $G^{\mathbb{C}}/Q$ and use the formula

$$\gamma(g^{\mathbb{C}}Q) = (g^{\mathbb{C}}, \tilde{\varphi}_\gamma(g^{\mathbb{C}})) \in G^{\mathbb{C}} \times_Q V$$

to define $\tilde{\varphi}_\gamma$ on the open set $p^{-1}(U) \subseteq G^{\mathbb{C}}$. The function $\tilde{\varphi}_\gamma$ satisfies

$$\tilde{\varphi}_\gamma(g^{\mathbb{C}}q) = \rho(q)^{-1}(\tilde{\varphi}_\gamma(g^{\mathbb{C}})) \tag{1.9a}$$

for $g^{\mathbb{C}} \in p^{-1}(U)$ and $q \in Q$, and therefore also

$$(Z\tilde{\varphi}_\gamma)(g^{\mathbb{C}}) = -\rho(Z)(\tilde{\varphi}_\gamma(g^{\mathbb{C}})) \tag{1.9b}$$

for $g^{\mathbb{C}} \in p^{-1}(U)$ and $Z \in \mathfrak{q}$, with Z acting as a *real* left-invariant vector field. We recover φ_γ by restricting $\tilde{\varphi}_\gamma$ to $G \cap p^{-1}(U)$. By definition of the complex structure, γ is holomorphic if and only if $\tilde{\varphi}_\gamma$ is holomorphic. We are thus to show that $\tilde{\varphi}_\gamma$ is holomorphic if and only if φ_γ satisfies (1.8).

Suppose $\tilde{\varphi}_\gamma$ is holomorphic. The Cauchy-Riemann equations say that

$$(iZ)\tilde{\varphi}_\gamma = i(Z\tilde{\varphi}_\gamma) \tag{1.10}$$

for $Z \in \mathfrak{g}$, with Z acting as a real left-invariant vector field. If Z is in $\mathfrak{q}$, write $Z = X + iY$ with X and Y in $\mathfrak{g}_0$. For $g \in G$ we have

$$
\begin{aligned}
-\rho(Z)(\varphi_\gamma(g)) &= -\rho(Z)(\tilde{\varphi}_\gamma(g)) \\
&= (Z\tilde{\varphi}_\gamma)(g) && \text{by (1.9b)} \\
&= (X\tilde{\varphi}_\gamma)(g) + (iY)\tilde{\varphi}_\gamma(g) \\
&= (X\tilde{\varphi}_\gamma)(g) + i(Y\tilde{\varphi}_\gamma)(g) && \text{by (1.10)} \\
&= (X\varphi_\gamma)(g) + i(Y\varphi_\gamma)(g) \\
&= (Z\varphi_\gamma)(g).
\end{aligned}
\tag{1.11}
$$

Thus φ_γ satisfies (1.8).

Conversely suppose φ_γ satisfies (1.8). Unwinding (1.11), we obtain

$$(iY)\tilde{\varphi}_\gamma(g) = i(Y\tilde{\varphi}_\gamma)(g)$$

whenever $Z = X + iY$ is in $\mathfrak{q}$. Replacing Z by iZ, we obtain also

$$(iX)\tilde{\varphi}_\gamma(g) = i(X\tilde{\varphi}_\gamma)(g).$$

Suitable linear combinations of these two equations give

$$(iZ)\tilde{\varphi}_\gamma(g) = i(Z\tilde{\varphi}_\gamma)(g) \qquad \text{and} \qquad (i\overline{Z})\tilde{\varphi}_\gamma(g) = i(\overline{Z}\tilde{\varphi}_\gamma)(g)$$

for Z in $\mathfrak{q}$. Since $\mathfrak{q} + \overline{\mathfrak{q}} = \mathfrak{g}$, (1.10) holds for all $Z \in \mathfrak{g}$ for the special case that g is in $p^{-1}(U) \cap G$. A general member of $p^{-1}(U)$ in $G^{\mathbb{C}}$ is of the form gq with $g \in p^{-1}(U) \cap G$ and $q \in Q$. Taking Z in $\mathfrak{g}$ and letting a dot indicate where a vector field is to act, we have

$$
\begin{aligned}
(iZ)\tilde{\varphi}_\gamma(gq\,\cdot) &= (\mathrm{Ad}(q)(iZ))\tilde{\varphi}_\gamma(g\cdot q) \\
&= \rho(q)^{-1}((\mathrm{Ad}(q)(iZ))\tilde{\varphi}_\gamma(g\,\cdot)) && \text{by (1.9a)} \\
&= \rho(q)^{-1}i(\mathrm{Ad}(q)Z)\tilde{\varphi}_\gamma(g\,\cdot) && \text{by the special case} \\
&= i(\mathrm{Ad}(q)Z)\tilde{\varphi}_\gamma(g\cdot q) && \text{by (1.9a)} \\
&= i(Z\tilde{\varphi}_\gamma)(gq\,\cdot).
\end{aligned}
$$

Thus $\tilde{\varphi}_\gamma$ satisfies (1.10) everywhere on $p^{-1}(U)$ and is holomorphic.

REFERENCE. See Griffiths-Schmid [**6**, pp. 258-259].

In typical applications to representation theory, ρ in the proposition is given on L and extends holomorphically to $L^{\mathbb{C}}$. The extension to Q is taken to be trivial on the unipotent radical of Q. Equation (1.8) holds for $Z \in \mathfrak{l}_0$ for any C^∞ section, and it extends to $Z \in \mathfrak{l}$ by complex linearity. Thus (1.8) may be replaced in this situation by the condition

$$Z\varphi_\gamma = 0 \qquad \text{for all } Z \in \mathfrak{u}. \tag{1.12}$$

The special case $\rho = 1$ shows how to recognize holomorphic functions on open subsets of G/L.

1.3. Constructions with tangent bundle.

Let M be a complex manifold, and let p be in M. We denote by $T_p(M)$ the tangent space of M (considered as a C^∞ manifold) at p, consisting of derivations of the algebra of smooth germs at p, and we let $T(M)$ be the tangent bundle. Also we denote by $T_{\mathbb{C},p}(M)$ the complex tangent space of M at p, consisting of derivations of the algebra of holomorphic germs at p, and we let $T_{\mathbb{C}}(M)$ be the corresponding complex tangent bundle. There is a canonical $\mathbb{R}$ isomorphism

$$T_p(M) \xrightarrow{\sim} T_{\mathbb{C},P}(M) \tag{1.13a}$$

given by

$$\xi \mapsto \zeta, \quad \text{where } \zeta(u+iv) = \xi(u) + i\xi(v). \tag{1.13b}$$

Let J_p be the member of $GL(T_p(M))$ that corresponds under (1.13) to multiplication by i in $T_{\mathbb{C},p}(M)$. Then $J = \{J_p\}$ is a bundle map from $T(M)$ to itself whose square is -1.

The following proposition allows us to relate these considerations to associated vector bundles.

PROPOSITION 2. *There are canonical bundle isomorphisms*

$$T(G/L) \cong G \times_L (\mathfrak{g}_0/\mathfrak{l}_0) \tag{1.14a}$$

$$T_{\mathbb{C}}(G^{\mathbb{C}}/Q) \cong G^{\mathbb{C}} \times_Q (\mathfrak{g}/\mathfrak{q}) \tag{1.14b}$$

with L and Q acting on $\mathfrak{g}_0/\mathfrak{l}_0$ and $\mathfrak{g}/\mathfrak{q}$, respectively, by Ad.

The inclusion $G/L \subseteq G^{\mathbb{C}}/Q$ allows us to regard

$$T_{\mathbb{C}}(G/L) \cong GQ \times_Q (\mathfrak{g}/\mathfrak{q}). \tag{1.15}$$

At any point $p = gL$ of G/L, the left sides of (1.14a) and (1.15), namely $T(G/L)$ and $T_{\mathbb{C}}(G/L)$, are $\mathbb{R}$ isomorphic via (1.13). It is easy to check that the corresponding isomorphism of the right sides of (1.14a) and (1.15) at p is given by

$$(g, X + \mathfrak{l}_0) \mapsto (g, X + \mathfrak{q}) \qquad \text{for } g \in G,\ X \in \mathfrak{g}_0.$$

This result allows us to compute the effect of J.

Complexifying (1.14a), we have

$$T(G/L)^{\mathbb{C}} \cong G \times_L (\mathfrak{g}_0/\mathfrak{l}_0)^{\mathbb{C}},$$

and J acts in the fiber at each point. We let $T(G/L)^{1,0}$ and $T(G/L)^{0,1}$ be the submodules of $T(G/L)^{\mathbb{C}}$ corresponding to the respective eigenvalues i and $-i$ of J, so that

$$T(G/L)^{\mathbb{C}} \cong T(G/L)^{1,0} \oplus T(G/L)^{0,1}. \tag{1.16a}$$

We have

$$(\mathfrak{g}_0/\mathfrak{l}_0)^{\mathbb{C}} \cong \mathfrak{g}/\mathfrak{l} \cong \bar{\mathfrak{u}} \oplus \mathfrak{u} \tag{1.16b}$$

as L modules, and a little calculation shows that (1.16b) gives the decomposition of the fibers under J corresponding to (1.16a). In other words

$$\begin{aligned} T(G/L)^{1,0} &\cong G \times_L \bar{\mathfrak{u}} \\ T(G/L)^{0,1} &\cong G \times_L \mathfrak{u}. \end{aligned} \tag{1.16c}$$

Taking duals in (1.16a) and forming alternating tensors, we have

$$\wedge^{p,q} T^*(G/L)^{\mathbb{C}} \cong G \times_L ((\wedge^p \bar{\mathfrak{u}})^* \otimes (\wedge^q \mathfrak{u})^*). \tag{1.17}$$

Via (1.17), members of $\mathcal{E}(\wedge^{p,q}T^*(G/L)^{\mathbb{C}})$ correspond to functions from G to $(\wedge^p\bar{\mathfrak{u}})^* \otimes (\wedge^q\mathfrak{u})^*$ transforming on the right under L by $\mathrm{Ad}^* \otimes \mathrm{Ad}^*$.

1.4. $\bar\partial$ operator.

The scalar $\bar\partial$ operator for a complex manifold M is an operator

$$\bar\partial : \mathcal{E}(\wedge^{p,q}T^*(M)^{\mathbb{C}}) \to \mathcal{E}(\wedge^{p,q+1}T^*(M)^{\mathbb{C}}),$$

and it has $\bar\partial^2 = 0$. For the case that $M = G/L$, we can interpret $\bar\partial$ in terms of (1.17).

Also we can construct a vector-valued version of $\bar\partial$. Namely let $G \times_L V$ be a holomorphic vector bundle as above. We introduce $\bar\partial_V = \bar\partial \otimes 1$ as an operator

$$\bar\partial_V : \mathcal{E}(\wedge^{p,q}T^*(G/L)^{\mathbb{C}} \otimes (G \times_L V)) \to \mathcal{E}(\wedge^{p,q+1}T^*(G/L)^{\mathbb{C}} \otimes (G \times_L V));$$

$\bar\partial_V$ is well defined because the transition functions for $G \times_L V$ are holomorphic. Also $\bar\partial_V^2 = 0$. Using (1.17) and dropping the subscript "V" on $\bar\partial_V$, we can interpret $\bar\partial_V$ as an operator

$$\bar\partial : \mathcal{E}(G \times_L ((\wedge^p\bar{\mathfrak{u}})^* \otimes (\wedge^q\mathfrak{u})^* \otimes V)) \to \mathcal{E}(G \times_L ((\wedge^p\bar{\mathfrak{u}})^* \otimes (\wedge^{q+1}\mathfrak{u})^* \otimes V)).$$

In representation theory one works with the case $p = 0$. We define

$$C^{0,q}(G/L, V) = \mathcal{E}(G \times_L ((\wedge^q\mathfrak{u})^* \otimes V)).$$

As always, this is the representation space for a continuous representation of G. The operator $\bar\partial$ is continuous and the kernel is closed. Whether or not the image of $\bar\partial$ is closed, we can define the **Dolbeault cohomology space** $H^{0,q}(G/L, V)$ as

$$H^{0,q}(G/L, V) = \ker(\bar\partial|_{C^{0,q}(G/L,V)})/\mathrm{image}(\bar\partial|_{C^{0,q-1}(G/L,V)}). \tag{1.18}$$

Since $\bar\partial$ commutes with G, the vector space $H^{0,q}(G/L, V)$ carries a representation of G, but possibly the topology is not Hausdorff.

The question whether the image of $\bar\partial$ is closed turns out to play a major role in the theory. Sometimes, partly to get around this question, one works with the subspace $\ker\bar\partial \cap \ker\bar\partial^*$ of $C^{0,q}(G/L, V)$, for a suitably defined "adjoint" $\bar\partial^*$, in place of $H^{0,q}(G/L, V)$. In addition to its technical simplicity, this subspace has other advantages that will not be discussed here. Members of $C^{0,q}(G/L, V)$ in $\ker\bar\partial \cap \ker\bar\partial^*$ are said to be **strongly harmonic**.

2. Bott-Borel-Weil Theorem

The Bott-Borel-Weil Theorem identifies the spaces $H^{0,q}(G/L, V)$ of (1.18) in the case that G is compact. In this situation $\bar\partial$ always has closed image, and (1.18) can be computed alternatively as the representation on the strongly harmonic forms in $C^{0,q}(G/L, V)$. (See [**19**, Chapter V].)

In setting up the complex structure for the case that G is compact, there are two possible ways of proceeding. One is to fix the complex structure (i.e., fix the parabolic subgroup Q) and identify $H^{0,q}(G/L, V)$ for all q and V. The other is to fix V, adapt the complex structure to V, and identify $H^{0,q}(G/L, V)$ for all q. The first way leads to a more general result when G is compact, but the relationship between the two ways is more complicated when G is noncompact.

We begin with the most important special case, where $L = T$. For the first approach, where the complex structure is fixed before V is given, the notation is

$$\begin{aligned}
&G = \text{compact connected Lie group}\\
&T = \text{maximal torus } (= L)\\
&\Delta = \{\text{roots of } (\mathfrak{g}, \mathfrak{t})\}\\
&\Delta^+ = \text{a positive system for } \Delta\\
&\delta = \tfrac{1}{2}\textstyle\sum_{\alpha\in\Delta^+}\alpha\\
&W = \text{Weyl group of } \Delta\\
&B = \text{Borel subgroup built from } \textit{negative} \text{ roots } (= Q).
\end{aligned} \tag{2.1}$$

The inclusion $G/T \hookrightarrow G^{\mathbb{C}}/B$ is onto since the compactness of G makes the image closed, as well as open. Thus we write $G/T = G^{\mathbb{C}}/B$. If $\lambda \in \mathfrak{t}^*$ is an integral parameter and ξ_λ is the corresponding character of T, we abbreviate the representation of T on $\mathbb{C}$ by ξ_λ as $\mathbb{C}_\lambda$. The role of V is played by $\mathbb{C}_\lambda$.

THEOREM 3. *Let $\lambda \in \mathfrak{t}^*$ be integral.*

(a) *If $\langle\lambda+\delta, \alpha\rangle = 0$ for some $\alpha \in \Delta$, then $H^{0,k}(G/T, \mathbb{C}_\lambda) = 0$ for all k.*

(b) *If $\langle\lambda+\delta, \alpha\rangle \neq 0$ for all $\alpha \in \Delta$, let*

$$q = \#\{\alpha \in \Delta^+ \mid \langle\lambda+\delta, \alpha\rangle < 0\}. \tag{2.2}$$

Choose $w \in W$ with $w(\lambda+\delta)$ dominant, and put $\mu = w(\lambda+\delta) - \delta$. Then

$$H^{0,k}(G/T, \mathbb{C}_\lambda) = \begin{cases} 0 & \text{if } k \neq q \\ F^\mu & \text{if } k = q, \end{cases} \tag{2.3}$$

where F^μ is a finite-dimensional irreducible representation of G with highest weight μ.

REFERENCES. For an exposition, see Baston-Eastwood [**3**, pp. 44-48]. The original paper is Bott [**4**].

For the second approach, in which V is given and then the complex structure is fixed, we let G, T, and Δ be as in (2.1). Let $\lambda_0 \in \mathfrak{t}^*$ be a given nonsingular parameter (λ_0 corresponds to $\lambda + \delta$ in Theorem 1), and suppose that $\lambda_0 - \delta_0$

is integral for the half sum δ_0 of positive roots in some (or equivalently each) positive system. Take

$$\begin{aligned} &\Delta^+ = \{\alpha \in \Delta \mid \langle \lambda_0, \alpha \rangle > 0\} \\ &\delta = \tfrac{1}{2} \textstyle\sum_{\alpha \in \Delta^+} \alpha \\ &B \text{ built from } \Delta^+ \text{ instead of } -\Delta^+. \end{aligned} \tag{2.4}$$

Again we have $G/T = G^{\mathbb{C}}/B$.

COROLLARY 4. *Let $\lambda_0 \in \mathfrak{t}^*$ be nonsingular with $\lambda_0 - \delta_0$ integral. With Δ^+, δ, and B defined as in (2.4),*

$$H^{0,k}(G/T, \mathbb{C}_{\lambda_0+\delta}) = \begin{cases} 0 & \text{if } k \neq \dim_{\mathbb{C}}(G/T) \\ F^{\lambda_0 - \delta_0} & \text{if } k = \dim_{\mathbb{C}}(G/T). \end{cases}$$

PROOF. If we put $\lambda = \lambda_0 - \delta$, this becomes a special case of Theorem 3.

The Bott-Borel-Weil Theorem extends to the general G/L with G compact and L the centralizer of a torus. We state the generalization of Theorem 3, omitting the generalization of Corollary 4. The notation is

$$\begin{aligned} &G = \text{compact connected Lie group} \\ &T = \text{a torus in } G \\ &L = Z_G(T) \\ &T \text{ extended to a maximal torus } \tilde{T} \text{ in } L \\ &\Delta = \{\text{roots of } (\mathfrak{g}, \tilde{\mathfrak{t}})\} \\ &\Delta(\mathfrak{l}) = \{\text{roots of } (\mathfrak{l}, \tilde{\mathfrak{t}})\} \subseteq \Delta \\ &\Delta^+ \text{ chosen with } \Delta(\mathfrak{l}) \text{ generated by simple roots} \\ &\delta = \tfrac{1}{2} \textstyle\sum_{\alpha \in \Delta^+} \alpha \\ &W = \text{Weyl group} \\ &Q = \text{built from } \mathfrak{l} \text{ and } \textit{negative} \text{ roots.} \end{aligned} \tag{2.5}$$

Then we have $G/L \cong G^{\mathbb{C}}/Q$.

THEOREM 5. *Let V^λ be irreducible for L with highest weight λ.*

(a) *If $\langle \lambda + \delta, \alpha \rangle = 0$ for some $\alpha \in \Delta$, then $H^{0,k}(G/L, V^\lambda) = 0$ for all k.*

(b) *If $\langle \lambda + \delta, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$, define q as in (2.2), choose $w \in W$ so that $w(\lambda + \delta)$ is dominant, and put $\mu = w(\lambda + \delta) - \delta$. Then*

$$H^{0,k}(G/L, V^\lambda) = \begin{cases} 0 & \text{if } k \neq q \\ F^\mu & \text{if } k = q. \end{cases}$$

3. Discrete series

For a unimodular group G, an irreducible unitary representation π is in the **discrete series** if it is a direct summand of the right regular representation on $L^2(G)$, or equivalently if some nonzero matrix coefficient $(\pi(g)v_1, v_2)$ is in $L^2(G)$. (See Godement [**5**].)

Let G be linear connected reductive, and let K be a maximal compact subgroup. For G compact (so that $K = G$), every irreducible unitary representation is in the discrete series. For G noncompact, the discrete series representations were parametrized by Harish-Chandra. His work can be summarized in the following two theorems.

THEOREM 6. *G has discrete series if and only if rank G =rank K.*

REFERENCES. For an exposition, see [**8**, p. 454]. The original paper is Harish-Chandra [**7**].

The condition on ranks means that a maximal torus T in K is maximal abelian in G. For simple groups that are not complex, the equal rank condition is usually satisfied, but not always. For example, it is satisfied for $SO_e(p,q)$ if p or q is even, for $SU(p,q)$ and $Sp(p,q)$ always, and for $Sp(n,\mathbb{R})$. It is not satisfied for $SL(n,\mathbb{R})$ for $n \geq 3$.

Now let us assume that the equal rank condition is satisfied. Fix a maximal torus T in K. Let Δ_K and Δ be the respective root systems of $\mathfrak{k}$ and $\mathfrak{g}$ with respect to $\mathfrak{t}$, and let W_K and W_G be their Weyl groups. The members of Δ_K are called **compact**, and the other members of Δ are called **noncompact**. Fix a positive system Δ^+ for Δ, and let δ be half the sum of the members of Δ^+.

THEOREM 7. *Assume rank G = rank K. Suppose $\lambda_0 \in \mathfrak{t}^*$ is nonsingular and $\lambda_0 - \delta$ is integral. Then there exists a discrete series representation π_{λ_0} of G such that the global character of π_{λ_0} is given on the conjugates of T by the function*

$$\pm \frac{\sum_{w\in W_K}(sgn\ w)e^{w\lambda_0}}{\sum_{w\in W_G}(sgn\ w)e^{w\delta}}. \tag{3.1}$$

Every discrete series is obtained this way, and two such are equivalent if and only if their parameters λ_0 are conjugate under W_K.

REFERENCES. For an exposition, see [**8**, pp. 310, 436, 454]. The original paper is Harish-Chandra [**7**].

REMARKS. For G compact, λ_0 is equal to the sum of the highest weight and δ when λ_0 is dominant; compare with λ_0 in Corollary 4. Also for G compact, (3.1) reduces to the Weyl Character Formula. For any G, if δ is not integral, the numerator and denominator of (3.1) are not separately well defined. But we can replace $e^{w\lambda_0}$ and $e^{w\delta}$ by $e^{w\lambda_0-\delta}$ and $e^{w\delta-\delta}$, respectively, and then the numerator and denominator are well defined.

Harish-Chandra's proof of Theorem 7 does not give an explicit realization of each discrete series. Instead it produces a discrete series representation for each parameter by finding a subspace of $L^2(G)$ with suitable properties. Soon after Harish-Chandra's work became known, Kostant [**11**] and Langlands [**12**] independently conjectured generalizations of the Bott-Borel-Weil Theorem that would realize all discrete series. Over a period of years beginnning with his thesis, Schmid settled these conjectures. The particular conjecture by Langlands avoids some analytic problems by replacing $H^{0,k}(G/T, \mathbb{C}_\lambda)$ by the space $\mathcal{H}^{0,k}(G/T, \mathbb{C}_\lambda)$ of strongly harmonic square integrable forms. One analytic problem that is avoided in this way is whether the image of $\bar\partial$ is closed; another is how to incorporate square integrability into the hypothesis. The notation is

$$\begin{aligned} &\text{rank } G = \text{rank } K \\ &T = \text{maximal torus in } K\text{, hence maximal in } G \\ &\Delta = \{\text{roots of } (\mathfrak{g}, \mathfrak{t})\} \\ &\Delta^+ = \text{a positive system for } \Delta \\ &\delta = \tfrac{1}{2}\textstyle\sum_{\alpha\in\Delta^+}\alpha \\ &B = \text{Borel subgroup built from } \textit{negative} \text{ roots.} \end{aligned} \tag{3.2}$$

THEOREM 8. *Let $\lambda \in \mathfrak{t}^*$ be integral.*

(a) *If $\langle\lambda+\delta, \alpha\rangle = 0$ for some $\alpha \in \Delta$, then $\mathcal{H}^{0,k}(G/T, \mathbb{C}_\lambda) = 0$ for all k.*
(b) *If $\langle\lambda+\delta, \alpha\rangle \neq 0$ for all $\alpha \in \Delta$, let*

$$\begin{aligned} q = \#\{\alpha\in\Delta^+ \mid \alpha \textit{ is compact and } \langle\lambda+\delta, \alpha\rangle < 0\} \\ + \#\{\alpha\in\Delta^+ \mid \alpha \textit{ is noncompact and } \langle\lambda+\delta, \alpha\rangle > 0\}. \end{aligned} \tag{3.3}$$

Then

$$\mathcal{H}^{0,k}(G/T, \mathbb{C}_\lambda) = \begin{cases} 0 & \textit{if } k \neq q \\ \pi_{\lambda+\delta} & \textit{if } k = q, \end{cases}$$

where $\pi_{\lambda+\delta}$ is the discrete series representation of G with Harish-Chandra parameter λ.

REFERENCE. Schmid [**15**].

The particular conjecture by Kostant works with the actual Dolbeault cohomology space $H^{0,k}(G/T, \mathbb{C}_\lambda)$. In this case one has to arrange that the nonzero cohomology appears in the highest possible degree. The result is then a generalization of Corollary 4.

Thus let $\lambda_0 \in \mathfrak{t}^*$ be a given nonsingular parameter, and suppose that $\lambda_0 - \delta_0$ is integral for the half sum of positive roots in some positive system. Take

$$\begin{aligned} &\Delta^+ = \{\alpha\in\Delta \mid \langle\lambda_0, \alpha\rangle > 0\} \\ &\delta = \tfrac{1}{2}\textstyle\sum_{\alpha\in\Delta^+}\alpha \\ &B \text{ built from } \Delta^+. \end{aligned} \tag{3.4}$$

THEOREM 9. *Let $\lambda_0 \in \mathfrak{t}^*$ be nonsingular with $\lambda_0 - \delta_0$ integral. With Δ^+, δ, and B defined as in (3.4),*

(a) *$H^{0,k}(G/T, \mathbb{C}_{\lambda_0+\delta}) = 0$ if $k \neq \dim_{\mathbb{C}}(K/T)$;*
(b) *$H^{0,k}(G/T, \mathbb{C}_{\lambda_0+\delta})$ is a Frechet space if $k = \dim_{\mathbb{C}}(K/T)$ (i.e., $\bar\partial$ has closed image in $C^{0,k}(G/T, \mathbb{C}_{\lambda_0+\delta})$), and the underlying space of K finite vectors of $H^{0,k}(G/T, \mathbb{C}_{\lambda_0+\delta})$ is equivalent with the space of K finite vectors of $\pi_{\lambda_0+\delta}$.*

REFRENCES. Schmid proved this result in [13], except for the identification of $\pi_{\lambda_0+\delta}$, under the additional assumption that λ_0 is very nonsingular. He gave the identification with $\pi_{\lambda_0+\delta}$ in [14], with the same additional assumption. Aguilar-Rodriguez [1] extended the theorem to the form stated here.

4. Schmid's Penrose transform

In his 1967 thesis, Schmid [13] introduced an operator for passing from the top-degree cohomology space in Theorem 9 to the space of sections of a complex vector bundle over G/K, and Wells and Wolf [20] developed the operator further. In fact, the operator readily generalizes to the setting in §1.1, and there it played an important role in [2]. When G/K is complex, the generalized operator reduces to the G equivariant Penrose transform as described in Baston-Eastwood [3].

The setting for this section will be like the one in Corollary 4 or Theorem 9, except that we allow a general L, possibly noncompact, in place of T. That is, we shall in effect adapt the positive system Δ^+ to our given parameter so that the cohomology of interest occurs in the maximum possible degree.

In this exposition we shall assume that rank $G =$ rank K in order to keep matters simple. The notation is

$$\begin{aligned}
&G = \text{linear connected reductive Lie group}\\
&\text{rank } G = \text{rank } K\\
&T = \text{a torus in } G\\
&L = Z_G(T)\\
&T \text{ extended to a maximal torus } \tilde{T} \text{ in } L\\
&\Delta = \{\text{roots of } (\mathfrak{g}, \tilde{\mathfrak{t}})\}\\
&\Delta(\mathfrak{l}) = \{\text{roots of } (\mathfrak{l}, \tilde{\mathfrak{t}})\} \subseteq \Delta
\end{aligned} \tag{4.1}$$

Instead of fixing the parameter of a representation of L and then introducing Δ^+, we shall fix Δ^+ and say what parameters are allowed. Thus we use the following additional notation:

$$\begin{aligned}
&\Delta^+ \text{ chosen with } \Delta(\mathfrak{l}) \text{ generated by simple roots}\\
&Q \text{ built from } \mathfrak{l} \text{ and } \textit{positive} \text{ roots}\\
&\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}\\
&\Delta(\mathfrak{u}) = \{\text{roots contributing to } \mathfrak{u}\}\\
&\Delta(\mathfrak{u}\cap\mathfrak{p}) = \{\text{noncompact roots contributing to } \mathfrak{u}\}\\
&\delta(\mathfrak{u}) = \tfrac{1}{2}\textstyle\sum_{\alpha\in\Delta(\mathfrak{u})} \alpha\\
&\delta(\mathfrak{u}\cap\mathfrak{p}) = \tfrac{1}{2}\textstyle\sum_{\alpha\in\Delta(\mathfrak{u}\cap\mathfrak{p})} \alpha.
\end{aligned} \tag{4.2}$$

We work with an integral parameter $\lambda \in \tilde{\mathfrak{t}}^*$ that satisfies

$$\begin{aligned}
\langle\lambda,\alpha\rangle &\geq 0 \qquad \text{for all } \alpha \in \Delta^+\\
\langle\lambda,\alpha\rangle &= 0 \qquad \text{for all } \alpha \in \Delta(\mathfrak{l}).
\end{aligned} \tag{4.3}$$

Then λ is the (unique) weight of a one-dimensional representation ξ_λ of L, and we write $\mathbb{C}_\lambda$ for the action of ξ_λ on $\mathbb{C}$. For such a parameter λ, the degree of interest for cohomology is

$$s = \dim_{\mathbb{C}}(K/(L \cap K)).$$

The complex manifold $G/L \subseteq G^{\mathbb{C}}/Q$ has s-dimensional compact complex submanifolds, namely $\{g \cdot K/(L \cap K)\}$, and the operator of interest will come from a kind of integration of $(0, s)$ forms over these submanifolds.

First we formulate the operator abstractly. It will help to identify sections γ and functions φ_γ under the isomorphism (1.7). Using the Bott-Borel-Weil Theorem, we have an isomorphism

$$H^{0,s}(K/(K \cap L), \mathbb{C}_{\lambda+2\delta(\mathfrak{u})}) \xrightarrow{\sim} F^{\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})}, \tag{4.4}$$

the object on the right being a K representation with the indicated highest weight. (This instance of the theorem is a specialization of Theorem 5 in the same way that Corollary 4 is a specialization of Theorem 3.) Noting that the space of top-degree forms $C^{0,s}(K/(K \cap L), \cdot)$ consists only of cocycles, let

$$P : C^{0,s}(K/(K \cap L), \mathbb{C}_{\lambda+2\delta(\mathfrak{u})}) \to F^{\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})} \tag{4.5}$$

be the map implementing (4.4). Let $R : K/(L \cap K) \to G/L$ be inclusion, let R^* be the pullback to $(0, s)$ forms, and let

$$\mathcal{P} : C^{0,s}(G/L, \mathbb{C}_{\lambda+2\delta(\mathfrak{u})}) \to \mathcal{E}(G \times_K F^{\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})}) \tag{4.6a}$$

be given by

$$\mathcal{P}f(x) = P(R^*(f(x\,\cdot))). \tag{4.6b}$$

More concretely let $\{\phi_i\}$ be an orthonormal basis of $F^{\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})}$, and let ϕ be a nonzero highest weight vector. Fix a nonzero $\omega_0 \in \wedge^s(\overline{\mathfrak{u}} \cap \mathfrak{k})^*$, and let

$$\varphi_i = (F^{\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})}(k)\phi,\, \phi_i)\omega_0.$$

This is in $C^{s,0}(K/(L \cap K), \mathbb{C}_{-\lambda-2\delta(\mathfrak{u})})$, and its product with a member h of $C^{0,s}(K/(L \cap K), \mathbb{C}_{\lambda+2\delta(\mathfrak{u})})$ is a volume form on $K/(L \cap K)$. For such an h we can therefore define

$$P(h) = \sum_i \Big(\int_{K/(L\cap K)} h\varphi_i \Big) \phi_i.$$

This version of P coincides with the one in (4.5), and then $\mathcal{P}$ is defined in terms of P by (4.6).

PROPOSITION 10. *The operator $\mathcal{P}$ defined by (4.6) descends to a well defined operator*

$$\mathcal{P} : H^{0,s}(G/L, \mathbb{C}_{\lambda+2\delta(\mathfrak{u})}) \to \mathcal{E}(G \times_K F^{\lambda+2\delta(\mathfrak{u}\cap\mathfrak{p})}).$$

REFERENCES. Schmid [**13**], Wells-Wolf [**20**], and Barchini-Knapp-Zierau [**2**, §10].

5. Zuckerman functors

Zuckerman functors provide an algebraic analog of the analytic construction in §1. They were introduced by Zuckerman [**21**] in a series of lectures and were developed further by Vogan [**18**]. Their full theory requires relating them to ring theory, and this step was carried out initially in [**10**]; for more detail, see [**9**]. For this section we use the following notation:

$$\begin{aligned} G &= \text{linear connected reductive Lie group}\\ K &= \text{a maximal compact subgroup}\\ T &= \text{a torus in } G\\ L &= Z_G(T)\\ Q &= \text{parabolic subgroup in } G^{\mathbb{C}} \text{ as in §1}\\ \mathfrak{q} &= \mathfrak{l}\oplus\mathfrak{u}\\ (\sigma,V) &= \text{smooth representation of } L. \end{aligned} \tag{5.1}$$

The space V can be infinite-dimensional, but the reader may wish to regard it as finite-dimensional for purposes of motivation. The representation (σ, V) gives us a representation of $\mathfrak{l}$, and we extend this to a representation of $\mathfrak{q}$ by making $\mathfrak{u}$ act as 0. It will be helpful for purposes of motivation to think of the representation of $\mathfrak{q}$ on V as coming from a holomorphic representation of Q on V, but this assumption can be avoided.

In the analytic setting, $\bar\partial$ is an operator

$$\bar\partial : \mathcal{E}(G\times_L((\wedge^m\mathfrak{u})^*\otimes V)) \to \mathcal{E}(G\times_L((\wedge^{m+1}\mathfrak{u})^*\otimes V)). \tag{5.2}$$

Using the isomorphism (1.7), we regard $\bar\partial$ as an operator with domain the space of smooth functions φ from G into $(\wedge^m\mathfrak{u})^*\otimes V$ satisfying

$$\varphi(gl) = (\mathrm{Ad}(l)^{-1}\otimes\sigma(l)^{-1})\varphi(g) \qquad \text{for } g\in G,\ l\in L \tag{5.3}$$

and with range the corresponding space of functions into $(\wedge^{m+1}\mathfrak{u})^*\otimes V$.

In the algebraic analog we try to construct only the K finite vectors of $H^{0,m}$, thus obtaining a $(\mathfrak{g}, K)$ module. (Recall that a $(\mathfrak{g}, K)$ **module** consists of compatible representations of $\mathfrak{g}$ and K on the same vector space with every vector K finite. Let $\mathcal{C}(\mathfrak{g}, K)$ be the category of all $(\mathfrak{g}, K)$ modules.)

The idea is to work with the Taylor coefficients at $g = 1$ of the function φ in (5.3), regarding each coefficient as attached to a left-invariant complex derivative (of some order) of φ at $g = 1$. Thus the idea of passing to Taylor coefficients gives us a linear map

$$\varphi\mapsto\varphi^\#\in\mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}),(\wedge^m\mathfrak{u})^*\otimes V).$$

The transformation law (5.3) forces

$$\varphi^\#\in\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}),(\wedge^m\mathfrak{u})^*\otimes V), \tag{5.4}$$

where $\mathfrak{l}$ acts on $U(\mathfrak{g})$ on the right. If we assume that φ is K finite, then the action of $L \cap K$ on the left of φ gives an action of $L \cap K$ on $\varphi^{\#}$ by $\operatorname{Hom}(\operatorname{Ad}, \operatorname{Ad}^* \otimes \sigma)$, and $\varphi^{\#}$ will be $L \cap K$ finite. Thus $\varphi^{\#}$ lies in a subspace that we denote

$$\operatorname{Hom}_{\mathfrak{l}}(U(\mathfrak{g}), (\wedge^m \mathfrak{u})^* \otimes V)_{L\cap K} \tag{5.5}$$

to indicate the $L \cap K$ finiteness. On (5.5) we have a representation of $\mathfrak{g}$ (via the action of $U(\mathfrak{g})$ on the left) and the representation of $L \cap K$, and (5.5) is a $(\mathfrak{g}, L \cap K)$ module.

The passage from the space of φ's as in (5.3) to the space of $\varphi^{\#}$'s in (5.5) loses information because

(a) formal power series do not have to converge and
(b) convergent power series do not have to globalize.

The modification that gets around the difficulties in (a) and (b) is to define away the problem. Let Γ be the functor

$$\Gamma : \mathcal{C}(\mathfrak{g}, L \cap K) \to \mathcal{C}(\mathfrak{g}, K)$$

given by

$$\begin{aligned} \Gamma(V) &= \text{sum of all } \mathfrak{k} \text{ invariant subspaces of } V \text{ for which} \\ &\qquad \text{the action of } \mathfrak{k} \text{ globalizes to } K, \\ \Gamma(\psi) &= \psi|_{\Gamma(V)} \qquad \text{if } \psi \in \operatorname{Hom}(V, W). \end{aligned}$$

The functor Γ is covariant and left exact and is called the **Zuckerman functor**.

IDEA #1. *Impose $\bar\partial$ between spaces*

$$\Gamma(\operatorname{Hom}_{\mathfrak{l}}(U(\mathfrak{g}), (\wedge^m \mathfrak{u})^* \otimes V)_{L\cap K}), \tag{5.6}$$

and take the kernel/image as a $(\mathfrak{g}, K)$ module analog of $H^{0,m}(G/L, V)$.

Now we bring in homological algebra. We *assume temporarily* that $L \subseteq K$. Then we make the following observations:

1) For the case $m = 0$ at least when V is finite-dimensional, the condition that $\varphi^{\#}$ come from a section γ with $\bar\partial\gamma = 0$ is that γ be holomorphic, hence that $Z\varphi = 0$ for all $Z \in \mathfrak{u}$, by (1.12). Thus the kernel/image space for $m = 0$ should be regarded as

$$\Gamma(\operatorname{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K}). \tag{5.7}$$

2) Identification of (5.7) as the space of interest for $m = 0$ suggests looking at the sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K} &\longrightarrow \operatorname{Hom}_{\mathfrak{l}}(U(\mathfrak{g}), (\wedge^0 \mathfrak{u})^* \otimes V)_{L\cap K} \\ &\longrightarrow \operatorname{Hom}_{\mathfrak{l}}(U(\mathfrak{g}), (\wedge^1 \mathfrak{u})^* \otimes V)_{L\cap K} \longrightarrow \cdots \end{aligned} \tag{5.8}$$

in the category $\mathcal{C}(\mathfrak{g}, L \cap K)$. In fact, (5.8) is an injective resolution of $\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K}$ in the category $\mathcal{C}(\mathfrak{g}, L \cap K)$. The maps will be made explicit before Theorem 11 below, and a proof that (5.8) is an injective resolution will be given in that theorem.

3) The category $\mathcal{C}(\mathfrak{g}, L \cap K)$ has enough injectives. Combining (2) and Idea #1, we see that the m^{th} space of interest, namely the m^{th} kernel/image of (5.6), is

$$\Gamma^m(\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K}), \tag{5.9}$$

where Γ^m is the m^{th} right derived functor of Γ. (In fact, (5.9) is defined as the m^{th} cohomology of the complex obtained by applying Γ to (5.8), since (5.8) is an injective resolution.)

4) The space (5.9) gives the underlying $(\mathfrak{g}, K)$ module of K finite vectors of $H^{0,m}(G/L, V)$ for the cases of compact groups and the discrete series. These results are due essentially to Zuckerman [**21**] and are proved in Vogan [**18**]. See also [**9**].

These observations lead us to the second crucial idea.

IDEA #2. *Even when L is not compact,* define *the m^{th} space of interest to be* (5.9).

In short, the Zuckerman construction is to pass from V in $\mathcal{C}(\mathfrak{l}, L \cap K)$ first to $\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K}$ in $\mathcal{C}(\mathfrak{g}, L \cap K)$ and then to $\Gamma^m(\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K})$ in $\mathcal{C}(\mathfrak{g}, K)$.

Finally let us return to the details of observation (2) above. First we need a formula for the differential in (5.8). From §1.4 we have $\bar\partial_V = \bar\partial \otimes 1$, and thus it is enough to understand $\bar\partial$ for $V = \mathbb{C}$. A formula for $\bar\partial$ on a function φ as in (5.3) (but with $V = \mathbb{C}$) is given in Griffiths-Schmid [**6**, (1.6)]. It works with an expansion

$$\varphi(g) = \sum_{i_1 < \cdots < i_m} f_{i_1 \cdots i_m}(g)\, \omega_{i_1} \wedge \cdots \wedge \omega_{i_m},$$

where the ω_i are left-invariant complex 1-forms obtained from a dual basis to a basis $\{Y_i\}$ of $\mathfrak{u}$. Let us regard the corresponding $\varphi^\#$ as in $\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}) \otimes \wedge^m \mathfrak{u}, \mathbb{C})$. If $Y_{j_1}, \ldots, Y_{j_m}$ have increasing indices, then

$$\varphi^\#(u \otimes Y_{j_1} \wedge \cdots \wedge Y_{j_m}) = u f_{j_1 \cdots j_m}(1),$$

where $\mathfrak{u}$ acts by left-invariant complex differentiation. A little computation with the formula for $\bar\partial$ in [**6**] shows that the element $\bar\partial\varphi$ corresponds to the element

$(\bar\partial\varphi)^\#$ given on elements of $U(\mathfrak{g})$ and monomials in $\wedge^{m+1}\mathfrak{u}$ by

$$\begin{aligned}
&(\bar\partial\varphi)^\#(u\otimes X_1\wedge\cdots\wedge X_{m+1})\\
&=\sum_{i=1}^{m+1}(-1)^{i+1}\varphi^\#(uX_i\otimes X_1\wedge\cdots\wedge\widehat{X}_i\wedge\cdots\wedge X_{m+1})\\
&\quad+\sum_{r<s}(-1)^{r+s}\varphi^\#(u\otimes[X_r,X_s]\wedge X_1\wedge\cdots\wedge\widehat{X}_r\wedge\cdots\wedge\widehat{X}_s\wedge\cdots\wedge X_{m+1}).
\end{aligned}\tag{5.10}$$

The formula for general V looks the same; the only difference is that the terms in (5.10) have values in V.

Formula (5.10) defines the maps in (5.8) except for the map going out from $\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}),V)_{L\cap K}$, which we take to be the obvious inclusion.

THEOREM 11. *Under the assumption that $L\subseteq K$, (5.8) is an injective resolution of $Hom_{\mathfrak{q}}(U(\mathfrak{g}),V)_{L\cap K}$.*

PROOF. In the notation of [9], we have

$$\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}),\,(\wedge^m\mathfrak{u})^*\otimes V)_{L\cap K}=I_{\mathfrak{l},L\cap K}^{\mathfrak{g},L\cap K}((\wedge^m\mathfrak{u})^*\otimes V).\tag{5.11}$$

Since L is compact, $(\wedge^m\mathfrak{u})^*\otimes V$ is injective in $\mathcal{C}(\mathfrak{l},L\cap K)$. The functor $I_{\mathfrak{l},L\cap K}^{\mathfrak{g},L\cap K}$ carries injectives to injectives, and therefore the members of (5.8) after $\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}),V)_{L\cap K}$ are injective in $\mathcal{C}(\mathfrak{g},L\cap K)$.

For exactness we note that exactness at $\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}),V)_{L\cap K}$ is trivial since the map out is defined as an inclusion. For exactness at

$$\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}),\,(\wedge^0\mathfrak{u})^*\otimes V)_{L\cap K}\cong\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}),V)_{L\cap K},\tag{5.12}$$

we see from (5.10) that the map out has

$$(\bar\partial\varphi)^\#(u\otimes X)=\varphi^\#(uX)\qquad\text{for }u\in U(\mathfrak{g}),\ X\in\mathfrak{u}.$$

This is 0 for all u and X if and only if $\varphi^\#$ respects the $\mathfrak{u}$ action, hence is in $\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}),V)_{L\cap K}$. Hence we have exactness at (5.12).

For exactness at the other members of (5.8), we rewrite the right side of (5.11) as

$$\begin{aligned}
&\cong I_{\mathfrak{q},L\cap K}^{\mathfrak{g},L\cap K}(I_{\mathfrak{l},L\cap K}^{\mathfrak{q},L\cap K}((\wedge^m\mathfrak{u})^*\otimes V))\\
&\cong I_{\mathfrak{q},L\cap K}^{\mathfrak{g},L\cap K}(\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{q}),\,(\wedge^m\mathfrak{u})^*\otimes V)_{L\cap K}).
\end{aligned}$$

Now $I_{\mathfrak{q},L\cap K}^{\mathfrak{g},L\cap K}$ is exact, and hence it is enough to prove that

$$\cdots\longrightarrow\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{q}),\,(\wedge^m\mathfrak{u})^*\otimes V)_{L\cap K}\longrightarrow\cdots$$

is exact if the differentials are given as in (5.10). Since passage to $L\cap K$ finite vectors is exact in such a Hom, it is enough to prove that

$$\cdots\longrightarrow\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{q}),\,(\wedge^m\mathfrak{u})^*\otimes V)\longrightarrow\cdots\tag{5.13}$$

is exact. But

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{q}), (\wedge^m\mathfrak{u})^* \otimes V) &\cong \mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{u}) \otimes U(\mathfrak{l}), (\wedge^m\mathfrak{u})^* \otimes V) \\
&\cong \mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{l}), \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{u}), (\wedge^m\mathfrak{u})^* \otimes V)) \\
&\cong \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{u}), (\wedge^m u)^* \otimes V) \\
&\cong \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{u}) \otimes \wedge^m u, V) \\
&= \mathrm{Hom}_{\mathbb{C}}(X_m, V),
\end{aligned}$$

where X_m is the Koszul (projective) resolution of $\mathbb{C}$ in $\mathcal{C}(\mathfrak{u}, 1)$. (See [**9**, Theorem 4.6].) It is easy to check that the differentials for (5.13) are the ones induced from the differentials for X_m, and hence (5.13) is exact. This completes the proof.

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