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# The Gindikin–Karpelevič Formula and Intertwining Operators

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*In Honor of F. I. Karpelevič*

Although I never met F. I. Karpelevič in person, I was greatly influenced early in my career by a paper that he and S. G. Gindikin published in 1962 as [**GiK1**] and improved upon a little later as [**GiK2**]. These papers obtain a formula for the evaluation of a certain definite integral, with parameters, that arises in the harmonic analysis of $L^2$ of a Riemannian symmetric space of noncompact type. Such a symmetric space is of the form $X = G/K$, where $G$ is a noncompact connected semisimple Lie group with finite center and $K$ is a maximal compact subgroup.

For current purposes, $G$ may be assumed to be any closed connected subgroup of real or complex matrices that is stable under conjugate transpose and has finite center. The group $G = SL(n, \mathbb{R})$ is an example, and in this case we can take $K = SO(n)$, with $X$ equal to the set of positive definite real symmetric matrices and the action of $G$ on $X$ given by $g \cdot x = gxg^{\mathrm{tr}}$.

Among other things, the papers [**GiK1**] and [**GiK2**] made possible the full development of a theory of intertwining operators for the standard induced representations of $G$, and that development is the way that those papers affected my own research.

## 1. Bhanu Murthy's Work

T. S. Bhanu Murthy was officially a student of I. M. Gelfand and F. A. Berezin, and Berezin and Karpelevič had jointly obtained [**BeK**] explicit integral formulas for the "zonal spherical functions" (now simply called "spherical functions") associated to the symmetric spaces of certain classical groups. The definite integral of interest arises from limiting properties of spherical functions, and Gindikin reports that Berezin suggested to Bhanu Murthy the problem of evaluating this definite integral for $SL(n, \mathbb{R})$. Bhanu Murthy worked with Karpelevič as unofficial supervisor on solving this problem, so much so that he thanked Karpelevič but not Berezin when he published the result in [**BM1**]. Bhanu Murthy mentions in [**BM1**] that Karpelevič was able to evaluate the definite integral for $SL(3, \mathbb{R})$, though apparently not by the method that Bhanu Murthy eventually found for

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$SL(n,\mathbb{R})$; Karpelevič did not publish his own result for $SL(3,\mathbb{R})$, and it is not clear what Karpelevič's method was. After Bhanu Murthy had solved the problem for $SL(n,\mathbb{R})$, Karpelevič suggested the additional problem of evaluating the corresponding definite integral for the real symplectic groups. Bhanu Murthy solved this problem as well, publishing the result in [**BM2**].

Before coming to the integral in any generality, it is instructive to consider the integral written out concretely in the special case $SL(3,\mathbb{R})$, where it is

$$\iiint_{\mathbb{R}^3} (1+x^2+z^2)^{-a}(1+y^2+(xy-z)^2)^{-b}\,dx\,dy\,dz. \tag{1.1}$$

The parameters $a$ and $b$ are complex numbers, and we shall see that (1.1) is convergent when $a$ and $b$ have real parts greater than $1/2$. Any attempt to estimate (1.1) is likely to reveal how subtle the integral is, even though there is no cancellation if $a$ and $b$ are real. More to the point, it is far from apparent that it is even possible to give an exact value for (1.1) in elementary terms.

The key to understanding (1.1) lies in an ingenious substitution that Bhanu Murthy or Karpelevič found. To evaluate (1.1), one regards $y$ as the variable in the expression $1+y^2+(xy-z)^2$, completes the square, changes variable in $y$ to eliminate the translation, and finds that (1.1) becomes

$$= \iiint_{\mathbb{R}^3} (1+x^2+z^2)^{-a}(1+x^2)^{-b}\Big(y^2+\frac{1+x^2+z^2}{(1+x^2)^2}\Big)^{-b}\,dx\,dy\,dz.$$

This integral is much simpler. Replacing $y$ by $y(1+x^2+z^2)^{1/2}(1+x^2)^{-1}$ shows that it is

$$= \iiint_{\mathbb{R}^3} (1+x^2+z^2)^{-a-b+\frac{1}{2}}(1+x^2)^{b-1}(1+y^2)^{-b}\,dx\,dy\,dz.$$

In turn, replacing $z$ by $z(1+x^2)^{1/2}$ shows that it is

$$= \iiint_{\mathbb{R}^3} (1+x^2)^{-a}(1+y^2)^{-b}(1+z^2)^{-a-b+\frac{1}{2}}\,dx\,dy\,dz.$$

Thus (1.1) is exhibited as the product of three 1-dimensional integrals of the form $\int_{-\infty}^{\infty}(1+x^2)^{-c}\,dx$. Use of the formula $\int_{-\infty}^{\infty}(1+x^2)^{-c}\,dx = \pi^{1/2}\Gamma(c-\frac{1}{2})\Gamma(c)^{-1}$, which dates back to Euler and is valid for Re $c > 1/2$, therefore completes the evaluation of (1.1).

To state the analog of (1.1) for $SL(n,\mathbb{R})$, it is helpful to introduce some matrices. In terms of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}, \tag{1.2}$$

the expression $1+x^2+z^2$ is the sum of the squares of the three entries in the first column, and $1+y^2+(xy-z)^2$ is the sum of the squares of the three 2-by-2 minors obtained from the first two columns. For $SL(n,\mathbb{R})$, one forms the analog of (1.2), namely a lower triangular matrix $(x_{ij})_{i,j=1}^n$ with 1's on the diagonal. For each $\ell$ with $1 \le \ell \le n-1$, an ingredient of the analog of (1.1) is obtained by forming the sum of the squares of the $\binom{n}{\ell}$ minors of size $\ell$-by-$\ell$ obtained from the first $\ell$ columns of $(x_{ij})_{i,j=1}^n$. The result is raised to a power depending on $\ell$, and the analog of (1.2) is the integral over $\mathbb{R}^{n(n-1)/2}$ of the product of the $n-1$ expressions raised to their respective powers. Bhanu Murthy evaluated this integral inductively in [**BM1**] by a sequence of steps that handle the off-diagonal entries of the last row, then of the next-to-last row, and so on. For the $i^{\text{th}}$ row, one focuses on the entries $x_{ij}$ from right to left, i.e., for $j$ decreasing from $i-1$ to 1. The $(i,j)^{\text{th}}$ step involves a change of variables that isolates the role of $x_{ij}$ in the integral.

Use of a little theory explains better where the integral comes from. For $G = SL(n,\mathbb{R})$, we are taking $K = SO(n)$. Let $A$ be the subgroup of $G$ consisting of all diagonal matrices whose diagonal entries are positive, and let $N$ be the subgroup of $G$ of upper triangular matrices with ones on the diagonal. Examining the Gram–Schmidt orthogonalization process, one readily sees that $G = KAN$ in the sense that each element of $G$ has a unique decomposition as a product from the indicated subgroups. The action of $G$ by matrix multiplication on the space $\mathbb{C}^n$ of $n$-dimensional column vectors extends naturally to an action of $G$ on the space $\bigwedge^\ell \mathbb{C}^n$ of alternating tensors of rank $\ell$. Let us use the notation $\pi_\ell$ to refer to this representation of $G$, and let $g = kan$ as above. If $\{e_j\}_{j=1}^n$ denotes the standard basis of $\mathbb{C}^n$ and if $a$ has diagonal entries $d_1, \ldots, d_n$, then it is easy to check that

$$\|\pi_\ell(kan)(e_1 \wedge \cdots \wedge e_\ell)\|^2 = d_1^2 \cdots d_\ell^2 \, \|(e_1 \wedge \cdots \wedge e_\ell)\|^2$$

and therefore that

$$\|\pi_\ell(g)(e_1 \wedge \cdots \wedge e_\ell)\|^2 = e^{2\lambda_\ell H(g)} \|(e_1 \wedge \cdots \wedge e_\ell)\|^2, \tag{1.3}$$

where $H(g)$ is the function from $G$ to diagonal matrices that extracts the $A$ part of $g$ and then takes its logarithm, and where $\lambda_\ell$ is the linear functional on diagonal matrices that yields the sum of the first $\ell$ diagonal entries. From (1.3) we see that $e^{2\lambda_\ell H(g)}$ is the sum of the squares of the $\binom{n}{\ell}$ minors of size $\ell$-by-$\ell$ formed from the first $\ell$ columns of $g$. If $\overline{N}$ denotes the subgroup of $SL(n,\mathbb{R})$ consisting of lower triangular matrices with ones on the diagonal, then the analog of (1.1) for $SL(n,\mathbb{R})$ is $\int_{\overline{N}} e^{-\nu H(\bar{n})}\, d\bar{n}$ for a suitable linear functional $\nu$ that encodes the exponent parameters we are using.

## 2. Spherical Functions and Harmonic Analysis

The integral we are discussing arises in the analysis done by Harish-Chandra in [**HC2**] and [**HC3**] in investigating $L^2$ of a Riemannian symmetric space of noncompact type. The end of the previous section gives an indication that this integral looks fairly tame when expressed in terms of group-theoretic constructs, but it is of course still the same complicated integral of which (1.1) is a special case. In order to describe the formula of Gindikin and Karpelevič, let us introduce the group-theoretic constructs for an arbitrary Riemannian symmetric space of noncompact type. For more details about terminology and notation beyond what is given here, see [**Kn2**].

Let $G$ be a noncompact connected semisimple Lie group with finite center, and let $K$ be a maximal compact subgroup. The associated symmetric space is $X = G/K$, and Harish-Chandra was interested in giving an explicit Fourier-type decomposition of $L^2(X)$ modeled on either the classical Fourier inversion formula for Euclidean space or the classical Plancherel formula in Euclidean space. In the setting of $X = G/K$, an abstract theorem due to I. Segal [**Se**] and a proof by Harish-Chandra [**HC1**] that $G$ is of type I together yield a Plancherel formula that gives, for any suitably regular $L^2$ function $f$ on $G/K$, an expression for the square of the $L^2$ norm of $f$ as an integral over $\pi$ of the Hilbert–Schmidt norms squared of $\pi(f)$, $\pi$ varying over the irreducible unitary representations of $G$ and $\pi(f)$ denoting the operator $\pi(f)v = \int_G f(x)\pi(x)v\,dx$. Only very special representations $\pi$ can enter into the formula, namely the ones with a nonzero $K$-fixed vector $\phi_\pi$. If $\mathcal{E}$ denotes the set of these special representations, then the abstract theory says that there exists a unique measure $d\mu$ on $\mathcal{E}$ such that an identity holds of the kind mentioned above. Harish-Chandra's goal in [**HC2**] and [**HC3**] was to find $\mathcal{E}$ and $d\mu$ explicitly in terms of the structure of $G$; it was not actually necessary to find all of $\mathcal{E}$, only $\mathcal{E}$ apart from subsets of $d\mu$ measure 0.

Harish-Chandra knew the simple identity $\|f\|_2^2 = (f^* * f)(1)$, where $f^*(x) = \overline{f(x^{-1})}$, and he observed from this identity with $f$ defined on $G/K$ that it is necessary and sufficient to obtain a Fourier inversion formula at the identity for sufficiently nice bi-$K$-invariant functions $F$ on $G$. If, as we may, we assume that each $\phi_\pi$ has norm 1, then the Fourier inversion formula takes the shape

$$F(1) = \int_{\mathcal{E}} (\pi(F)\phi_\pi, \phi_\pi)\, d\mu(\pi) \quad \text{for } F \text{ bi-}K\text{-invariant,} \tag{2.1}$$

and $\mathcal{E}$ and $d\mu$ are to be determined explicitly.

Examples of representations $\pi$ in $\mathcal{E}$ were known from the book [**GeN**] of I. M. Gelfand and M. A. Naimark and were studied by F. Bruhat in his thesis [**Br**]: they are the ones in what is now called the "spherical principal series." Let $G = KAN$ be an Iwasawa decomposition of $G$, let $\mathfrak{a}$ be the Lie algebra of $A$, and let $M$ be the centralizer of $A$ in $K$. We shall make use of *restricted roots*, the nonzero simultaneous eigenvalues of the members of $\mathfrak{a}$ in their action by left bracket on the Lie algebra of $G$. The group $N$ is the exponential of the sum of these eigenspaces for the *positive* restricted roots. We write $e^{H(g)}$ for the $A$ component of an element $g$ in the Iwasawa decomposition.

The *spherical principal series* consists of representations induced from $MAN$ to $G$, the inducing representation being of the form $1 \otimes e^\lambda \otimes 1$ on $MAN$, where $\lambda$ is a complex-valued linear functional on $\mathfrak{a}$ and where $e^\lambda$ is the corresponding 1-dimensional representation of $A$. Apart from technicalities, the space of the $\lambda^{\text{th}}$ representation is

$$\{f : G \to \mathbb{C} \mid f(xman) = e^{-(\lambda+\rho)H(a)} f(x) \text{ for } x \in G \text{ and } man \in MAN\}, \tag{2.2}$$

the norm is the $L^2$ norm of the restriction to $K$ with respect to normalized Haar measure, and the action by $G$ is given by

$$(U_\lambda(g)f)(x) = f(g^{-1}x).$$

In (2.2), $\rho$ is a real-valued linear functional on $\mathfrak{a}$ with the property that Haar measure on $N$ satisfies $d(ana^{-1}) = e^{2\rho \log a}\, dn$; its value is half the sum of the positive restricted roots, repeated according to their multiplicities. The representation $U_\lambda$ is unitary if $\lambda$ is imaginary-valued. Bruhat [**Br**] proved that $U_\lambda$ is irreducible for almost every imaginary-valued $\lambda$. (Later Kostant [**Ko**] proved that $U_\lambda$ is irreducible for *every* imaginary-valued $\lambda$.)

The constant function 1 on $K$ can be extended via $G = KAN$ to an element of the space for $U_\lambda$, and the result is a $K$-invariant member of norm 1 in the space. For $\pi = U_\lambda$, we can write $\phi_\lambda$ for the element $\phi_\pi$, and the result is that $\phi_\lambda(x) = e^{-(\lambda+\rho)H(x)}$. Then

$$(U_\lambda(x)\phi_\lambda, \phi_\lambda) = \int_K e^{-(\lambda+\rho)H(x^{-1}k)} 1(k)\, dk = \int_K e^{-(\lambda+\rho)H(x^{-1}k)}\, dk.$$

The function

$$\varphi_\lambda(x) = \int_K e^{-(\lambda+\rho)H(x^{-1}k)}\,dk$$

is the $\lambda^{\text{th}}$ *spherical function*, and any integrable function $F$ on $G$ satisfies

$$(U_\lambda(F)\phi_\lambda, \phi_\lambda) = \int_G F(x)(U_\lambda(x)\phi_\lambda, \phi_\lambda)\,dx = \int_G F(x)\varphi_\lambda(x)\,dx.$$

When $F$ is bi-$K$-invariant, we denote the right side by $\widehat{F}(\lambda)$; the function $\widehat{F}$ is the spherical Fourier transform of the bi-$K$-invariant function $F$. Harish-Chandra soon presumed and ultimately proved that the measure $d\mu$ in (2.1) is carried on the subset of $\mathcal{E}$ consisting of the spherical principal series with $\lambda$ imaginary-valued. Thus the desired Fourier inversion formula (2.1) can be rewritten as

$$F(1) = \int_{i\mathfrak{a}'} \widehat{F}(\lambda)\,d\mu(\lambda), \tag{2.3}$$

where $\mathfrak{a}'$ denotes the dual of the real vector space $\mathfrak{a}$.

Harish-Chandra's guess for what $d\mu$ would be was based on results of H. Weyl concerning ordinary differential equations. This analogy led him to examine the asymptotic behavior of $\varphi_\lambda(x)$ for fixed imaginary-valued $\lambda$ as $x$ tends to infinity through $G$ in a certain way, as follows. Members of $G$ have a nonunique decomposition according to $G = KAK$; in fact, only the members $a$ of $A$ with $\alpha(\log a) \geq 0$ for every positive restricted root $\alpha$ are needed in the decomposition. The set of $a$ where $\alpha(\log a) > 0$ for all positive $\alpha$ is called the open positive Weyl chamber $A^+$, and we write $\overline{A^+}$ for its closure. Since $\varphi_\lambda$ is bi-$K$-invariant, its values on $G$ are completely determined by its values on $\overline{A^+}$. The way in which the asymptotic values of $\varphi_\lambda(x)$ are studied is that $x$ is taken to be a member $a$ of $\overline{A^+}$ and $a$ gets large in the sense that $\alpha(\log a)$ tends to $+\infty$ for each positive restricted root $\alpha$. We write $a \to +\infty$ for this behavior of $a$.

Harish-Chandra obtained an expansion, as $a \to +\infty$ with $\lambda$ fixed, of $\varphi_\lambda(a)$ for almost every imaginary-valued $\lambda$; the coefficients in the expansion determined a certain function $\mathbf{c}$ defined and holomorphic on a dense subset of $\lambda \in i\mathfrak{a}'$; cf. Lemma 37 of [**HC2**]. Harish-Chandra's conjecture for $d\mu$ on the basis of the analogy with Weyl's theory was that $d\mu(\lambda) = |\mathbf{c}(\lambda)|^{-2}\,d|\lambda|$ for suitable normalizations of Haar measures, the measure being carried on $i\mathfrak{a}'$. The goal of [**HC2**] and [**HC3**] was therefore to prove that, for suitable normalizations of Haar measures,

$$F(1) = \int_{i\mathfrak{a}'} \widehat{F}(\lambda)|\mathbf{c}(\lambda)|^{-2}\,d|\lambda|. \tag{2.4}$$

This goal was not completely reached in those two papers, and we will return to this point in a moment.

For $\lambda$ complex-valued with real part strictly dominant, Harish-Chandra showed in Theorem 4 of [**HC2**] that the analytic continuation of $\mathbf{c}(\lambda)$ was given by

$$\mathbf{c}(\lambda) = \int_{\overline{N}} e^{-(\lambda+\rho)H(\bar{n})}\,d\bar{n}, \tag{2.5}$$

where $\overline{N} = \Theta N$, $\Theta$ being the global Cartan involution of $G$ corresponding to $K$. In the special case of $SL(n,\mathbb{R})$, this is the integral we considered in §1.

The paper [**GiK1**] of Gindikin and Karpelevič took $\mathbf{c}(\lambda)$ as a fundamental function in harmonic analysis, and the point of the paper was to obtain an explicit expression for $\mathbf{c}(\lambda)$ in terms of values of standard transcendental functions. The proof involved an inductive argument that generalized the three stages for reducing (1.1) that were written out in §1. To obtain a final formula, one needed an explicit evaluation in the cases in which the symmetric space had rank one. The formula in the rank-one case had been obtained by Harish-Chandra in §13 of [**HC2**] by means of an argument with classical hypergeometric functions. The paper [**GiK1**] gave a different-looking rank-one formula, saying that Harish-Chandra's formula contained a small mistake; however, use of the duplication formula $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\frac{1}{2})$ shows that the two rank-one formulas are the same except for a multiplicative constant. A more elementary derivation of the rank-one formula appears in [**He2**], pp. 437–438. In any event, substitution of the rank-one formula into the inductive argument yields one version of what is now called the *Gindikin–Karpelevič formula*. Two other versions of the Gindikin–Karpelevič formula, as well as the rank-one formula, will be recited in §3.

The paper [**GiK2**] sought to be a little more systematic and a little more general about the inductive argument, and it advanced part way toward an application of the Gindikin–Karpelevič formula to intertwining operators that would come out later in the work of G. Schiffmann. We return to this matter in §4.

At the end of [**HC3**], Harish-Chandra stated two conjectures that he was unable to prove and that would complete the proof of the desired inversion formula (2.4) for recovering $F(1)$ from $\widehat{F}$. The first of these conjectures concerns the global behavior of $\mathbf{c}(\lambda)$ for $\lambda$ imaginary valued. The Gindikin–Karpelevič formula gave such a good handle on $\mathbf{c}(\lambda)$ that one can well imagine settling this conjecture quickly. Indeed, in a 1964 paper Helgason ([**He1**], §3) gave an affirmative answer to the conjecture, making use of the Gindikin–Karpelevič formula.

The second conjecture at the end of [**HC3**] is that $F_f(a) = e^{\rho(\log a)}\int_{\overline{N}} f(\bar{n}h)\,d\bar{n}$ does not vanish identically on $A$ for any nonzero smooth bi-$K$-invariant function $f$ on $G$ that satisfies a symmetric-space analog of the conditions for a function on Euclidean space to be a Schwartz function. Harish-Chandra proved this conjecture in §21 of [**HC4**]. This work completed the proof of the Plancherel formula for $G/K$.

## 3. Gindikin–Karpelevič Formula

The formula of Gindikin and Karpelevič may be written in various ways, and we select two of them to reproduce in this section, following [**Kn1**]. For the first one we work with all Iwasawa decompositions $G = KAN$ with $K$ and $A$ fixed. The choice of $N$ is determined by deciding which restricted roots are to be considered as positive. Let $M$ be the centralizer of $A$ in $K$. For any choice of $N$, the closed subgroup $MAN$ is a *minimal parabolic subgroup* of $G$. We write $\mathfrak{n}$, $\mathfrak{n}'$, etc., for the Lie algebras of $N$, $N'$, etc., and we write $\overline{N}$, $\overline{N}'$, etc., for the images under the global Cartan involution $\Theta$ of $N$, $N'$, etc.

Here is a first version of the Gindikin–Karpelevič formula.

THEOREM 1. *Let $MAN$, $MAN'$, and $MAN''$ be minimal parabolic subgroups with the same $MA$, and suppose that $\mathfrak{n}''\cap\mathfrak{n}\subseteq\mathfrak{n}'\cap\mathfrak{n}$. Define $H(\cdot)$ and $\rho$ relative to the two decompositions $G = KAN$ and $G = KAN'$, calling them $H(\cdot)$ and $\rho$, $H'(\cdot)$ and $\rho'$. If $\lambda\in(\mathfrak{a}')^{\mathbb{C}}$ is real-valued and if Haar measures are suitably normalized, then*

$$\int_{\overline{N}\cap N''} e^{-(\lambda+\rho)H(\bar{n})}\,d\bar{n} = \Big[\int_{\overline{N}'\cap N''} e^{-(\lambda+\rho')H'(\bar{n}')}\,d\bar{n}'\Big]\Big[\int_{\overline{N}\cap N'} e^{-(\lambda+\rho)H(\bar{n})}\,d\bar{n}\Big].$$

*This formula is valid also for any complex-valued* $\lambda$ *for which the integrals in question are convergent for* $\operatorname{Re}\lambda$.

REMARK. The tools that we develop below will show that the integrals in question are convergent when $\langle \operatorname{Re}\lambda, \beta\rangle > 0$ for every $N$-positive restricted root that is negative for $N''$.

Theorem 1 permits an inductive evaluation of $\int_{\overline{N}} e^{-(\lambda+\rho)H(\bar n)}\,d\bar n$ and some related integrals, and the result is a product formula similar to what was obtained with (1.1). We discuss some background for Theorem 1 here, but postpone most of the discussion of Theorem 1 to §4 because the meaning of Theorem 1 is best understood in the context of the intertwining operators that will be discussed in that section. One begins with a very general lemma.

LEMMA 1. *Let* $N$ *be a simply connected nilpotent analytic group with Lie algebra* $\mathfrak{n}$, *and let* $\mathfrak{n}_i$, $0 \le i \le r$, *be a nonincreasing sequence of ideals in* $\mathfrak{n}$ *such that* $\mathfrak{n}_0 = \mathfrak{n}$, $\mathfrak{n}_r = 0$, *and* $[\mathfrak{n}, \mathfrak{n}_i] \subseteq \mathfrak{n}_{i+1}$ *for* $0 \le i < r$. *Suppose that* $\mathfrak{s}$ *and* $\mathfrak{t}$ *are vector subspaces of* $\mathfrak{n}$ *such that* $\mathfrak{n} = \mathfrak{s} \oplus \mathfrak{t}$ *and* $\mathfrak{n}_i = (\mathfrak{s}\cap\mathfrak{n}_i)\oplus(\mathfrak{t}\cap\mathfrak{n}_i)$ *for all* $i$. *Then the map* $\mathfrak{s}\oplus\mathfrak{t} \to N$ *given by* $(X, Y) \mapsto \exp X \exp Y$ *is a diffeomorphism onto.*

The proof is Problem 20 at the end of Chapter I of [**Kn2**] and may be found in the chapter of hints in that book. In our application, $\mathfrak{s}$ and $\mathfrak{t}$ will actually be Lie subalgebras of $\mathfrak{n}$.

Let $\Sigma$ be the set of restricted roots, and let $\langle\,\cdot\,,\,\cdot\,\rangle$ be the complex bilinear form on $(\mathfrak{a}')^{\mathbb{C}}$ obtained in the usual way from a given invariant bilinear form on $\mathfrak{g}$ that is invariant under the Cartan involution on $\mathfrak{g}$ and is positive definite on $\mathfrak{a}' \times \mathfrak{a}'$. The various choices for $N$ are in one-one correspondence with the choices of a positive system $\Pi$ within $\Sigma$.

COROLLARY. *Let* $N$ *and* $N'$ *be two choices of the nilpotent group* $N$ *in the Iwasawa decomposition of* $G$, *the two choices corresponding to choices* $\Pi$ *and* $\Pi'$ *of a positive system within* $\Sigma$, *and let* $V$ *be an analytic subgroup of* $N$ *whose Lie algebra is the sum of the full restricted root spaces for some subset of* $\Pi$. *Then*

$$V = (V \cap N')(V \cap \overline{N}') \tag{3.1}$$

*in the sense that multiplication from the product of the two groups on the right is a diffeomorphism onto* $V$.

PROOF. The Lie algebra $\mathfrak{v}$ of $V$ is the sum of certain full restricted root spaces corresponding to some subset $\Omega$ of $\Pi$. Let $\mathfrak{v} = \mathfrak{s} \oplus \mathfrak{t}$ be the decomposition of $\mathfrak{v}$ corresponding to $\Omega = (\Omega \cap \Pi') \cup (\Omega \cap (-\Pi'))$. Define $\mathfrak{v}_i$ to be the subspace of $\mathfrak{v}$ corresponding to those members of $\Omega$ that are sums of $\ge i$ simple restricted roots in $\Pi$. Then Lemma 1 gives the required decomposition of $V$ if we take into account that the exponential map is a diffeomorphism from $\mathfrak{v}$ onto $V$.

A decomposition of the form (3.1) always leads to a corresponding product decomposition of Haar measure, namely

$$dv = dn'\,d\bar n', \tag{3.2}$$

if the Haar measures are suitably normalized. This is quite a general fact about Haar measures; see Theorem 8.32 of [**Kn2**].

We are now in a position to see the role of the inclusion $\mathfrak{n}'' \cap \mathfrak{n} \subseteq \mathfrak{n}' \cap \mathfrak{n}$ in Theorem 1. The following lemma explains matters.

LEMMA 2. *Let $MAN$, $MAN'$, and $MAN''$ be minimal parabolic subgroups with the same $MA$, and suppose that $\mathfrak{n}'' \cap \mathfrak{n} \subseteq \mathfrak{n}' \cap \mathfrak{n}$. Then*

$$\overline{N} \cap N'' = (\overline{N}' \cap N'')(\overline{N} \cap N') \tag{3.3}$$

*in the sense that multiplication from the product of the two groups on the right is a diffeomorphism onto $\overline{N} \cap N''$.*

PROOF. From $\mathfrak{n}'' \cap \mathfrak{n} \subseteq \mathfrak{n}' \cap \mathfrak{n}$ we have $\bar{\mathfrak{n}}'' \cap \mathfrak{n} \supseteq \bar{\mathfrak{n}}' \cap \mathfrak{n}$, and thus

$$\overline{N}' \cap N \cap N'' \subseteq \overline{N}'' \cap N \cap N'' = 1. \tag{3.4}$$

An application of (3.1) gives

$$\overline{N}' \cap N'' = (\overline{N}' \cap \overline{N} \cap N'')(\overline{N}' \cap N \cap N''),$$

and we conclude from (3.4) that

$$\overline{N}' \cap N'' = \overline{N}' \cap \overline{N} \cap N''. \tag{3.5a}$$

Application of $\Theta$ to (3.4) shows that $N' \cap \overline{N} \cap \overline{N}'' = 1$ and yields similarly

$$N' \cap \overline{N} = N' \cap \overline{N} \cap N''. \tag{3.5b}$$

One more application of (3.1), followed by substitution from (3.5), gives

$$\overline{N} \cap N'' = (\overline{N}' \cap \overline{N} \cap N'')(N' \cap \overline{N} \cap N'') = (\overline{N}' \cap N'')(N' \cap \overline{N}),$$

which is the identity that was to be proved.

Let us turn to a second version of the Gindikin–Karpelevič formula. Fix a positive system $\Pi$ for $\Sigma$ (and therefore also the corresponding $\mathfrak{n}$ and $N$). Let $\mathfrak{g}$ be the Lie algebra of $G$. If $\beta$ is in $\Pi$, we say that $\beta$ is *reduced* if $\frac{1}{2}\beta$ is not in $\Pi$. In this case let $\mathfrak{g}^{(\beta)}$ be the Lie subalgebra of $\mathfrak{g}$ generated by the restricted root spaces for $\beta$, $2\beta$, $-\beta$, and $-2\beta$. The subalgebra $\mathfrak{g}^{(\beta)}$ is stable under the Cartan involution of $\mathfrak{g}$, and is simple. Let $G^{(\beta)}$ be the corresponding analytic subgroup of $G$. The $K$ and $A$ for $G^{(\beta)}$ may be taken to be the connected groups $K^{(\beta)} = K \cap G^{(\beta)}$ and $A^{(\beta)} = A \cap G^{(\beta)}$, and the $M$ group is then $M^{(\beta)} = M \cap K^{(\beta)}$. The group $A^{(\beta)}$ is 1-dimensional, its Lie algebra being generated by the element $H_\beta$ in $\mathfrak{a}$ such that $\lambda(H_\beta) = \langle \lambda, \beta \rangle$ for all $\lambda \in (\mathfrak{a}')^{\mathbb{C}}$. Consequently the symmetric space $G^{(\beta)}/K^{(\beta)}$ has rank one, and the group $G^{(\beta)}$ is said to have *real rank one*.

The set of restricted roots of $G^{(\beta)}$ is either $\{\beta, 2\beta, -\beta, -2\beta\}$ or $\{\beta, -\beta\}$, according as $2\beta$ is or is not in $\Sigma$. If we decree that $\beta$ is to be positive for $G^{(\beta)}$, then $\mathfrak{n}^{(\beta)}$ is the sum of the restricted root spaces for $\beta$ and $2\beta$, and $\bar{\mathfrak{n}}^{(\beta)}$ is the sum of the restricted root spaces for $-\beta$ and $-2\beta$. Let us write $H^{(\beta)}(\,\cdot\,)$ and $\rho^{(\beta)}$ for the analogs of $H(\,\cdot\,)$ and $\rho$ for $G^{(\beta)}$. The function $H^{(\beta)}(\,\cdot\,)$ is given by restriction of $H(\,\cdot\,)$. For $\beta$ simple, one can show that $\rho^{(\beta)}$ is the restriction of $\rho$, but this equality does not necessarily hold for nonsimple $\beta$. For example, if $G = SL(3, \mathbb{R})$, then, in the notation of §1, the positive restricted roots are given by $\beta_1 = e_1 - e_2$, $\beta_2 = e_2 - e_3$, and $\beta_1 + \beta_2 = e_1 - e_3$, where $e_j$ denotes evaluation of the $j^{\text{th}}$ diagonal entry of the diagonal matrices. Then we have

$$\rho^{(\beta_1)} = \tfrac{1}{2}\beta_1, \quad \rho^{(\beta_2)} = \tfrac{1}{2}\beta_2, \quad \text{and} \quad \rho^{(\beta_1+\beta_2)} = \tfrac{1}{2}(\beta_1 + \beta_2).$$

On the other hand, $\rho$ is equal to $\beta_1 + \beta_2$, and thus

$$\rho\big|_{\mathbb{R}H_{\beta_1}} = \tfrac{1}{2}\beta_1, \quad \rho\big|_{\mathbb{R}H_{\beta_2}} = \tfrac{1}{2}\beta_2, \quad \text{and} \quad \rho\big|_{\mathbb{R}H_{\beta_1+\beta_2}} = \beta_1 + \beta_2.$$

We need one more ingredient. The members of the normalizer $N_K(A)$ of $A$ in $K$ conjugate one choice of $N$ to another, while the members of the centralizer $M = Z_K(A)$ conjugate each choice of $N$ to itself. The quotient group $W(A) = N_K(A)/Z_K(A)$ is called the *Weyl group* and is known to act simply transitively on the set of $N$'s. If $w$ is in $N_K(A)$, we write $[w]$ for the class of $w$ in $W(A)$.

Now we can state the second version of the Gindikin–Karpelevič formula.

THEOREM 2. *Let $MAN$ be a parabolic subgroup of $G$, and let $w$ be in the normalizer $N_K(A)$. For $\lambda$ in $(\mathfrak{a}')^{\mathbb{C}}$ and $\beta$ in $\Pi$, let $\lambda_\beta = \frac{\langle\lambda,\beta\rangle}{|\beta|^2}\,\beta$. Then*

$$\int_{\overline{N}\cap w^{-1}Nw} e^{-(\lambda+\rho)H(\bar{n})}\,d\bar{n} = \prod_{\beta\in\Pi,\,\frac{1}{2}\beta\notin\Pi,\,w\beta\notin\Pi} \int_{\overline{N}^{(\beta)}} e^{-(\lambda_\beta+\rho^{(\beta)})H^{(\beta)}(\bar{n})}\,d\bar{n}$$

*for any real-valued $\lambda$, and the formula remains valid for any complex-valued $\lambda$ such that the integrals in question are convergent for $\operatorname{Re}\lambda$, namely those with $\langle\operatorname{Re}\lambda,\beta\rangle > 0$ for all $\beta$ in $\Pi$ such that $w\beta$ is not in $\Pi$.*

One particular Weyl group element has $w_0^{-1}Nw_0 = \overline{N}$ for any representative $w_0$ in $W_K(A)$, and then the condition $w_0\beta \notin \Pi$ is automatic. This special case gives us a formula for (2.5), as follows.

COROLLARY. *Let $MAN$ be a parabolic subgroup of $G$. For $\lambda$ in $(\mathfrak{a}')^{\mathbb{C}}$ and $\beta$ in $\Pi$, let $\lambda_\beta = \frac{\langle\lambda,\beta\rangle}{|\beta|^2}\,\beta$. If $\langle\operatorname{Re}\lambda,\beta\rangle > 0$ for all $\beta\in\Pi$, then*

$$\int_{\overline{N}} e^{-(\lambda+\rho)H(\bar{n})}\,d\bar{n} = \prod_{\beta\in\Pi,\,\frac{1}{2}\beta\notin\Pi} \int_{\overline{N}^{(\beta)}} e^{-(\lambda_\beta+\rho^{(\beta)})H^{(\beta)}(\bar{n})}\,d\bar{n},$$

*with all the integrals in question convergent.*

For $G = SL(3,\mathbb{R})$, there are three factors on the right side, and each is of the form $\int_{-\infty}^{\infty}(1+x^2)^{-c}\,dx$, as in §1.

References for how to evaluate each integral on the right side of the formula in the corollary were given at the end of §2. Let us be content to state the result: Let $G$ be simple with real rank one, let $\beta$ be the positive restricted root such that $\frac{1}{2}\beta$ is not a restricted root, let $p$ be the multiplicity of $\beta$, and let $q$ be the multiplicity of $2\beta$. Then

$$\mathbf{c}(\lambda) = \frac{c\,\Gamma(\frac{1}{4}(p+2\lambda_\beta))\Gamma(\lambda_\beta)}{\Gamma(\frac{1}{2}(p+2\lambda_\beta))\Gamma(\frac{1}{4}(p+2q+2\lambda_\beta))} \qquad \text{if } \operatorname{Re}\lambda_\beta > 0, \tag{3.6}$$

where $\lambda_\beta = \frac{\langle\lambda,\beta\rangle}{|\beta|^2}\,\beta$ and where $c$ is a multiplicative constant independent of $\lambda$. For real-rank-one groups it is customary to use the normalization $\int_{\overline{N}} e^{-2\rho H(\bar{n})}\,d\bar{n} = 1$, i.e., $\mathbf{c}(\rho) = 1$. For $\lambda = \rho$, we have $\lambda_\beta = \frac{1}{2}(p+2q)$, and therefore the value of $c$ is $\Gamma(p+q)/\Gamma(\frac{1}{2}(p+q))$ with this normalization.

## 4. Intertwining Operators

The spherical principal series is part of the full *principal series*, in which the representations are still induced from the minimal parabolic subgroup $S = MAN$ to $G$ but the inducing representation is now of the form $\sigma\otimes e^{\lambda}\otimes 1$ on $MAN$, where $\sigma$ is an irreducible unitary representation of $M$ and $\lambda$ remains equal to a complex-valued linear functional on $\mathfrak{a}$. Let us suppose that $\sigma$ acts on the inner-product space

$V^\sigma$. Apart from technicalities, the space of the representation with parameters $\sigma$ and $\lambda$ is

(4.1)
$$\{f : G \to V^\sigma \mid f(xman) = e^{-(\lambda+\rho)H(a)}\sigma^{-1}(f(x)) \text{ for } x \in G \text{ and } man \in MAN\},$$

the norm squared is $\int_K |f(k)|^2_{V^\sigma}\, dk$, and the action by $G$ remains as

$$(U(S,\sigma,\lambda)(g)f)(x) = f(g^{-1}x).$$

The representation $U(S,\sigma,\lambda)$ is unitary if $\lambda$ is imaginary-valued. Bruhat [**Br**] proved that $U(S,\sigma,\lambda)$ is irreducible for almost every imaginary-valued $\lambda$. When $\sigma$ is nontrivial, there are occasionally imaginary-valued parameters $\lambda$ where reducibility occurs.

We have included $S$ as a parameter in the notation for the representation in order to examine later the dependence of the representation on the choice of $N$. Long ago Harish-Chandra recognized that the global distribution character of $U(S,\sigma,\lambda)$, which can easily be computed explicitly and turns out to be a function, is independent of the choice of $N$. Also, if $w$ is in $N_K(A)$ and if $w\sigma$ and $w\lambda$ are defined by $w\sigma(m) = \sigma(w^{-1}mw)$ and $w\lambda(a) = \lambda(w^{-1}aw)$, then the global character of $U(S,\sigma,\lambda)$ is unchanged by replacing $(\sigma,\lambda)$ by $(w\sigma,w\lambda)$. At least when $\lambda$ is imaginary-valued, so that the induced representation is unitary, it follows that the corresponding representations are equivalent when $N$ is changed and when $(\sigma,\lambda)$ is replaced by $(w\sigma,w\lambda)$. Bruhat [**Br**] examined the equivalence of $U(S,\sigma,\lambda)$ and $U(S,w\sigma,w\lambda)$ closely in his thesis.

Inspired perhaps by other cases historically in which the explicit identification of an operator implementing an equivalence led to a rich new theory, R. A. Kunze and E. M. Stein took up the question of identifying actual operators that implemented the equivalences studied by Bruhat. In [**KuS1**], [**KuS2**], and [**KuS3**], they worked successively with $SL(2,\mathbb{R})$, with $SL(n,\mathbb{C})$, and with general semisimple groups. In all cases they found that it was a good idea to allow $\lambda$ initially to be complex-valued even if the apparent eventual goal was to study only $\lambda$ imaginary-valued. On the basis of a heuristic computation that was regrettably cut from [**KuS3**] during the refereeing process, they found that an operator $A_S(w,\sigma,\lambda)$ carrying $U(S,\sigma,\lambda)$ to $U(S,w\sigma,w\lambda)$ and commuting with the action by $G$ (i.e., an *intertwining operator*) ought to be given formally by

$$A_S(w,\sigma,\lambda)f(x) = \int_{\overline{N}\cap w^{-1}Nw} f(xw\bar{n})\, d\bar{n}. \tag{4.2}$$

In addition, when the class $[w]$ in $W(A)$ is the reflection in a simple restricted root $\beta$, this operator in a certain sense could be regarded as a tensor product of an operator for $G^{(\beta)}$ with the identity operator. Such operators they could handle somewhat.

The paper [**KuS2**] had studied the dependence of $A_S(w,\sigma,\lambda)$ on $w$. To the extent that the space of intertwining operators for a particular choice of parameters is 1-dimensional, the nicest behavior would be for the operators (4.2) to satisfy a cocycle relation

$$A(w_1w_2,\sigma,\lambda) = A(w_1, w_2\sigma, w_2\lambda)A(w_2,\sigma,\lambda). \tag{4.3}$$

The work in [**KuS2**] with $SL(n,\mathbb{C})$ showed that this relation did not hold for all $w_1$ and $w_2$, but the use of abelian Fourier analysis enabled the authors, in the case of $SL(n,\mathbb{C})$, to normalize the operators in such a way that the cocycle relation always holds. The cocycle relation is then to be interpreted on any $f$ as an identity of meromorphic functions in $\lambda$ that are holomorphic for $\lambda$ imaginary-valued. With the normalized operators in place, the authors went on to deduce certain consequences.

G. Schiffmann [**Sc**] took up the problem of making sense of (4.2) for general $G$ to the extent possible. There were at least three separate problems to address: the convergence of the defining integral (4.2) for an open set of $\lambda$, the cocycle relation (4.3), and the problem of analytic continuation of (4.2). In considering the cocycle relation, he was led to make use of the notion of a "minimal product" in $W(A)$. The *length* $\ell(s)$ of a member $s$ of $W(A)$ is the number of reduced positive restricted roots that $s$ sends into negative restricted roots, and a product $s_1 \cdots s_n$ is *minimal* if $\ell(s_1 \cdots s_n)$ is as large as conceivable, namely equal to $\ell(s_1) + \cdots + \ell(s_n)$. He wanted to prove (4.3) when $[w_1 w_2] = [w_1][w_2]$ is a minimal product in the Weyl group. For example, in the case of $SL(3, \mathbb{R})$, the reflection $s_{13}$ in $e_1 - e_3$ in the Weyl group has length 3 and is the product $s_{13} = s_{12}s_{23}s_{12}$ of three simple reflections, each necessarily of length 1. Thus this decomposition of $s_{13}$ is a minimal product. One needs to understand the corresponding product decomposition of the operators.

In the case that $\sigma = 1$, one knows a function in the space on which $U(S, 1, \lambda)$ operates, namely $\phi_\lambda(x) = e^{-(\lambda+\rho)H(x)}$. Since any left-$K$-invariant function on $K$ is constant, $\phi_\lambda$ is the only $K$-invariant function in the representation space, up to a multiplicative constant. Hence (4.3) may be interpreted on $\phi_\lambda$ as follows: $A(w_2, \sigma, \lambda)$ sends $e^{-(\lambda+\rho)H(x)}$ to a multiple $c_{w_2}(\lambda)$ of $e^{-(w_2\lambda+\rho)H(x)}$, and $A(w_1 w_2, \sigma, \lambda)$ sends $e^{-(w_2\lambda+\rho)H(x)}$ to a multiple $c_{w_1}(w_2\lambda)$ of $e^{-(w_1 w_2\lambda+\rho)H(x)}$. Meanwhile $A(w_1 w_2, \sigma, \lambda)$ sends $e^{-(\lambda+\rho)H(x)}$ to a multiple $c_{w_1 w_2}(\lambda)$ of $e^{-(w_1 w_2\lambda+\rho)H(x)}$. If (4.3) is valid, then

$$c_{w_1 w_2}(\lambda) = c_{w_1}(w_2\lambda)c_{w_2}(\lambda). \tag{4.4}$$

What Schiffmann realized is that the decomposition (4.4) is exactly the one carried out by Gindikin and Karpelevič, $c_w(\lambda)$ is the left side of the identity in Theorem 2, and $\mathbf{c}(\lambda)$ is just $c_{w_0}(\lambda)$, where $[w_0]$ is the member of $W(A)$ such that $w_0^{-1} N w_0 = \overline{N}$. Consequently, apart from the question of convergence and the question of being able to iterate formulas suitably, Theorem 2 follows if one can prove (4.3) for minimal products and also the formula

$$A_S(w, \sigma, \lambda)e^{-(\lambda+\rho)H(\cdot)} = c_w(\lambda)e^{-(\lambda+\rho)H(\cdot)}. \tag{4.5}$$

It turns out that this analysis, including the proof of the Gindikin–Karpelevič formula, is more transparent when one views $A_S(w, \sigma, \lambda)$ in a natural way as a composition of two operators and separates the effect on (4.3) of each of these operators. This separation is carried out partially in [**KnS3**] and more fully in Chapter VII of [**Kn1**], and we give a sketch of some of it here.

Let us remember that $U(S, \sigma, \lambda)$ and $U(S', \sigma, \lambda)$ have the same global character if $S = MAN$ and $S' = MAN'$. A formal operator carrying the space of the first representation to the space of the second, commuting with the action of $G$, is

$$A(S'{:}S{:}\sigma{:}\lambda)f(x) = \int_{\overline{N} \cap N'} f(x\bar{n})\, d\bar{n}. \tag{4.6}$$

Let us give some indication of the fact that (4.6) does indeed carry the space of $U(S, \sigma, \lambda)$ to the space of $U(S', \sigma, \lambda)$, because the same tools will arise again later.

Let $f$ be in the space of $U(S,\sigma,\lambda)$. For the behavior under $A$, we have

$$\begin{aligned} A(S'{:}S{:}\sigma{:}\lambda)f(xa) &= \int_{\overline{N}\cap N'} f(xa\bar{n})\,d\bar{n} = \int_{\overline{N}\cap N'} f(x(a\bar{n}a^{-1})a)\,d\bar{n} \\ &= \det(\mathrm{Ad}(a)\big|_{\bar{\mathfrak{n}}\cap\mathfrak{n}'})^{-1}\int_{\overline{N}\cap N'} f(x\bar{n}a)\,d\bar{n} \\ &= e^{-(\lambda+\rho)\log a}\det(\mathrm{Ad}(a)\big|_{\bar{\mathfrak{n}}\cap\mathfrak{n}'})^{-1}\int_{\overline{N}\cap N'} f(x\bar{n})\,d\bar{n} \\ &= e^{-(\lambda+\rho')\log a}A(S'{:}S{:}\sigma{:}\lambda)f(x), \end{aligned}$$

the last equality holding since

$$\det(\mathrm{Ad}(a)\big|_{\bar{\mathfrak{n}}\cap\mathfrak{n}'})^{-1} = e^{(\rho-\rho')\log a}. \tag{4.7}$$

The behavior under $M$ is handled by a similar change of variables, but more easily. For the behavior under $N'$, we use the decomposition $N' = (N'\cap\overline{N})(N'\cap N)$ of (3.1) to make the identification

$$N'\cap\overline{N} \leftrightarrow N'/(N'\cap N). \tag{4.8}$$

After a suitable normalization, the Haar measure for $N'\cap\overline{N}$ matches the invariant measure on $N'/(N'\cap N)$, according to (3.2). If $n_0'$ is in $N'$, then we have

$$\begin{aligned} A(S'{:}S{:}\sigma{:}\lambda)f(xn_0') &= \int_{N'/(N'\cap N)} f(xn_0'\dot{n})\,d\dot{n} \\ &= \int_{N'/(N'\cap N)} f(x\dot{n})\,d\dot{n} \\ &= A(S'{:}S{:}\sigma{:}\lambda)f(x), \end{aligned}$$

as required. Thus, if we ignore convergence questions, $A(S'{:}S{:}\sigma{:}\lambda)$ has the asserted intertwining property.

Comparison of (4.2) and (4.6) shows that formally we have

$$A_S(w,\sigma,\lambda) = R(w)A(w^{-1}Sw{:}S{:}\sigma{:}\lambda), \tag{4.9}$$

where $R(w)$ is the operator given by

$$R(w)f(x) = f(xw) \quad \text{for } w\in N_K(A).$$

The operator $R(w)$ carries the space of $U(w^{-1}Sw,\sigma,\lambda)$ to the space of $U(S,w\sigma,w\lambda)$, commuting with the action of $G$. To approach (4.3) by writing each operator as a composition, we have to know the effect on an operator (4.6) of conjugating by $R(w)$. The relevant formal identity is

$$A(S_2{:}S_1{:}\sigma{:}\lambda) = R(w)^{-1}A(wS_2w^{-1} : wS_1w^{-1} : w\sigma : w\lambda)R(w). \tag{4.10}$$

The analog of the cocycle relation (4.3) for the operators $A(S'{:}S{:}\sigma{:}\lambda)$ is the product formula

$$A(S''{:}S{:}\sigma{:}\lambda) = A(S''{:}S'{:}\sigma{:}\lambda)A(S'{:}S{:}\sigma{:}\lambda). \tag{4.11}$$

Under the hypothesis $\mathfrak{n}''\cap\mathfrak{n}\subseteq\mathfrak{n}'\cap\mathfrak{n}$, (4.11) follows immediately from Lemma 2 and the corresponding formula (3.2) for the Haar measures.

The analog of (4.5) is

$$(A(S'{:}S{:}1{:}\lambda)e^{-(\lambda+\rho)H(\cdot)})(x) = e^{-(\lambda+\rho)H(x)}\int_{\overline{N}\cap N'} e^{-(\lambda+\rho)H(x\bar{n})}\,d\bar{n}. \tag{4.12}$$

This is a straightforward consequence of formulas that we have already seen. In detail, write $x = kan_0'$ relative to $G = KAN'$. The left side of (4.12) is

$$
\begin{aligned}
&= \int_{\overline{N} \cap N'} e^{-(\lambda+\rho)H(x\bar{n})}\, d\bar{n} \\
&= \int_{N'/(N' \cap N)} e^{-(\lambda+\rho)H(xn')}\, d\dot{n}' && \text{by (4.8) and (3.2)} \\
&= \int_{N'/(N' \cap N)} e^{-(\lambda+\rho)H(an_0'n')}\, d\dot{n}' && \text{since } k \text{ drops out} \\
&= \int_{N'/(N' \cap N)} e^{-(\lambda+\rho)H(an')}\, d\dot{n}' && \text{by translation} \\
&= \int_{\overline{N} \cap N'} e^{-(\lambda+\rho)H(a\bar{n}a^{-1})} e^{-(\lambda+\rho)\log a}\, d\bar{n} && \text{by (4.8) and (3.2)} \\
&= (\det \operatorname{Ad}(a)\big|_{\bar{\mathfrak{n}} \cap \mathfrak{n}'})^{-1} e^{-(\lambda+\rho)\log a} \int_{\overline{N} \cap N'} e^{-(\lambda+\rho)H(\bar{n})}\, d\bar{n} && \text{under } a\bar{n}a^{-1} \to \bar{n} \\
&= e^{-(\lambda+\rho')\log a} \int_{\overline{N} \cap N'} e^{-(\lambda+\rho)H(x\bar{n})}\, d\bar{n} && \text{by (4.7),}
\end{aligned}
$$

and this equals the right side of (4.12).

Except for questions of convergence, the combination of (4.12) and (4.11), the latter under the hypothesis $\mathfrak{n}'' \cap \mathfrak{n} \subseteq \mathfrak{n}' \cap \mathfrak{n}$, proves Theorem 1. Then we use (4.9) and (4.10) to convert (4.11) into (4.3) and to convert (4.12) into (4.5). It is easy to check that the hypothesis $\mathfrak{n}'' \cap \mathfrak{n} \subseteq \mathfrak{n}' \cap \mathfrak{n}$ transforms into the hypothesis of minimal product for Weyl group elements. Then (4.12), (4.5), and an easy induction prove Theorem 2; see p. 180 of [**Kn1**] for details. For a careful treatment of the normalization of Haar measures, see §2 of [**KnS2**].

Finally, we have to address the convergence questions. The integrands in Theorem 2 are nonnegative if $\lambda$ is real, and the equality holds whether or not the sides are finite. Formula (3.6) shows that the right side is finite when $\langle \operatorname{Re} \lambda, \beta \rangle > 0$ for each $\beta$ that appears. This handles the convergence question in Theorem 2, and the region of convergence in Theorem 1 follows as a consequence. We can then use these convergence results to settle convergence of the intertwining operators. The point is that if $f$ is in the space for $U(S, \sigma, \lambda)$, then

$$|f(x)|_{V^\sigma} \leq e^{-(\operatorname{Re}\lambda+\rho)H(x)} \sup_{k \in K} |f(k)|_{V^\sigma}.$$

Thus any estimates for integrals involving $e^{-(\operatorname{Re}\lambda+\rho)H(x)}$ imply estimates for integrals obtained from intertwining operators.

## 5. Concluding Remarks

Thus the Gindikin–Karpelevič formula provided tools and ideas for unblocking the theory of intertwining operators in the late 1960s. Conversely, the theory of intertwining operators suggested that the Gindikin–Karpelevič formula be put in a certain particular group-theoretic setting where it became fairly natural to prove.

Some of my own work early in my career was with the above intertwining operators, beginning for groups of real rank one. In [**KnS1**], Stein and I proved the meromorphic continuation of the operators in the real-rank-one case without any Fourier analysis, avoiding the limitation that the Kunze–Stein approach in

[**KuS2**] had imposed of requiring $N$ to be abelian for all relevant real-rank-one cases. From there, we succeeded in normalizing the operators and using them to settle irreducibility/reducibility of the unitary principal series. Also we produced complementary series by showing how to use the intertwining operators to change the inner product for certain nonimaginary $\lambda$'s to make the representation unitary. Schiffmann independently obtained this analytic continuation, and he produced his own normalization, but he did not carry the theory any further.

The effect of the results in §§3 and 4 was that Stein and I were able to extend our theory from real rank one to general real rank. A theory for general real rank in turn led us directly to the discovery of the "$R$ group," introduced in [**KnS2**]; the $R$ group described the reducibility of unitary principal series.

In later work [**KnS3**], Stein and I studied representations induced from other parabolic subgroups $MAN$, in which $M$ need not be compact, and the use of operators of the form (4.6), rather than (4.2), enabled us to get around the problem that the relevant Weyl group need not act transitively on the set of $N$'s. The theory of the $R$ group extended to this situation as long as the representation $\sigma$ on $M$ was in the discrete series of $M$. Subsequently G. J. Zuckerman and I found in [**KnZ**] how to use the $R$ group to classify the irreducible tempered representations of $G$, and that classification has, in the hands of other people, led to significant progress in the theory of automorphic forms.

Part of the legacy of the Gindikin–Karpelevič formula is the idea of proving theorems by reducing matters to real rank one. That idea was already present in special cases in Bhanu Murthy's work and in [**KuS2**], but the idea became part of the standard arsenal of tools for researchers in the field as a result of [**GiK1**] and [**GiK2**]. I have seen the idea used with great success a number of times since the original uses with the **c** function and intertwining operators.

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