FUNCTIONS BEHAVING LIKE ALMOST AUTOMORPHIC FUNCTIONS[1]

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Let G be a discrete group and let ℓ^∞ be the Banach algebra of all bounded complex-valued functions on G with the supremum norm. A G-subalgebra of ℓ^∞ is a left-invariant conjugate-closed subalgebra of ℓ^∞ which contains the constants and is closed under uniform limits.

This paper is divided into two parts, and the connection between the parts is indicated only at the end. The first part gives some simple properties of a certain G-subalgebra L of ℓ^∞. The functions in L arise naturally from Ellis' work in

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[2] and the author's in [5], and they seem to be closely related both to the almost automorphic functions of Veech [7] and to the distal functions of [6].

□ The second part of the paper contains a theorem and an application about disjointness in the sense of Furstenberg [4], together with some remarks about how this theorem might apply to L if more were known about L.

1. THE ALGEBRA L

The notation we use is essentially that in [5]. The maximal ideal space of a G-algebra B is denoted M(B); M(B) is a flow in a natural way, and G maps canonically onto a dense orbit of M(B). If $\{g_n\}$ is a net in G such that $\lim f(gg_n)$ exists for all f in ℓ^∞ and all g in G, then $\{g_n\}$ converges in $M(\ell^\infty)$ to some α and we write $T_\alpha f(g) = \lim f(gg_n)$. T_α is called a <u>shift operator</u>. Conversely, if $\{g_n\}$ converges to some α in $M(\ell^\infty)$, then $\lim f(gg_n)$ exists and depends only on α. Thus the shift operator T_α depends only on α and not on the net which defines it. Right translations are examples of shift operators. The set of shift operators is a semigroup under composition, and the compact Hausdorff topology it inherits from $M(\ell^{\infty})$ is such that the map $T_\alpha \to T_\alpha T_\beta$ for fixed T_β is continuous.

A shift operator T_u is minimal if it lies in a minimal left ideal, and T_u is idempotent if $T_u T_u = T_u$. Define

$$L = \{ f \mid T_u f = f \quad \text{for every minimal idempotent} \quad T_u \}$$

L is the intersection of all the algebras A which arise in [5], or it is the intersection of all maximal G-subalgebras of minimal functions in the sense of [5], or it is the intersection of all the algebras $\mathcal{A}(u)$ in the notation of Ellis [2]. Although no maximal G-subalgebra of minimal functions need be invariant under right translation, L is right-invariant because a right translate of such a maximal subalgebra is another such subalgebra.

We recall that an f in ℓ^{∞} is distal if the equality $T_\alpha T_\beta f = T_\alpha T_\gamma f$ implies $T_\beta f = T_\gamma f$. Distal functions are exactly those functions lying in algebras whose maximal ideal spaces are distal flows. Almost automorphic functions are defined as follows. If a net $\{g_n\}$ defines a shift operator T_α, then it is easy to see that $\{g_n^{-1}\}$ does define a shift operator, and we call this operator $T_{\alpha-1}$. Call an f in ℓ^∞ almost automorphic if $T_{\alpha}{-1}T_\alpha f = f$ for all T_α. Almost periodic functions are both almost automorphic and distal, and an almost automorphic distal function is almost periodic. L contains all almost automorphic functions and all distal functions.

PROPOSITION 1.1. If f is in ℓ^∞, f is in L if and only if the equality $T_\alpha T_\beta f = T_\alpha f$ implies $T_\beta f = f$.

<u>Proof</u>. If the condition holds and T_u is idempotent, then the equality $T_u T_u f = T_u f$ implies $T_u f = f$. Conversely, let f be in L and let $T_\alpha T_\beta f = T_\alpha f$. By Corollary 3-5 of [5], there is a minimal idempotent T_v, say in the left ideal I, such that $T_v T_\beta f = T_\beta f$. Since $T_\alpha T_v$ is in I, find by Lemma 2 of [1] members T_δ and T_u of I with $T_u(T_\alpha T_v) = T_\alpha T_v$ and $T_\delta(T_\alpha T_v) = T_u$. Since $T_v T_u = T_v$, we obtain

$$\begin{aligned} T_\beta f &= T_v T_\beta f = T_v T_u T_\beta f = T_v(T_\delta T_\alpha T_v)T_\beta f \\ &= T_v T_\delta T_\alpha (T_\beta f) = T_v T_\delta T_\alpha f = T_v T_\delta T_\alpha T_v f \end{aligned}$$

$$= T_v T_u f = T_v f = f$$

as required.

We wish to point out some similarities between the role of functions in L relative to distal functions and the role of almost automorphic functions relative to almost periodic functions. In [7] Veech proved, among other things, the following five facts about almost automorphic functions. In them, (1a) and (1b) follow easily from each other, and (2), (3), and (4) are corollaries of (1b).

(1a) If G is given the relative topology from the orbit Ge in the universal equicontinuous flow, then the almost automorphic functions are exactly the bounded continuous functions on G.

(1b) A G-subalgebra A consists entirely of almost automorphic functions if and only if M(A) is proximally equicontinuous and the fiber above e in the equicontinuous quotient has just one point in it.

(2) f is almost periodic if and only if $T_\alpha f$ is almost automorphic for all T_α.

(3) The composition of an almost automorphic function followed by a bounded continuous function on its range is again almost automorphic.

(4) A nontrivial G-subalgebra of almost automorphic functions contains nonconstant almost periodic functions.

PROPOSITION 1.2. Let $\{T_\alpha\}$ be a set of shift operators, and let A be the G-subalgebra of all f in ℓ^∞ for which $T_\alpha f = f$ for all the given α's. Then any bounded continuous function on the orbit Ge of M(A) is a member of A. Conversely, if B is a G-subalgebra of ℓ^∞ and A is the G-subalgebra of all bounded continuous functions on the orbit Ge of M(B), then there is a set $\{T_\alpha\}$ of shift operators such that A is all functions in ℓ^∞ for which $T_\alpha f = f$ for all the α's.

Proof. Let the set of shift operators be given, let T_α be in the set and be defined by a net $\{g_n\}$, and let f be bounded and continuous on the orbit Ge of M(A). Every continuous h on M(A) satisfies

$$\lim h(g_n) = T_\alpha h(e) = h(e)$$

and it follows that $g_n e$ converges to e in M(A). Then $gg_n e \to ge$ also, and hence

$$T_\alpha f(g) = \lim f(gg_n e) = f(ge) = f(g)$$

the middle equality holding by the continuity of f on Ge. Thus $T_\alpha f = f$.

Conversely let B and A be given, let $\{T_\alpha\}$ be the set of all shift operators such that $T_\alpha h = h$ for all h in B, and suppose f is a function in ℓ^∞ with $T_\alpha f = f$ for all these α's. We are to show f is in A or that f is continuous on the orbit Ge in M(B). To see f is continuous at e in M(B), let $\{g_n e\}$ be a net converging to e in M(B) and suppose, by taking a subnet if necessary, that $\{g_n\}$ defines a shift operator T_α. Since $g_n e \to e$ in M(B), we have $T_\alpha h = h$ for all h in B and consequently $T_\alpha f = f$. That is,

$$\lim f(g_n e) = T_\alpha f(e) = f(e)$$

Since $\{g_n\}$ is arbitrary, f is continuous at e. To obtain continuity at ge in M(B), we observe that the left translate f_g satisfies $T_\alpha f_g = f_g$ and hence f_g is continuous at e. In other words f is continuous at ge.

PROPOSITION 1.3. L has the following properties:

(1') If G is given the relative topology from the orbit Ge in the universal distal flow, then every bounded continuous function on G is in L. There is a set of shift operators $\{T_\alpha\}$ such that the bounded continuous functions on G in this topology are those functions in ℓ^∞ with $T_\alpha f = f$ for these α's.

(2') f is distal if and only if $T_\alpha f$ is in L for every shift operator T_α.

(3') The composition of a function in L followed by a bounded continuous function on its range is again in L.

Proof. (1') is a special case of Proposition 1.2. (2') is immediate from the definition of L and from the fact that an f in ℓ^∞ is distal if and only if $T_u T_\alpha f = T_\alpha f$ for every T_α and every minimal idempotent T_u. This fact is Theorem 3.7 of [6]. (3') is a consequence of Proposition 1.2 and the fact that the composition of continuous functions is continuous.

For an example, $\cos 2\pi n^2 \theta$ is a well-known distal function on the integers and its range does not include 0 if θ is irrational. Since signum is continuous on the line with 0 deleted, signum $\cos 2\pi n^2\theta$ is in L.

One might conjecture as a converse to the first half of (1') that every member of L is continuous on G in the relative topology from the universal distal flow. But Furstenberg pointed out that M(L) should be closed under isometric extensions (see Proposition 1.4), whereas this closure property is not apparent for the algebra of bounded functions on G continuous in this relative topology. A safer conjecture would be that there is an analogue to property (4), namely that any nontrivial G-subalgebra of L contains nonconstant distal functions. The answer to this question is not known.

PROPOSITION 1.4. If B is a G-subalgebra of L and A is a G-subalgebra of ℓ^∞ for which M(A) is an isometric extension (see [3]) of M(B), then A is contained in L.

Proof. Let ρ exhibit the isometric extension, let u be an idempotent in the Ellis semigroup of

M(A), and let π be the projection of M(A) onto M(B). Then $\pi(u)$ is idempotent on M(B) and must satisfy $\pi(u)\pi(e) = \pi(e)$ since $B \subseteq L$. That is, ue and e are in the same fiber and $\rho(ue, e)$ is defined. Since ρ is G-invariant and continuous,

$$\rho(ue, e) = \rho(u^2e, ue) = \rho(ue, ue) = 0$$

Since ρ is a metric on each fiber, ue = e. This condition on M(A) means that $A \subseteq L$.

2. A DISJOINTNESS THEOREM

We say that two flows X and Y under G have a common factor if for some flow Z with more than one point there are homomorphisms of X and Y onto Z. The flows X and Y are disjoint if the only closed G-invariant subset of X x Y which projects onto all of X and all of Y is X x Y itself. If X and Y are both minimal, it is easy to see that X and Y are disjoint if and only if X x Y is minimal. In particular, this observation applies to maximal ideal spaces of G-subalgebras of minimal functions, since these flows are minimal. The maximal ideal spaces of G-subalgebras A and B have no common factor if and only if $T_\alpha(A) \cap T_\beta(B)$ = constants for every T_α and T_β. If M(A) and M(B) are minimal, they have no common factor provided $A \cap T_\beta(B)$ = constants for every T_β.

Disjointness of X and Y implies that X and Y have no common factor, and Furstenberg asked in [4, p. 25] about the converse. The converse is false in general, and we give an example which both settles the question and illuminates the relation between the two notions.

EXAMPLE. Let G be a compact topological group, and restrict attention to G-subalgebras of continuous functions. The maximal ideal spaces of such algebras are left coset spaces of G and are in one-to-one correspondence with the closed subgroups of G. Let H_1 and H_2 be two closed subgroups. Then G/H_1 and G/H_2 have no common factor if and only if $[gH_1g^{-1}H_2]$, the closed subgroup generated by gH_1g^{-1} and H_2, is all of G for each g in G. Also, G/H_1 and G/H_2 are disjoint if and only if H_1H_2, the set of products of a member of H_1 by a member of H_2, is all of G.

If $H_1H_2 = G$, then $gH_1g^{-1}H_2 = G$ also. Thus the inclusion $[H_1H_2] \supseteq H_1H_2$ shows that disjointness implies no common factor. The reverse implication is false. For instance, take G to be the symmetric group on four letters, let H_1 be the powers of a 4-cycle, and let H_2 be the powers of a 3-cycle. Then $[H_1H_2] = G$. Since gH_1g^{-1} is again the powers of a 4-cycle, we have $[gH_1g^{-1}H_2] = G$ for every g. Hence G/H_1 and G/H_2 have no common factor. On the other hand, H_1H_2 has at most twelve elements and therefore cannot be all of G. Thus G/H_1 and G/H_2 are not disjoint.

When G is compact, the Ellis semigroup (or group, actually) of G/H is $G/\bigcup_{g \in G} gHg^{-1}$. Let E(G/H) be the Ellis group of G/H. Even if G/H_1 and G/H_2 are disjoint, it does not follow that $E(G/H_1)$ and $E(G/H_2)$ are disjoint. For an example, take G to be the symmetric group S_n with $n > 2$, H_1 to be the alternating group A_n, and H_2 to be the identity and a transposition. Then $H_1H_2 = G$, $E(G/H_1)$ is a two-element group $\{+1, -1\}$ and $E(G/H_2) = S_n$. The closed orbit of (+1, e) in $E(G/H_1) \times E(G/H_2)$ contains no points of the form (-1, even permutation), and $E(G/H_1)$ and $E(G/H_2)$ are therefore not disjoint.

Return to the case of a G not necessarily compact. The theorem to follow comes from an idea suggested by the above example. We know that $[H_1 H_2] = H_1 H_2$ when either H_1 or H_2 is normal. Now H is normal exactly when the algebra of functions lifted to G from G/H is right-invariant, and this condition suggests that we assume one of the flows is closed under the operation of all shift operators.

We say a G-subalgebra A is shift-invariant if $T_\alpha f$ is in A whenever f is in A and T_α is a shift operator. For M(A), shift-invariance of A means that there is a transitive set of homeomorphisms of M(A) commuting with G.

THEOREM 2.1. Let A and B be G-subalgebras of ℓ^∞ and suppose B is distal and shift-invariant. If M(A) and M(B) have no common factors and if M(A) is minimal, then M(A) and M(B) are disjoint and, consequently, M(A) x M(B) is minimal.

<u>Proof</u>. M(A) x M(B) is semisimple because all distal functions occur in every maximal G-subalgebra of minimal functions on G. Thus we are to prove that the orbit of (e_A, e_B) in M(A) x M(B) is dense.

Let C be the least G-subalgebra of ℓ^∞ containing both A and B. It is well known that M(C) is canonically isomorphic with the closure of the orbit of (e_A, e_B) in M(A) x M(B) and that e_C is identified with (e_A, e_B). Let E(C) be the Ellis semigroup of C, let π_A be the projection of M(C) on M(A) or E(C) on E(A), and define π_B analogously.

Define X_e to be the set of points $\pi_A(r)e_A$ in M(A) as r ranges through all members of E(C) for which $\pi_B(r)$ is the identity on M(B). We prove that if $X_e = M(A)$, then $G(e_A, e_B)$ is dense in M(A) x M(B) and that if X_e is not M(A), then M(A) and M(B) have a common factor.

If $X_e = M(A)$, let (x, y) be given in M(A) x M(B) and let U be any neighborhood of y. Since Ge_B is dense in M(B), we can choose $y' = ge_B$ in U for some g in G. Then $g^{-1}x$ is in M(A), hence in X_e. Hence $g^{-1}x = \pi_A(r)e_A$ with $\pi_B(r)$ equal to the identity on M(B), and

$$\begin{aligned} gr(e_A, e_B) &= (g\pi_A(r)e_A, g\pi_B(r)e_B) \\ &= (gg^{-1}x, ge_B) = (x, y') \end{aligned}$$

That is, the closure of $G(e_A, e_B)$ meets (x, U).

Since U is arbitrary, the orbit of (e_A, e_B) is dense.

Now suppose X_e is not all of M(A). For t in E(B), we define X_t to be all points $\pi_A(r)e_A$ in M(A) as r ranges through the members of E(C) with $\pi_B(r) = t$. Every element of M(A) is in some X_t. We claim that either $X_s = X_t$ or $X_s \cap X_t$ is empty. Let x be in $X_s \cap X_t$ and let y be in X_s. It suffices to show y is in X_t. Let

$$x = \pi_A(r_s)e_A = \pi_A(r_t)e_A \quad \text{and} \quad y = \pi_A(p_s)e_A$$

with

$$\pi_B(r_s) = \pi_B(p_s) = s \quad \text{and} \quad \pi_B(r_t) = t$$

Since M(C) is minimal, we can find members u and r_s^{-1} of E(C) such that $u^2 = u$, $ue_C = e_C$, and $r_s^{-1}r_su = u$. Then

$$\begin{aligned} y &= \pi_A(p_s)e_A = \pi_A(p_s)\pi_A(u)e_A \\ &= \pi_A(p_sr_s^{-1})\pi_A(r_s)\pi_A(u)e_A = \pi_A(p_sr_s^{-1})\pi_A(r_s)e_A \\ &= \pi_A(p_sr_s^{-1})\pi_A(r_t)e_A = \pi_A(p_sr_s^{-1}r_t)e_A \end{aligned}$$

Since E(B) is a group, $\pi_B(u)$ is the identity and $\pi_B(r_s^{-1}) = s^{-1}$. Thus $\pi_B(p_sr_s^{-1}r_t) = ss^{-1}t = t$, and y is in X_t.

If x and y are in M(A), define $x \sim y$ if x and y are in the same X_t. We have just proved that this is an equivalence relation. There exist inequivalent pairs since X_e is not all of M(A). The equivalence relation is group-invariant since $gX_t = X_{gt}$. To see the relation is closed, let x_n be in X_{t_n}, let $x_n \to x$, and let $t_n \to t$. We are to show x is in X_t. For suitable r_n we have

$$x_n = \pi_A(r_n)e_A \quad \text{and} \quad \pi_B(r_n) = t_n$$

Passing to a subnet if necessary, we may assume r_n converges, say to r. Then by continuity of π_A and π_B we obtain

$$x = \pi_A(r)e_A \quad \text{and} \quad \pi_B(r) = t$$

That is, x is in X_t.

Let $(Y, G) = \sigma_A(M(A), G)$ be the quotient. (Y, G) is not the one-point flow. It is straightforward to verify that (Y, G) is a quotient flow of (E(B), G) under the definition $\sigma_B(t) = \sigma_A(X_t)$ for t in E(B). Since B is shift-invariant, E(B) and M(B) are isomorphic flows. Therefore M(A) and M(B) have Y as a common factor. The proof is complete.

COROLLARY 2.2. If C is a G-subalgebra of minimal functions, then any directed system of G-subalgebras of C with no nonconstant distal functions has an upper bound.

Proof. Let D be the algebra of all distal functions. It is shift-invariant. By the theorem, M(D) is disjoint from the maximal ideal space of each member of the system. An easy calculation shows that Furstenberg's definition (in [4]) of disjointness from a given flow is preserved under inverse limits. The inverse limit of the system in question is therefore disjoint from M(D) and can have no common factors with it. That is, the closure of the union of the given algebras has no distal functions other than the constants, and the proof is complete.

Let A and B be G-subalgebras of minimal functions and distal functions, respectively. By the theorem and Proposition II.1 of [4], M(A) and M(B) are disjoint if A contains no nonconstant distal functions. This fact may help settle the question whether every nontrivial G-subalgebra of L contains nonconstant distal functions. If the answer to the question is yes, is the M(A) in the statement above disjoint from M(L)?

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