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L-Indistinguishability and R Groups for the Special Linear Group

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In an earlier paper [7] we showed how to realize explicitly the irreducible constituents of unitary principal series representations of the special linear group $SL_n(k)$ when k is a nondiscrete locally compact field of characteristic 0. We saw that the set of constituents has a canonical simply transitive group action on it by a finite abelian group, and we interpreted this finite abelian group as the Galois group of a canonical finite abelian Galois field extension of k.

The various irreducible constituents of a single unitary principal series representation are L-indistinguishable in the sense of Langlands [2, 24, 25]. In fact, it is expected that they exhaust a whole equivalence class under L-indistinguishability. Our purpose in this paper is to generalize our parametrization of the members of these special L-packets, exhibiting for an arbitrary L-packet a simply transitive group action by a finite abelian group that is the Galois group of a canonical finite abelian Galois field extension of k.

However, we can give only a relative proof of our results, because the final definition of L-indistinguishability for $SL_n(k)$ awaits completion of results about the discrete series of $GL_n(k)$. We shall therefore cast our results as theorems in the presence of two widely believed "Working Hypotheses" about $GL_n(k)$. These hypotheses are stated in Section 4.

Our main theorems are that these Working Hypotheses imply

(1) L-distinguishability is well-defined for $SL_n(k)$ and each L-packet for $SL_n(k)$ consists precisely of the irreducible constituents of the restriction of a single irreducible admissible representation of $GL_n(k)$ (Theorem 4.1), and

(2) each L-packet for $SL_n(k)$ is acted upon canonically by the aforementioned abelian Galois group in simply transitive fashion (Theorem 4.2).

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The plan of the paper is as follows. In Section 1, we summarize the results of [7] on reducibility of unitary principal series and reinterpret them so that they appear generalizable. The reinterpreted results of [7] lead naturally to an analysis in Section 2 of the reducibility that occurs when one restricts an irreducible admissible representation from a totally disconnected group to an open normal subgroup whose quotient is finite abelian. In Section 3 we summarize relevant parts of the Langlands conjectural parametrization of irreducible admissible representations of a group G by admissible homomorphisms of the Weil group of k into the L group of G, and we make matters more explicit for the general linear group $GL_n(k)$. Then we state our Working Hypotheses in Section 4 and prove our main theorems there.

J. Rosenberg raised the question of whether our abelian field extensions play a genuine role in the theory or whether it is just the Galois groups that are relevant. We expect that the field extensions themselves play a role in the actual realization of a member of an L-class, and we assemble some evidence for such a conjecture in Section 5.

Our work in this paper has been influenced greatly by remarks from Casselman, Jacquet, Kutzko, Langlands, and Silberger, by the approach of Labesse and Langlands [23] to L-indistinguishability for $SL_2(k)$, and by the work of MacDonald [26] on L-functions for representations of general linear groups over finite fields.

1. Reducible Principal Series of $SL_n(k)$

In this section we shall summarize the results of [7] concerning parametrization of the irreducible constituents of unitary principal series of special linear groups. Then, in preparation for Section 4, we shall match this parametrization with the more familiar one given in terms of the R group of [17, 21, 27, 35].

Let k be a nonarchimedean nondiscrete locally compact field of characteristic 0, and let $k^\times$ be the multiplicative group. We use the notation GL for the linear group $GL_n(k)$ and SL for the special linear group, the subgroup of matrices in GL of determinant one. If A is a locally compact abelian group, $\hat{A}$ will denote the dual group of continuous homomorphisms into the circle.

Let T and T_s be the diagonal subgroups of GL and SL, respectively. We write $(a_1, \ldots, a_n)$ for the member of T with diagonal entries $a_1, \ldots, a_n$. Let χ_s be a member of $\hat{T}_s$. We recall the unitary principal series representation of SL with parameter χ_s. (See [7] for details.) If N is the group of upper triangular matrices with ones on the diagonal, the unitary principal series representation $U(\chi_s)$ is given by unitary induction as

$$U(\chi_s) = \operatorname{ind}_{T_s N}^{SL}(\chi_s \otimes 1).$$

Now let us recall the action of the Weyl group W for this situation. The group W can be regarded as the group of all permutations on n letters; it operates on $\hat{T}_s$ (and also on $\hat{T}$) by

$$\sigma\chi_s(a_1, \ldots, a_n) = \chi_s(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

For each pair (i, j) with $1 \leqslant i < j \leqslant n$, let $\psi_{ij}: k^\times \to T_s$ be defined by the recipe that $\psi_{ij}(a)$ is a in the ith entry, a^{-1} in the jth entry, and 1 in the other entries.

Next we choose a particular extension χ of χ_s from T_s to T. Namely, define $\chi_i \in (k^\times)\hat{}$ for $1 \leqslant i \leqslant n-1$ by $\chi_i = \chi_s \circ \psi_{in}$, and define $\chi_n = 1$. Our choice is χ in $\hat{T}$ given by

$$\chi(a_1, \ldots, a_n) = \prod_{i=1}^{n} \chi_i(a_i) = \chi_s(a_1, \ldots, a_{n-1}, a_1^{-1} \cdot \cdots \cdot a_{n-1}^{-1}), \tag{1.1}$$

and it is clear that the restrcition of χ to T_s is χ_s. As in [7] we are led by the work of Labesse and Langlands [23] to define

$$L(\chi) = \{\sigma \in W \mid \sigma\chi = (\omega \circ \det)\chi \text{ for some } \omega \text{ in } (k^\times)\hat{}\},$$
$$\bar{L}(\chi) = \{\omega \in (k^\times)\hat{} \mid (\omega \circ \det)\chi = \sigma\chi \text{ for some } \sigma \text{ in } W\}.$$

Then $L(\chi)$ and $\bar{L}(\chi)$ are finite groups, $\bar{L}(\chi)$ is abelian, and there is an obvious homomorphism φ of $L(\chi)$ *onto* $\bar{L}(\chi)$ given by associating the unique ω that goes with σ in the definition of $L(\chi)$.

If N_χ denotes the intersection of $\ker\omega$ for all ω in $\bar{L}(\chi)$, we have a canonical isomorphism

$$\bar{L}(\chi)\hat{} \cong k^\times/N_\chi. \tag{1.2}$$

It is shown explicitly in [7] how $k^\times/N_\chi$ has a canonical simply transitive group action on the set of irreducible constituents of $U(\chi_s)$.

There is another known way of describing the reducibility of $U(\chi_s)$, namely, that the commuting algebra of $U(\chi_s)$ has a basis of operators parametrized by a finite group $R(\chi_s)$. The exact connection between $R(\chi_s)$ and $\bar{L}(\chi)$ will play a role in Section 4, and we exhibit that connection now.

We recall the definition of $R(\chi_s)$ as in [27], [35], [21], or [17]. Let

$$W(\chi_s) = \{\sigma \in W \mid \sigma\chi_s = \chi_s\}.$$

Define

$$W'(\chi_s) = (\text{group generated by transpositions } \sigma_{ij} \text{ such that } \chi_s \circ \psi_{ij} = 1)$$

and

$$R(\chi_s) = \{\sigma \in W(\chi_s) \mid i < j \text{ and } \chi_s \circ \psi_{ij} = 1 \text{ implies } \sigma(i) < \sigma(j)\}.$$

Then $W'(\chi_s)$ and $R(\chi_s)$ are both subgroups of $W(\chi_s)$, $W'(\chi_s)$ is normal, and we have a semidirect product decomposition

$$W(\chi_s) = R(\chi_s)\, W'(\chi_s). \tag{1.3}$$

THEOREM 1.1. *$W(\chi_s) = L(\chi)$, and the kernel of φ: $L(\chi) \to \bar{L}(\chi)$ is $W'(\chi_s)$. Consequently the finite group $R(\chi_s)$ is canonically isomorphic with $\bar{L}(\chi)$.*

Proof. Let σ be in $L(\chi)$ with $\sigma\chi = (\omega \circ \det)\chi$. Restricting to T_s, we obtain $\sigma\chi = \chi$ since $\det = 1$ on T_s. Thus σ is in $W(\chi_s)$. In the reverse direction, let $\sigma\chi_s = \chi_s$. Then

$$\begin{aligned}
\sigma\chi(a_1,..., a_n) &= \sigma\chi(a_1,..., a_{n-1}, a_1^{-1} \cdot \cdots \cdot a_{n-1}^{-1})\, \sigma\chi(1,..., 1, a_1 \cdot \cdots \cdot a_n)\\
&= \sigma\chi_s(a_1,..., a_{n-1}, a_1^{-1} \cdot \cdots \cdot a_{n-1}^{-1})\, \sigma\chi(1,..., 1, a_1 \cdot \cdots \cdot a_n)\\
&= \chi_s(a_1,..., a_{n-1}, a_1^{-1} \cdot \cdots \cdot a_{n-1}^{-1})\, \sigma\chi(1,..., 1, a_1 \cdot \cdots \cdot a_n)\\
&= \chi(a_1,..., a_{n-1}, a_n)\, \sigma\chi(1,..., 1, a_1 \cdot \cdots \cdot a_n).
\end{aligned}$$

Hence χ is in $L(\chi)$ with ω equal to the composition of $\sigma\chi$ and the nth inclusion. Thus $W(\chi_s) = L(\chi)$.

The kernel of φ: $L(\chi) \to \bar{L}(\chi)$ is $\{\sigma \mid \sigma\chi = \chi\}$. It is clear that any σ with this property is a product of transpositions σ_{ij} with this property (and conversely). Suppose $\sigma_{ij}\chi = \chi$ for the transposition σ_{ij}. Then $\chi_i = \chi_j$ and so

$$\chi \circ \psi_{ij}(a) = \chi_i(a)\, \chi_j(a^{-1}) = 1; \tag{1.4}$$

thus σ_{ij} is in $W'(\chi_s)$, and it follows that σ is in $W'(\chi_s)$. Conversely if σ_{ij} is in $W'(\chi_s)$, then (1.4) shows $\chi_i = \chi_j$ and hence $\sigma_{ij}\chi = \chi$; thus σ_{ij} is in the kernel of φ. Thus the kernel of φ is $W'(\chi_s)$.

Since φ is onto $\bar{L}(\chi)$, $\bar{L}(\chi)$ is canonically isomorphic with $W(\chi_s)/W'(\chi_s)$, which is canonically isomorphic with $R(\chi_s)$ because of (1.3). Consequently $R(\chi_s)$ is canonically isomorphic with $\bar{L}(\chi)$.

Presently we shall define groups $\bar{L}$ in a wider context, and we conclude this section by showing how to redefine $\bar{L}(\chi)$ for the unitary principal series so that the definition appears generalizable. First of all, $U(\chi_s)$ is equivalent with the restriction to SL of the representation

$$U(\chi) = \operatorname{ind}_{TN}^{GL}(\chi \otimes 1).$$

By global character theory, $U(\chi)$ is equivalent with $U(\chi')$ if and only if $\chi' = \sigma\chi$ for some σ in W. Taking $\chi' = (\omega \circ \det)\chi$, we find

$$\bar{L}(\chi) = \{\omega \in (k^\times)\hat{}\ \mid U((\omega \circ \det)\chi) \cong U(\chi)\}.$$

But $U((\omega \circ \det)\chi) \cong U(\chi) \otimes (\omega \circ \det)$, and thus

$$\bar{L}(\chi) = \{\omega \in (k^\times)\hat{}\ \mid U(\chi) \otimes (\omega \circ \det) \cong U(\chi)\}. \tag{1.5}$$

This formula for $\bar{L}(\chi)$ does not require that $U(\chi)$ be an induced representation and will be used in Section 4 to define $\bar{L}$ groups more generally. The condition on $\omega \circ \det$ in (1.5) will be a theme in Section 2, where we develop general results to apply in Section 4.

2. Restrictions of Representations to Subgroups

A *totally disconnected group* is a separable locally compact group whose open compact subgroups form a neighborhood base at the identity. For such a group there is a theory of admissible representations; e.g., see Section 1.5 of [32]. Lemma 2.1a below is stated without proof in [16], and the statement does not seem to occur elsewhere in the literature in this generality. The remainder of Lemma 2.1 is inspired by [16] and by Section 2 of [23]. Lemma 2.3 is an adaptation of Lemma 2.5 of [23]. The lemmas use the notation $g\pi$ to denote the representation $h \to \pi(g^{-1}hg)$.

Lemma 2.1. *Let π be an irreducible admissible representation of a totally disconnected group G, and let H be an open normal subgroup of G such that G/H is finite abelian. Then*

(a) *$\pi|_H$ is the finite direct sum of irreducible admissible representations of H.*

(b) *When the irreducible constituents of $\pi|_H$ are grouped according to their equivalence classes as*

$$\pi|_H \cong \sum_{i=1}^{M} \oplus\, m_i \pi_i \tag{2.1}$$

with the π_i irreducible and inequivalent, the integers m_i are all equal.

(c) *The subgroup G_{π_1} defined by*

$$G_{\pi_1} = \{g \in G \mid g\pi_1 \cong \pi_1\} \tag{2.2}$$

has the property that G/G_{π_1} permutes the classes of the π_i's simply transitively.

(d) *The number of one-dimensional (continuous) characters* ν *of* G *in the collection*

$$X_H(\pi) = \{\nu = \text{one-dimensional character of } G \mid \nu|_H = 1 \text{ and } \pi \otimes \nu \cong \pi\} \tag{2.3}$$

is $m^2 M$, *where* m *is the common multiplicity of the* π_i.

(e) *Every one-dimensional character of* G *that is trivial on* G_{π_1} *is in* $X_H(\pi)$.

Proof. For (a) an easy induction allows us to assume that G/H is cyclic, say with $g_0 H$ as a generator. If $v \neq 0$ is in the representation space V of π, then V is finitely generated as an H-module, with v, $\pi(g_0)v$, $\pi(g_0)^2 v, \ldots, \pi(g_0)^{|G/H|-1} v$ as generators. By Lemma 1.5.3 of [32], $\pi|_H$ has an irreducible quotient. The smooth contragredient $\tilde{\pi}$ of π has π as its smooth contragredient and is admissible and irreducible, by Lemma 1.5.2 of [32], and the above argument shows its restriction to H has an irreducible quotient. Passing to annihilators, we see that $\pi|_H$ has an irreducible subspace, say W.

Each space $\pi(g_0)^k W$ is left stable by $\pi|_H$, and $\pi|_H$ acts irreducibly on it. The irreducibility of π on V implies that

$$\sum_{k=1}^{|G/H|} \pi(g_0)^k W = V.$$

Thus we can choose a minimal subset $\{k_i\}$ of $\{1, 2, \ldots, |G/H|\}$ such that

$$\sum_{k_i} \pi(g_0)^{k_i} W = V. \tag{2.4}$$

We claim the sum (2.4) is direct. In fact, suppose

$$\pi(g_0)^{k_1} W \cap \left(\sum_{k_i \neq k_1} \pi(g_0)^{k_i} W \right)$$

is not 0. Then it must be $\pi(g_0)^{k_1} W$, and so

$$\pi(g_0)^{k_1} W \subseteq \sum_{k_i \neq k_1} \pi(g_0)^{k_i} W$$

and

$$\sum_{k_i \neq k_1} \pi(g_0)^{k_i} W = V,$$

in contradiction to minimality. Thus (2.4) is a direct sum, and (a) follows.

For (b) and (c), let V_i be the subspace of V corresponding to $m_i \pi_i$.

Schur's lemma is applicable in the context of admissible representations [5, p. 65], and it follows that each subspace of V that is stable and irreducible under $\pi|_H$ is contained in some V_i. Therefore for each g in G and each index i there is an index j such that $\pi(g)\, V_i \subseteq V_j$. Applying this fact to g^{-1}, we see that $\pi(g)^{-1} V_j \subseteq V_i$ and hence $\pi(g)\, V_i = V_j$. In particular, $g\pi_i \cong \pi_j$. Also $m_i = m_j$ in (b).

Thus G permutes the classes of the π_i's. The action is transitive since G acts irreducibly on V. This proves (b). The isotropy subgroup at π_1 for the permutation action is G_{π_1}, and G_{π_1} is normal since G/H is abelian. Then (c) follows.

For (d), Schur's lemma shows that the dimension of the intertwining algebra for $\pi|_H$ is m^2M. Then $|X_H(\pi)| = m^2M$ is proved just as in Lemma 3.2 of [7]; the argument in [7] goes through since Frobenius reciprocity is valid in the present context (Theorem 1.7.10 of [32]).

For (e), let ν be a one-dimensional character of G that is trivial on G_{π_1}. For each j, (c) allows us to choose g_j in G so that $\pi(g_j)\, V_1 = V_j$. Then $\{g_j\}$ is a complete system of coset representatives for G/G_{π_1}. Define an operator A from V into itself to be the scalar $\nu(g_j)^{-1}$ on V_j. It is readily checked that $A\pi(g) = \pi(g)\, \nu(g) A$ for all g in G. Thus $\pi \otimes \nu \cong \pi$, and ν is in $X_H(\pi)$.

COROLLARY 2.2. *Let π be an irreducible admissible representation of a totally disconnected group G, and let H be an open normal subgroup of G such that G/H is finite abelian. Let $\pi|_H$ decompose as in* (2.1) *and suppose that the irreducible constituents $\pi_1, \ldots, \pi_M$ have multiplicity one. With G_{π_1} and $X_H(\pi)$ as in* (2.2) *and* (2.3), *let*

$$N = \{g \in G \mid \nu(g) = 1 \textit{ for all } \nu \textit{ in } X_H(\pi)\}.$$

Then $N = G_{\pi_1}$, and G/N acts simply transitively on $\{\pi_1, \ldots, \pi_M\}$.

Proof. First we observe that $N \subseteq G_{\pi_1}$. In fact, if g is in N, then $\nu(g) = 1$ for all ν in $X_H(\pi)$. By Lemma 2.1e, $\nu(g) = 1$ for all ν that are trivial on G_{π_1}. Since the characters of G/G_{π_1} separate points, g is in G_{π_1}. Thus $N \subseteq G_{\pi_1}$.

Now $(G/N)^\wedge \cong X_H(\pi)$, and Lemma 2.1d gives

$$|G/N| = |(G/N)^\wedge| = |X_H(\pi)| = M,$$

since $m = 1$ by assumption. Lemma 2.1c implies that $|G/G_{\pi_1}| = M$ also, and we thus conclude $N = G_{\pi_1}$. Then G/N acts simply transitively on $\{\pi_1, \ldots, \pi_M\}$ by Lemma 2.1c.

LEMMA 2.3. *Let G be a totally disconnected group, let H be an open normal subgroup of G such that G/H is finite abelian, and let π be an* *irreducible admissible representation of H. Then there exists an irreducible admissible representation $\tilde{\pi}$ of G such that $\tilde{\pi}|_H$ contains π as a constituent.*

Proof. An easy induction allows us to assume that G/H is cyclic of prime order p. Let g_0H be a generator of G/H and put $\sigma = \mathrm{ind}_H^G \pi$. The operator A defined by

$$A(f) = (f(1), f(g_0),..., f(g_0^{p-1}))$$

exhibits an equivalence

$$\sigma|_H \cong \sum_{k=0}^{p-1} \oplus g_0^k \pi. \tag{2.5}$$

If σ is irreducible, then we may take $\tilde{\pi} = \sigma$, and (2.5) shows that $\tilde{\pi}|_H$ contains π. Thus we may assume σ is reducible.

In this case we shall prove that $g_0\pi \cong \pi$. Assume $g_0\pi$ and π are inequivalent. Then π, $g_0\pi$,..., $g_0^{p-1}\pi$ are pairwise inequivalent, since p is prime. Say $g_0^k\pi$ operates on V_k and $\sum_{k=0}^{p-1} V_k = V$. It follows from Schur's lemma that any H-stable subspace of V is the sum of a subcollection of the V_k. In particular, there exists a subspace W of V that is stable and irreducible under $\sigma(G)$, and W is of the form $W = \sum \oplus V_{k_i}$. Let σ_0 be the restriction of σ to W, and apply Lemma 2.1c to σ_0. The group G_{π_1} of the lemma is intermediate between H and G and must equal H or G since p is prime. It cannot be G since $g_0^{k_i+1}\pi$ is not equivalent with $g_0^{k_i}\pi$. Hence $G_{\pi_1} = H$, and the lemma shows that the number of summands in W is p. That is, $W = V$, and σ was irreducible, contradiction.

Thus $g_0\pi \cong \pi$. Since Schur's lemma is valid in this context [5, p. 65], we can run through the argument of Lemma 7.9 of [18] to see that π extends to a representation $\tilde{\pi}$ of G on the same space in which π operates. The extension $\tilde{\pi}$ then has the required properties.

LEMMA 2.4. *Let G be a totally disconnected group, let H be an open normal subgroup of G such that G/H is finite abelian, and let π be an irreducible admissible representation of H. Suppose that $\tilde{\pi}$ and $\tilde{\pi}'$ are irreducible admissible representations of G whose restrictions to H are multiplicity-free and contain π. Then $\tilde{\pi}|_H$ is equivalent with $\tilde{\pi}'|_H$, and also $\tilde{\pi}$ is equivalent with $\tilde{\pi}' \otimes \nu$ for some one-dimensional character ν of G that is trivial on H.*

Proof. Lemma 2.1c immediately shows that $\tilde{\pi}|_H \cong \tilde{\pi}'|_H$. Unwinding the equivalences, we may assume that $\tilde{\pi}$ and $\tilde{\pi}'$ act on the same space V and that

$$\tilde{\pi}|_H = \sum_{i=1}^{M} \oplus \pi_i = \tilde{\pi}'|_H, \tag{2.6}$$

with the π_i irreducible and inequivalent and with π_i acting in V_i. For definiteness, let us say $\pi_1 = \pi$. For g in the subgroup G_{π_1} defined by (2.2), $\tilde{\pi}(g)$ and $\tilde{\pi}'(g)$ leave V_1 stable. For such g and for h in H, easy computation using (2.6) gives

$$\tilde{\pi}'(g)^{-1}\tilde{\pi}(g)\,\tilde{\pi}(h) = \tilde{\pi}(h)\,\tilde{\pi}'(g)^{-1}\tilde{\pi}(g).$$

Restricting both sides of this equation to V_1 and applying Schur's lemma, we see that $\tilde{\pi}'(g)^{-1}\tilde{\pi}(g)$ acts as a scalar on V_1; let us call this scalar $\nu(g)$. We have

$$\tilde{\pi}(g) = (\tilde{\pi}' \otimes \nu)(g) \qquad \text{on } V_1, \text{ for } g \text{ in } G_{\pi_1}. \tag{2.7}$$

We claim that ν is a character of G_{π_1} and is trivial on H. In fact, it is trivial on H by (2.6). If g_1 and g_2 are in G_{π_1}, then we multiply the equation

$$\nu(g_1)I = \tilde{\pi}'(g_1)^{-1}\tilde{\pi}(g_1) \qquad \text{on } V_1$$

on the left by $\tilde{\pi}'(g_2)^{-1}|_{V_1}$ and on the right by $\tilde{\pi}(g_2)|_{V_1}$ to obtain

$$\begin{aligned} \nu(g_1)\,\nu(g_2)I &= \nu(g_1)\,\tilde{\pi}'(g_2)^{-1}\tilde{\pi}(g_2) && \text{on } V_1 \\ &= \tilde{\pi}'(g_2)^{-1}\tilde{\pi}'(g_1)^{-1}\tilde{\pi}(g_1)\,\tilde{\pi}(g_2) && \text{on } V_1 \\ &= \tilde{\pi}'(g_1 g_2)^{-1}\tilde{\pi}(g_1 g_2) && \text{on } V_1 \\ &= \nu(g_1 g_2)I. \end{aligned}$$

Hence ν is a character of G_{π_1}.

We can regard ν as a character of G_{π_1}/H and extend it to a character of G/H since G/H is finite abelian. We use the same notation ν to denote such an extension, regarded as a character of G that is trivial on H.

Let g be in G. Then $g\pi_1 \cong \pi_i$ for some unique i, and we must have $\tilde{\pi}(g)\,V_1 = V_i$ and $\tilde{\pi}'(g)\,V_1 = V_i$. The operator on V_i given by

$$\nu(g)\,\tilde{\pi}'(g)\,\tilde{\pi}(g)^{-1}$$

commutes with $\tilde{\pi}|_H$ and, by Schur's lemma, must be a scalar $b(g)$. Equation (2.7) shows that $b(gg_0) = b(g)$ for g_0 in G_{π_1}, and thus we can write b_i for $b(g)$ when $g\pi_1 \cong \pi_i$. Define an operator B on V to be the scalar b_i on V_i. To complete the proof, we shall show that

$$B\tilde{\pi}(g) = \nu(g)\,\tilde{\pi}'(g)B \qquad \text{on } V, \text{ for } g \text{ in } G. \tag{2.8}$$

Thus suppose $g\pi_1 \cong \pi_i$. To show (2.8) on V_j, choose g_j with $g_j\pi_1 \cong \pi_j$. We are to show that

$$B\tilde{\pi}(g) = b_j\nu(g)\,\tilde{\pi}'(g) \qquad \text{on } V_j,$$

hence that

$$B\tilde{\pi}(g)\,\tilde{\pi}(g_j) = b_j \nu(g)\,\tilde{\pi}'(g)\,\tilde{\pi}(g_j) \qquad \text{on} \quad V_1. \tag{2.9}$$

The left side of (2.9) is $B\tilde{\pi}(gg_j)$ and the right side is

$$\begin{aligned} &= \nu(g)\,\tilde{\pi}'(g)\,b_j\tilde{\pi}(g_j) \\ &= \nu(g)\,\tilde{\pi}'(g)\,\nu(g_j)\,\tilde{\pi}'(g_j)\,\tilde{\pi}(g_j)^{-1}\tilde{\pi}(g_j) \\ &= \nu(gg_j)\,\tilde{\pi}'(gg_j) \qquad \text{on } V_1. \end{aligned}$$

The two sides of (2.9) are therefore equal by definition of B on V_k, where $gg_j\pi_1 \cong \pi_k$. This proves (2.8) and completes the proof of the lemma.

3. Langlands Conjecture for $GL_n(k)$

Let k be a nondiscrete locally compact field of characteristic 0, let $\bar{k}$ be an algebraic closure of k, and let W_k be the Weil group of $\bar{k}$ over k. See [34], especially p. 6, for the definition of W_k. If k is nonarchimedean, W_k is a dense subgroup of $\mathrm{Gal}(\bar{k}/k)$ and is suitably retopologized. For any k, there is a natural continuous homomorphism of W_k onto $k^\times$ that exhibits the continuous one-dimensional representations ("quasicharacters") of W_k (over $\mathbb{C}$) as in one–one correspondence with the continuous one-dimensional representations of $k^\times$.

Let G be a reductive algebraic group over k. Langlands has conjectured a correspondence between homomorphisms of W_k and irreducible admissible representations of G. We shall sketch the conjecture for general G, referring to Borel [2] for details, and shall then make matters more precise for the case $G = GL_n(k)$.

Let ${}^LG$ be the L group of G [2, pp. 29–30]. This is the semidirect product of a connected complex reductive group ${}^LG^0$ by $\mathrm{Gal}(\bar{k}/k)$. Let $\Phi_0(G)$ be the set of admissible homomorphisms of W_k into ${}^LG$, modulo inner automorphisms of ${}^LG$. (See [2, p. 40], for the definition of "admissible.") We use the notation φ to denote either an admissible homomorphism or its class in $\Phi_0(G)$. Let $\Pi(G)$ be the set of equivalence classes of irreducible admissible representations of G. We use the notation π to denote either an irreducible admissible representation or its class in $\Pi(G)$.

It is conjectured that one can associate in a natural way to each class φ in $\Phi_0(G)$ a finite subset Π_φ of $\Pi(G)$ such that $\Pi_\varphi \cap \Pi_{\varphi'} = \varnothing$ if $\varphi \neq \varphi'$ and such that a number of other properties hold.

One does not expect the correspondence $\varphi \to \Pi_\varphi$ to exhaust $\Pi(G)$, because special representations ought not occur in the image. To make a conjecture for which all of $\Pi(G)$ is obtained, one modifies the definition of the Weil group, introducing the Weil–Deligne group W_k' defined in [34, pp. 19–20]. Then $\Phi_0(G)$ is replaced by $\Phi(G)$, the set of equivalence classes of admissible homomorphisms of W_k' into ${}^LG$ (as in [2, p. 40], or [6, pp. 276–277]), and then the conjecture includes the statement that the sets Π_φ exhaust $\Pi(G)$ as φ runs through $\Phi(G)$. *For simplicity of exposition, we shall work just with the Weil group W_k and with $\Phi_0(G)$.* The union of the Π_φ for φ in $\Phi_0(G)$ will be denoted $\Pi_0(G)$.

One knows how to associate an L-function and an ε-factor to any equivalence class of finite-dimensional representations of the Weil group W_k. Thus if we have a correspondence $\varphi \to \Pi_\varphi$, we can associate to each pair (π, ρ), where π is in $\Pi_0(G)$ and ρ is a finite-dimensional representation of ${}^LG$, an L-function and an ε-factor by this prescription: let π be in Π_φ and attach to π the L-function and ε-factor for $\rho \circ \varphi$. The possibility of having such an assignment of L-functions and ε-factors is part of the rationale for the conjectured correspondence $\varphi \to \Pi_\varphi$. (See Section 12 of [2].)

Under the construction in the previous paragraph, two equivalence classes of representations π and π' in the same finite subset Π_φ of $\Pi_0(G)$ will always get the same L-functions. For this reason, we say that the members of a single Π_φ are *L-indistinguishable*, and we call Π_φ an *L-packet*.

There are many conditions one might impose on the correspondence $\varphi \to \Pi_\varphi$. These are discussed in Section 10 of [2]. Conditions of the following kind are appropriate:

(1) compability relative to tensoring ("twisting") by a one-dimensional representation,

(2) compatibility relative to the central character of π,

(3) consistency of assignments of L-functions and ε-factors,

(4) characterization of square-integrable representations,

(5) characterization of tempered representations,

(6) functoriality relative to suitable homomorphisms.

Langlands [24] showed how to construct a correspondence $\varphi \to \Pi_\varphi$ for $k = \mathbb{R}$. The construction is summarized in Section 11 of [2]. In this case $\Pi_0(G)$ is all of $\Pi(G)$. Explicit understanding of the set Π_φ comes only after the additional step of classifying irreducible tempered representations; this step was carried out in [19].

When $k = \mathbb{C}$, the construction is easier because the members of the unitary principal series are irreducible and exhaust the irreducible tempered representations. One has $\Pi_0(G) = \Pi(G)$, and each Π_φ is a singleton set. See Section 11 of [2].

The construction has been completed for $GL_2(k)$ by Kutzko [22]. For

$GL_n(k)$, only parts of the construction are done; this point will be discussed further below.

For the remainder of this section we shall specialize to $G = GL_n(k)$. The Galois group in the definition of ${}^LG$ acts trivially, and the effect is that we can take ${}^LG = GL_n(\mathbb{C})$. The members of $\Phi_0(G)$ are then just equivalence classes of continuous representations of W_k into $GL_n(\mathbb{C})$ with image contained in the set of semisimple matrices. The list of desired conditions on the correspondence $\varphi \to \Pi_\varphi$, $\varphi \in \Phi_0(G)$, is as follows:

(0′) Each Π_φ is a singleton set $\{\pi_\varphi\}$.

(1′) If ω is a quasicharacter of $k^\times$ and φ maps to π_φ, then $\omega\varphi$ (regarded as a homomorphism on W_k) maps to $\pi_\varphi \otimes (\omega \circ \det)$.

(2′) The quasicharacter of $k^\times$ obtained by restricting π_φ to the center of G coincides with the quasicharacter of $k^\times$ obtained by regarding the one-dimensional representation $\det\varphi$ of W_k as a representation of $k^\times$.

(3′) The L-function and ε-factor associated to (π_φ, I) by carrying over the Artin–Weil data from W_k (see Section 3 of [34]) coincide with the L-function and ε-factor associated to π_φ by Godement and Jacquet [12].

(4′) π_φ is square-integrable if and only if φ is irreducible.

(5′) π_φ is tempered if and only if φ has relatively compact image.

Parts of a construction of a correspondence $\varphi \to \Pi_\varphi$ for φ in $\Phi_0(G)$ can be made for $G = GL_n(k)$. The construction is compatible with induction from parabolic subgroups, and the Langlands classification proved in Section 4 of [24] for real groups works in the p-adic case (Silberger [33] and Wallach). Thus, as Borel points out in Section 11 of [2], the problem of constructing the correspondence boils down to handling the discrete series. Deciding whether each Π_φ is a singleton set involves both knowledge of the discrete series and the classification of irreducible tempered representations; for results on this classification see [1, 15]. Progress on dealing with L-functions is summarized in Section 12 of [2].

One knows how to construct a wide class of discrete series (see Howe [14] and Gerardin [10]), and when n is prime to the residual characteristic of k, one expects a completeness result (see Carayol [3]). For these good discrete series, the construction involves parameters that index irreducible members of $\Phi_0(G)$, and Gerardin [10] describes how the correspondence respects L-functions and ε-factors. For the remaining discrete series, the construction and correspondence still involve open problems. However, as already remarked, these delicate points have been treated for $GL_2(k)$ by Kutzko [22].

4. L-INDISTINGUISHABILITY FOR $SL_n(k)$

As noted in condition (0′) of Section 3, one expects that problems of L-indistinguishability do not arise for $GL_n(k)$. The group $SL_n(k)$ appears to be a comparatively easy case where such problems do arise. This fact provides the motivation for the problem below.

Let k be a nondiscrete locally compact field of characteristic 0, and assume that k is nonarchimedean. Let $GL = GL_n(k)$, $SL = SL_n(k)$, and $Z = \text{center}(GL)$. We denote by H the product of SL and Z in GL. The L group of SL may be taken as $PGL_n(\mathbb{C})$.

PROBLEM. Assuming that a Langlands correspondence $\tilde{\varphi} \to \tilde{\Pi}_{\tilde{\varphi}}$ has been constructed from $\Phi_0(GL)$ to finite subsets of $\Pi(GL)$ satisfying an appropriate list of properties, construct a compatible Langlands correspondence $\varphi \to \Pi_\varphi$ from $\Phi_0(SL)$ to finite subsets of $\Pi(SL)$. Analyze the L-indistinguishability that results for SL, i.e., describe the internal structure of L-packets.

As was noted in Section 3, construction of a correspondence for GL involves unsolved questions. Thus our solution will involve certain assumptions about GL. These assumptions we isolate now into two "Working Hypotheses," which are believed to be true by many experts.

WORKING HYPOTHESES 1. There exists a Langlands correspondence $\tilde{\varphi} \to \tilde{\Pi}_{\tilde{\varphi}}$ from $\Phi_0(GL)$ to disjoint finite subsets of $\Pi(GL)$ such that conditions (0′) and (1′) of Section 3 hold. Fix such a correspondence, and let $\Pi_0(GL)$ be the union of the sets $\tilde{\Pi}_{\tilde{\varphi}}$.

WORKING HYPOTHESIS 2. The restriction to SL of any member of $\tilde{\Pi}_0$ is multiplicity-free.

These hypotheses are known to be valid for $n = 2$, according to Kutzko [22] and Labesse and Langlands [23], and Working Hypothesis 2 was proved for the unitary principal series of GL by Howe and Silberger (cf. [7]). Under these hypotheses the construction of a correspondence for SL is settled by Theorem 4.1 below. Analysis of the resulting L-indistinguishability will be taken up after the proof of the theorem.

THEOREM 4.1. *Under the two Working Hypotheses, there exists a unique Langlands correspondence $\varphi \to \Pi_\varphi$ of $\Phi_0(SL)$ to disjoint finite subsets of $\Pi(SL)$ such that each member of some Π_φ is a direct summand of the restriction to SL of some member of $\Pi_0(GL)$ and such that the following condition of functoriality holds*:

(6′) *Let η be the quotient mapping of ${}^L GL = GL_n(\mathbb{C})$ onto*

${}^LSL = PGL_n(\mathbb{C})$. For any $\tilde{\varphi}$ in $\Phi_0(GL)$, the restriction to SL of the unique member $\tilde{\pi}_{\tilde{\varphi}}$ of $\tilde{\Pi}_{\tilde{\varphi}}$ is the finite direct sum of members of $\Pi_{\eta\circ\tilde{\varphi}}$.

Specifically $\Pi_{\eta\circ\tilde{\varphi}}$ consists of the set of irreducible constituents of $\tilde{\pi}_{\tilde{\varphi}}$, and the union $\Pi_0(SL)$ of all Π_φ for φ in $\Phi_0(SL)$ consists of the set of all irreducible constituents of all members of $\Pi_0(GL)$.

Proof of existence. Let $\varphi: W_k \to PGL_n(\mathbb{C})$ be admissible. From Henniart [13], there exists an admissible lifting $\tilde{\varphi}: W_k \to GL_n(\mathbb{C})$ such that $\varphi = \eta \circ \tilde{\varphi}$. By (0′) in Working Hypothesis 1, $\tilde{\Pi}_{\tilde{\varphi}}$ is a singleton set $\{\tilde{\pi}_{\tilde{\varphi}}\}$. The subgroup H, the product of SL and Z, is an open normal subgroup of GL with finite abelian quotient. By Lemma 2.1a, $\tilde{\pi}_{\tilde{\varphi}}|_H$ is the finite direct sum of irreducible admissible representations of H. Then $\tilde{\pi}_{\tilde{\varphi}}|_{SL}$ must be the finite direct sum of irreducible admissible representations of SL since Z acts as scalars, and these constituents must be inequivalent by Working Hypothesis 2. We define Π_φ to be the set of classes of these constituents.

First we check that Π_φ is well-defined. If $\tilde{\varphi}'$ is another lift of φ, then $\tilde{\varphi}' = \omega\tilde{\varphi}$ for some quasicharacter ω of $k^\times$. By (1′) of Working Hypothesis 1, $\tilde{\pi}_{\tilde{\varphi}'} = \tilde{\pi}_{\tilde{\varphi}} \otimes (\omega \circ \det)$. Thus $\tilde{\pi}_{\tilde{\varphi}'}$ and $\tilde{\pi}_{\tilde{\varphi}}$ have the same restriction to SL, and Π_φ is well-defined.

The various sets Π_φ have been shown to be finite, and (6′) clearly holds. It is clear that the union of all Π_φ is the set of all irreducible constituents of all members of $\Pi_0(GL)$. We prove that the various Π_φ are disjoint.

Thus suppose $\Pi_\varphi \cap \Pi_{\varphi'} \neq \varnothing$. Let π have equivalence class in the intersection, and let $\tilde{\varphi}$ and $\tilde{\varphi}'$ be lifts of φ and φ', respectively, to homomorphisms into $GL_n(\mathbb{C})$. Selecting representatives of equivalence classes appropriately, we may assume π is an irreducible constituent of both $\tilde{\pi}_{\tilde{\varphi}}|_{SL}$ and $\tilde{\pi}_{\tilde{\varphi}'}|_{SL}$. If ν and ν' denote the restrictions of $\tilde{\pi}_{\tilde{\varphi}}$ and $\tilde{\pi}_{\tilde{\varphi}'}$ to Z, then $\pi_{\tilde{\varphi}}|_H$ and $\tilde{\pi}_{\tilde{\varphi}'}|_H$ contain $\pi \otimes \nu$ and $\pi \otimes \nu'$, respectively, as constituents. The quasicharacter ν/ν' of Z is trivial on $Z \cap SL$ and therefore extends to a quasicharacter $\omega \circ \det$ of GL. Put $\tilde{\varphi}'' = \omega\tilde{\varphi}'$. By (1′) of Working Hypothesis 1, we may take

$$\tilde{\pi}_{\tilde{\varphi}''} = \tilde{\pi}_{\tilde{\varphi}'} \otimes (\omega \circ \det)$$

and therefore $\tilde{\pi}_{\tilde{\varphi}''}|_H$ contains as a constituent

$$(\pi \otimes \nu') \otimes (\omega \circ \det) = \pi \otimes \nu.$$

Thus $\tilde{\pi}_{\tilde{\varphi}}|_H$ and $\tilde{\pi}_{\tilde{\varphi}''}|_H$ have $\pi \otimes \nu$ as a common constituent. By Lemma 2.4 and Working Hypothesis 2, $\tilde{\pi}_{\tilde{\varphi}} \cong \tilde{\pi}_{\tilde{\varphi}''} \otimes (\omega' \circ \det)$ for some character $\omega' \circ \det$ of GL that is trivial on H. Putting $\tilde{\varphi}''' = \omega'\tilde{\varphi}''$ and again applying (1′) of Working Hypothesis 1, we obtain $\tilde{\pi}_{\tilde{\varphi}} \cong \tilde{\pi}_{\tilde{\varphi}'''}$. By the assumed disjointness for GL, we conclude $\tilde{\varphi} \cong \tilde{\varphi}'''$. That is, $\tilde{\varphi} \cong \omega'\omega\tilde{\varphi}'$. Composing with η, we obtain $\varphi \cong \varphi'$. This proves the required disjointness for SL.

Proof of uniqueness. Let $\varphi: W_k \to PGL_n(\mathbb{C})$ be given. If $\tilde{\varphi}: W_k \to GL_n(\mathbb{C})$ is any lifting of φ, then (6′) says that Π_φ must contain every irreducible constituent of $\tilde{\pi}_{\tilde{\varphi}}|_{SL}$. Suppose that π is another member of Π_φ. By assumption π is a direct summand of $\tilde{\pi}|_{SL}$ for some $\tilde{\pi}$ in $\Pi_0(GL)$. By (0′) we can write $\tilde{\pi} = \tilde{\pi}_{\tilde{\varphi}'}$. Then π must be a member of $\Pi_{\eta \circ \tilde{\varphi}'}$. By disjointness, $\eta \circ \tilde{\varphi}' = \varphi$. Thus $\tilde{\varphi}' = \omega\tilde{\varphi}$ for some quasicharacter ω of $k^\times$. By (1′), $\tilde{\pi}_{\tilde{\varphi}'} = \tilde{\pi}_{\tilde{\varphi}} \otimes (\omega \circ \det)$. Thus $\tilde{\pi}_{\tilde{\varphi}'}$ and $\tilde{\pi}_{\tilde{\varphi}}$ have the same restriction to SL, and π is a constituent of $\tilde{\pi}_{\tilde{\varphi}}|_{SL}$. That is, Π_φ contains only the irreducible constituents of $\tilde{\pi}_{\tilde{\varphi}}|_{SL}$.

Remark. If our Working Hypotheses had been couched in terms of the Weil–Deligne group W_k' and all of $\Phi(GL)$, we would have assumed that the correspondence is onto $\Pi(GL)$. Then Lemma 2.3 would be a tool in showing the correspondence for SL is onto $\Pi(SL)$.

Continuing with our Working Hypotheses, we now analyze the structure of the individual L-packets for SL.

THEOREM 4.2. *Under the two Working Hypotheses, let $\varphi \to \Pi_\varphi$ be the Langlands correspondence for SL given in Theorem* 4.1. *For any φ in $\Phi_0(G)$, let $\tilde{\pi}$ represent any member of $\Pi_0(GL)$ whose restriction to SL contains a member of Π_φ, and let*

$$\bar{L}(\varphi) = \{\omega \in (k^\times)\hat{}\ |\ \tilde{\pi} \cong \tilde{\pi} \otimes (\omega \circ \det)\}.$$

Then $\bar{L}(\varphi)$ is unambiguously defined. If N_φ denotes the intersection of the kernels of all ω in $\bar{L}(\varphi)$, then $\bar{L}(\varphi)$ is canonically isomorphic with the dual of the finite abelian group

$$\Gamma_\varphi = k^\times / N_\varphi,$$

and Γ_φ has a canonical simply transitive group action on the L-packet Π_φ.

Remarks. Suppose we start with a unitary principal series representation $U(\chi_s)$ of SL, as in Section 1. Then $U(\chi_s)$ is the restriction to SL of the unitary principal series representation $U(\chi)$ of GL, with χ as in (1.1), and $U(\chi)$ is irreducible, by Gelfand and Neumark [9]. Suppose $U(\chi)$ is in $\Pi_0(GL)$ (which had better be the case but is not a consequence of our assumptions). Then it follows from Theorem 4.1 that the irreducible constituents of $U(\chi_s)$ comprise a single L-packet for SL. Let φ be the $\Phi_0(SL)$ parameter for this L-packet. In Theorem 4.2 we can take $\tilde{\pi}$ to be $U(\chi)$. Then (1.5) shows that $\bar{L}(\varphi)$ coincides with the group $\bar{L}(\chi)$ in Section 1. By (1.2) the group $\Gamma_\varphi = k^\times / N_\varphi$ coincides with $k^\times / N_\chi$, which parametrizes the irreducible constituents of $U(\chi_s)$. Thus Theorem 4.2 may be regarded as a generalization of the parametrization given in [7] for reducibility of unitary principal series (under the assumptions that the Working Hypotheses hold and that $U(\chi)$ is in $\Pi_0(GL)$).

Proof of Theorem 4.2. The assumption on $\tilde{\pi}$ is that $\tilde{\pi} = \tilde{\pi}_{\tilde{\varphi}}$ with $\varphi = \eta \circ \tilde{\varphi}$, in view of Theorem 4.1. A second choice $\tilde{\pi}' = \tilde{\pi}_{\tilde{\varphi}'}$ with $\varphi \cong \eta \circ \tilde{\varphi}'$ forces $\tilde{\varphi}' = \omega_0 \tilde{\varphi}$ for some quasicharacter ω_0 of $k^\times$. By Working Hypothesis 1, we must have

$$\tilde{\pi}' \cong \tilde{\pi} \otimes (\omega_0 \circ \det).$$

Then it is clear that $\bar{L}(\varphi)$ is unambiguously defined.

It follows from duality theory for finite abelian groups that $\bar{L}(\varphi)$ is canonically the dual of $\Gamma_\varphi = k^\times / N_\varphi$. To complete the proof, we exhibit the canonical simply transitive group action of Γ_φ on Π_φ. By Theorem 4.1, Π_φ is the set of irreducible constituents of $\tilde{\pi}|_{SL}$, and this set stands in canonical one–one correspondence with the set of irreducible constituents of $\tilde{\pi}|_H$, since Z acts as scalars under $\tilde{\pi}$. We shall apply Corollary 2.2. The multiplicity-one assumption is part of Working Hypothesis 2. Our definition of $\bar{L}(\varphi)$ makes $X_H(\tilde{\pi})$, in the notation of (2.3), equal to $\bar{L}(\varphi) \circ \det$. The group N in the corollary is then $\{g \in GL \mid \det g \in N_\varphi\}$. The corollary says that GL/N permutes the irreducible constituents of $\tilde{\pi}|_H$ simply transitively. Since GL/N is canonically isomorphic with $k^\times / N_\varphi$, the theorem follows.

As we remarked before the proof, the $\bar{L}$ group in the case of unitary principal series coincides with the $\bar{L}$ group of Section 1, which in turn is canonically isomorphic with the R group, by Theorem 1.1. In an early attempt to understand the connection between reducibility and L-indistinguishability for real groups, Langlands introduced another group, the S group, and proved that $S \cong R$ in certain cases where one expected reducibility to account fully for L-indistinguishability (cf. [20]). Shelstad [31] then proved for tempered L-classes of real groups that the dual of S (S being abelian here) acts simply transitively on the L-class.

The definition of the S group in general is

$$S(\varphi) = Z_{{}^LG^0}(\text{Image } \varphi) / (Z^W \cdot Z^0_{{}^LG^0}(\text{Image } \varphi)),$$

the quotient of the centralizer of the image of φ by the product of the centralizer Z^W of ${}^LG$ in ${}^LG^0$ and the identity component of the centralizer of Image φ. It is noted in [21] that $S \cong R$ for the unitary principal series of any Chevalley group for general k (see [25] for a proof). Proceeding by analogy with the real case, Langlands addresses in [25] the question of whether $S(\varphi)$ is in some sense dual to Π_φ for nonarchimedean fields k, particularly in the case of $SL_n(k)$. For $SL_n(k)$ he constructs a map of $S(\varphi)$ into $(k^\times)\hat{}$ but does not complete the parametrization of Π_φ by means of $S(\varphi)$. We shall show below that Langlands' argument can be easily completed if our Working Hypotheses are assumed.

One further remark is in order. Although R has not yet been defined for nonarchimedean k except in the case of unitary principal series for Chevalley groups, there is reason to expect that the general relation between $S(\varphi)$ and R will be more subtle than that for real groups. Indeed, Langlands has introduced a subgroup F of S such that for real groups $S/F \cong R$ for the standard induced representations (cf. [20, Theorem 3.4]). But for $SL_2(k)$ with k nonarchimedean there are supercuspidal representations for which $S = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $F = \mathbb{Z}_2$ (see p. 199 of [30]), and one would certainly want $R = \{1\}$ for all supercuspidal representations.

THEOREM 4.3. *Under the two Working Hypotheses, there is a canonical isomorphism $S(\varphi) \cong \bar{L}(\varphi)$ for each φ in $\Phi_0(SL)$. Consequently $S(\varphi)$ is finite abelian and $S(\varphi)\hat{}$ has a canonical simply transitive group action on Π_φ.*

Proof. Fix an admissible homomorphism φ: $W_k \to PGL_n(\mathbb{C})$, and let $\tilde{\varphi}$ be a lift to $GL_n(\mathbb{C})$. Then $\varphi = \eta \circ \tilde{\varphi}$. Let s be in $Z_{PGL}(\text{Image}\,\varphi)$, and fix $\tilde{s}$ to be any member of $GL_n(\mathbb{C})$ with $\eta(\tilde{s}) = s$. Then define a function $\chi_s : W_k \to GL_n(\mathbb{C})$ by

$$\chi_s(w) = \tilde{s}\tilde{\varphi}(w)\,\tilde{s}^{-1}\tilde{\varphi}(w)^{-1} \qquad \text{for} \quad w \in W_k. \tag{4.1}$$

It is clear that $\chi_s(w)$ is independent of the choice of representative $\tilde{s}$. Since $\eta(\chi_s(w))$ is evidently 1, $\chi_s(w)$ is a scalar times the identity. We use the same notation $\chi_s(w)$ for the matrix and the scalar. Let us rewrite (4.1) in the form

$$\tilde{s}\tilde{\varphi}(w)\,\tilde{s}^{-1} = \chi_s(w)\,\tilde{\varphi}(w). \tag{4.2}$$

Running through the argument given earlier in connection with (2.7), we see that χ_s is a quasicharacter of W_k. From (4.2), we then see that $s \to \chi_s$ is a homomorphism of $Z_{PGL}(\text{Image}\,\varphi)$ into quasicharacters of $k^\times$.

In fact, χ_s is in $\bar{L}(\varphi)$. Namely, (4.2) shows that $\tilde{\varphi} \cong \chi_s\tilde{\varphi}$. By (0′) of Working Hypothesis 1, $\tilde{\pi}_{\tilde{\varphi}} \cong \tilde{\pi}_{\chi_s\tilde{\varphi}}$. And by (1′),

$$\tilde{\pi}_{\chi_s\tilde{\varphi}} \cong \tilde{\pi}_{\tilde{\varphi}} \otimes (\chi_s \circ \det).$$

Hence χ_s is trivial on Z, is a genuine character, and is in $\bar{L}(\varphi)$. Thus $s \to \chi_s$ is a homomorphism of $Z_{PGL}(\text{Image}\,\varphi)$ into $\bar{L}(\varphi)$.

The homomorphism is onto $\bar{L}(\varphi)$. In fact, if ω is in $\bar{L}(\varphi)$, then $\tilde{\pi}_{\tilde{\varphi}} \cong \tilde{\pi}_{\tilde{\varphi}} \otimes (\omega \circ \det)$ and Working Hypothesis 1 gives us $\tilde{\varphi} \cong \omega\tilde{\varphi}$. If $\tilde{E}$ implements the equivalence, then

$$\tilde{E}\tilde{\varphi}(w)\,\tilde{E}^{-1} = \omega(w)\,\tilde{\varphi}(w).$$

Thus $\eta(\tilde{E}) = E$ is such that $\chi_E = \omega$, and $s \to \chi_s$ is onto $\bar{L}(\varphi)$.

Since Z^W is trivial and since $\bar{L}(\varphi)$ is discrete, the map $s \to \chi_s$ is trivial on $Z^W \cdot Z_{PGL}(\text{Image}\,\varphi)$ and descends to a homomorphism of $S(\varphi)$ onto $\bar{L}(\varphi)$.

Next, we show this homomorphism of $S(\varphi)$ onto $\bar{L}(\varphi)$ is one–one. Thus suppose $\chi_s(w) = 1$ for all w in W_k. Then $\tilde{s}$ commutes with all $\tilde{\varphi}(w)$, i.e., $\tilde{s}$ centralizes the image of $\tilde{\varphi}$ in $GL_n(\mathbb{C})$. The centralizer of a group of semisimple matrices in $GL_n(\mathbb{C})$ is connected, and it follows that s is in $Z^0_{PGL}(\text{Image}\,\varphi)$. Hence the map of $S(\varphi)$ onto $\bar{L}(\varphi)$ is one–one. This establishes the isomorphism $S(\varphi) \cong \bar{L}(\varphi)$.

From Theorem 4.2, $\bar{L}(\varphi) \cong \hat{\Gamma}_\varphi$ and Γ_φ has a canonical simply transitive action on Π_φ. Hence $S(\varphi)\hat{}$ has a canonical simply transitive action on Π_φ.

5. Role of the Extension Field Corresponding to Γ_φ

For this section we assume the two Working Hypotheses of Section 4. Let $\Gamma_\varphi = k^\times / N_\varphi$ be the finite abelian group given in Theorem 4.2 as having a simply transitive group action on the L-class Π_φ of representations of $SL_n(k)$. By the fundamental theorem of local class field theory (see [28, Chap. XIV]) there is a unique finite abelian Galois extension K_φ of k whose norm group is N_φ. Then $\Gamma_\varphi \cong \mathrm{Gal}(K_\varphi/k)$ canonically. J. Rosenberg raised the question of whether K_φ plays any genuine role in the theory or whether it is just Γ_φ that is relevant. We expect that K_φ plays a role in the actual realization of some member of Π_φ; which member of Π_φ is realized depends on some initial choices, such as of an additive character of k. In this section we give some evidence for such a conjecture.

1. *Supercuspidal representations for* $n = 2$. (See [4, 8, 23, 29, 30].) One starts with data (K, ψ, ρ), where K is a quadratic extension of k, ψ is an additive character of k, and ρ is a character of the (compact) subgroup of $K^\times$ in the kernel NK^1_k of the norm map $N_{K/k}$. The data (K, ψ) lead to a Weil representation of SL in the space of Schwartz functions on K, and one studies the invariant subspace of functions f with $f(sx) = \rho(s) f(x)$ for s in NK^1_k and x in K.

Let σ be the nontrivial element of $\mathrm{Gal}(K/k)$. If ρ is nontrivial and ρ^σ/ρ does not have order 2, this invariant subspace is irreducible and gives a discrete series representation. The group Γ_φ has order 2 and is exactly $k^\times / N_{K/k}(K)$. Thus K_φ is K.

If ρ is nontrivial and ρ^σ/ρ has order 2, the invariant subspace is reducible, decomposing into two pieces, each in the discrete series. One of the irreducible constituents consists of those functions in the subspace that are supported in the set $N_{\tilde{K}/K}(\tilde{K})$, where $\tilde{K}$ is the quadratic extension of K corresponding to the character of order 2 of $K^\times$ given by $x \to \rho(x^\sigma/x)$. In this case reference to [23] shows that Γ_φ has order 4 and is exactly $k^\times/N_{\tilde{K}/k}(\tilde{K})$. Thus K_φ is $\tilde{K}$, not K!

Accordingly we expect $\tilde{K}$ to play a role in the realization. In fact, all we have to do is pull back the functions in the representation space via the norm map $N_{\tilde{K}/K}$ to realize them on $\tilde{K}$. The functions on $\tilde{K}$ are the those such that

$$F(sX) = \rho(N_{\tilde{K}/K}(s))\, F(X) \qquad \text{for} \quad s \in N\tilde{K}^1_k,\ X \in \tilde{K}. \tag{5.1}$$

The advantage of this realization is the symmetry it incorporates. The representation in question arises also from a Weil-type construction by means of either of the other two quadratic extensions of k contained in $\tilde{K}$. In all three cases we are led to the same space of functions on $\tilde{K}$, by a comparison of (5.1) with Shelstad [30, p. 199]. Moreover, Casselman's proof [4] of the equivalence of the representations in their quadratic-extension settings amounts to a proof that the identity operator is an intertwining operator for the representations in the setting of $\tilde{K}$.

2. *Supercuspidal representations for general n.* (See [10, 14].) Many discrete series representations $\tilde{\pi}$ of $GL_n(k)$ are obtained from a construction that begins with an abelian Galois extension K of degree n over k. Let

$$G_K = \{ g \in GL \mid \det g \in N_{K/k}(K) \}.$$

P. Kutzko pointed out to us that these discrete series representations $\tilde{\pi}$ have the property that $\tilde{\pi}|_{G_K}$ is the sum of n inequivalent irreducible pieces $\pi_1, \ldots, \pi_n$, with $\tilde{\pi} \cong \operatorname{ind}_{G_K}^{GL}(\pi_j)$ for each j. So suppose the discrete series representation π of SL is obtained as a constituent of $\tilde{\pi}|_{SL}$. Since G_K is normal, the distribution character $\chi_{\tilde{\pi}}$ is supported on G_K, and $\chi_{\tilde{\pi}}(\omega \circ \det) = \chi_{\tilde{\pi}}$ for any ω in $(k^\times)\hat{}$ that is trivial on $N_{K/k}(K)$. Hence $\tilde{\pi} \otimes (\omega \circ \det) \cong \tilde{\pi}$ for such ω, and every such ω is in the $\bar{L}$ group for π (if we assume π is in $\Pi_0(SL)$). If $\pi = \pi_j|_{SL}$ for some j, then this is the whole $\bar{L}$ group for π, and the field extension we would associate with π is just K itself.

On the other hand, if the $\pi_j|_{SL}$ are reducible, then the $\bar{L}$ group for π points to a larger field extension than K, and presumably the larger field extension deals with π in a more symmetric manner than K does, just as in the case $n = 2$.

3. *Unitary principal series.* (See [7, 8, 29].) For $n = 2$, the reducible unitary principal series representations arise from characters of order 2 of $k^\times$. Each such character leads to a quadratic extension of k. After we fix an additive character of k, one of the two irreducible constituents is then realized as the space of radial functions in the Weil representation for that quadratic extension. The group Γ_φ has order 2 and points to the same quadratic extension.

Of course, the irreducible constituents can also be realized as functions on $N_{K/k}(K)$, if K is the relevant quadratic extension. Such a realization may be said not to involve K explicitly. The advantage of the realization by means of the Weil representation, which does involve K explicitly, is that the group action takes a much simpler form.

The irreducible constituents of unitary principal series for general n, according to [7], may be realized as vector-valued functions supported on the respective cosets of $k^\times$ modulo a subgroup N_χ. Let K be the finite abelian Galois extension of k with $N_\chi = N_{K/k}(K)$. Then the realization may be pulled back, via $N_{K/k}$, to a realization by means of functions on K. We expect that the group action takes a much simpler form when K is used, but we have not succeeded in carrying out the details.

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