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NORMALIZING FACTORS, TEMPERED REPRESENTATIONS, AND L-GROUPS

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Several lecturers have alluded to the intertwining operators associated with principal series representations, particularly for SL(2, $\boldsymbol{R}$). These operators and their normalizations play a role in the trace formula, in reducibility questions for principal series, and in the construction of an inner product that exhibits certain representations as unitary. We shall review the development of these operators and their normalizations in the context of a linear real reductive group G (as defined in Wallach's lectures) whose identity component has compact center.

We shall be especially interested in the application to reducibility questions for unitary principal series and other continuous series of representations coming from parabolic subgroups, since the answers to these questions lead to a classification of irreducible tempered representations and thereby complement the Langlands classification [**12**]. The answers concerning reducibility are reviewed in §2 in terms of three easy-to-calculate finite groups, denoted W, W', and R.

In lectures during 1975–76, the authors mentioned how, in some special cases, the groups W, W', R, initially defined in terms of roots, could be defined in terms of co-roots. Building on this presentation, Langlands [**14**] was able to redefine these groups in general in terms of the L-group. We present his definitions, along with an example, in §3. In §4 we summarize earlier work [**8**], [**9**] that leads from the R-group to the classification of irreducible tempered representations.

1. Intertwining operators and normalizing factors. In the group G, fix a maximal compact subgroup K and Cartan involution θ. To each parabolic subgroup P, we associate the Langlands decomposition $P = MAN$ with MA θ-stable and with M a linear real reductive group whose identity component has compact center. To the pair (ξ, ν), where ξ is an irreducible unitary representation of M and ν is a complex-valued linear functional on the Lie algebra $\mathfrak{a}$ of A, we associate the representation $U_P(\xi, \nu, x)$, with x in G, given by

$$U_P(\xi, \nu) = \operatorname{ind}_{MAN}^G(\xi \otimes e^\nu \otimes 1). \tag{1.1}$$

We adopt the convention that G acts on the left in the induced representation. A member f of the representation space satisfies

$$f(xman) = \exp(-(\nu + \rho_P) \log a)\, \xi(m)^{-1} f(x),$$

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where ρ_P is the usual half-sum of restricted roots associated with P. If $P = P_0$ is minimal parabolic, these representations comprise the *nonunitary principal series*. In formula (1.1), we have assumed that ξ is an irreducible unitary representation of M, but we shall allow also that ξ is a nonunitary principal series representation of M, provided we are not working with formulas involving adjoints of operators.

For SL(2, $\boldsymbol{R}$), one can restrict the functions in the representation space for the nonunitary principal series to the lower triangular group $\bar{N} = \theta N = \{(\begin{smallmatrix}1 & 0\\ x & 1\end{smallmatrix})\}$ and realize the representations in spaces of functions on $\boldsymbol{R}$. The representations become

$$\begin{aligned}\mathscr{P}^{\pm,\zeta}(g)f(x) &= |-bx+d|^{-1-\zeta} f\left(\frac{ax-c}{-bx+d}\right) && \text{if } +,\\ &= \operatorname{sgn}(-bx+d)\,|-bx+d|^{-1-\zeta} f\left(\frac{ax-c}{-bx+d}\right) && \text{if } -,\end{aligned}$$

if $g = (\begin{smallmatrix}a & b\\ c & d\end{smallmatrix})$. Kunze and Stein [**10**] showed that the operator

$$f \to \int_{-\infty}^{\infty} \frac{f(x-y)\,dy}{|y|^{1-\zeta}} \qquad \text{for } \mathscr{P}^{+,\zeta}, \tag{1.2a}$$

$$\to \int_{-\infty}^{\infty} \frac{(\operatorname{sgn} y)\,f(x-y)\,dy}{|y|^{1-\zeta}} \qquad \text{for } \mathscr{P}^{-,\zeta}, \tag{1.2b}$$

intertwines $\mathscr{P}^{\zeta}$ with $\mathscr{P}^{-\zeta}$ when it is convergent, namely for $\operatorname{Re}\zeta > 0$. Later [**11**] they found a formula in the induced picture, namely

$$f \to \int_{\bar{N}} f(gw\bar{n})\,d\bar{n}, \quad \text{where } w = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

This is the composition of two operators, $f \to \int_{\bar{N}} f(g\bar{n})\,d\bar{n}$ and a relatively trivial translation operator by w. The first operator intertwines the representation induced from $P = MAN$ with the one induced from $\bar{P} = MA\bar{N}$ and the same data on MA.

In the general case, let $P_1 = MAN_1$ and $P_2 = MAN_2$ be two parabolics with the same MA, and define

$$A(P_2 : P_1 : \xi : \nu) f(x) = \int_{\bar{N}_1 \cap N_2} f(x\bar{n})\,d\bar{n}. \tag{1.3}$$

Formally

$$U_{P_2}(\xi, \nu, g) A(P_2 : P_1 : \xi : \nu) = A(P_2 : P_1 : \xi : \nu) U_{P_1}(\xi, \nu, g). \tag{1.4}$$

In general, the integral (1.3) will not converge but will be defined by analytic continuation. To accomplish this analytic continuation, we need the representations to occur in a single space, as ν varies. This space is obtained by restricting the functions to K. Say ξ operates in the space H^ξ. We consider functions $f: K \to H^\xi$ satisfying $f(km) = \xi(m)^{-1} f(k)$ for $k \in K$, $m \in M \cap K$. Under the action by g, we replace $f(k)$ by $f(g^{-1}k)$ and restrict back to K. Thus we are led to define

$$U_P(\xi, \nu, g) f(k) = \exp(-(\rho_P + \nu)\,H(g^{-1}k))\,\xi(\mu(g^{-1}k))^{-1} f(\kappa(g^{-1}k)), \tag{1.5}$$

where x decomposes within $G = KMAN$ as $\kappa(x)\mu(x)e^{H(x)}n$.

The analytic continuation of (1.3) is accomplished in three stages. In the first stage the essence of the argument can be seen by continuing (1.2a) if f is smooth and compactly supported. Since f has compact support, we can integrate over a finite interval. Expand $f(x-y)$ about $y=0$ in a finite Taylor series with remainder. Each of the main terms can then be integrated explicitly and continues meromorphically to the whole plane. The error term is integrable for $\operatorname{Re}\zeta > -n-1$ if n is the order of the Taylor series. Hence the integral continues meromorphically to the whole plane.

In Schiffmann **[15]** and in **[6]**, it is shown that this style of argument yields a meromorphic continuation of (1.3) if P_1 and P_2 are minimal parabolic and A is one-dimensional. In the second stage, for minimal parabolics with $\dim A > 1$, Schiffmann shows how to decompose $\bar{N}_1 \cap N_2$ into a product of subgroups and to write (1.3) as an iterated integral; the analytic continuation is essentially then reduced to the case $\dim A = 1$.

The third and final stage of the analytic continuation was obtained independently by Wallach **[17]** and in **[7]**. We can use Casselman's subrepresentation theorem to imbed ξ as an $(\mathfrak{m}, K \cap M)$ module in a nonunitary principal series representation of M. It follows from the double induction theorem that (1.1) is a subrepresentation of a nonunitary principal series and (1.3) is a restriction of an intertwining operator for the nonunitary principal series. Then (1.3) has an analytic continuation for K-finite f, and (1.4) holds if g is replaced by a member of K or of the Lie algebra $\mathfrak{g}$ of G. Moreover, with Haar measures normalized suitably, we obtain

$$A(P_2 : P_1 : \xi : \nu)^* = A(P_1 : P_2 : \xi : -\bar{\nu}) \tag{1.6}$$

for ξ unitary, if the adjoint is defined K-space by K-space, and

$$A(P_2 : P_1 : \xi : \nu) = R(w^{-1})\, A(wP_2w^{-1} : wP_1w^{-1} : w\xi : w\nu)\, R(w),$$

where $R(w)$ denotes right translation by an element w in K representing a member of the Weyl group $W(\mathfrak{a})$.

These operators tend to have poles at many interesting values of ν. We introduce scalar normalizing factors—in part to eliminate some of these poles, in part to make the operators unitary for ν imaginary, and in part to make the operators behave nicely under composition. The normalizing factors are not unique, and different choices are useful for different purposes.

Again the construction is in several stages. We impose the condition that ξ have a real infinitesimal character. Matters are based on the following lemma **[6**, p. 544**]**, proved using Weierstrass canonical products.

LEMMA 1.1. *If $\eta(z)$ is a meromorphic function in the plane such that*

(i) *$\eta(z)$ is real on the real axis,*

(ii) *$\eta(z) \geqq 0$ on the imaginary axis,*

(iii) *$\eta(z) = \eta(-z)$ for all z,*

then there exists a meromorphic function $\gamma(z)$ in the plane such that $\eta(z) = \gamma(-z)\gamma(z)$ and $\gamma(z)$ is real for real z.

The first stage of the construction deals with $P = MAN$ a minimal parabolic and G of real rank one ($\dim A = 1$). In this case $A(P : \bar{P} : \xi : \nu)\, A(\bar{P} : P : \xi : \nu)$ is a self-intertwining operator for $U_P(\xi, \nu)$. It is a result due to Bruhat [3] that, for ν nonzero imaginary, the unitary representation $U_P(\xi, \nu)$ is irreducible. It follows readily that

$$A(P : \bar{P} : \xi : \nu) A(\bar{P} : P : \xi : \nu) = \eta(\bar{P} : P : \xi : \nu) I \tag{1.7}$$

with $\nu = z\rho_P$ meromorphic and scalar valued for z in C. One checks that

(i) $\eta(\bar{P}:P:\xi:\nu)$ is real for ν real.

(ii) $\eta(\bar{P}:P:\xi:\nu) \geqq 0$ for ν imaginary.

(iii) $\eta(\bar{P}:P:\xi:\nu) = \eta(\bar{P}:P:\xi:-\nu)$ for all ν.

(iv) $\eta(\bar{P}:P:\xi:\nu)$ depends only on the class of ξ, and if φ is an automorphism of G leaving K and P stable, $\eta(\bar{P}:P:\xi^\varphi:\nu) = \eta(\bar{P}:P:\xi:\nu)$. Then we can apply the lemma to obtain $\gamma(\bar{P}:P:\xi:\nu)$ and to define normalized operators by

$$\mathscr{A}(\bar{P}:P:\xi:\nu) = \gamma(\bar{P}:P:\xi:\nu)^{-1} A(\bar{P}:P:\xi:\nu).$$

The normalized operators satisfy

$$\mathscr{A}(P:\bar{P}:\xi:\nu)\,\mathscr{A}(\bar{P}:P:\xi:\nu) = I$$

and

$$\mathscr{A}(\bar{P}:P:\xi:\nu)^* = \mathscr{A}(P:\bar{P}:\xi:-\bar{\nu}).$$

These two relations together imply $\mathscr{A}$ is unitary for ν imaginary.

The second stage is to handle a minimal parabolic for general G. Use Schiffmann's decomposition of a general intertwining operator into operators that are essentially rank one operators, and use the product of the normalizing factors as normalizing factor for the given operator. Then one proves the relation

$$\mathscr{A}(P_3:P_1:\xi:\nu) = \mathscr{A}(P_3:P_2:\xi:\nu)\,\mathscr{A}(P_2:P_1:\xi:\nu). \tag{1.8}$$

The third stage is to handle a general parabolic $P = MAN$ with $\dim A = 1$. We again use the trick of imbedding ξ as a subrepresentation of a nonunitary principal series representation of M. If we combine this trick with formula (1.8) for minimal parabolics, we are led to the conclusion that (1.7) holds for our P with $\dim A = 1$ and that η has the same properties as before. Again we apply the lemma to obtain a normalizing factor γ, and we set $\mathscr{A} = \gamma^{-1}A$.

The final stage is for a general parabolic with G general and is handled in the same way as with a minimal parabolic and G general. If the γ's are chosen compatibly, the result is as follows [7, p. 2460].

THEOREM 1.2. *The normalized intertwining operators satisfy*

(i) $\mathscr{A}(P_3:P_1:\xi:\nu) = \mathscr{A}(P_3:P_2:\xi:\nu)\,\mathscr{A}(P_2:P_1:\xi:\nu)$.

(ii) $\mathscr{A}(P_2:P_1:\xi:\nu)^* = \mathscr{A}(P_1:P_2:\xi:-\bar{\nu})$, *$K$-space by K-space.*

(iii) $\mathscr{A}(P_2:P_1:\xi:\nu)$ *is unitary for ν imaginary.*

(iv) *If w in K represents a member of the Weyl group $W(\mathfrak{a})$, then*

$$\mathscr{A}(P_2:P_1:\xi:\nu) = R(w^{-1})\,\mathscr{A}(wP_2w^{-1}:wP_1w^{-1}:w\xi:w\nu)R(w).$$

For w in K representing a member of $W(\mathfrak{a})$, let

$$\mathscr{A}_P(w,\xi,\nu) = R(w)\mathscr{A}(w^{-1}Pw:P:\xi:\nu).$$

From (i) and (iv) we obtain the cocycle relation

$$\mathscr{A}_P(w_1w_2,\xi,\nu) = \mathscr{A}_P(w_1,w_2\xi,w_2\nu)\mathscr{A}_P(w_2,\xi,\nu). \tag{1.9}$$

From (ii) we find that

$$\mathscr{A}_P(w,\xi,\nu)^* = \mathscr{A}_P(w^{-1},w\xi,-w\bar{\nu}) \tag{1.10}$$

and hence that $\mathscr{A}_P(w,\xi,\nu)$ is unitary for ν imaginary. The intertwining relation is

(1.11) $$U_P(w\xi, w\nu, g)\mathscr{A}_P(w, \xi, \nu) = \mathscr{A}_P(w, \xi, \nu)U_P(\xi, \nu, g)$$

for g in K or in $\mathfrak{g}$.

For application to adelic situations, a more specific normalization is needed. At almost all places the representation has a K-fixed vector, and the infinite tensor product of local normalized intertwining operators will be defined only if the local operator fixes the K-fixed vector. This condition determines the normalizing factor. For example, with P_1 and P_2 minimal parabolic and $\xi = 1$, we are led to normalize (1.3) by taking

$$\gamma(P_2 : P_1 : \xi : \nu) = \int_{\bar{N}_1 \cap N_2} \exp(-(\rho_P + \nu)\, H(\bar{n}))\, d\bar{n},$$

which can be computed in terms of Harish-Chandra's $\boldsymbol{c}$-functions. Schiffmann pursued this idea further at an early stage in his work leading to **[15]**. Langlands was led to conjecture **[13**, p. 282**]** in general that a valid normalization is obtained by using the quotient of two L-functions:

(1.12) $$\gamma(P_2 : P_1 : \xi : \nu) = L(0, \bar{\rho}_{P_2|P_1} \circ \varphi)/L(1, \bar{\rho}_{P_2|P_1} \circ \varphi).$$

Here φ is the homomorphism of the Weil group of $\boldsymbol{R}$ into the L-group ${}^L(MA)$ of MA corresponding to $\xi \otimes e^\nu$, and $\bar{\rho}_{P_2|P_1}$ is the contragredient of the representation of ${}^L(MA)$ on the Lie algebra of ${}^L\bar{N}_1 \cap {}^LN_2$. Arthur **[1]** proved that (1.12) is a valid normalization if the Haar measures are normalized suitably.

2. The R-group, a first formulation. Fix a discrete series representation ξ of M, and consider the corresponding continuous series representation $U_P(\xi, \nu)$ with ν imaginary. In [7], it is stated how the problem of determining the algebra $\mathscr{C}(\xi, \nu)$ of operators that commute with $U_P(\xi, \nu)$ can be reduced to algebraic problems involving certain finite groups.

Fix ν. Let W be the subgroup of elements s in $W(\mathfrak{a})$ that fix ν and the class of ξ. If w is a representative in K of an s in W, then one can define $\xi(w)$ in such a way that ξ extends to a representation of the smallest group containing M and w; the definition of $\xi(w)$ is unique up to a scalar factor equal to a root of unity. Then $\xi(w)\, \mathscr{A}_P(w, \xi, \nu)$ is independent of the representative w, and we can write

(2.1) $$\xi(s)\, \mathscr{A}_P(s, \xi, \nu)$$

for it. One sees from (1.11) that the unitary operator (2.1) intertwines $U_P(\xi, \nu)$ with itself. The essence of the next theorem is due to Harish-Chandra **[4]**.

THEOREM 2.1. *The operators* (2.1), *for s in W, span the commuting algebra* $\mathscr{C}(\xi, \nu)$.

Despite formula (1.9), it does not follow that the map

(2.2) $$s \in W \to \xi(s)\mathscr{A}_P(s, \xi, \nu)$$

is a homomorphism into unitary operators; the $\xi(s)$ factors need to be chosen compatibly, and there may a priori be an obstruction to making such a choice. However, the map (2.1) is at least a homomorphism into the projective unitary group. Let W' be its kernel.

THEOREM 2.2. *The group W' is the Weyl group of a root system Δ' contained in the* *set of roots of* $(\mathfrak{g}, \mathfrak{a})$; *consequently* $W = W'R$ *is a semidirect product if* R *is defined as the subgroup of* W *leaving stable the positive roots in* Δ'. *The image of the map* (2.2) *consists of linearly independent operators*; *consequently the dimension of* $\mathscr{C}(\xi, \nu)$ *equals the order of* R.

It turns out that the elements of R are characterized in W as those elements for which the normalizing factor of (2.1) is holomorphic at ν. Thus W' and R are intimately connected with the functions $\eta(P_2 : P_1 : \xi : \nu)$ defined by

$$A(P_1 : P_2 : \xi : \nu) A(P_2 : P_1 : \xi : \nu) = \eta(P_2 : P_1 : \xi : \nu) I.$$

In turn, these functions can be connected by analytic methods with the Plancherel measures of subgroups of G; these measures have been determined by Harish-Chandra. Consequently we can describe Δ' very concretely, as follows.

Adjoin to $\mathfrak{a}$ a compact Cartan subalgebra of the Lie algebra $\mathfrak{m}$ of $\mathscr{M}$, and denote a typical root by β. Let λ be the Harish-Chandra parameter for the discrete series ξ. Let α be an $\mathfrak{a}$-root and suppose 2α is not an $\mathfrak{a}$-root. If α has even multiplicity and 2α is not an $\mathfrak{a}$-root, define

$$\mu_{\xi,\alpha}(\nu) = \prod_{\beta|\mathfrak{a}=\alpha} \langle \lambda + \nu, \beta \rangle.$$

If α has odd multiplicity, define

$$\mu_{\xi,\alpha/2}(\nu) = \mu_{\xi,\alpha}(\nu) = \Big(\prod_{\beta|\mathfrak{a}=c\alpha,\, c>0} \langle \lambda + \nu, \beta \rangle \Big) f_{\xi,\alpha}(\nu),$$

where $f_{\xi,\alpha}(\nu) = \tan(\pi\langle \nu, \alpha \rangle / |\alpha|^2)$ or $\cot(\pi \langle \nu, \alpha \rangle / |\alpha|^2)$ according as

$$\xi(\gamma_\alpha) = -(-1)^{2\langle \rho_\alpha, \alpha \rangle / |\alpha|^2} I \quad \text{or} \quad +(-1)^{2\langle \rho_\alpha, \alpha \rangle / |\alpha|^2} I.$$

Here α is a real root and there is a corresponding homomorphism of $SL(2, \boldsymbol{R})$ into G, and γ_α denotes the image of $-I$.

Now we can characterize Δ' as

$$\Delta' = \{\text{roots } \alpha \text{ of } \mathfrak{a} \,|\, s_\alpha \text{ is in } W \text{ and } \mu_{\xi,\alpha}(\nu) = 0\},$$

where ξ_α denotes the reflection corresponding to α. As before,

$$R = \{p \in W \,|\, p\alpha > 0 \text{ for every } \alpha > 0 \text{ in } \Delta'\}.$$

Then we have the following result [5], [**8**].

THEOREM 2.3. (a) *R is a finite direct sum of 2-element groups* $\boldsymbol{Z}_2$.

(b) *For s in R the operators $\xi(s)$ can be selected so that the mapping* (2.2) *is a homomorphism into unitary operators*; *consequently the algebra $\mathscr{C}(\xi, \nu)$ is commutative.*

(c) *There exists a set of positive orthogonal real roots* $\mathscr{H} = \{\alpha_1, \ldots, \alpha_q\}$ *such that* (i) *the only roots in the span of $\mathscr{H}$ are the* $\pm\alpha_j$; (ii) *each r in R is of the form* $s_{\alpha_{j_1}} \cdots s_{\alpha_{j_n}}$; *and* (iii) *each α_j occurs in the decomposition of some r in R.*

3. The *R*-group, a second formulation. An admissible representation of G is *tempered* if its K-finite matrix coefficients are in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$. It is known that every irreducible tempered representation is a summand in some representation of the type considered in §2, induced from discrete series on M and a unitary character on A. The R group can be used to decompose these induced representations and give a classification of the irreducible tempered representations. We return to this point in §4.

For now, we want to consider W, W', and R from a different point of view. We begin with some motivation in the case that G is split and our parabolic is minimal. Then M is finite abelian and is generated by the elements γ_α that are the images of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in the SL(2, $\boldsymbol{R}$)'s that correspond to each root. Here ξ is a character of M, and we assume $\nu = 0$. Let the root system be Δ. It is a simple matter to see that $\Delta' = \{\alpha \in \Delta \mid \xi(\gamma_\alpha) = 1\}$. The elements γ_α transform under Weyl group elements differently from what one might expect. The correct rule is

$$s_\beta \gamma_\alpha s_\beta^{-1} = \gamma_{s_\beta \alpha} = \gamma_\alpha \gamma_\beta^{2\langle\beta,\alpha\rangle/|\alpha|^2}.$$

As a result, Δ' need not be closed under addition within the set of all roots. It is in the co-root system that there is closure under addition. The co-root of α is $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$, and $\Delta'^\vee$ is closed under addition within $\Delta^\vee$. We introduce a kind of dual group $G^\sim$, not the L-group just yet. $G^\sim$ is a connected real group with a compact Cartan subgroup and root system $\Delta^\vee$, arranged so that the roots of $\Delta'^\vee$ are compact and the others are noncompact; it is to have a centerless complexification $G^\sim_C$, a Lie algebra $\mathfrak{g}^\sim$, and a Cartan decomposition $\mathfrak{g}^\sim = \mathfrak{k}^\sim \oplus i\mathfrak{p}^\sim$. Elements of W' lead to elements in the Weyl group of $\mathfrak{k}^\sim$, with representatives in $G^\sim$. Elements of W lead to elements in the Weyl group of $\mathfrak{g}^\sim_C$ that leave the compact roots stable, hence normalize $\mathfrak{k}^\sim_C$ and $\mathfrak{k}^\sim \oplus i\mathfrak{p}^\sim$, hence normalize $\mathfrak{k}^\sim$ and $\mathfrak{p}^\sim$ and therefore $\mathfrak{g}^\sim$. It follows that R injects onto the quotient $\mathrm{Norm}_{G^\sim_C}(\mathfrak{g})/G^\sim$. This style of argument can be pushed to yield a proof that $R = \sum Z_2$ in this case.

Langlands [**14**] built on these ideas and gave a formulation in general of W, W', and R in terms of the L-group. We use the notation of Borel [**2**] approximately and will follow the L-group constructions given by Langlands [**12**] when we need them.

First let us see what the above example has to do with the L-group. Of course, ${}^LG^0$ is just $G^\sim_C$, up to coverings. The character ξ is specified by giving $\xi(\gamma_\alpha)$, and γ_α behaves like the (character associated to the) co-root $\alpha^\vee$. To know $\alpha^\vee$ on ξ for each $\alpha^\vee$ is to know ξ as an element of the dual torus ${}^LT^0$. There is a corresponding homomorphism φ of the Weil group into ${}^LG$ given by $\varphi(z) = 1$ for $z \in C^\times$ and $\varphi(\tau) = \xi \times \sigma \in {}^LT$; here σ is the trivial outer automorphism of ${}^LG^0$, trivial since G is split. The elements of the Weyl group that centralize the image $\{1, \xi \times \sigma\}$ of φ are those of W; this statement motivates Theorem 3.1 below. The roots whose root vectors are centralized by the image of φ are those in Δ'; this statement motivates Theorem 3.3 below. The observation above that R is isomorphic to $\mathrm{Norm}_{G^\sim_C}(\mathfrak{g})/G^\sim$ could have been stated in terms of $K^\sim$ and then motivates Theorem 3.4 below.

We pass to the general case. Fix a minimal parabolic $P_0 = M_0A_0N_0$ in G, and let T be a Cartan subgroup of G containing A_0. Let ${}^LG^0 \rtimes \{1, \sigma\}$ be the L-group of G, with its maximal torus ${}^LT^0$ and Borel subgroup ${}^LB^0$. The "standard relevant parabolics" ${}^LP$, in the sense of [**2**], correspond to the standard parabolics $P \supseteq P_0$ of G. Let ${}^L(MA)$ be the Levi component of ${}^LP$. According to [**12**], the classes of L-indistinguishable irreducible admissible representations of G correspond to conjugacy classes (suitably defined) of certain kinds of homomorphisms φ of the Weil group into ${}^LG$. A representation is tempered if and only if the image of φ is bounded. It comes from the series of §2 for P if, after conjugation, φ has image in the Levi component ${}^L(MA)$ of ${}^LP$ but not in any smaller such Levi component. We may and shall assume that $\varphi(C^\times) \subseteq {}^LT^0$ and $\varphi(\tau)$ is in the normalizer of ${}^LT^0$. We shall give an example of these notions at the end of this section, illustrating the three theorems below.

Fix a summand in a representation $U_P(\xi, \nu)$. The corresponding φ goes with a whole L-class of representations in G. Since the image of φ is in ${}^L(MA)$, the L-class consists of those representations obtained by inducing from a single L-class for MA. The latter class corresponds with the map φ^{MA} obtained by regarding φ as having image in ${}^L(MA)$, and it is characterized by the data ν, the central character of ξ, and the infinitesimal character of ξ. But the definitions of W, W', and R depend only on ν, the central character of ξ, and the infinitesimal character of ξ. Thus the question arises how to define W, W', and R in terms of φ directly.

[*Digression.* A simple consequence of the discussion above and a lemma due to D. Shelstad is that the cardinalities of R and the L-classes corresponding to φ and φ^{MA} are related by the formula

$$|L\text{-class for } \varphi| = |R|\,|L\text{-class for } \varphi^{MA}|.$$

In fact, the representations in the L-class for φ are all the irreducible constituents of all representations $U_P(\xi', \nu)$ with $\xi' \otimes \nu$ in the L-class for φ^{MA}. The R-group for all these $U_P(\xi', \nu)$ is the same as ξ' varies. Moreover, Shelstad's lemma says that $U_P(\xi_1, \nu)$ and $U_P(\xi_2, \nu)$ are disjoint if $\xi_1 \otimes \nu$ and $\xi_2 \otimes \nu$ are L-equivalent but not equivalent. The formula follows.]

Let $S = \text{Cent(Image } \varphi)$, the centralizer being taken in ${}^LG^0$, and let S^0 be the identity component of S and $\mathfrak{s}$ its Lie algebra.

The Weyl group $W(\mathfrak{a})$ imbeds in the Weyl group of $({}^LG^0, {}^LT^0)$. Namely any element of $W(\mathfrak{a})$ has a representative that normalizes the Cartan subalgebra of $\mathfrak{m}$ and preserves positive roots of $\mathfrak{m}$; in this way an element of $W(\mathfrak{a})$ leads to a unique member of the Weyl group of $({}^LG^0, {}^LT^0)$. With this identification in mind, we have the following result.

THEOREM 3.1 (LANGLANDS). $W = (\text{Norm}({}^LT^0) \cap S)/({}^LT^0 \cap S)$, *the normalizer being taken in* ${}^LG^0$.

We denote the lattices that are given as part of the L-group data by L and $L^\vee$:

$$L = \text{Hom}\,(T_C, C^\times) \cong \text{Hom}\,(C^\times, {}^LT^0),$$
$$L^\vee = \text{Hom}\,({}^LT^0, C^\times) \cong \text{Hom}\,(C^\times, T_C).$$

These are in duality as follows: If λ is in $L = \text{Hom}(C^\times, {}^LT^0)$ and $\lambda^\vee$ is in $L^\vee = \text{Hom}({}^LT^0, C^\times)$, then $\lambda^\vee \circ \lambda$ is a power of z in $C^\times$, and $\langle \lambda, \lambda^\vee \rangle$ denotes this power. The homomorphism φ on $C^\times$, with values in ${}^LT^0$, can be written symbolically as

$$\varphi(z) = z^\mu \bar{z}^\nu = z^{\mu-\nu}(z\bar{z})^\nu, \tag{3.1}$$

where μ and ν are in $L \otimes C$ and $\mu - \nu$ is in L. Formula (3.1) means that

$$\lambda^\vee(\varphi(z)) = z^{\langle \mu-\nu, \lambda^\vee \rangle}(z\bar{z})^{\langle \nu, \lambda^\vee \rangle}$$

for all $\lambda^\vee$ in $L^\vee$.

The condition that the image of φ lies in no proper parabolic within P means that $\varphi(\tau)\,\alpha^\vee = -\alpha^\vee$ exactly for the roots of ${}^L(MA)$ and that $\langle \mu, \alpha^\vee \rangle \neq 0$ for all roots of ${}^L(MA)$. (See Lemma 3.3 and the paragraph before Lemma 3.1 in [**12**].)

Let ${}^L\mathfrak{t}$ be the Lie algebra of ${}^LT$ and decompose ${}^L\mathfrak{t}$ into the $+1$ and -1 eigenspaces for $\varphi(\tau)$ as ${}^L\mathfrak{t} = {}^L\mathfrak{t}^+ \oplus {}^L\mathfrak{t}^-$.

LEMMA 3.2 (LANGLANDS). *The Lie algebra ${}^L\mathfrak{t}^+$, which lies in $\mathfrak{S}$, is a Cartan subalgebra of $\mathfrak{S}$.*

PROOF. If X is in $\mathfrak{S}$, then the condition that $X^{\varphi(\tau)} = X$ means that X is the sum of a member of ${}^L\mathfrak{t}^+$ and a sum Y of root vectors. Moreover, Y must centralize $\varphi(C^\times)$. If $Y = \sum X_{\alpha^\vee}$, then

$$\sum Y_{\alpha^\vee} = Y = Y^{\varphi(z)} = \sum z^{\langle\mu-\nu,\alpha^\vee\rangle}(z\bar{z})^{\langle\nu,\alpha^\vee\rangle}\, Y_{\alpha^\vee}.$$

Hence each $\alpha^\vee$ with $Y_{\alpha^\vee} \neq 0$ satisfies $\langle\mu, \alpha^\vee\rangle = \langle\nu, \alpha^\vee\rangle = 0$. Thus

$$\mathfrak{S} \subseteq {}^L\mathfrak{t}^+ + \sum_{\langle\mu,\alpha^\vee\rangle=\langle\nu,\alpha^\vee\rangle=0} CX_{\alpha^\vee}.$$

We are to prove that ${}^L\mathfrak{t}^+$ is maximal abelian in $\mathfrak{S}$. It is enough to see that no $X_{\alpha^\vee}$ with $\langle\mu, \alpha^\vee\rangle = \langle\nu, \alpha^\vee\rangle = 0$ centralizes ${}^L\mathfrak{t}^+$, i.e., that no such $\alpha^\vee$ vanishes on ${}^L\mathfrak{t}^+$. However, $\langle\mu, \alpha^\vee\rangle = 0$ implies $\alpha^\vee$ is not a root of ${}^L(MA)$, as we noted above, and then $\varphi(\tau)\,\alpha^\vee \neq -\alpha^\vee$. Thus $\alpha^\vee$ does not vanish on ${}^L\mathfrak{t}^+$.

THEOREM 3.3 (LANGLANDS). *The group W' is canonically isomorphic with the Weyl group of $(\mathfrak{S}, {}^L\mathfrak{t}^+)$.*

THEOREM 3.4 (LANGLANDS). *Suppose G_C is semisimple and simply-connected. Then the group $F = ({}^LT^0 \cap S)/({}^LT^0 \cap S^0)$ injects into S/S^0, and the quotient is isomorphic to the R group. Moreover, $F = \{1\}$ if G is a split group and P is a minimal parabolic.*

EXAMPLE. Let G be $SU(2, 1)$, which we conjugate for convenience by

$$g = \begin{pmatrix} 2^{-1/2} & 0 & 2^{-1/2} \\ 0 & 1 & 0 \\ -2^{-1/2} & 0 & 2^{-1/2} \end{pmatrix},$$

so as to be able to take

$$T = \begin{pmatrix} e^{t+i\theta} & & \\ & e^{-2i\theta} & \\ & & e^{t-i\theta} \end{pmatrix}.$$

The Borel subgroup B we use is the upper triangular group. The group G is quasi-split but not split, and the L-group ${}^LG$ is

$${}^LG = {}^LG^0 \rtimes Z_2 = \mathrm{PGL}(3, C) \rtimes \{1, \sigma\},$$

where σ is a particular realization of the nontrivial outer automorphism of $\mathrm{PGL}(3, C)$. Specifically ${}^LT^0$ is the diagonal group, ${}^LT$ is ${}^LT^0 \rtimes \{1, \sigma\}$, and ${}^LB^0$ is the upper triangular group. We are led to the standard simple co-roots and use standard root vectors in $\mathrm{PGL}(3, C)$; the root vectors for the simple co-roots are

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad\text{and}\quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and we require $X_1^\sigma = X_2$, $X_2^\sigma = X_1$. Explicitly the action of σ on ${}^LG^0$ is $x^\sigma = w_0(x^{\mathrm{tr}})^{-1}w_0^{-1}$, where

$$w_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and $\sigma^2 = 1$. On the Lie algebra ${}^L\mathfrak{t}$, we have $\mathrm{diag}(c_1, c_2, c_3)^\sigma = (-c_3, -c_2, -c_1)$.

We assume that the homomorphism φ maps $\boldsymbol{C}^\times$ into ${}^LT^0$ and maps τ into $w \times \sigma$, where w is in $\mathrm{Norm}({}^LT^0)$. Up to conjugation of φ, there are two possibilities for w as a member of $\mathrm{Norm}({}^LT^0)/{}^LT^0$, namely the cosets of 1 and w_0. In either case, $\varphi(\tau)$ yields an involution of ${}^L\mathfrak{t}$, with $+1$ and -1 eigenspaces ${}^L\mathfrak{t} = {}^L\mathfrak{t}^+ \oplus {}^L\mathfrak{t}^-$. If $[w] = [w_0]$, then ${}^L\mathfrak{t}^+ = 0$ and we are led to discrete series. We shall be interested in the contrary case $[w] = [1]$. Say $w = x^\vee$ with $x^\vee$ in ${}^LT^0$. The action of $\varphi(\tau)$ on diagonal matrices ${}^LT^0$ is in this case the same as the action of σ.

We write members of $L = \mathrm{Hom}(T_{\boldsymbol{C}}, \boldsymbol{C}^\times)$ by abbreviating

$$\lambda(\mathrm{diag}(z, z^{-1}w^{-1}, w)) = z^a w^b$$

as

$$\lambda = a[z] + b[w].$$

Here $\{[z], [w]\}$ is a basis of L. Let $\{\delta_z, \delta_w\}$ denote the dual basis of $L^\vee = \mathrm{Hom}({}^LT^0, \boldsymbol{C}^\times)$. If

$$\lambda^\vee(\mathrm{diag}(a_1, a_2, a_3) \bmod \text{center}) = (a_1/a_2)^c\,(a_3/a_2)^d, \tag{3.2}$$

then

$$\lambda^\vee = c\delta_z + d\delta_w.$$

The action of $\varphi(\tau)$ on ${}^LT^0$ yields an action on $L^\vee$ and then one on L, by duality, namely

$$(a[z] + b[w])^{\varphi(\tau)} = -b[z] - a[w].$$

Let $\varphi(z)$ be as in (3.1), for z in $\boldsymbol{C}^\times$. Since $\bar{z} = \tau z \tau^{-1}$ is in the Weil group, we obtain $\varphi(\bar{z}) = \varphi(z)^{\varphi(\tau)}$, and then (3.1) implies $\nu = \mu^{\varphi(\tau)}$. (Here the action of $\varphi(\tau)$ on $L \otimes \boldsymbol{C}$ occurs only in the L part.)

Write

$$\mu = c_1[z] + c_2[w] \quad \text{with } c_1, c_2 \text{ in } \boldsymbol{C};$$
$$\nu = \mu^{\varphi(\tau)} = -c_2[z] - c_1[w].$$

The condition that $\mu - \nu$ be in L says that $c_1 + c_2$ is in $\boldsymbol{Z}$. From p. 27 of **[12]**, we see that the character of T that is to provide data for a principal series representation is in this case

$$e^H \to e^{\langle \mu, H + \sigma\bar{H}\rangle/2} = e^{\langle \mu, H\rangle}.$$

If $H = \mathrm{diag}(t + i\theta, -2i\theta, t - i\theta)$, we are led to the character

$$\mathrm{diag}(e^t e^{i\theta}, e^{-2i\theta}, e^{t-i\theta}) \to e^{t(c_1 - c_2)}\, e^{i\theta(c_1 + c_2)}.$$

If $\mathrm{Re}(c_1 - c_2) > 0$, we are led to nontempered representations. We shall specialize to the principal series with $c_1 - c_2 = 0$, which corresponds to $e^{in\theta}$ on the M part and trivial on the A part, by taking $\mu = n[z]/2 + n[w]/2$. These principal series representations are reducible (splitting into two parts) if and only if n is even and $\neq 0$. To see what $\varphi(z)$ is, we realize ${}^LT^0$ as

$${}^LT^0 = \operatorname{diag}(a_1, a_2, a_3) \text{ mod center.}$$

Apply (3.2) to the element $\varphi(z)$ with $c = 1$, $d = 0$, and then with $c = 0$, $d = 1$. Then

$$\begin{aligned}(a_1/a_2)(z) = \delta_z\varphi(z) &= z^{\langle \mu-\nu, \delta_z\rangle}(z\bar{z})^{\langle \nu, \delta_z\rangle} \\ &= z^{\langle 2\mu, \delta_z\rangle}(z\bar{z})^{\langle -\mu, \delta_z\rangle} = z^n/|z|^n\end{aligned}$$

and

$$(a_3/a_2)(z) = z^n/|z|^n.$$

Hence

$$\varphi(z) = \begin{pmatrix} z^n/|z|^n & 0 & 0 \\ 0 & (z^n/|z|^n)^{-2} & 0 \\ 0 & 0 & z^n/|z|^n \end{pmatrix}^{1/3} \quad \text{mod center,}$$

in the sense that if the cube roots are extracted compatibly, their ambiguity disappears when we pass to $PGL(3, C)$.

To compute the image of φ completely, there is one more step—to determine $\varphi(\tau) = x^\vee \times \sigma$. We must have $\varphi(\tau)^2 = \varphi(-1)$, and we conclude $(x^\vee)(x^\vee)^\sigma = \varphi(-1)$. We can take

$$\begin{aligned} x^\vee &= 1 && \text{if } n \text{ is even,} \\ &= \operatorname{diag}(i, 1, -i) && \text{if } n \text{ is odd.}\end{aligned}$$

Now we can compute the groups S and S^0 that are the subjects of the theorems of this section. There are three distinct cases for the image of φ, corresponding to $n = 0$, n odd, and n nonzero even. The pattern of the computation is to compute $S_1 = \text{Cent}\,\varphi(C^\times)$, which is connected, being the centralizer of a torus, and then to compute the centralizer of $\varphi(\tau)$ in S_1. The idea is to work as much as possible on the Lie algebra level. For $n = 0$ and n odd, we use a trick to obtain S; S is contained in the normalizer in ${}^LG^0$ of $\mathfrak{s}$, and $\mathfrak{s}$ is found to be $\mathfrak{sl}(2, C)$. Since $\mathfrak{sl}(2, C)$ has no outer automorphisms, S must conjugate S^0 by inner automorphisms. Then it is easy to see that $S = S^0$. The results are as follows.

$$n = 0: \qquad \mathfrak{s} = \begin{pmatrix} a & b & 0 \\ c & 0 & b \\ 0 & c & -a \end{pmatrix}, \quad S = S^0,$$

$$n \text{ odd}: \qquad \mathfrak{s} = \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ c & 0 & -a \end{pmatrix}, \quad S = S^0,$$

$$n \text{ even} \neq 0: \quad \mathfrak{s} = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad S = S^0 \rtimes \{1, w_0\}.$$

These results are compatible with the theorems of this section and the facts that

$$n = 0 \text{ or odd}: \quad W' = \boldsymbol{Z}_2, \quad R = \{1\},$$

$$n \text{ even} \neq 0: \quad W' = \{1\}, \quad R = \boldsymbol{Z}_2.$$

4. Irreducible tempered representations. We continue to investigate the representations $U_P(\xi, \nu)$ discussed in §2. Here ξ is a discrete series representation of M and ν is imaginary on $\mathfrak{A}$. We need a parametrization for the discrete series, and we use Harish-Chandra's. Let T^- be a compact Cartan subgroup of M, let $\mathfrak{t}^-$ be its Lie algebra, and let Z_M be the center of M. Each discrete series representation of M is determined by a nonsingular linear form λ on $i\mathfrak{t}^-$ and a character χ on Z_M. The form λ satisfies the integrality condition that $\lambda - \rho$ (with ρ equal to half the sum of the positive roots of M in some order) lifts to a character $e^{\lambda-\rho}$ on T^-, and λ and χ satisfy the compatibility condition that $e^{\lambda-\rho}$ and χ agree on their common domain. We write $\xi = \xi(\lambda, C, \chi)$, where C is the unique Weyl chamber of $i\mathfrak{t}^-$ with respect to which λ is dominant. Two such ξ's are equivalent if and only if $\chi = \chi'$ and there is some w in the Weyl group $W(T^-, M) = \mathrm{Norm}_M(T^-)/\mathrm{Cent}_M(T^-)$ with $w\lambda = \lambda'$ and $wC = C'$.

Representations that are "limits of discrete series" are discussed in **[18]**. We can parametrize them in the same fashion, writing $\xi = \xi(\lambda, C, \chi)$, except that λ is allowed to be singular and there is more than one choice of C that makes λ dominant. These representations are irreducible or zero, and the criterion for equivalence of nonzero ones is the same as for discrete series.

A *basic representation* is an induced representation $U_P(\xi(\lambda, C, \chi), \nu)$, with $\xi(\lambda, C, \chi)$ a limit of discrete series on M. If λ is nonsingular, so that ξ is in the discrete series, then we say that the basic representation is *induced from discrete series*. A basic representation $U_P(\xi(\lambda, C, \chi), \nu)$, has *nondegenerate data* if for each root α of $(\mathfrak{m}, \mathfrak{t}^-)$, $\langle\lambda, \alpha\rangle = 0$ implies that s_α is not in $W(T^-: M)$. A representation induced from discrete series automatically has nondegenerate data.

Nondegeneracy accomplishes several things. For one thing, it eliminates the 0 representation from consideration. For a second thing, it assigns a definite parabolic to the data for a basic representation. For example, in $\mathrm{SL}^{\pm}(2, \boldsymbol{R})$, there is a single limit of discrete series representation, and it imbeds as a full principal series. The nondegeneracy assumption requires that this representation be viewed in the principal series. In general, it requires that a basic representation be attached to as small a parabolic as possible. A third thing that nondegeneracy accomplishes is to allow the whole discussion of W, W', Δ', and R in §2 to extend to basic representations with nondegenerate data. The theorem from **[8]**, **[9]** is as follows.

THEOREM 4.1. (a) *A basic representation with nondegenerate data is necessarily tempered, and it is irreducible if and only if its R-group is trivial.*

(b) *Two irreducible basic representations $U_P(\xi(\lambda, C, \chi), \nu)$ and $U_{P'}(\xi(\lambda', C', \chi'), \nu')$ with nondegenerate data are equivalent if and only if there is an element w in G carrying M to M', A to A', $\mathfrak{t}$ to $\mathfrak{t}'$, and (λ, C, χ, ν) to $(\lambda', C', \chi', \nu')$.*

(c) *Every irreducible tempered representation is basic and can be written with nondegenerate data.*

The R-group can be used to point to those basic representations with nondegenerate data that are needed to exhibit the reducibility of a representation induced from discrete series. The constituents of a given representation induced from discrete series, or even of all induced from an L-class of discrete series, are L-indistinguishable. A precise description of how this reducibility may be read off from the data, from the R-group, and from the set of orthogonal roots in Theorem 2.3(c) is given in [**8**].

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