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COMMUTATIVITY OF INTERTWINING OPERATORS FOR SEMISIMPLE GROUPS

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To Susan

In an earlier paper [13], E.M. Stein and the author developed a theory of intertwining operators for representations of real reductive groups G induced from parabolic subgroups. In part of that paper, we dealt specifically with the major unitary representations that contribute to the Plancherel formula for G and determined their irreducibility/reducibility in terms of a finite group known as the R group.

The present paper makes a detailed study of the R group, concluding with a structure theorem for the commuting algebra for each of these major unitary representations. The theorem shows in particular that each such representation splits into the direct sum of inequivalent irreducible representations. However, our results will be obtained only for a more limited class of groups G than are the subject of [13]. The limited class includes all linear connected semisimple groups. Each group G in the limited class is assumed to have a faithful matrix representation and to have some other properties that restrict the disconnectedness of G; precise axioms are given in §1. An example due to Vogan [16], discussed further below, shows that some such limitation is necessary.

We introduce a minimum of notation needed to give a precise statement of the main theorem. Let G be a linear reductive group satisfying the axioms of §1, let K be a maximal compact subgroup, and let MAN be the Langlands decomposition of a parabolic subgroup of G such that M has a discrete series. To each discrete series

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representation ξ of M and each imaginary-valued linear functional Λ on the Lie algebra of A, we associate the induced representation of G given by

$$U(\xi, \Lambda, \cdot) = \operatorname*{ind}_{MAN \uparrow G} (\xi \otimes (\exp \Lambda) \otimes 1).$$

Let $W_{\xi,\Lambda}$ be the subgroup of the Weyl group of A of elements that fix Λ and the class of ξ. To each element w of $W_{\xi,\Lambda}$, §8 of [13] associates a unitary self-intertwining operator $\xi(w)\mathcal{A}(w, \xi, \Lambda)$ for $U(\xi, \Lambda, \cdot)$. These operators multiply according to the group law of $W_{\xi,\Lambda}$, except for scalar factors. Let $R_{\xi,\Lambda}$ be the subgroup of $W_{\xi,\Lambda}$ defined in §6 below or in §13 of [13]. Then the operators associated just with the elements of $R_{\xi,\Lambda}$ form a linear basis of the commuting algebra of $U(\xi, \Lambda, \cdot)$, by Theorem 13.4 of [13].

MAIN THEOREM: *The group $R_{\xi,\Lambda}$ is the direct sum of a number r of copies of $\mathbb{Z}_2$ with $r \leq \dim A$, and the operators $\xi(w)\mathcal{A}(w, \xi, \Lambda)$ associated to the elements of $R_{\xi,\Lambda}$ commute with one another. In particular, the commuting algebra of $U(\xi, \Lambda, \cdot)$ is commutative and its dimension is a power of two; therefore, $U(\xi, \Lambda, \cdot)$ decomposes into the direct sum of inequivalent irreducible representations.*

This theorem has two parts – the fact that $R = \Sigma\, \mathbb{Z}_2$ (given below as Theorem 6.1) and the commutativity of the operators (given below as Theorem 7.1). Vogan's example shows that the second conclusion is not automatic from the first, even for linear groups: Let G be the semidirect product of $SL(2, \mathbb{R}) \oplus SL(2, \mathbb{R})$ by the eight-element quaternion group H_8 (of standard basis elements of the real quaternions, together with their negatives), where i operates on the first $SL(2, \mathbb{R})$ by conjugation by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and j operates on the second $SL(2, \mathbb{R})$ by conjugation by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. For the minimal parabolic subgroup, M is $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus H_8$, and we use $\Lambda = 0$ and $\xi = \operatorname{sgn} \otimes \operatorname{sgn} \otimes \sigma$ with σ an irreducible two-dimensional representation of H_8. Then $R_{\xi,\Lambda} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, but the commuting algebra of $U(\xi, \Lambda, \cdot)$ is isomorphic with a full 2-by-2 matrix algebra.

The trouble encountered in Vogan's example is that the disconnectedness of G is too wild. In §1 we introduce axioms for a hereditary class of linear reductive groups G whose disconnectedness is more limited. A key property of such groups, not shared by Vogan's example, is given in Lemma 4.4 and is used critically in Lemma 7.6.

The main difficulty in proving that $R = \Sigma\, \mathbb{Z}_2$ is in understanding the subgroup of the Weyl group that fixes ξ. In §§2–3 we make the necessary detailed analysis of how the Weyl group of A acts on a compact Cartan subalgebra of the Lie algebra of M; this analysis is of independent interest and centers about Theorem 3.7. The proof that $R = \Sigma\, \mathbb{Z}_2$ begins in earnest in §4.

Once one knows that $R = \Sigma\, \mathbb{Z}_2$, it is not too difficult to deduce the commutativity of the operators. The proof that we give here of Theorem 7.1 departs from the announced proof ([9] and [10]) and is shorter than the announced proof. It uses the concept of 'superorthogonality of roots' introduced by Gregg Zuckerman and the author in joint work.

In the case that MAN is a minimal parabolic subgroup, parts of the paper simplify considerably: The results of §2 reduce to easy facts in [11], part of §4 is not needed, and §5 is almost completely unnecessary.

The results of this paper were announced in [10]. In the case that MAN is minimal parabolic, they had been obtained earlier, and brief sketches of proofs had been given in [8] and [9]. The press of other matters has delayed publication of complete proofs until now.

§1. Assumptions on G

The groups G in this paper are real Lie groups *of matrices* satisfying the following axioms:

(i) The identity component G_0 of G has a reductive Lie algebra $\mathfrak{g}$.
(ii) G_0 has compact center.
(iii) G has finitely many components.
(iv) If $G^{\mathbb{C}}$ denotes the analytic group of matrices with Lie algebra the complexification $\mathfrak{g}^{\mathbb{C}}$ and if $Z(G)$ denotes the centralizer of G in the total general linear group of matrices, then $G \subseteq G^{\mathbb{C}} \cdot Z(G)$.

From (iv) it follows that each $Ad(g)$, for g in G, is in $Ad(G^{\mathbb{C}})$; this latter statement is the 4th axiom in [13]. Thus the present axioms are a specialization of those in [13], which in turn are a specialization of those of Harish-Chandra in [4]. All finite groups satisfy the axioms of [13], whereas the only finite groups that satisfy (iv) above are the abelian ones.

Since G satisfies the axioms of [4] or [13], all of the basic notation of [13] and group decompositions of [4] make sense. The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is defined relative to a Cartan involution θ, and $\mathfrak{m}_p \oplus \mathfrak{a}_p \oplus \mathfrak{n}_p$ and $M_pA_pN_p$ refer to minimal parabolics constructed in the standard way from the Cartan decomposition. If $\mathfrak{b}_0$ denotes a maximal abelian subspace of $\mathfrak{m}_p$, then $\mathfrak{a}_p \oplus \mathfrak{b}_0$ is a Cartan subalgebra of $\mathfrak{g}$. Roots relative to this Cartan subalgebra are real on $\mathfrak{a}_p + i\mathfrak{b}_0$. The $\mathfrak{a}_p$-roots (roots relative to $\mathfrak{a}_p$) are the restrictions to $\mathfrak{a}_p$ of such roots. If α is an $\mathfrak{a}_p$-root, the root space for α with $\mathfrak{g}$ is denoted $\mathfrak{g}_\alpha$. We assume given an $Ad(G)$-invariant, θ-invariant, nondegenerate, symmetric bilinear form B on $\mathfrak{g} \times \mathfrak{g}$ such that $B_\theta(X, Y) = -B(X, \theta Y)$ is positive definite, and from B we can construct in the standard way an inner product $\langle \cdot, \cdot \rangle$ on the dual space $(\mathfrak{a}_p + i\mathfrak{b}_0)'$.

As a consequence of the axioms of [4] or [13], we have $G = G_0M_p$, i.e., every component of G meets the compact group M_p. Further properties and notation are given in [13].

LEMMA 1.1: *If z is in $Z_G(G_0)$, the centralizer of G_0 in G, then z is in Z_G, the center of G.*

PROOF: Let g be in G and write $g = x\zeta$ by (iv), with x in G^{C} and ζ in $Z(G)$. Then z commutes with x since $Ad(z) = 1$, and z commutes with ζ since ζ is in $Z(G)$. Hence z commutes with g.

Define

$$F = Z_{M_p} \cap Z(G) \exp i\mathfrak{a}_p.$$

The group F is compact and abelian, often finite.

LEMMA 1.2: *$M_p = (M_p)_0F$, where $(M_p)_0$ is the identity component of M_p.*

PROOF: Let m in M_p be given. Then $\mathfrak{b}_0$ and $Ad(m)\mathfrak{b}_0$ are two maximal abelian subspaces of $\mathfrak{m}_p$ and are conjugate by $Ad((M_p)_0)$. Adjusting m by a member of $(M_p)_0$ and changing notation, we may assume that $Ad(m)\mathfrak{b}_0 = \mathfrak{b}_0$. Since the Weyl group of $(M_p)_0$ is transitive on the Weyl chambers, we may, after introducing an ordering, assume $Ad(m)$ preserves the set of positive roots of $(M_p)_0$. Now $Ad(m)$ leaves $\mathfrak{a}_p$ pointwise fixed, and we may assume that $(\mathfrak{a}_p + i\mathfrak{b}_0)'$ is ordered with $\mathfrak{a}_p'$ before $(i\mathfrak{b}_0)'$. Then $Ad(m)$ normalizes $\mathfrak{a}_p + i\mathfrak{b}_0$ and preserves the set of positive roots. Write $m = gz$ by (iv) with g in G^{C}, z in $Z(G)$. Then $Ad(g) = Ad(m)$ normalizes $\mathfrak{a}_p + i\mathfrak{b}_0$ and preserves the set of positive roots, so that $g = \exp H$ with H in $(\mathfrak{a}_p + i\mathfrak{b}_0)^{\mathbb{C}}$. Since m is in M_p, $Ad(g)$ has to be unitary on $\mathfrak{g}^{\mathbb{C}}$, and thus $ad\,H$ has only imaginary eigenvalues. Consequently $H = H_1 + H_2$ with H_1 in $\mathfrak{b}_0$ and H_2 in $i\mathfrak{a}_p$. Here $\exp H_1$ is in $(M_p)_0$. Thus m, after a left multiplication by a member of $(M_p)_0$, may be assumed to be of the form $(\exp H_2)z$ with H_2 in $i\mathfrak{a}_p$ and z in $Z(G)$. This m is in M_p, clearly commutes with M_p, and is exhibited as in $Z(G)\exp i\mathfrak{a}_p$; hence it is in F. The lemma follows.

Let MAN be a parabolic subgroup containing a minimal parabolic subgroup built from M_pA_p. The Lie algebra is written $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, and we have $M \supseteq M_p$ and $\mathfrak{a} \subseteq \mathfrak{a}_p$. M satisfies the axioms of [13]. We write $\mathfrak{a}_p = \mathfrak{a} \oplus \mathfrak{a}_M$, orthogonal sum. Then $\mathfrak{a}_M$ plays the same role for M that $\mathfrak{a}_p$ does for G, and it is shown in §1 of [13] that the M_p group for M is the same as the M_p group for G.

LEMMA 1.3: *M satisfies the present axioms, and $M = M_0F$, where M_0 is the identity component of M.*

REMARK: F is central in M_p but often not central in M. This circumstance is responsible for certain complications in §4 below.

PROOF: Since M is linear and satisfies the axioms of [13], only axiom (iv) is at issue. Since M satisfies the axioms of [13] and its M_p group is the same as for G, we have $M = M_0M_p$. Then Lemma 1.2 gives

$$M = M_0M_p = M_0(M_p)_0F = M_0F.$$

Now M_0 is contained in $M^{\mathbb{C}}$, and F satisfies

$$F \subseteq (\exp i\mathfrak{a}_p)Z(G) = (\exp i\mathfrak{a}_M)(\exp i\mathfrak{a})Z(G)$$
$$\subseteq M^{\mathbb{C}}(\exp i\mathfrak{a})Z(G) \subseteq M^{\mathbb{C}}Z(M).$$

Therefore M satisfies axiom (iv).

Roots relative to $\mathfrak{a}$ are restrictions to $\mathfrak{a}$ of $\mathfrak{a}_p$-roots. The root space for an $\mathfrak{a}$-root ϵ is denoted $\mathfrak{g}_\epsilon$. If H_ϵ is the member of $\mathfrak{a}$ dual to ϵ, then

$$\mathfrak{g}^{(\epsilon)} = \mathfrak{m} + \mathbb{R}H_\epsilon + \sum_{c \neq 0} \mathfrak{g}_{c\epsilon}$$

is a reductive Lie algebra, and one can show that the group $G^{(\epsilon)} =$

$MG_0^{(\epsilon)}$ satisfies the axioms. An $\mathfrak{a}$-root ϵ is called *reduced* if $c\epsilon$ is not an $\mathfrak{a}$-root for $0 < c < 1$.

We define Weyl groups $W(\mathfrak{a}_p)$, $W(\mathfrak{a})$, and $W(\mathfrak{a}_M)$ as the obvious normalizers-in-K-divided-by-centralizers. Members of $W(\mathfrak{a})$ always have extensions to members of $W(\mathfrak{a}_p)$; cf. Lemma 8 of [11].

Now suppose, as will henceforth be the case in this paper, that $\mathfrak{m}$ has a compact Cartan subalgebra (which can be assumed to be in $\mathfrak{k} \cap \mathfrak{m}$). We shall construct a compact Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{m}$ with the property that F normalizes $\mathfrak{b}$ (see Proposition 4.5). First we remark that $\mathfrak{a} + \mathfrak{b}$ will have to be a Cartan subalgebra of $\mathfrak{g}$, and roots relative to it will be real on $\mathfrak{a} + i\mathfrak{b}$. Since any two Cartan subalgebras of $\mathfrak{g}^{\mathbb{C}}$ are conjugate via $G^{\mathbb{C}}$, we will be able to find a member $\mathbf{c}$ of $Ad(G^{\mathbb{C}})$ with

$$\mathbf{c}(\mathfrak{a}_p + i\mathfrak{h}_0) = \mathfrak{a} + i\mathfrak{b}.$$

It is this map $\mathbf{c}$, the Cayley transform, that we construct. By Lemma 4 of [11], we can find an orthogonal system $\delta_1, \ldots, \delta_n$ of roots of $(\mathfrak{m}^{\mathbb{C}}, (\mathfrak{a}_M + i\mathfrak{h}_0)^{\mathbb{C}})$ that vanish on $\mathfrak{h}_0$ and span $\mathfrak{a}_M'$; such a system may be constructed so as to be strongly orthogonal (no $\delta_i \pm \delta_j$ is a root). For each δ_j fix a root vector X_j in $\mathfrak{m}$ so that $[X_j, \theta X_j] = -2|\delta_j|^{-2} H_{\delta_j}$. Set $\mathbf{c}_j = \exp \frac{\pi i}{4}(X_j - \theta X_j)$. The various $\mathbf{c}_j$ commute, and then $\mathbf{c} = \Pi_{j=1}^n \mathbf{c}_j$ has the required properties. Note that $\mathbf{c}$ leaves $\mathfrak{a}$ and $\mathfrak{h}_0$ pointwise fixed.

§2. Odd and even roots of $\mathfrak{a}$

Form the decomposition into $M_pA_pN_p$ of a minimal parabolic subgroup of G, compatibility with the Cartan decomposition of $\mathfrak{g}$. Fix a parabolic subgroup MAN containing $M_pA_pN_p$, and let notations be as in §1. We shall assume that MAN is cuspidal, i.e., that $\mathfrak{m}$ has a compact Cartan subalgebra. Let $\mathfrak{b}$ be the compact Cartan subalgebra of $\mathfrak{m}$ constructed in §1, and let $\mathbf{c}\colon \mathfrak{a}_p + i\mathfrak{h}_0 \to \mathfrak{a} + i\mathfrak{b}$ be the Cayley transform.

We have $\mathfrak{a}_p = \mathfrak{a} \oplus \mathfrak{a}_M$, and $\mathfrak{a}$ dominates $\mathfrak{a}_M$ in the ordering. If α is an $\mathfrak{a}_p$-root, we shall often write $\alpha = \alpha_R + \alpha_I$ as the decomposition into the projections on $\mathfrak{a}'$ and $\mathfrak{a}_M'$. As in §3 of [11], the hypothesis that MAN is cuspidal implies that $\bar{\alpha} = \alpha_R - \alpha_I$ is also an $\mathfrak{a}_p$-root. Following [11], we say that α is a *useful* $\mathfrak{a}_p$-root if $2\langle \alpha, \bar{\alpha}\rangle / |\alpha|^2 \neq +1$. Whenever Proposition 10b of [11] applies, this notion depends only on α_R (i.e., it holds for all α_I or else for none); thus we can speak unambiguously of *useful* $\mathfrak{a}$-roots. In the remaining cases, we say α_R is *useful* if it is the restriction to $\mathfrak{a}$ of some useful $\mathfrak{a}_p$-root.

Proposition 10b applies except when $\mathfrak{g}$ has a factor that is some form of the exceptional group G_2. A group G_2 can arise as a complex group, with minimal parabolics as the only cuspidal ones, or as an $\mathbb{R}$-split group, with all parabolics cuspidal. All of these cases have all $\mathfrak{a}_p$-roots useful except for the two maximal parabolic subgroups in the $\mathbb{R}$-split G_2, in which case $\mathfrak{a}$ is built from one root and can be examined directly. When $\mathfrak{a}$ is built from the long root and α_R is the reduced $\mathfrak{a}$-root, then the positive $\mathfrak{a}$-roots are α_R and $2\alpha_R$, both of which are useful; however, there are two not useful $\mathfrak{a}_p$-roots that restrict to α_R. When $\mathfrak{a}$ is built from the short root and α_R is the reduced $\mathfrak{a}$-root, then the positive $\mathfrak{a}$-roots are α_R and $2\alpha_R$ and $3\alpha_R$, with α_R and $2\alpha_R$ useful and $3\alpha_R$ not useful; the only not useful positive $\mathfrak{a}_p$-roots are those restricting to $3\alpha_R$.

Proposition 10c of [11] says that, apart from G_2, if α_R is useful, so is $c\alpha_R$ for every $c \neq 0$. Moreover the only possible positive multiples of a reduced $\mathfrak{a}$-root that can be an $\mathfrak{a}$-root are $\{1, 2\}$, according to the Corollary to Proposition 12. These statements remain valid for G_2, with the exception of the two cases noted above.

By Proposition 12 of [11], the useful $\mathfrak{a}$-roots form a root system Δ in a subspace of $\mathfrak{a}$, and the Main Theorem of [11] says that the group $W(\mathfrak{a})$ is just the Weyl group of Δ. We shall use these results starting in §3. But first, we relate usefulness, multiplicities, and properties of roots relative to $\mathfrak{a} + i\mathfrak{b}$.

If $\alpha = \alpha_R + \alpha_I$ is the decomposition of an $\mathfrak{a}_p$-root according to $\mathfrak{a} \oplus \mathfrak{a}_M$, recall $\bar{\alpha}$ is defined as $\alpha_R - \alpha_I$. It is shown in the proof of Lemma 9 of [11] that this conjugation is implemented by a member of $W(\mathfrak{a}_p)$. Consequently if α is an $\mathfrak{a}_p$-root, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{\bar{\alpha}}$ have the same dimension.

Lemma 2.1: *The following conditions on an $\mathfrak{a}$-root α_R are equivalent*:

(i) *The multiplicity of α_R as an $\mathfrak{a}$-root is odd.*
(ii) *The multiplicity of α_R as an $\mathfrak{a}_p$-root (when extended by 0 on $\mathfrak{a}_M$) is odd.*
(iii) *α_R is a root of $\mathfrak{a} + i\mathfrak{b}$ when extended by 0 on $i\mathfrak{b}$.*

Proof: The root space for α_R as an $\mathfrak{a}$-root is the sum

$$\sum_{\alpha_I \in (\mathfrak{a}_M)'} \mathfrak{g}_{\alpha_R + \alpha_I} = \mathfrak{g}_{\alpha_R} + \sum_{\alpha_I > 0} (\mathfrak{g}_{\alpha_R + \alpha_I} \oplus \mathfrak{g}_{\alpha_R - \alpha_I}),$$

and the remarks above show that the sum $\Sigma_{\alpha_I > 0}(-)$ on the right is even-dimensional. Hence (i) and (ii) are equivalent. According to the remarks after Lemma 3 of [11], (ii) holds if and only if α_R is a root of $\mathfrak{a}_p + i\mathfrak{b}_0$ when extended by 0 on $i\mathfrak{b}_0$. Applying the Cayley transform $\mathbf{c}$, we see that (ii) and (iii) are equivalent.

We shall call an $\mathfrak{a}$-root α_R *odd* or *even* according as the dimension of $\Sigma_{c>0}\, \mathfrak{a}_{c\alpha_R}$ is odd or even.[1]

LEMMA 2.2: *The following three conditions on an $\mathfrak{a}$-root α_R are equivalent*:

(i) *α_R is odd.*
(ii) *Some multiple of α_R is a root of $\mathfrak{a} + i\mathfrak{b}$ when extended by 0 on $i\mathfrak{b}$.*
(iii) *The reductive Lie algebra $\mathfrak{g}^{(\alpha_R)}$ has a compact Cartan subalgebra.*

PROOF: If (i) holds, then dim $\mathfrak{g}_{c_0\alpha_R}$ is odd for some $c_0 > 0$, and then Lemma 2.1 applied to $c_0\alpha_R$ shows that (ii) holds. Conversely if (ii) holds, then $c_0\alpha_R$ is a root for some $c_0 > 0$. Moreover, $c\alpha_R$ is not a root for positive c other than c_0, since a nontrivial multiple of a root cannot be a root. Applying Lemma 2.1 to $c\alpha_R$ for each c, we obtain (i). For the equivalence of (ii) and (iii), it is well known that $\mathfrak{g}^{(\alpha_R)}$ has a compact Cartan subalgebra if and only if there exists a root of $\mathfrak{a} + i\mathfrak{b}$ that takes on only real values, i.e., vanishes on $\mathfrak{b}$. Such a root must be a multiple of $\mathfrak{g}_R$, and the equivalence follows.

LEMMA 2.3: *If $\alpha_R \pm \alpha_I$ are not-useful roots of $\mathfrak{a}_p$, then $\alpha_R + \alpha_I + \gamma$ is not a root of $\mathfrak{a}_p + i\mathfrak{b}_0$ for any $\gamma \neq 0$ in $(i\mathfrak{b}_0)'$. Moreover, $2\alpha_I$ is a root of $\mathfrak{a}_p + i\mathfrak{b}_0$ but $2\alpha_I + \gamma$ is not a root of $\mathfrak{a}_p + i\mathfrak{b}_0$ for any $\gamma \neq 0$ in $(i\mathfrak{b}_0)'$.*

PROOF: First suppose that the roots in question are not in any G_2 factor. If $\alpha_R + \alpha_I + \gamma$ is a root, then $|\alpha_R + \alpha_I|^2 = |\gamma|^2$ by Lemma 2 of [11], and $\alpha_R + \alpha_I + \gamma$ and $\alpha_R - \alpha_I - \gamma$ are both roots because conjugation relative to $\mathfrak{a}$ carries roots to roots. Then

$$|\alpha_R + \alpha_I + \gamma|^2 = 2|\alpha_R + \alpha_I|^2 = 4\langle \alpha_R + \alpha_I, \alpha_R - \alpha_I \rangle$$

[1] The nomenclature *inessential* and *essential* was used in [11] but will be abandoned now since it is misleading.

with the second equality holding since $\alpha_R + \alpha_I$ is not useful; hence

$$\frac{2\langle \alpha_R + \alpha_I + \gamma, \alpha_R - \alpha_I - \gamma \rangle}{|\alpha_R + \alpha_I + \gamma|^2} = \frac{-|\alpha_R + \alpha_I|^2}{2|\alpha_R + \alpha_I|^2} = -\frac{1}{2},$$

which is not an integer, contradiction. Thus $\alpha_R + \alpha_I + \gamma$ cannot be a root of $\mathfrak{a}_p + i\mathfrak{h}_0$.

Since $\alpha_R + \alpha_I$ is a root of $\mathfrak{a}_p$, the only remaining possibility is that $\alpha_R + \alpha_I$ itself is a root of $\mathfrak{a}_p + i\mathfrak{h}_0$. Similarly $\alpha_R - \alpha_I$ is a root, and their difference $2\alpha_I$ must be a root of $\mathfrak{a}_p + i\mathfrak{h}_0$. If also $2\alpha_I + \gamma$ is a root, then the sum

$$\alpha_R + \alpha_I + \gamma = (\alpha_R - \alpha_I) + (2\alpha_I + \gamma)$$

is a root, contradiction. Thus $2\alpha_I + \gamma$ is not a root of $\mathfrak{a}_p + i\mathfrak{h}_0$.

If the roots in question are in a G_2 factor, the G_2 must be split over $\mathbb{R}$. No roots in the factor then have a component in $(i\mathfrak{h}_0)'$. If $\alpha_R \pm \alpha_I$ are not useful, then $2\alpha_I$ is an $\mathfrak{a}_p$-root, hence a root of $\mathfrak{a}_p + i\mathfrak{h}_0$, and the rest of the lemma is vacuous.

LEMMA 2.4: *Let $\alpha_R \pm \beta$ be roots of $\mathfrak{a} + i\mathfrak{b}$ with $\alpha_R \neq 0$, $\beta \neq 0$, and $2\alpha_R$ not a root of $\mathfrak{a} + i\mathfrak{b}$. Then α_R is a useful $\mathfrak{a}$-root if and only 2β is not a root of $\mathfrak{a} + i\mathfrak{b}$.*

PROOF: For both directions of the proof, form the roots of $\mathfrak{a}_p + i\mathfrak{h}_0$ given by

$$\mathbf{c}^{-1}(\alpha_R \pm \beta) = \alpha_R \pm \mathbf{c}^{-1}(\beta) = \alpha_R \pm (\alpha_I + \gamma).$$

If α_R is not useful as an $\mathfrak{a}$-root, then $a_R \pm \mathfrak{a}_I$ are not useful as $\mathfrak{a}_p$-roots. Hence Lemma 2.3 shows that $\gamma = 0$ and that $2\alpha_I$ is a root of $\mathfrak{a}_p + i\mathfrak{h}_0$. Thus $2\beta = \mathbf{c}(2\alpha_I)$ is a root of $\mathfrak{a} + i\mathfrak{b}$.

Conversely suppose that α_R is useful and that 2β is indeed a root. In the expression $\alpha_I + \gamma = \mathbf{c}^{-1}(\beta)$, we cannot have $\alpha_I = 0$, by Lemma 1 of [11]. Since $\alpha_R + \alpha_I$ has to be useful (even in G_2, under our hypotheses), the only possibility is that

$$|\alpha_R|^2 \leq |\alpha_I|^2. \tag{2.1}$$

By assumption the root string $\{2\beta, \alpha_R + \beta\}$ does not extend to $2\alpha_R$, and therefore $|\alpha_R + \beta|^2 \geq |2\beta|^2$, from which it follows that $|\alpha_R|^2 \geq 3|\beta|^2$.

Applying (2.1), we obtain

$$3|\beta|^2 \leq |\alpha_I|^2. \tag{2.2}$$

By Lemma 1 of [11] applied to $2(\alpha_I + \gamma)$, 4γ is not a root of $\mathfrak{a}_p + i\mathfrak{h}_0$, and thus

$$\begin{aligned} 0 &\geq \langle 2(\alpha_I + \gamma), -\theta(2(\alpha_I + \gamma))\rangle = 4\langle \alpha_I + \gamma, \alpha_I - \gamma\rangle \\ &= 4(|\alpha_I|^2 - |\gamma|^2) = 4(2|\alpha_I|^2 - |\beta|^2). \end{aligned}$$

Inequality (2.2) shows that the right side is >0, and we have a contradiction.

LEMMA 2.5: *If $a_R \pm \beta$ are roots of $\mathfrak{a} + i\mathfrak{b}$ with α_R useful and with $\alpha_R \neq 0$, $\beta \neq 0$, and $2\alpha_R$ not a root of $\mathfrak{a} + i\mathfrak{b}$, then $\langle \alpha_R + \beta, \alpha_R - \beta\rangle = 0$ and consequently $|\alpha_R|^2 = |\beta|^2$.*

PROOF: The difference 2β of $\alpha_R + \beta$ and $\alpha_R - \beta$ is not a root of $\mathfrak{a} + i\mathfrak{b}$ by Lemma 2.4, and the sum $2\alpha_R$ is not a root by assumption. Hence $\alpha_R + \beta$ and $\alpha_R - \beta$ are orthogonal. Since $|\alpha_R + \beta|^2 = |\alpha_R - \beta|^2$, it follows that $|\alpha_R|^2 = |\beta|^2$.

LEMMA 2.6: *If $\alpha_R \pm \beta$ are roots of $\mathfrak{a} + i\mathfrak{b}$ with $\alpha_R \neq 0$ and if $2\alpha_R$ is an $\mathfrak{a}$-root, then $2\alpha_R$ is a root of $\mathfrak{a} + i\mathfrak{b}$ when extended by 0 by $i\mathfrak{b}$.*

PROOF: Assume on the contrary that $2\alpha_R$ is not a root of $\mathfrak{a} + i\mathfrak{b}$. Proposition 10c of [11], together with an examination of the various G_2 cases, shows that α_R is useful, and hence we conclude from Lemma 2.5 that $|\alpha_R|^2 = |\beta|^2$. Choose $\gamma \neq 0$ in $(i\mathfrak{b})'$ so that $2\alpha_R + \gamma$ is a root of $\mathfrak{a} + i\mathfrak{b}$. Since $4\mathfrak{a}_R$ is not an $\mathfrak{a}$-root (corollary to Proposition 12 of [11]), Lemma 2.5 gives $|2\alpha_R|^2 = |\gamma|^2$. Form the inner product

$$\langle 2\alpha_R + \gamma, \alpha_R \pm \beta\rangle = 2|\alpha_R|^2 \pm \langle \beta, \gamma\rangle, \tag{2.3}$$

and without loss of generality choose the sign of β so that

$$2|\alpha_R|^2 + \langle \beta, \gamma\rangle \neq 0. \tag{2.4}$$

By the Schwarz inequality

$$1 \geq \frac{\langle \beta, \gamma\rangle^2}{|\beta|^2|\gamma|^2} = \frac{\langle \beta, \gamma\rangle^2}{|\alpha_R|^2 4|\alpha_R|^2}, \tag{2.5}$$

and thus $|\langle\beta, \gamma\rangle| \leq 2|\alpha_R|^2$, with equality if and only if $\beta = c\gamma$. From (2.4), we then have

$$2|\alpha_R|^2 + \langle\beta, \gamma\rangle > 0,$$

and we can conclude from (2.3) that

$$\alpha_R + (\gamma - \beta) = (2\alpha_R + \gamma) - (\alpha_R + \beta) \tag{2.6}$$

is a root of $\mathfrak{a} + i\mathfrak{b}$. Again we can apply Lemma 2.5, with the result that $|\alpha_R|^2 = |\gamma - \beta|^2$ or else $\gamma = \beta$. In the former case,

$$|\beta|^2 = |\alpha_R|^2 = |\gamma - \beta|^2 = |\gamma|^2 - 2\langle\beta, \gamma\rangle + |\beta|^2,$$

whence $2\langle\beta, \gamma\rangle = |\gamma|^2 = 4|\alpha_R|^2$ and equality holds in the Schwarz inequality (2.5). Thus $\beta = c\gamma$ in both cases.

If $c \neq 1$, then $|\gamma|^2 = 2\langle\beta, \gamma\rangle$ and $\beta = c\gamma$ says that $c = 1/2$ and $\gamma = 2\beta$. That is, $\alpha_R + \beta$ and $2(\alpha_R + \beta)$ are roots of $\mathfrak{a} + i\mathfrak{b}$, in contradiction to the fact that twice a root is not a root.

If $c = 1$, then α_R must be a root of $\mathfrak{a} + i\mathfrak{b}$ by (2.6); thus β is a root, and $2\alpha_R = (2\alpha_R + \beta) - \beta$ is a root, contradiction. This completes the proof.

LEMMA 2.7: *Let α_R be a reduced $\mathfrak{a}$-root.*

(a) *If α_R and $2\alpha_R$ are both $\mathfrak{a}$-roots, then α_R has even multiplicity as an $\mathfrak{a}$-root, $2\alpha_R$ has odd multiplicity as an $\mathfrak{a}$-root, $2\alpha_R$ is a root of $\mathfrak{a} + i\mathfrak{b}$ when extended by 0 on $i\mathfrak{b}$, α_R and $2\alpha_R$ are both useful, and α_R is odd. If also $3\alpha_R$ is an $\mathfrak{a}$-root, it has even multiplicity and is not useful.*

(b) *If α_R is odd and $2\alpha_R$ is not an $\mathfrak{a}$-root, then α_R is useful and, when extended by 0 on $i\mathfrak{b}$, is a root of $\mathfrak{a} + i\mathfrak{b}$.*

(c) *If α_R is not useful, then α_R is even.*

(d) *$W(\mathfrak{a})$ carries odd roots to odd roots and even roots to even roots.*

PROOF: In (a), Lemma 2.6 shows that $2\alpha_R$ is a root of $\mathfrak{a} + i\mathfrak{b}$. Since twice a root is not a root, $\mathfrak{a}_R$ cannot be a root. The conclusions about multiplicities of α_R and $2\alpha_R$ then follow from Lemma 2.1. If $3\alpha_R$ is not an $\mathfrak{a}$-root, then the rest of (a) follows immediately. If $3\alpha_R$ is an $\mathfrak{a}$-root, the roots in question lie in a split G_2 factor with $\mathfrak{a}$ built from a short root, and (a) follows by looking at this case directly.

For (b), Lemma 2.1 says α_R extends to be a root of $\mathfrak{a} + i\mathfrak{b}$, and this extension exhibits α_R as useful. Conclusion (c) follows from (a) and (b). Finally in (d), if w represents a member of $W(\mathfrak{a})$, then $Ad(w)$ carries the root space for α_R to the root space for $w\alpha_R$, and the conclusion follows.

LEMMA 2.8: *If α_R and α_R' are nonorthogonal useful roots of the same length, then α_R and α_R' are either both even or both odd.*

PROOF: We may assume that α_R and α_R' are linearly independent. Then the hypotheses imply that $p_{\alpha_R} p_{\alpha_R'} \alpha_R = \alpha_R'$, with the indicated root reflections existing in $W(\mathfrak{a})$ since α_R and α_R' are useful. The result therefore follows from Lemma 2.7d.

§3. Action of $W(\mathfrak{a})$ on compact Cartan subalgebra of $\mathfrak{m}$

We continue with the notation of §2. In order to understand the action of $W(\mathfrak{a})$ on discrete series of M, we shall first introduce in Theorem 3.7 an action of $W(\mathfrak{a})$ on the compact Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{m}$. For this purpose let us recall the main theorem of [11] – that $W(\mathfrak{a})$ is exactly the Weyl group of the system Δ of useful roots of $\mathfrak{a}$.

Ultimately we shall decompose $W(\mathfrak{a})$ into a semidirect product, in order to analyze the action on $\mathfrak{b}$, and the semidirect product decomposition and action will depend upon choices of orderings. Thus we suppose that $\mathfrak{a}'$ and $(i\mathfrak{b})'$ have been ordered lexicographically in some fashion. Let

Π_e = {those simple roots of Δ that are even}

$$\mathfrak{a}_e = \sum_{\alpha_R \in \Pi_e} \mathbb{R} H_{\alpha_R}$$

W_e = subgroup of $W(\mathfrak{a})(= W(\Delta))$ generated by reflections in the members of Π_e.

Notice that simplicity has been defined relative to the set Δ of *useful* roots of $\mathfrak{a}$, not to the set of all roots of $\mathfrak{a}$. The key result behind the action of $W(\mathfrak{a})$ on $\mathfrak{b}$ is the following imbedding theorem.

PROPOSITION 3.1: *It is possible to choose β in $(i\mathfrak{b})'$ corresponding to each α_R in Π_e so that $\alpha_R + \beta$ is a root of $\mathfrak{a} + i\mathfrak{b}$, so that the reflection p_β preserves the set of positive roots of $(\mathfrak{m}, \mathfrak{b})$, and so that the linear extension of the mapping given by $\alpha_R \to J(\alpha_R) = \beta$ is an isometry of $\mathfrak{a}_e'$ into $(i\mathfrak{b})'$.*

Some explanation is appropriate. Suppose $\mathfrak{g}$ is complex semisimple and the parabolic subgroup is minimal. Then $\mathfrak{m}_p = i\mathfrak{a}_p$, and the action of $W(\mathfrak{a}_p)$ on $\mathfrak{m}_p$ is just the action on $\mathfrak{a}_p$ transported to $\mathfrak{m}_p$ via i. On the other hand, if $\mathfrak{g}$ is real split and if the parabolic subgroup is minimal, then $\mathfrak{m}_p = 0$ and $W(\mathfrak{a}_p)$ acts trivially. The general case behaves like a mixture of these two, with a distinction made according as whether roots are even or odd. Reflections in even roots are analogous to those in the complex case, and reflections in odd roots are analogous to those in real split groups. To capture this action, we first imbed the even simple roots into $(i\mathfrak{b})'$ by a generalization of the multiplication-by-i map of the complex semisimple case.

LEMMA 3.2:

(a) *Let α_R be an odd $\mathfrak{a}$-root. Then there is a representative w in K_0 of the reflection p_{α_R} on $\mathfrak{a}$ such that w is in the analytic subgroup corresponding to $\mathfrak{g}^{(\alpha_R)}$ and such that $Ad(w)$ is the identity on $\mathfrak{m}$.*

(b) *Let α_R be an even useful $\mathfrak{a}$-root, and let $\alpha_R \pm \beta$ be roots of $\mathfrak{a} + i\mathfrak{b}$ restricting to α_R. Then there is a representative w in K_0 of the reflection p_{α_R} on $\mathfrak{a}$ such that w is in the analytic subgroup corresponding to $\mathfrak{g}^{(\alpha_R)}$ and such that $Ad(w)$ is -1 on $\mathbb{R}H_\beta$ and is $+1$ on the orthocomplement of H_β in $i\mathfrak{b}$.*

PROOF: (a) Since α_R is odd, Lemma 2.2 shows that $\mathfrak{g}^{(\alpha_R)}$ has a compact Cartan subalgebra. By Lemma 4 of [11], there exists w_1 in the analytic subgroup with Lie algebra $\mathfrak{g}^{(\alpha_R)} \cap \mathfrak{k}$ such that $Ad(w_1)$ is 1 on $\mathfrak{g}^{(\alpha_R)} \cap \mathfrak{k}$ and -1 on $\mathfrak{g}^{(\alpha_R)} \cap \mathfrak{p}$. Applying Lemma 4 of [11] to $\mathfrak{m}$, we obtain w_2 in the analytic subgroup with Lie algebra $\mathfrak{m} \cap \mathfrak{k}$ such that $Ad(w_2)$ is 1 on $\mathfrak{m} \cap \mathfrak{k}$ and -1 on $\mathfrak{m} \cap \mathfrak{p}$. Then $w = w_1 w_2$ has the required properties.

(b) Let $\alpha = \alpha_R + \beta$, and let X_α be a root vector in $\mathfrak{g}^C$. With conjugation defined relative to $\mathfrak{g}$, $\bar{X}_\alpha$ is then a root vector for $\bar{\alpha} = \alpha_R - \beta$, since roots are imaginary on $\mathfrak{b}$. The complexification of the Cartan involution θ is 1 on $i\mathfrak{b}$ and -1 on $\mathfrak{a}$, and hence

$$\theta\bar{\alpha} = \theta(\alpha_R - \beta) = -\alpha_R - \beta = -\alpha.$$

Thus $\theta\bar{X}_\alpha$ is a root vector for $-\alpha$. Now $B(X, -\theta\bar{X})$ is >0 for all $X \neq 0$ in $\mathfrak{g}^C$ and in particular for X_α. Multiply X_α by a constant so that $B(X_\alpha, -\theta\bar{X}_\alpha) = 2/|\alpha|^2$, set $H = 2|\alpha|^{-2}H_\alpha$, and set $X_{-\alpha} = -\theta\bar{X}_\alpha$. Then

$$[H, X_\alpha] = 2X_\alpha, \quad [H, X_{-\alpha}] = -2X_{-\alpha}, \quad [X_\alpha, X_{-\alpha}] = H,$$

and it follows from the theory of $SL(2,\mathbb{C})$ that

$$w^+ = \exp\frac{\pi}{2}(X_\alpha - X_{-\alpha}) = \exp\frac{\pi}{2}(X_\alpha + \theta\bar{X}_\alpha)$$

represents the Weyl group reflection for α in $(\mathfrak{a}+i\mathfrak{b})^{\mathbb{C}}$. Similarly

$$w^- = \exp\frac{\pi}{2}(\bar{X}_\alpha + \theta X_\alpha)$$

represents the Weyl group reflection for $\bar{\alpha}$. The roots α and $\bar{\alpha}$ are strongly orthogonal by Lemmas 2.5 and 2.4, from which it follows that

$$w^+w^- = \exp\frac{\pi}{2}(X_\alpha + \bar{X}_\alpha + \theta X_\alpha + \overline{\theta X}_\alpha).$$

Consequently w^+w^- is exhibited as in the (real) analytic subgroup corresponding to $\mathfrak{g}^{(\alpha_R)}$. Evidently $Ad(w^+w^-)$ is -1 on $\mathbb{C}H_\alpha + \mathbb{C}H_{\bar{\alpha}}$ and is $+1$ on the orthocomplement in $(\mathfrak{a}+i\mathfrak{b})^{\mathbb{C}}$. Hence $Ad(w^+w^-)$ is -1 on $\mathbb{R}H_{\alpha_R} + \mathbb{R}H_\beta$ and is $+1$ on the orthocomplement in $\mathfrak{a}+i\mathfrak{b}$. Thus $w = w^+w^-$ has the required properties.

LEMMA 3.3: *Let α_1 and α_2 be distinct members of Π_e such that $\alpha_1+\alpha_2$ is an $\mathfrak{a}$-root. Then $2(\alpha_1+\alpha_2)$ is not an $\mathfrak{a}$-root.*

PROOF: An exceptional G_2 factor has no even useful roots. Thus we may disregard these cases in the lemma. If $2(\alpha_1+\alpha_2)$ is an $\mathfrak{a}$-root, then $\alpha_1+\alpha_2$ and $2(\alpha_1+\alpha_2)$ are useful (hence in Δ) by Proposition 10c of [11]. Without loss of generality let $|\alpha_1|^2 \geq |\alpha_2|^2$ and form the α_1-string in Δ through $2(\alpha_1+\alpha_2)$. Then

$$\frac{2\langle 2(\alpha_1+\alpha_2), \alpha_1\rangle}{|\alpha_1|^2} = 4 + \frac{4\langle \alpha_2, \alpha_1\rangle}{|\alpha_1|^2} = 4 \text{ or } 2$$

since α_1 and α_2 are distinct and simple within Δ and since $|\alpha_1|^2 \geq |\alpha_2|^2$. Therefore

$$2\alpha_2 = 2(\alpha_1+\alpha_2) - 2\alpha_1$$

is a member of Δ, and α_2 must be odd by Lemma 2.7a, contradiction.

LEMMA 3.4: *Let $\alpha = \alpha_1+\beta_1$ and $\alpha' = \alpha_2+\beta_2$ be roots of $\mathfrak{a}+i\mathfrak{b}$ such* *that α_1 and α_2 are useful even $\mathfrak{a}$-roots, and suppose $\alpha + \alpha'$ is a root of $\mathfrak{a} + i\mathfrak{b}$ but $\alpha_1 + \alpha_2$ is not useful. Then $|\alpha_1| = |\alpha_2|$, $\frac{2\langle\alpha_1, \alpha_2\rangle}{|\alpha_1|^2} = +1$, and α is orthogonal to α'.*

PROOF: Without loss of generality, let $|\alpha_2| \geq |\alpha_1|$. Since the restriction to $\mathfrak{a}$ of

$$\alpha + \alpha' = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)$$

is not useful, Lemma 2.4 shows that $2(\beta_1 + \beta_2)$ is a root of $\mathfrak{a} + i\mathfrak{b}$. Moreover,

$$|2(\beta_1 + \beta_2)|^2 = |\alpha + \alpha'|^2 = \frac{4}{3}|\alpha_1 + \alpha_2|^2. \tag{3.1}$$

Expanding the right side and using the equality $|\alpha|^2 = 2|\alpha_1|^2$ given by Lemma 2.5, we obtain

$$\frac{|2(\beta_1 + \beta_2)|^2}{|\alpha|^2} = \frac{1}{2}\frac{|2(\beta_1 + \beta_2)|^2}{|\alpha_1|^2} = \frac{1}{2}\cdot\frac{4}{3}\left(1 + \frac{2\langle\alpha_1, \alpha_2\rangle}{|\alpha_1|^2} + \frac{|\alpha_2|^2}{|\alpha_1|^2}\right).$$

That is,

$$\frac{3}{2}\frac{|2(\beta_1 + \beta_2)|^2}{|\alpha|^2} = 1 + \frac{2\langle\alpha_1, \alpha_2\rangle}{|\alpha_1|^2} + \frac{|\alpha_2|^2}{|\alpha_1|^2}. \tag{3.2}$$

Since α_1 and α_2 are in Δ and $|\alpha_2| \geq |\alpha_1|$, the right side is an integer. The left side is 3/2 of the ratio of the length squared of two roots. Thus both sides of (3.2) are 3:

$$|2(\beta_1 + \beta_2)|^2 = 2|\alpha|^2 \tag{3.3}$$

$$1 + \frac{2\langle\alpha_1, \alpha_2\rangle}{|\alpha_1|^2} + \frac{|\alpha_2|^2}{|\alpha_1|^2} = 3. \tag{3.4}$$

Let $|\alpha_2|^2 = n|\alpha_1|^2$ with $1 \leq n \leq 4$. Then (3.4) gives $2\langle\alpha_1, \alpha_2\rangle/|\alpha_1|^2 = 2 - n$, which is impossible for $n = 3$ if α_2 is long and α_1 is short. If $n = 4$, then $\alpha_2 = -2\alpha_1$; hence $\alpha_1 + \alpha_2 = -\alpha_1$ is useful, contrary to hypothesis.

Suppose $n = 2$, so that $|\alpha_2|^2 = 2|\alpha_1|^2$. By Lemma 2.5, $|\alpha'|^2 = 2|\alpha|^2$.

Thus (3.1) and (3.3) lead to the conclusion that

$$|\alpha + \alpha'|^2 = |\alpha'|^2 = 2|\alpha|^2,$$

which contradicts the fact that the sum of a short root and a long root is necessarily short.

Thus $n = 1$, so that $|\alpha_2|^2 = |\alpha_1|^2$ and $2\langle\alpha_1, \alpha_2\rangle/|\alpha_1|^2 = +1$. By Lemma 2.5, $|\alpha'|^2 = |\alpha|^2$. Thus (3.1) and (3.3) lead to the conclusion that

$$|\alpha + \alpha'|^2 = 2|\alpha|^2 = 2|\alpha'|^2.$$

Expanding the left side, we see immediately that $\langle\alpha, \alpha'\rangle = 0$.

LEMMA 3.5: *Let $\alpha_1 + \beta_1$ and $\alpha_2 + \beta_2$ be roots of $\mathfrak{a} + i\mathfrak{b}$ that restrict on $\mathfrak{a}$ to distinct α_1 and α_2 in Π_e. If α_1 and α_2 are orthogonal, then so are $\alpha_1 + \beta_1$ and $\alpha_2 \pm \beta_2$. If α_1 and α_2 are not orthogonal, then exactly one of $\langle\alpha_1 + \beta_1, \alpha_2 + \beta_2\rangle$ and $\langle\alpha_1 + \beta_1, \alpha_2 - \beta_2\rangle$ is 0. The first one is 0 if and only if $\langle\beta_1, \beta_2\rangle$ is >0, and the second one is 0 if and only if $\langle\beta_1, \beta_2\rangle$ is <0.*

PROOF: First suppose α_1 and α_2 are orthogonal. Since α_1 and α_2 are simple for Δ, $\alpha_1 - \alpha_2$ is not a useful $\mathfrak{a}$-root. But nor can $\alpha_1 - \alpha_2$ be an $\mathfrak{a}$-root that is not useful, by Lemma 3.4. Hence $(\alpha_1 + \beta_1) - (\alpha_2 \pm \beta_2)$ are not roots. Also $\alpha_1 - \alpha_2$ not in Δ and α_1 orthogonal to α_2 imply $\alpha_1 + \alpha_2$ is not in Δ, and $\alpha_1 + \alpha_2$ cannot be an $\mathfrak{a}$-root that is not useful, again by Lemma 3.4. Hence $(\alpha_1 + \beta_1) + (\alpha_2 \pm \beta_2)$ are not roots. Thus $\alpha_1 + \beta_1$ is orthogonal to $\alpha_2 \pm \beta_2$.

Now suppose that α_1 and α_2 are not orthogonal. Since they are simple for Δ, we must have $\langle\alpha_1, \alpha_2\rangle < 0$. Then one of the two inner products $\langle\alpha_1 + \beta_1, \alpha_2 \pm \beta_2\rangle$ is $\neq 0$. Say $\langle\alpha_1 + \beta_1, \alpha_2 + \beta_2\rangle \neq 0$. We shall show that this inner product is <0. In the contrary case, $(\alpha_1 + \beta_1) - (\alpha_2 + \beta_2)$ is a root and the orthogonality conclusion of Lemma 3.4 shows that $\alpha_1 - \alpha_2$ is useful, contradicting the fact that α_1 and α_2 are simple in the system Δ of useful roots of $\mathfrak{a}$. Thus $\langle\alpha_1 + \beta_1, \alpha_2 + \beta_2\rangle$ is <0.

It follows that the sum

$$(\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)$$

is a root of $\mathfrak{a} + i\mathfrak{b}$. By Lemma 3.3, $2(\alpha_1 + \alpha_2)$ is not an $\mathfrak{a}$-root, and we know $\alpha_1 + \alpha_2$ is useful since $\langle\alpha_1, \alpha_2\rangle < 0$ and α_1 and α_2 are in Δ. We shall show shortly that $\beta_1 + \beta_2 \neq 0$. Then it follows from Lemma 2.5

that $|\alpha_1+\alpha_2|^2=|\beta_1+\beta_2|^2$. Since Lemma 2.5 also gives $|\alpha_1|^2=|\beta_1|^2$ and $|\alpha_2|^2=|\beta_2|^2$, we conclude that $\langle\alpha_1,\alpha_2\rangle=\langle\beta_1,\beta_2\rangle$. Therefore the other inner product under consideration is

$$\langle\alpha_1+\beta_1,\alpha_2-\beta_2\rangle=\langle\alpha_1,\alpha_2\rangle-\langle\beta_1,\beta_2\rangle=0.$$

In this case $\langle\beta_1,\beta_2\rangle=\langle\alpha_1,\alpha_2\rangle$ is <0. If instead we had started with $\langle\alpha_1+\beta_1,\alpha_2-\beta_2\rangle\neq 0$, we could use this argument with β_2 replaced by $-\beta_2$ to conclude $\langle\alpha_1+\beta_1,\alpha_2+\beta_2\rangle=0$ and $\langle\beta_1,\beta_2\rangle>0$.

Thus to complete the proof, we show $\beta_1+\beta_2\neq 0$. If $\beta_2=-\beta_1$, then $|\alpha_1|^2=|\beta_1|^2=|\beta_2|^2=|\alpha_2|^2$ says that α_1 and α_2 are distinct nonorthogonal simple roots in Δ of the same length; thus $2\langle\alpha_1,\alpha_2\rangle/|\alpha_1|^2=-1$. Hence

$$\begin{aligned}\langle\alpha_1+\beta_1,\alpha_2-\beta_2\rangle&=-\frac{1}{2}|\alpha_1|^2-\langle\beta_1,\beta_2\rangle=-\frac{1}{2}|\alpha_1|^2+|\alpha_1|^2\\&=\frac{1}{2}|\alpha_1|^2>0,\end{aligned}$$

and $(\alpha_1+\beta_1)-(\alpha_2-\beta_2)$ is a root. The nonorthogonality of $(\alpha_1+\beta_1)$ and $-(\alpha_2-\beta_2)$, because of Lemma 3.4, implies that $\alpha_1-\alpha_2$ is useful, and then we have a contradiction to the simplicity of α_1 and α_2.

LEMMA 3.6: *Let α_R be an even useful $\mathfrak{a}$-root and choose $\beta>0$ in $(i\mathfrak{b})'$ so that $\alpha_R+\beta$ is a root of $\mathfrak{a}+i\mathfrak{b}$ and β is as small as possible. If γ is any positive root of $(\mathfrak{m},\mathfrak{b})$, then $p_\beta\gamma$ is a root of $(\mathfrak{m},\mathfrak{b})$ and is positive.*

PROOF: The roots $\alpha_R+\beta$ and $\alpha_R-\beta$ are orthogonal by Lemma 2.5, and thus

$$p_\beta\gamma=p_{\alpha_R+\beta}p_{\alpha_R-\beta}\gamma,$$

from which it follows that $p_\beta\gamma$ is a root. This root clearly vanishes on $\mathfrak{a}$ and is thus a root of $(\mathfrak{m},\mathfrak{b})$. We still need to show that $p_\beta\gamma>0$.

Suppose on the contrary that

$$0>p_\beta\gamma=\gamma-2\langle\gamma,\beta\rangle|\beta|^{-2}\beta. \tag{3.5}$$

Then we must have $\langle\gamma,\beta\rangle>0$. Now $p_\gamma(\alpha_R+\beta)=\alpha_R+p_\gamma\beta$ is a root and so is $\alpha_R-p_\gamma\beta$. The inequality on $\langle\gamma,\beta\rangle$ implies that

$$p_\gamma\beta=\beta-2\langle\beta,\gamma\rangle|\gamma|^{-2}\gamma<\beta.$$

The minimality of β implies that $p_\gamma\beta \leq -\beta$. Therefore

$$2\beta \leq 2\langle\beta, \gamma\rangle|\gamma|^{-2}\gamma. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$\gamma < \frac{2\langle\gamma, \beta\rangle}{|\beta|^2}\beta \leq \frac{2\langle\beta, \gamma\rangle^2}{|\beta|^2|\gamma|^2}\gamma. \tag{3.7}$$

Now $\langle\gamma, \beta\rangle > 0$ says $\alpha_R + \beta - \gamma$ is a root. Here $\beta - \gamma$ is not 0 since α_R is even, and thus Lemma 2.5 gives $|\alpha_R|^2 = |\beta - \gamma|^2$. Also $|\alpha_R|^2 = |\beta|^2$ by Lemma 2.5, and so $2\langle\beta, \gamma\rangle = |\gamma|^2$. Substituting in (3.7), we obtain

$$\gamma < \frac{\langle\beta, \gamma\rangle}{|\beta|^2}\gamma = \frac{2\langle\alpha_R + \beta, \gamma\rangle}{|\alpha_R + \beta|^2}\gamma.$$

If N is the coefficient of γ on the right, we conclude $N > 1$. Then

$$p_{\alpha_R+\beta}\gamma = \gamma - N(\alpha_R + \beta) = -N\alpha_R + (\gamma - N\beta)$$

is a root with $N > 1$. Hence $2\alpha_R$ is an $\mathfrak{a}$-root. From Lemma 2.7a we conclude that α_R is odd, contradiction. Thus we must have $p_\beta\gamma > 0$.

PROOF OF PROPOSITION 3.1: For each α_R in Π_e, let $J(\alpha_R) = \pm\beta$, with β as in Lemma 3.6 and with the sign to be determined shortly. The signs will be determined so that $\alpha_R \neq \alpha_R'$ implies

$$\langle\alpha_R + J\alpha_R, \alpha_R' - J\alpha_R'\rangle = 0. \tag{3.8}$$

Here $\langle\alpha_R, \alpha_R'\rangle = \langle J\alpha_R, J\alpha_R'\rangle$. Since also $|\alpha_R|^2 = |J\alpha_R|^2$ from Lemma 2.5, the isometry will follow. The preservation of the positive roots of $\mathfrak{m}$ is immediate from Lemma 3.6.

Thus we want to choose the sign of each $J\alpha_R$ so that (3.8) holds. Since the Dynkin diagram of Δ has no loops [7, p. 130], we can number the simple roots of Δ (including therefore the members of Π_e) so that each one is immediately connected with only one previous one. We choose the signs inductively, following this numbering. Choose the sign arbitrarily for the first member of Π_e. Assuming that the signs have been chosen for the first $j-1$ members of Π_e in such a way that (3.8) holds, look at the j^{th} case. If $\langle(\alpha_R)_j, (\alpha_R)_i\rangle = 0$, Lemma 3.6 shows that (3.8) will hold for j and i, no matter which sign is used. There is at most one $i < j$ for which $\langle(\alpha_R)_j, (\alpha_R)_i\rangle$ is $\neq 0$. For this i, Lemma 3.6 says that $(\alpha_R)_i + J(\alpha_R)_i$ is orthogonal to either $(\alpha_R)_j + \beta_j$ or $(\alpha_R)_j - \beta_j$. In the first case define $J(\alpha_R)_j = -\beta_j$. In the second case define $J(\alpha_R)_j = \beta_j$. Then the orthogonality is proved with j signs chosen, and the choice is completed by induction. This completes the proof.

Fix $J: \mathfrak{a}'_e \to i\mathfrak{b}$ as in Proposition 3.1. Let W_Π be the set of simple reflections relative to Δ, so that $W_\Pi \subseteq W(\mathfrak{a})$. Then J defines a map of W_Π into the orthogonal group $O(i\mathfrak{b})'$ as follows: If α_R is in Π_e, map p_{α_R} into $p_{J\alpha_R}$. If α_R is a simple root in Δ not in Π_e, map p_{α_R} into the identity.

THEOREM 3.7: *The mapping of W_Π into $O(i\mathfrak{b})'$ defined by J extends to a group homomorphism of $W(\mathfrak{a})$ into $O(i\mathfrak{b})'$. The resulting action of $W(\mathfrak{a})$ on $(i\mathfrak{b})'$ has the properties that*

(a) *each p in $W(\mathfrak{a})$ has a representative w in the normalizer of $\mathfrak{a}$ in K_0 such that $Ad(w)$ agrees on $i\mathfrak{b}$ with the action of p.*

(b) *for w in W_e, $Jw = wJ$ on $\mathfrak{a}'_e$, and*

(c) *for w in $W(\mathfrak{a})$, if γ is a positive root of $(\mathfrak{m}, \mathfrak{b})$, then so is $w\gamma$.*

PROOF: The Main Theorem of [11] says that W_Π generates $W(\mathfrak{a})$. Let $F(W_\Pi)$ be the free group on W_Π. The mapping of W_Π into $O(i\mathfrak{b})'$ extends to a group homomorphism φ of $F(W_\Pi)$ into $O(i\mathfrak{b})'$, and we shall show that the relation subgroup of $F(W_\Pi)$ maps to the identity. Then the rest follows from Lemma 3.2 and Proposition 3.1. Thus we are to show that the basic relations (see [1], pages 11–12, 74)

$$p_{\alpha_i}^2 = 1, (p_{\alpha_i} p_{\alpha_j})^{2, 3, 4, \text{or } 6} = 1, \ \alpha_i \text{ and } \alpha_j \text{ simple in } \Delta,$$

map to the identity. Clearly $\varphi(p_{\alpha_i})^2 = 1$. Let i and j be given. If α_i and α_j are both in Π_e, then

$$[\varphi(p_{\alpha_i})\varphi(p_{\alpha_j})]^{2, 3, 4, \text{or } 6} = 1 \tag{3.9}$$

on $J(\mathfrak{a}'_e)$ because it is true on $\mathfrak{a}'_e$; also each factor on the left side of (3.9) is the identity on $J(\mathfrak{a}'_e)^\perp$. So (3.9) holds for such i and j. If neither α_i nor α_j is in Π_e, then (3.9) holds because every factor is 1. If, say, α_i is in Π_e and α_j is not, then the left side of (3.9) reduces to $\varphi(p_{J\alpha_i})^{2, 3, 4, \text{or } 6}$, and this is 1 if the exponent is even. If the exponent is odd, then it is 3 and α_i and α_j are nonorthogonal and of equal length. Lemma 2.8 rules out this situation as a possibility and completes the proof.

LEMMA 3.8: *Let* ϵ *be a useful* $\mathfrak{a}$*-root, and fix an ordering on* $(i\mathfrak{b})'$. *Then the action of* p_ϵ *on* $(i\mathfrak{b})'$ *given in Theorem* 3.7 *is as follows*:

(a) *it is* $p_{\epsilon'}$ *if* ϵ *is even and* ϵ' *is minimal among positive elements of* $(i\mathfrak{b})'$ *such that* $\epsilon + \epsilon'$ *is a root of* $\mathfrak{a} + i\mathfrak{b}$.

(b) *it is* 1 *if* ϵ *is odd.*

PROOF: (a) Let q_ϵ be the action on $(i\mathfrak{b})'$ given in Theorem 3.7. Then Theorem 3.7 and the first part of the proof of Lemma 3.6 show that $p_\epsilon q_\epsilon$ and $p_\epsilon p_{\epsilon'}$ are both in the complex Weyl group of $\mathfrak{a} + i\mathfrak{b}$. Hence so is $q_\epsilon p_{\epsilon'}$. But $q_\epsilon p_{\epsilon'}$ fixes $\mathfrak{a}$ and by Chevalley's Lemma must be in the complex Weyl group of $i\mathfrak{b}$. By Theorem 3.7c and Lemma 3.6, $q_\epsilon p_{\epsilon'}$ leaves the positive roots of $i\mathfrak{b}$ stable. Hence $q_\epsilon p_{\epsilon'} = 1$ and $q_\epsilon = p_{\epsilon'}$.

(b) We use the same argument as in (a) except that we use 1 in place of $p_{\epsilon'}$ and drop the reference to Lemma 3.6. The element p_ϵ is in the complex Weyl group of $\mathfrak{a} + i\mathfrak{b}$ by Lemma 3.2a.

Let S be the subgroup of elements w of $W(\mathfrak{a})$ that act as 1 on $(i\mathfrak{b})'$ in the action of Theorem 3.7. Recall that W_e was defined at the beginning of §3.

PROPOSITION 3.9: S *is normal in* $W(\mathfrak{a})$, *and* $W(\mathfrak{a})$ *is the semidirect product* $W(\mathfrak{a}) = W_e S$.

PROOF: Clearly S is normal. If w is in $S \cap W_e$, then w is in W_e and $w = 1$ on $J(\alpha_e')$. By Theorem 3.7b, $w = 1$ on $\mathfrak{a}_e'$ and so $w = 1$. To see that $W(\mathfrak{a}) = W_e S$, let w be in $W(\mathfrak{a})$. Consider the action of w on $(i\mathfrak{b})'$. Since w is a product of reflections in even roots that act as members of W_e and reflections in odd roots that act as the identity (by definition of the action), w has the same effect on $(i\mathfrak{b})'$ as a member w_e of W_e. Then $w_e^{-1}w$ acts as the identity on $(i\mathfrak{b})'$ and so is in S. Thus $W = W_e S$.

The set Δ of useful $\mathfrak{a}$-roots is a root system, possibly nonreduced. Let

$$\Delta_0 = \{\epsilon \text{ in } \Delta \mid \epsilon \text{ has odd multiplicity}\}. \tag{3.10}$$

PROPOSITION 3.10: Δ_0 *is a reduced root system, and* S *is its Weyl group.*

PROOF: In view of Lemma 2.7, Δ_0 picks out one positive multiple of each reduced odd $\mathfrak{a}$-root, together with its negative. If α is a reduced odd $\mathfrak{a}$-root, the positive multiple is α if 2α is not an $\mathfrak{a}$-root, and it is 2α otherwise. Then it follows from Lemma 2.7d that Δ_0 is closed under its own reflections. Since it is a subset of a root system, it is itself a root system. Δ_0 is reduced, according to Lemma 2.7a.

Its Weyl group $W(\Delta_0)$ is contained in S, by Lemma 3.8b. To prove equality, it is enough to prove that the only w in S that leaves stable the set of positive roots in Δ_0 is $w = 1$. By Proposition 3.9 it is enough to prove that any w in $W(\mathfrak{a})$ that leaves stable the set of positive roots in Δ_0 is in W_e.

We do the latter by induction on the length $\ell(w)$, the case $\ell(w) = 0$ being trivial. Suppose $\ell(w) > 0$ and suppose w leaves stable the set of positive roots in Δ_0. There must be some member β of Π_e such that $w\beta < 0$, since otherwise w permutes the positive roots in Δ. Then $\ell(wp_\beta) < \ell(w)$, p_β is in W_e, and we claim $wp_\beta\epsilon > 0$ for every $\epsilon > 0$ in Δ_0. [In fact, if $\epsilon > 0$ is in Δ_0, then so is $p_\beta\epsilon$ since the only positive roots in Δ mapped by p_β into negative roots are the multiples of β, and ϵ is not a multiple of β. Then $wp_\beta\epsilon$ is positive since we are assuming w carries positive roots of Δ_0 into positive roots.] These facts reduce the proof to showing that wp_β is in W_e, and the induction is complete.

§4. Action of $W(\mathfrak{a})$ on discrete series of M

We continue with the notation of §2. In this section we shall use Theorem 3.7 to analyze in part the action of $W(\mathfrak{a})$ on (equivalence classes of) discrete series representations of M.

We begin with the identity component M_0. Let ξ be a discrete series representation of M_0. Essentially as given in [2], the *Harish-Chandra parameter* μ of ξ is by definition the unique member of $(i\mathfrak{b})'$ such that

$$\langle \mu, \delta \rangle > 0 \text{ for every compact root } \delta > 0 \text{ of } (\mathfrak{m}, \mathfrak{b})$$

and such that the distribution character Θ_ξ of ξ is given on $\exp \mathfrak{b}$ as $\Theta_\xi = \epsilon(\mu)\Theta_\mu$, where $\epsilon(\mu)$ is a well-defined sign and where Θ_μ is the function

$$\Theta_\mu(h) = \frac{\sum_{s \in W_{K_M}} \epsilon(s)\xi_{s_\mu}(h)}{\xi_\rho(h) \prod_{\delta>0} (1 - \xi_\delta(h)^{-1})}.$$

Here ξ_λ is the character on $\exp \mathfrak{b}$ corresponding to λ, W_{K_M} is the Weyl group of the compact roots, and ρ is half the sum of the positive roots; if ρ fails to be integral, some of the ingredients of the formula are not well defined, but the formula as a whole is. The parameter μ is nonsingular (with respect to all roots of $(\mathfrak{m}, \mathfrak{b})$) and determines ξ up to unitary equivalence.

PROPOSITION 4.1: *Let φ be an automorphism of M_0 leaving $\mathfrak{k} \cap \mathfrak{m}$ stable and $\mathfrak{b}$ stable and the set of positive compact roots of $(\mathfrak{m}, \mathfrak{b})$ stable. For a discrete series ξ of M_0 with Harish-Chandra parameter μ, let $\xi^\varphi(m) = \xi(\varphi^{-1}m)$ and $\mu^\varphi(H) = \mu(\varphi^{-1}H)$. Then ξ^φ is a discrete series with Harish-Chandra parameter μ^φ.*

We omit the proof. One can give a straightforward proof by means of characters, or one can give a somewhat shorter proof that uses the theory of lowest K-types.

To pass to M, recall from §1 that M_0 has finite index in M and that

$$M = M_0F,$$

where $F = Z_{M_p} \cap Z(G) \exp i\mathfrak{a}_p$. Define a subgroup of M by

$$M^\# = M_0Z_M,$$

where Z_M is the center of M.

LEMMA 4.2: $M^\# = M_0 \cdot (F \cap Z_M)$

PROOF: Suppose z in Z_M decomposes as $z = m_0f$ with m_0 in M_0 and f in F. Then

$$fm_0f^{-1} = fm_0ff^{-2} = fzf^{-2} = zff^{-2} = zf^{-1} = m_0$$

and f commutes with m_0. Since F is abelian, it follows that f is in Z_M. Thus $z = m_0f$ exhibits z as in $M_0 \cdot (F \cap Z_M)$.

LEMMA 4.3: $Z_M(\mathfrak{b}) \subseteq M^\#$.

PROOF: Let $m = k \exp X$ be the Cartan decomposition of a member of $Z_M(\mathfrak{b})$. Then $(\theta m)^{-1}m = \exp 2X$ is in $Z_M(\mathfrak{b})$ and so

$$b(\exp 2X)b^{-1} = \exp 2X$$

for all b in $\exp \mathfrak{b}$. Differentiating, we obtain $[\mathfrak{b}, X] = 0$ and so X is in $\mathfrak{b}$. Since X is also in $\mathfrak{p}$, $X = 0$. Thus m is in $Z_{K \cap M}(\mathfrak{b})$. By the axioms of §1, write $m = m^{C}z$ with m^{C} in the connected complex group M^{C} and z in the centralizer $Z(M)$ of M. Since $Ad(m) = Ad(m^{C})$ is 1 on $\mathfrak{b}$, m^{C} is in $\exp \mathfrak{b}^{C}$. Write $m^{C} = m_1 m_2$ with m_1 in $\exp(i\mathfrak{b})$ and m_2 in $\exp \mathfrak{b}$. Now $Ad(m_2)$ and $Ad(m)$ are unitary on $\mathfrak{m}^{C}$ since m is in K, and $Ad(m_1)$ is unitary only if $m_1 = 1$. Thus $m_1 = 1$ and m^{C} is in $\exp \mathfrak{b} \subseteq M_0$. Then z must be in M and so z is in Z_M. Thus $m = m^{C}z$ is in $M_0 Z_M = M^{\#}$.

A discrete series representation of $M^{\#}$ is scalar on $F \cap Z_M$ (since $F \cap Z_M$ is central) and therefore determines a *central character* on $F \cap Z_M$. Moreover the restriction of the discrete series to M_0 is still irreducible. Thus every discrete series representation of $M^{\#}$ is determined by its central character and by the Harish-Chandra parameter of its restriction to M_0. In the context of the previous lemma, the following lemma is implicit in the work of Harish-Chandra. Its proof was communicated to us by G. Zuckerman.

LEMMA 4.4: *If ξ is a discrete series representation of M, then $\xi|_{M^{\#}}$ splits into the sum of inequivalent discrete series of $M^{\#}$.*

REMARK: See Lemma 5.3 of [13] concerning the existence of the splitting.

PROOF: We are to show that $M/M^{\#}$ acts without fixed points on the discrete series of $M^{\#}$. Thus let ω be a discrete series representation of $M^{\#}$, and let x be in M. Suppose that $\omega^x \cong \omega$. We show x is in $M^{\#}$.

Let $B = Z_{M^{\#}}(\mathfrak{b})$. The groups B and $x^{-1}Bx$ are two compact Cartan subgroups of $M^{\#}$ are thus conjugate: $x^{-1}Bx = m^{\#}Bm^{\#-1}$ for some $m^{\#}$ in $M^{\#}$. Then $xm^{\#}$ is a member s of the normalizer $N_M(B)$, and it is enough to show that s is in $M^{\#}$ under the assumption that $\omega^s \cong \omega$. Adjusting s by a member of the compact Weyl group of $(\mathfrak{m}, \mathfrak{b})$, we see that it is enough to show that if t is in $N_M(B)$ and $\omega^t \cong \omega$ and t leaves stable the positive compact roots, then t is in $M^{\#}$.

Applying Proposition 4.1, we see that $Ad(t)$ fixes the Harish-Chandra parameter of ω, which is a regular element. Since $Ad(t)$ is in the connected complex adjoint group, we conclude that $Ad(t)$ centralizes $\mathfrak{b}^{C}$. Thus t is in $Z_M(\mathfrak{b})$, and the result follows from Lemma 4.3.

We can reinterpret Lemma 4.4 in terms of Mackey theory [14]: If $\xi|_{M^{\#}} = \xi_1 + \cdots + \xi_n$ is the decomposition into irreducible pieces, then

$$\xi \cong \operatorname*{ind}_{M^{\#}\uparrow M} \xi_j \quad \text{for} \quad 1 \leq j \leq n. \tag{4.1}$$

This means that ξ determines and is determined by its central character on $F \cap Z_M$ and by the unordered set $\{\mu_j\}$ of Harish-Chandra parameters of the constituents of $\xi|_{M^{\#}}$. Lemma 4.4 implies that the μ_j are distinct, since the central character is the same for each ξ_j.

The constituents ξ_j can all be expressed in terms of one of them, up to unitary equivalence, as $\xi_j = f_j\xi_1$, where $f_1, \ldots, f_n$ is a set of coset representatives for $F/(F \cap Z_M)$. We shall translate this fact into a conclusion about the parameters $\{\mu_j\}$ in Corollary 4.6 below.

Proposition 4.5: *If f is in F, then f normalizes M_0, $\mathfrak{k} \cap \mathfrak{m}$, and $\mathfrak{b}$, and in fact f represents a member of the complex Weyl group of* $(\mathfrak{m}, \mathfrak{b})$. *If s is selected to represent the member of the compact Weyl group of* $(\mathfrak{m}, \mathfrak{b})$ *such that sf leaves stable the positive compact roots of* $(\mathfrak{m}, \mathfrak{b})$, *and if ξ_0 is a discrete series representation of M_0 with Harish-Chandra parameter μ, then $f\xi_0$ has parameter $sf\mu$, where $sf\mu(H) = \mu(Ad(sf)^{-1}H)$.*

Proof: Suppose we can show that f normalizes $\mathfrak{b}$. Then the fact that $Ad(f)$ is in $Ad(M^{\mathbb{C}})$ implies that f represents a member of the complex Weyl group, and the remaining parts of the proposition follow from Proposition 4.1.

By construction we have

$$\mathfrak{a} + i\mathfrak{b} = \mathfrak{c}(\mathfrak{a} + \mathfrak{a}_M + i\mathfrak{h}_0) = \mathfrak{a} + \mathfrak{c}(\mathfrak{a}_M) + i\mathfrak{h}_0,$$

with $\mathfrak{c}$ an explicit Cayley transform that carries H_{δ_j} in $\mathfrak{a}_M$ into a multiple of $i(X_{\delta_j} + \theta X_{\delta_j})$ in $\mathfrak{c}(\mathfrak{a}_M) \subseteq i\mathfrak{b}$, where $\{\delta_j\}$ is a particular basis of $\mathfrak{a}'_M$ given by strongly orthogonal roots. Now $Ad(f)$ is in $\exp(i\,ad\,\mathfrak{a}_p)$, which acts trivially on $\mathfrak{h}_0$, and we need to see that it normalizes $\mathfrak{c}(\mathfrak{a}_M)$. Let $Ad(f) = \exp(i\,adH)$, $H \in \mathfrak{a}_p$. Then

$$Ad(f)(X_{\delta_j} + \theta X_{\delta_j}) = e^{i\delta_j(H)}X_{\delta_j} + e^{-i\delta_j(H)}\theta X_{\delta_j}.$$

This has to be in $\mathfrak{g}$, and X_{δ_j} and θX_{δ_j} are in $\mathfrak{g}$. Since $\mathfrak{g} \cap i\mathfrak{g} = 0$, it follows that $\delta_j(H)$ is a multiple of π and that

$$Ad(f)(X_{\delta_j} + \theta X_{\delta_j}) = \pm(X_{\delta_j} + \theta X_{\delta_j}).$$

This completes the proof.

NOTATION: We write $m_f(\mu)$ for $sf\mu$ in the proposition. Then m_f is a member of the orthogonal group on $(i\mathfrak{b})'$.

COROLLARY 4.6: *Let ξ be a discrete series representation of M with Harish-Chandra parameter set $\{\mu_j\}$. Then the parameters μ_j are exactly the distinct values that $m_f(\mu_1)$ assumes as f ranges through coset representatives of $F/(F \cap Z_M)$.*

PROOF: This is immediate from Proposition 4.5 since the constituents ξ_j of ξ are characterized as all $f\xi_1$.

PROPOSITION 4.7: *With an ordering for $(i\mathfrak{b})'$ fixed, let ξ be a discrete series representation with central character χ on $F \cap Z_M$ and with Harish-Chandra parameter set $\{\mu_j\}$. If p is in $W(\mathfrak{a})$ and w is a representative of p in the normalizer of $\mathfrak{a}$ in K, then $w\xi$ is a discrete series representation with central character $p\chi$ and with Harish-Chandra parameter set $\{p\mu_j\}$, where $p\mu_j$ refers to the action of $W(\mathfrak{a})$ on $(i\mathfrak{b})'$ given in Theorem 3.7.*

REMARKS: Here $w\xi(m)$ is defined as $\xi(w^{-1}mw)$. In defining $p\chi$, the ambiguity in choosing a representative of p is by an element of M, and this ambiguity is harmless since χ is defined only on central elements of M.

PROOF: The central character of $w\xi$ is obviously $p\chi$. To get at the parameter set of $w\xi$, first assume that w is the special representative of p given in Theorem 3.7a. The automorphism $\varphi(m) = w^{-1}mw$ of M_0 leaves $\mathfrak{k} \cap \mathfrak{m}$ stable since w is in K and normalizes $\mathfrak{a}$, it leaves $\mathfrak{b}$ stable by Theorem 3.7a, and it leaves the positive roots stable by Theorem 3.7c. Hence Proposition 4.1 implies that the Harish-Chandra parameter of $w\xi_j|_{M_0}$ is $p\mu_j$ if μ_j is the parameter of ξ_j. This proves the proposition for special w.

For general w, we can write $w = mw'$ with w' of the special form above and with m in $K \cap M$. Applying the special case, we see that we are to show that if ξ has parameter set $\{\mu_j\}$, then so does $m\xi$. But $m\xi$ is equivalent with ξ and so has the same parameter set.

LEMMA 4.8: *Let p be in $W(\mathfrak{a})$. In terms of the action of Theorem 3.7, if f is in F, then $pm_fp^{-1} = m_g$ for some g in F.*

PROOF: Let p act by its special representative w in $N_k(\mathfrak{a})$ and let m_f act by sf with s in $N_{K\cap M_0}(\mathfrak{b})$. Since wfw^{-1} is in M, we can write

$wfw^{-1} = m_0 g$ with m_0 in M_0 and g in F. By Proposition 4.5, the elements w, f, and g all normalize $\mathfrak{b}$ and are in K. Hence the same thing is true of m_0, so that m_0 is in $N_{K\cap M_0}(\mathfrak{b})$. Now w normalizes M_0, so that wsw^{-1} is in M_0; since w and s normalize $\mathfrak{b}$ and are in K, wsw^{-1} is in $N_{K\cap M_0}(\mathfrak{b})$. Thus we can write

$$wsfw^{-1} = wsw^{-1}m_0 g$$

with $(wsw^{-1})m_0$ in $N_{K\cap M_0}(\mathfrak{b})$ and g in F, and the lemma follows.

PROPOSITION 4.9: *With an ordering for $(i\mathfrak{b})'$ fixed, let ξ be a discrete series representation of M with central character χ on $Z_{M_p} \cap Z_M$ and with Harish-Chandra parameter set $\{\mu_j\}$. Let p be in $W(\mathfrak{a})$ and let p act on $(i\mathfrak{b})'$ as in Theorem 3.7. Then*

(a) *$p\xi$ is equivalent with ξ is and only if $p\chi = \chi$ and $p\mu_1 = \mu_j$ for some j, and*

(b) *$p\xi|_{M_0}$ is equivalent with $\xi|_{M_0}$ if and only if $p\mu_1 = \mu_j$ for some j.*

PROOF: The representation $p\xi$ has central character $p\chi$ and parameter set $\{p\mu_j\}$ by Proposition 4.7. Hence the necessity in (a) and (b) is immediate. By Lemma 4.8 and Corollary 4.6,

$$p\mu_i = pm_f\mu_1 = pm_f p^{-1}p\mu_1 = m_g(p\mu_1).$$

Hence $p\mu_1 = \mu_j$ implies $\{p\mu_i\} = \{\mu_i\}$, and then it follows that $p\xi|_{M_0}$ is equivalent with $\xi|_{M_0}$. This proves the sufficiency in (b).

Suppose also that $p\chi = \chi$. From $p\mu_1 = \mu_j$, we conclude by the same argument as in Proposition 4.7 that $w\xi_1|_{M_0} \cong \xi_j|_{M_0}$, where w is the special representative of p in Theorem 3.7. Since $p\chi = \chi$, we obtain $w\xi_1|_{M^\#} \cong \xi_j|_{M^\#}$. Applying (4.1), we conclude $w\xi \cong \xi$, and this proves the sufficiency in (a).

THEOREM 4.10: *Let ξ be a discrete series representation of M, and let p be in the subgroup W_e of $W(\mathfrak{a})$. (See §3.) If $p\xi|_{M_0}$ is equivalent with $\xi|_{M_0}$, then p is the product of reflections p_{α_R} in W_e such that $p_{\alpha_R}\xi|_{M_0}$ is equivalent with $\xi|_{M_0}$.*

PROOF: An ordering in $\mathfrak{a}'$ has been specified to make W_e defined, but an ordering in $(i\mathfrak{b})'$ is at our disposal. First we arbitrarily define positivity for the compact roots of $(\mathfrak{m}, \mathfrak{b})$, and this definition is enough to determine the Harish-Chandra parameter set $\{\mu_j\}$ for ξ. Now choose the ordering in $(i\mathfrak{b})'$ in such a way that μ_1 is dominant with respect to all positive roots of $(\mathfrak{m}, \mathfrak{b})$.

We are given that $p\xi|_{M_0} \cong \xi|_{M_0}$. By Proposition 4.9b and Corollary 4.6, $p\mu_1 = m_f\mu_1$ in terms of the action of Theorem 3.7. Now Theorem 3.7 says that p leaves stable the positive roots of $(\mathfrak{m}, \mathfrak{b})$. Thus $\gamma > 0$ implies $p^{-1}\gamma > 0$ and

$$\langle p\mu_1, \gamma\rangle = \langle \mu_1, p^{-1}\gamma\rangle > 0,$$

μ_1 being dominant nonsingular. Thus $p\mu_1$ is dominant and $m_f\mu_1$ must be dominant. According to Proposition 4.5, m_f is in the complex Weyl group, and thus $m_f\mu_1 = \mu_1$. Thus $p\mu_1 = \mu_1$.

Theorem 3.7b says that the action of W_e on $(i\mathfrak{b})'$ is isomorphic with the standard Weyl group action of W_e. By Chevalley's Lemma, $p\mu_1 = \mu_1$ implies that p is the product of reflections p_{α_R} fixing μ_1. But $p_{\alpha_R}\mu_1 = \mu_1$ implies that $p_{\alpha_R}\xi|_{M_0} \cong \xi|_{M_0}$, by Proposition 4.9b. This completes the proof.

§5. Plancherel factors for parabolic rank one cases

We continue with the notation of §2. In this section we shall assemble all the tools needed to reduce the study of the R group to the case of a minimal parabolic in a group split over $\mathbb{R}$.

For this purpose we recall that we can associate an element γ_α in G_0 to each $\mathfrak{a}_p$-root α by the definition

$$\gamma_\alpha = \exp(2\pi i|\alpha|^{-2}H_\alpha). \tag{5.1}$$

Properties of the elements γ_α are assembled on page 279 of [11]; each γ_α satisfies $\gamma_\alpha^2 = 1$ and is in Z_{M_p}.

We recall also that if ξ is an irreducible unitary representation of M, then we can associate a "Plancherel factor" $\mu_{\xi,\alpha_R}(\nu)$ to each reduced positive $\mathfrak{a}$-root α_R. This is a meromorphic function of ν in $\mathfrak{a}'^{\mathbb{C}}$ obtained from intertwining operators and is holomorphic for ν imaginary. It depends only on the projection of ν into $(\mathfrak{a}^{(\alpha_R)})'^{\mathbb{C}}$, and it may therefore be treated as a function of one complex variable. If ξ is in the discrete series, it appears in the definition of the R group, and its vanishing properties are related to reducibility questions. See §§10–13 of [13].

The main result of this section is as follows.

Proposition 5.1: *Suppose ϵ is a reduced $\mathfrak{a}$-root and ξ is a discrete series representation of M.*

(a) *If* ϵ *is even and useful and if* $p_\epsilon\xi|_{M_0} \cong \xi|_{M_0}$, *then* $p_\epsilon\xi \cong \xi$ *and* $\mu_{\xi,\epsilon}(0) = 0$.

(b) *If* ϵ *is odd and* 2ϵ *is an* $\mathfrak{a}$*-root, then* $p_\epsilon\xi \cong \xi$.

(c) *If* ϵ *is odd and* 2ϵ *is not an* $\mathfrak{a}$*-root, then* $\mu_{\xi,\epsilon}(0) = 0$ *if* $\xi(\gamma_\epsilon) = +I$ *and* $\mu_{\xi,\epsilon}(0) \neq 0$ *if* $\xi(\gamma_\epsilon) = -I$. *If* $\xi(\gamma_\epsilon) = +I$, *then* $p_\epsilon\xi \cong \xi$.

(d) *If* ϵ *is odd and* 2ϵ *is an* $\mathfrak{a}$*-root and if* δ *is an odd* $\mathfrak{a}$*-root such that* $p_\delta\xi(\gamma_{2\epsilon}) \neq \xi(\gamma_{2\epsilon})$, *then* $\mu_{p_\delta\xi,\epsilon}(0)$ *is zero if and only if* $\mu_{\xi,\epsilon}(0)$ *is not zero.*

LEMMA 5.2: *Let* ϵ *be a useful* $\mathfrak{a}$*-root. If* z *is in* $Z_M \cap F$, *then*

$$zp_\epsilon^{-1}z^{-1}p_\epsilon = \gamma_\epsilon \text{ or } 1,$$

where p_ϵ, *the reflection in* ϵ, *is extended to be in* $W(\mathfrak{a}_p)$ *and where* γ_ϵ *is the element of* $Z_M \cap F$ *given by*

$$\gamma_\epsilon = \exp 2\pi i|\epsilon|^{-2}H_\epsilon. \tag{5.2}$$

If χ *is a character of* $Z_M \cap F$, *then* $p_\epsilon\chi = \chi$ *if* $\chi(\gamma_\epsilon) = 1$.

REMARK: Formula (5.2) deals with $\mathfrak{a}$-roots, and formula (5.1) deals with $\mathfrak{a}_p$-roots. Part of the conclusion here is that (5.2) necessarily leads to an element of $Z_M \cap F$.

PROOF: Since $\chi(z)(p_\epsilon\chi(z))^{-1} = \chi(zp_\epsilon^{-1}z^{-1}p_\epsilon)$ and since p_ϵ normalizes both Z_M and F, the second statement follows from the first. Since z is in F, write

$$z = z'\exp \pi i(H_\mathfrak{a} + H_{\mathfrak{a}_M}) \tag{5.3}$$

with z' in $Z(G)$, $H_\mathfrak{a}$ in $\mathfrak{a}$, and $H_{\mathfrak{a}_M}$ in $\mathfrak{a}_M$. By Lemma 4 of [11], -1 is in $W(\mathfrak{a}_M)$; let w_M be a representative. Since z is in Z_M, we have

$$z = w_M^{-1}zw_M = \theta(w_M^{-1}zw_M)$$

and therefore

$$\begin{aligned} Ad(\exp \pi i(H_\mathfrak{a} + H_{\mathfrak{a}_M})) &= Ad(\exp \pi i(H_\mathfrak{a} - H_{\mathfrak{a}_M})) \\ &= Ad(\exp \pi i(-H_\mathfrak{a} + H_{\mathfrak{a}_M})). \end{aligned}$$

Hence

$$Ad(\exp 2\pi i H_\mathfrak{a}) = 1 = Ad(\exp 2\pi i H_{\mathfrak{a}_M}). \tag{5.4}$$

Applying the first of these equalities to a root vector for the $\mathfrak{a}$-root ϵ, we obtain

$$\epsilon(H_{\mathfrak{a}}) \in \mathbb{Z}. \tag{5.5}$$

First suppose ϵ is odd. Since the conclusion of the lemma is independent of the representative of p_ϵ, we may extend p_ϵ so as to be the identity on $\mathfrak{a}_M$. If we write $H_{\mathfrak{a}}$ above as

$$H_{\mathfrak{a}} = n|\epsilon|^{-2}H_\epsilon + H_\epsilon^\perp$$

with $H_\epsilon^\perp$ orthogonal to H_ϵ, then (5.5) shows n is an integer. Moreover

$$(H_{\mathfrak{a}} + H_{\mathfrak{a}_M}) - p_\epsilon(H_{\mathfrak{a}} + H_{\mathfrak{a}_M}) = 2n|\epsilon|^{-2}H_\epsilon.$$

When we use (5.3) to form $zp_\epsilon^{-1}z^{-1}p_\epsilon$, the z' cancels and we obtain

$$zp_\epsilon^{-1}z^{-1}p_\epsilon = \exp(2n\pi i|\epsilon|^{-2}H_\epsilon) = \gamma_\epsilon^n,$$

as required.

Now let us suppose that ϵ is even and useful. If ϵ, extended by 0 on $\mathfrak{a}_M$, is an $\mathfrak{a}_p$-root, then the argument in the previous paragraph applies and gives the desired conclusion. Thus we shall assume that ϵ does not extend by 0 to become an $\mathfrak{a}_p$-root. From the first paragraph of §4 of [11], we see that there must exist orthogonal $\mathfrak{a}_p$-roots $\alpha = \epsilon + \epsilon'$ and $\bar{\alpha} = \epsilon - \epsilon'$ extending ϵ. If we apply the second equality of (5.4) to a root vector for the $\mathfrak{a}_p$-root α, we obtain

$$\epsilon'(H_{\mathfrak{a}_M}) \in \mathbb{Z}. \tag{5.6}$$

Then we have

$$H_{\mathfrak{a}} = n|\epsilon|^{-2}H_\epsilon + H_\epsilon^\perp$$

and

$$H_{\mathfrak{a}_M} = m|\epsilon|^{-2}H_{\epsilon'} + H_{\epsilon'}^\perp$$

with n and m both integers. Lemma 11 of [11] shows that we may use $p_\alpha p_{\bar{\alpha}}$ as an extension of p_ϵ. Then

$$(H_{\mathfrak{a}} + H_{\mathfrak{a}_M}) - p_\alpha p_{\bar{\alpha}}(H_{\mathfrak{a}} + H_{\mathfrak{a}_M}) = 2n|\epsilon|^{-2}H_\epsilon + 2m|\epsilon|^{-2}H_{\epsilon'}.$$

Thus

$$zp_\epsilon^{-1}z^{-1}p_\epsilon = \exp 2\pi i\left(\frac{n+m}{2|\epsilon|^2}(H_\epsilon + H_{\epsilon'}) + \frac{n-m}{2|\epsilon|^2}(H_\epsilon - H_{\epsilon'})\right)$$

$$(5.7) \qquad = \gamma_\alpha^{n+m}\gamma_{\bar\alpha}^{n-m}.$$

By (5.6), n and m are both integers and thus $n+m$ is congruent to $n-m$ modulo 2. The element (5.7) must therefore be $\gamma_\alpha\gamma_{\bar\alpha}$ or 1. When $n=1$ and $m=0$, we see from (5.7) that $\gamma_\alpha\gamma_{\bar\alpha} = \gamma_\epsilon$, with γ_ϵ defined as in (5.2). Hence (5.7) is always either γ_ϵ or 1, and the proof is complete.

LEMMA 5.3: *Let $\epsilon + \epsilon'$ be an extension to $\mathfrak{a} + i\mathfrak{b}$ of an even useful $\mathfrak{a}$-root ϵ. Then*

$$\exp 2\pi i|\epsilon|^{-2}H_\epsilon = \exp 2\pi i|\epsilon|^{-2}H_{\epsilon'}.$$

PROOF: The quotient of the left side by the right side is the element in G^{C} given by

$$\{\exp 2\pi i|\epsilon - \epsilon'|^{-2}H_{\epsilon-\epsilon'}\}^2,$$

since $|\epsilon - \epsilon'|^2 = 2|\epsilon|^2$ by Lemma 2.5. This is of the form γ_α^2 for α the root of G^{C} given by $\epsilon - \epsilon'$, and thus it equals 1.

LEMMA 5.4: *Suppose that H_0 is in $i\mathfrak{b}$ and $z = \exp 2\pi i H_0$ is in Z_M. Let ξ be a discrete series representation of M, and suppose μ_1 is one of the Harish-Chandra parameters of ξ and is dominant. Let ρ_M be half the sum of the positive roots of $(\mathfrak{m}, \mathfrak{b})$. Then $\xi(z)$ is the scalar $\exp 2\pi i(\mu_1 - \rho_M)(H_0)$.*

REMARKS: This result can be deduced from a careful reading of Harish-Chandra [2]. However, we give a proof that uses the subsequent work on discrete series by Schmid.

PROOF: Let ρ_n be half the sum of the positive noncompact roots of $(\mathfrak{m}, \mathfrak{b})$. Then $\mu_1 + 2\rho_n - \rho_M$ is the Blattner parameter (lowest highest weight) of the component of $\xi|_{M_0}$ with Harish-Chandra parameter μ_1. From the work of Schmid ([15], Theorem 1.3) there exists a vector in the representation space on which $\mathfrak{b}$ acts with weight $\mu_1 + 2\rho_n - \rho_M$. On this vector $\xi(z)$ acts by the scalar

$$(5.8) \qquad \exp(\mu_1 + 2\rho_n - \rho_M)(2\pi i H_0).$$

Since z is central, $\xi(z)$ acts by (5.8) everywhere. In this expression, $\exp 2\rho_n(2\pi i H_0) = 1$ since $2\rho_n$ is a sum of roots and since z is central. The lemma follows.

PROOF OF PROPOSITION 5.1a: Introduce an ordering in $(i\mathfrak{b})'$ as in the proof of Theorem 4.10 so that we can obtain the conclusion from Proposition 4.9b that $p_\epsilon\mu_1 = \mu_1$ for one of the Harish-Chandra parameters μ_1 of ξ. To obtain $p_\epsilon\xi \cong \xi$, we are to prove that the central character χ of ξ on $Z_M \cap F$ has $p_\epsilon\chi = \chi$, and Lemma 5.2 shows that it is enough to show $\chi(\gamma_\epsilon) = 1$. Define ϵ' as in Lemma 3.8a. Then Lemma 5.3 shows that

$$\gamma_\epsilon = \exp 2\pi i|\epsilon|^{-2}H_{\epsilon'}. \tag{5.9}$$

The action of p_ϵ on $(i\mathfrak{b})'$ being by $p_{\epsilon'}$, the equality $p_\epsilon\mu_1 = \mu_1$ means that $p_{\epsilon'}\mu_1 = \mu_1$. Since $p_{\epsilon'}$ leaves stable the positive roots of $(\mathfrak{m}, \mathfrak{b})$, we have $p_{\epsilon'}\rho_M = \rho_M$, where ρ_M is half the sum of the positive roots of $(\mathfrak{m}, \mathfrak{b})$. Thus ϵ' is orthogonal to $\mu_1 - \rho_M$. Applying Lemma 5.4 with $H_0 = |\epsilon|^{-2}H_{\epsilon'}$ and using (5.9), we find that $\xi(\gamma_\epsilon)$ is the identity, i.e., $\chi(\gamma_\epsilon) = 1$. Consequently $p_\epsilon\xi \cong \xi$.

Now we prove that $\mu_{\xi,\epsilon}(0) = 0$. In fact, Harish-Chandra's Lemma 18 in [3] shows that the representation of $G^{(\epsilon)}$ induced from $MA^{(\epsilon)}N^{(\epsilon)}$ with ξ on M and trivial on A is irreducible, since $G^{(\epsilon)}$ has no discrete series by Theorem 13 of [2] and by Lemma 2.2. We have just seen that $p_\epsilon\xi \cong \xi$. Hence Corollary 12.8 of [13] implies that $\mu_{\xi,\epsilon}(0) = 0$.

PROOF OF PROPOSITION 5.1b: Lemma 3.8b shows that p_ϵ fixes the parameter set of ξ. In Lemma 5.2 it is clear that $\gamma_\epsilon = \gamma_{2\epsilon}^2 = 1$, and hence that lemma shows that p_ϵ fixes the central character of ξ. By Proposition 4.9a, $p_\epsilon\xi$ is equivalent with ξ.

Before moving to the next part of Proposition 5.1, let us recall how Plancherel factors are computed in the spirit of [13]. If ξ is a discrete series representation of M, then ξ can be imbedded infinitesimally as a subrepresentation of a nonunitary principal series representation of M, by a theorem of Casselman (cf. §5 of [13]). Say ξ imbeds in the nonunitary principal series with parameters (σ, λ_M), where σ is a representation of the compact group M_p and λ_M is in $\mathfrak{a}_M'$. Writing p instead of μ for the Plancherel factor associated to a real-rank one group and letting ϵ be a reduced $\mathfrak{a}$-root, we have from Proposition 10.2d of [13]

$$\mu_{\xi,\epsilon}(\nu) = c \prod_{n=1}^{3} \prod_{\alpha|_{\mathfrak{a}}=n\epsilon} p_{\sigma,\alpha}((\nu + \lambda_M)|_{\mathfrak{a}^{(\alpha)}}), \tag{5.10}$$

with α in each product denoting $\mathfrak{a}_p$-roots with the indicated restrictions to $\mathfrak{a}$. Here c is a nonzero constant independent of ξ, and n assumes the value 3 only in one exceptional case arising from split G_2.

LEMMA 5.5: *Suppose that H is in $\mathfrak{a}_M$ and $z = \exp 2\pi i H$ is in Z_M. Let ξ be a discrete series representation of M that imbeds in the nonunitary principal series with parameters (σ, λ_M), and let ρ_M^+ be half the sum of the positive $\mathfrak{a}_M$-roots, with their multiplicities. Then*

$$(\lambda_M - \rho_M^+)(H)\begin{cases} \text{is in } \tfrac{1}{2}\mathbb{Z} \text{ always} \\ \text{is in } \mathbb{Z} \text{ if } \xi(z) = I. \end{cases}$$

PROOF: Since $z = \theta z = \exp(-2\pi i H) = z^{-1}$, we have $z^2 = 1$. Now recall that the Cayley transform $\mathbf{c}: (\mathfrak{a}_p + i\mathfrak{b}_0) \to \mathfrak{a} + i\mathfrak{b}$ is a member $Ad(\exp X)$ of $Ad(\exp \mathfrak{m}^C)$, by construction. Let $m = \exp X$ in M^C. Since z is in Z_M, $Ad(z) = 1$ on $\mathfrak{m}^C$ and

$$zmz^{-1} = z(\exp X)z^{-1} = \exp Ad(z)X = \exp X = m.$$

Then it follows that

$$\begin{aligned} z = mzm^{-1} &= m(\exp 2\pi i H)m^{-1} = \exp 2\pi i\, Ad(m)H \\ &= \exp 2\pi i \mathbf{c}(H). \end{aligned} \tag{5.11}$$

Let μ_1 be a Harish-Chandra parameter for ξ, and introduce an ordering for $(i\mathfrak{b})'$ that makes μ_1 dominant. Since the infinitesimal characters of ξ and the nonunitary principal series representation of M must be the same, we have

$$\mu_1 = w\mathbf{c}(\lambda_M + \Lambda^- + \rho^-) \tag{5.12}$$

for a suitable element w in the complex Weyl group of $(\mathfrak{m}, \mathfrak{b})$, where Λ^- is a highest weight of σ and ρ^- is half the sum of the positive roots of $i\mathfrak{b}_0$. If we regard w as in M^C, then $wzw^{-1} = z$ since z is in Z_M, and we obtain

$$z = \exp 2\pi i w\mathbf{c}(H) \tag{5.13}$$

from (5.11).

Let $\rho_{\mathfrak{b}}$ be half the sum of the positive roots of $(\mathfrak{m}, \mathfrak{b})$. Applying Lemma 5.4 with $H_0 = w\mathbf{c}(H)$, we see from (5.13) that $\xi(z)$ acts as the scalar

$$\exp 2\pi i(\mu_1 - \rho_{\mathfrak{b}})(wc(H)).$$

This scalar must be ± 1 since $z^2 = 1$. In view of (5.12), we therefore conclude that

$$(wc(\lambda_M + \Lambda^- + \rho^-) - \rho_{\mathfrak{b}})(wc(H)) \text{ is in } \tfrac{1}{2}\mathbb{Z},$$

and is actually in $\mathbb{Z}$ if $\xi(z) = I$. Write $\rho^- = \rho_{\mathfrak{a}_M + i\mathfrak{b}_0} - \rho_M^+$. Then $wc\rho_{\mathfrak{a}_M + i\mathfrak{b}_0} - \rho_{\mathfrak{b}}$ is an integral combination of roots and acts as an integer on $wc(H)$ since z is central. Hence

$$wc(\lambda_M + \Lambda^- - \rho_M^+)(wc(H)) \text{ is in} \begin{cases} \tfrac{1}{2}\mathbb{Z} \text{ always} \\ \mathbb{Z} \text{ if } \xi(z) = I. \end{cases}$$

Since $\Lambda^-(H) = 0$, the lemma follows.

Proof of Proposition 5.1c: Let $\epsilon + \epsilon' + \gamma$ be an $(\mathfrak{a}_p + i\mathfrak{b}_0)$-root with $\epsilon' \neq 0$. By Lemma 2 of [11] and Lemma 2.5, we have

$$|\epsilon'|^2 = |\epsilon|^2 = |\epsilon' + \gamma|^2.$$

Since γ is orthogonal to ϵ', $\gamma = 0$. Thus the $\mathfrak{a}_p$-roots of the form $\epsilon + \epsilon'$ with $\epsilon' \neq 0$ all have multiplicity one. Also ϵ must have odd multiplicity as an $\mathfrak{a}_p$-root. In (5.10), only the product for $n = 1$ is present, since 2ϵ is not an $\mathfrak{a}$-root, and we have

$$\mu_{\xi,\epsilon}(\nu) = cp_{\sigma,\epsilon}((\nu + \lambda_M)|_{\mathfrak{a}^{(\epsilon)}}) \times \prod_{\substack{\epsilon' > 0 \\ \alpha = \epsilon + \epsilon'}} \{p_{\sigma,\alpha}((\nu + \lambda_M)|_{\mathfrak{a}^{(\alpha)}}) p_{\sigma,\bar{\alpha}}((\nu + \lambda_M)|_{\mathfrak{a}^{(\bar{\alpha})}})\}.$$

Let us set $\nu = 0$, remembering that we should really do a passage to the limit. Each $p(\cdot)$ is an even function and the p's in braces correspond to $SL(2, \mathbb{R})$ since the $\mathfrak{a}$'s have multiplicity one. Letting

$$p_+(z) = z \tan \pi z/2$$

$$p_-(z) = z \cot \pi z/2, \tag{5.14}$$

we therefore have

$$\mu_{\xi,\epsilon}(0) = c' p_{\sigma,\epsilon}(0) \prod_{\substack{\epsilon'>0 \\ \alpha=\epsilon+\epsilon'}} \left\{ p_{\sigma(\gamma_\alpha)}\left(\frac{2\langle\lambda_M, \alpha\rangle}{|\alpha|^2}\right) \times \times p_{\sigma(\gamma_{\bar\alpha})}\left(\frac{2\langle\lambda_M, \alpha\rangle}{|\alpha|^2}\right)\right\}. \tag{5.15}$$

From the theory of $\mathfrak{so}(2n, 1)$ as summarized in §16 of [12], $p_{\sigma,\epsilon}(0)$ is zero or nonzero according as $\sigma(\gamma_\epsilon)$ is $+I$ or $-I$. Also $\xi(\gamma_\epsilon)$ has to be the same scalar as $\sigma(\gamma_\epsilon)$ since γ_ϵ is central. Thus we are to show that the factor $\Pi_{\epsilon'>0}\{-\}$ in (5.15) is regular and nonzero.

Letting $\alpha = \epsilon + \epsilon'$, we have

$$\gamma_\epsilon = \exp 2\pi i|\epsilon|^{-2}H_\epsilon = \exp 2\pi i|\alpha|^{-2}(H_\alpha + H_{\bar\alpha}) = \gamma_\alpha\gamma_{\bar\alpha} \tag{5.16a}$$

$$= \gamma_\alpha\gamma_{\bar\alpha}^{-1} = \exp 2\pi i|\alpha|^{-2}(H_\alpha - H_{\bar\alpha}) = \exp 2\pi i|\epsilon|^{-2}H_{\epsilon'}. \tag{5.16b}$$

From (5.16a),

$$\sigma(\gamma_{\bar\alpha}) = \sigma(\gamma_\alpha)\sigma(\gamma_\epsilon). \tag{5.17}$$

Also ϵ' is an $(\mathfrak{a}_M + i\mathfrak{b}_0)$-root and the infinitesimal character of ξ is nonsingular, so that $\langle\lambda_M, \epsilon'\rangle \neq 0$. If $\xi(\gamma_\epsilon) = -I$, then (5.14) and (5.17) show that the tangents in the factor $\Pi_{\epsilon'>0}\{-\}$ of (5.15) cancel the cotangents, and we obtain

$$p_{\sigma(\gamma_\alpha)}\left(\frac{2\langle\lambda_M, \alpha\rangle}{|\alpha|^2}\right)p_{\sigma(\gamma_{\bar\alpha})}\left(\frac{2\langle\lambda_M, \alpha\rangle}{|\alpha|^2}\right) = |\epsilon|^{-4}\langle\lambda_M, \epsilon'\rangle^2 \neq 0, \tag{5.18}$$

as required. If $\xi(\gamma_\epsilon) = +I$, Lemma 5.5 and (5.16b) say

$$(\lambda_M - \rho_M^+)(|\epsilon|^{-2}H_{\epsilon'}) \text{ is in } \mathbb{Z}.$$

That is,

$$\frac{2\langle\lambda_M - \rho_M^+, \alpha\rangle}{|\alpha|^2} \text{ is in } \mathbb{Z}. \tag{5.19}$$

If we show that

$$\frac{2\langle\rho_M^+, \alpha\rangle}{|\alpha|^2} = \frac{\langle\rho_M^+, \epsilon'\rangle}{|\epsilon'|^2} \text{ is in } \mathbb{Z} + \tfrac{1}{2} \tag{5.20}$$

for all ϵ', then we can conclude that all the tangent-cotangent factors are evaluated at odd multiples of $\pi/4$ and are harmless. Thus (5.18) holds in this case also, up to sign.

To prove (5.20), first let $\epsilon+\epsilon'$ and $\epsilon\pm\epsilon''$ be roots, with $\epsilon'\neq\pm\epsilon''$. By the Schwarz inequality, $\langle\epsilon',\epsilon''\rangle<|\epsilon|^2$, whence

$$\langle\epsilon+\epsilon',\epsilon+\epsilon''\rangle>0 \text{ and } \langle\epsilon+\epsilon',\epsilon-\epsilon''\rangle>0.$$

Thus $\epsilon'\pm\epsilon''$ are both roots and $|\epsilon'\pm\epsilon''|^2=|\epsilon+\epsilon'|^2=2|\epsilon'|^2$. Then

$$\frac{\langle\rho_M^+,\epsilon'\rangle}{|\epsilon'|^2}=\frac{2\langle\rho_M^+,\epsilon'-\epsilon''\rangle}{|\epsilon'-\epsilon''|^2}+\frac{\langle\rho_M^+,\epsilon''\rangle}{|\epsilon''|^2}\in\mathbb{Z}+\frac{\langle\rho_M^+,\epsilon''\rangle}{|\epsilon''|^2},$$

and the truth of (5.20) is independent of ϵ'.

If ϵ' is the smallest positive supplement for ϵ and if $\delta>0$ is a simple $\mathfrak{a}_M$-root ($\neq\epsilon'$), then $\epsilon+(\epsilon'-\delta)$ is not a root and so $\langle\epsilon',\delta\rangle\leq 0$. Consequently ϵ' is simple for $\mathfrak{a}_M$. Then $2\langle\rho_M^+,\epsilon'\rangle/|\epsilon'|^2=1$, and (5.20) follows.

LEMMA 5.6: *Let φ be an automorphism of M that leaves stable a minimal parabolic subgroup $S=M_pA_MN_M$ of M and fixes Haar measure on M and S. Let τ be a representation of S and set $\tau^\varphi(s)=\tau(\varphi^{-1}(s))$. Then $\operatorname{ind}_{S\uparrow M}\tau^\varphi$ is equivalent with $(\operatorname{ind}_{S\uparrow M}\tau)^\varphi$.*

PROOF: If F is in the first induced space, then $F\circ\varphi$ is in the second induced space. The rest consists of diagram-chasing.

PROOF OF PROPOSITION 5.1d: Choose the special representative w of p_δ given in Theorem 3.7 and let $\varphi(m)=w^{-1}mw$. Suppose ξ imbeds in the nonunitary principal series of M with parameters (σ,λ_M). Then Lemma 5.6 shows that a representation equivalent with $p_\delta\xi=\xi^\varphi$ imbeds in the nonunitary principal series of M with parameters $(\sigma^\varphi,\lambda_M^\varphi)=(\sigma^\varphi,\lambda_M)$.

Let us observe that when $n=3$ is possible in (5.10), no δ satisfies the hypotheses of Proposition 5.1d. Putting $\nu=0$ in (5.10), we therefore see that we are to show that when we replace σ by $p_\delta\sigma$ in the expression

$$\mu_{\xi,\epsilon}(0)=c\prod_{\alpha|_{\mathfrak{a}}=\epsilon}p_{\sigma,\alpha}(\lambda_M|_{\mathfrak{a}^{(\alpha)}})\prod_{\alpha|_{\mathfrak{a}}=2\epsilon}p_{\sigma,\alpha}(\lambda_M|_{\mathfrak{a}^{(\alpha)}}),\tag{5.21}$$

then the whole expression switches from zero to nonzero, or vice-versa.

The element $\gamma_{2\epsilon}$ is in the center of M. Thus the hypothesis $p_\delta\xi(\gamma_{2\epsilon}) \neq \xi(\gamma_{2\epsilon})$ implies that $p_\delta\xi$ and ξ are not equivalent. But Lemma 3.8 and Proposition 4.9b together imply that $p_\delta\xi|_{M_0}$ and $\xi|_{M_0}$ are equivalent. Thus $\gamma_{2\epsilon}$ is not in M_0.

Let $\epsilon + \beta$ be an extension of the $\mathfrak{a}$-root ϵ to a root of $\mathfrak{a} + i\mathfrak{b}$. We must have

$$\frac{2\langle \epsilon + \beta, \epsilon - \beta\rangle}{|\epsilon + \beta|^2} = 1 \text{ or } 0 \text{ or } -1. \tag{5.22}$$

We shall show that ± 1 are not possible values for (5.22). Define $\gamma_{\epsilon+\beta} = \exp 2\pi i|\epsilon + \beta|^{-2}H_{\epsilon+\beta}$ within the three-dimensional complex subgroup of $G^{\mathbb{C}}$ corresponding to the root $\epsilon + \beta$, and define $\gamma_{\epsilon-\beta}$ similarly. If (5.22) is ± 1, then $2\langle \epsilon - \beta, 2\epsilon\rangle/|\epsilon - \beta|^2$ is odd and

$$\gamma_{\epsilon+\beta} = \gamma_{p_{2\epsilon}(\epsilon-\beta)} = \gamma_{\epsilon-\beta}\gamma_{2\epsilon}^{2\langle\epsilon-\beta, 2\epsilon\rangle/|\epsilon-\beta|^2} = \gamma_{\epsilon-\beta}\gamma_{2\epsilon}.$$

Hence

$$\begin{aligned}\gamma_{2\epsilon} &= \gamma_{\epsilon+\beta}\gamma_{\epsilon-\beta} = \gamma_{\epsilon+\beta}\gamma_{\epsilon-\beta}^{-1} = \exp 2\pi i|\epsilon + \beta|^{-2}(H_{\epsilon+\beta} - H_{\epsilon-\beta})\\ &= \exp 2\pi i|\epsilon + \beta|^{-2}H_{2\beta}\end{aligned}$$

exhibits $\gamma_{2\epsilon}$ as in $\exp \mathfrak{b} \subseteq M_0$. Thus the left side of (5.22) is 0.

Now form $\mathbf{c}^{-1}(\beta)|_{\mathfrak{a}_M} = \epsilon'$. We cannot have $\epsilon' = 0$, since otherwise ϵ and 2ϵ would be $\mathfrak{a}_p$-roots and Proposition 5 of [11] would force $\gamma_{2\epsilon}$ to be in M_0. Then we must have $\mathbf{c}^{-1}(\beta) = \epsilon'$. In fact, otherwise

$$0 = \langle \epsilon + \beta, \epsilon - \beta\rangle = \langle \epsilon + \mathbf{c}^{-1}(\beta), \epsilon - \mathbf{c}^{-1}(\beta)\rangle < \langle \epsilon + \epsilon', \epsilon - \epsilon'\rangle,$$

and we conclude ϵ is not useful, in contradiction to Lemma 2.7a. Thus we conclude that if $\epsilon + \epsilon'$ is an extension of ϵ to an $\mathfrak{a}_p$-root, then $\epsilon' \neq 0$ and $\epsilon + \epsilon'$ has multiplicity 1. Moreover, $2\epsilon'$ is a root of $\mathfrak{a}_M + i\mathfrak{b}_0$, by (5.22).

Since we are assuming ϵ is an $\mathfrak{a}$-root, $\epsilon + \beta$ and 2ϵ provide roots of two lengths for $\mathfrak{a} + i\mathfrak{b}$, and there cannot be a longer root $2\epsilon + \beta'$. Thus 2ϵ has multiplicity one as an $\mathfrak{a}$-root.

Thus, in terms of (5.14), (5.21) is just

$$\mu_{\xi,\epsilon}(0) = c' p_{\sigma(\gamma_{2\epsilon})}(0) \prod_{\substack{\epsilon'>0\\ \alpha=\epsilon+\epsilon'}} p_{\sigma(\gamma_\alpha)}\left(\frac{2\langle\lambda_M, \alpha\rangle}{|\alpha|^2}\right) p_{\sigma(\gamma_{\bar\alpha})}\left(\frac{2\langle\lambda_M, \alpha\rangle}{|\alpha|^2}\right).$$

Now $\gamma_\alpha = \exp 2\pi i|\alpha|^{-2}H_{\epsilon+\epsilon'} = \exp 2\pi i|2\epsilon|^{-2}(H_{2\epsilon} + H_{2\epsilon'}) = \gamma_{2\epsilon}\gamma_{2\epsilon'}$

and similarly for $\gamma_{\alpha'}$. Thus

$$\mu_{\xi,\epsilon}(0) = c' p_{\xi(\gamma_{2\epsilon})}(0)\Big\{ \prod_{\substack{\epsilon'>0 \\ \alpha=\epsilon+\epsilon'}} p_{\xi(\gamma_{2\epsilon})\sigma(\gamma_{2\epsilon'})}\Big(\frac{2\langle\lambda_M, 2\epsilon'\rangle}{|2\epsilon'|^2}\Big)\Big\}^2. \tag{5.23}$$

Here $\langle\lambda_M, 2\epsilon'\rangle$ is not 0 since $2\epsilon'$ is a root of $\mathfrak{a}_M + i\mathfrak{b}_0$ and ξ has nonsingular infinitesimal character.

Let the expression in braces in (5.23) be P^+ and let the corresponding expression for $p_\delta\xi$ be P^-. Since $p_\delta\sigma(\gamma_{2\epsilon'}) = \sigma(\gamma_{p_\delta(2\epsilon')}) = \sigma(\gamma_{2\epsilon'})$ and since $\langle\lambda_M, 2\epsilon'\rangle$ is nonzero, we see that $P^- = 1/P^+$ except for a nonzero constant. Thus we are to compare

$$p_{\xi(\gamma_{2\epsilon})}(0)P^+ \tag{5.24a}$$

and

$$p_{-\xi(\gamma_{2\epsilon})}(0)/P^+, \tag{5.24b}$$

both of which must be holomorphic even functions before the evaluation at $\nu = 0$. Because of the known behavior of $p_+(0)$ and $p_-(0)$, P^+ must have a double pole or be regular nonvanishing or have a double zero at $\nu = 0$. If P^+ has a double pole, then $p_{\xi(\gamma_{2\epsilon})}(0)$ must have a double zero (by (5.24a)), and (5.24a) is nonvanishing. In this case (5.24b) clearly vanishes. Similarly if P^+ has a double zero, then (5.24a) vanishes and (5.24b) does not. Finally if P^+ is regular and nonvanishing, then we can drop P^+ in (5.24a) and (5.24b), and one factor vanishes and the other does not, by (5.14). This finishes the proof of Proposition 5.1.

§6. Reduction of $R = \Sigma\, \mathbf{Z}_2$ to split case and minimal parabolic

We are now in a position to treat questions of reducibility. The R group deals with induced representations

$$\operatorname*{ind}_{MAN\uparrow G} (\xi \otimes \exp\Lambda \otimes 1), \tag{6.1}$$

where ξ is a discrete series representation of M and $\exp\Lambda$ is a unitary character of A. In the notation of §13 of [13], let

$$W_{\xi,\Lambda} = \{s \in W(\mathfrak{a}) \mid s\xi \cong \xi \text{ and } s\Lambda = \Lambda\}$$

$$\Delta' = \{\beta \in \Delta \mid \mu_{\xi,\beta}(\Lambda|_{\mathfrak{a}^{(\beta)}}) = 0 \text{ and } p_\beta \in W_{\xi,\Lambda}\}$$

$$W'_{\xi,\Lambda} = \text{Weyl group of } \Delta'$$

$$R_{\xi,\Lambda} = \{r \in W_{\xi,\Lambda} \mid r\beta > 0 \text{ for every } \beta > 0 \text{ in } \Delta'\}.$$

Each member of $W_{\xi,\Lambda}$ leads to a unitary self-intertwining operator for the representation (6.1), and Theorem 13.4 of [13] says that the operators corresponding to the subgroup $R_{\xi,\Lambda}$ form a linear basis of the commuting algebra of (6.1).

THEOREM 6.1: *If ξ is a discrete series representation of M and* $\exp \Lambda$ *is a unitary character of A, then the group $R_{\xi,\Lambda}$ is a finite direct sum of copies of the two-element group $\mathbb{Z}_2$, with the number of copies bounded above by the dimension of A.*

According to Lemma 14.1 of [13], it is enough to prove this theorem for $\Lambda = 0$, and we shall therefore limit ourselves to this case for the remainder of this section. Theorem 15.1 of [13] establishes Theorem 6.1 if G is a connected split semisimple Lie group of matrices, and we shall proceed by reducing the general case to this special case.

Recall the notation of §§2–4. Let $\mu = \{\mu_j\}$ be the parameter set of ξ, and, by means of the action in Theorem 3.7, define

$$W_\mu = \{w \in W(\mathfrak{a}) \mid w\mu = \mu\}$$

$$W_{e,\mu} = W_e \cap W_\mu.$$

PROPOSITION 6.2: *$W_\mu = W_{e,\mu}S$, and $W_{e,\mu}$ is generated by its own reflections. Moreover,*

$$W_{e,\mu} \subseteq W'_{\xi,0} \subseteq W_{\xi,0} \subseteq W_\mu.$$

PROOF: Proposition 3.9 says $W(\mathfrak{a}) = W_eS$. Since $S \subseteq W_\mu$, $W_{e,\mu}S \subseteq W_\mu$. Conversely if w in W_μ decomposes as w_es, then w_e is in both W_e and W_μ, hence in $W_{e,\mu}$. Thus $W_\mu = W_{e,\mu}S$.

$W_{e,\mu}$ is generated by its own reflections, by Theorem 4.10 and Proposition 4.9b. Each such reflection is in $W'_{\xi,0}$ by Proposition 5.1a, and hence $W_{e,\mu} \subseteq W'_{\xi,0}$. The inclusion $W'_{\xi,0} \subseteq W_{\xi,0}$ holds by definition, and the inclusion $W_{\xi,0} \subseteq W_\mu$ follows from Proposition 4.9.

COROLLARY 6.3: *If every simple useful* $\mathfrak{a}$*-root is even, then* $R_{\xi,0} = \{1\}$. *Consequently the representation* (6.1) *is irreducible.*

PROOF: If every simple useful $\mathfrak{a}$-root is even, then $W_e = W(\mathfrak{a})$ and so $W_{e,\mu} = W_\mu$. We then see from the proposition that $W'_{\xi,0} = W_{\xi,0}$. Theorem 13.4 of [13] shows that $R_{\xi,0} \cong W_{\xi,0}/W'_{\xi,0}$, and the corollary follows.

We defined Δ_0 as a subset of Δ by (3.10), and we let

$$\Delta_{e,\mu} = \{\beta \in \Delta \mid p_\beta \in W_{e,\mu}\}$$

$$\Delta_\mu = \Delta_0 \cup S\Delta_{e,\mu}.$$

We may assume that Δ_0 is not empty, since otherwise Corollary 6.3 shows that $R_{\xi,0} = \{1\}$.

LEMMA 6.4: Δ_μ *is a reduced root system on (a subspace of)* $\mathfrak{a}$, *and its Weyl group is* W_μ.

PROOF: To show that Δ_μ is a root system in the sense of [1, p. 142], it is enough to show that Δ_μ is nonempty and is closed under its own reflections, since Δ_μ is a subset of the root system Δ. Δ_μ is nonempty since Δ_0 is now assumed nonempty. Lemma 2.7d shows that Δ_0 is closed under arbitrary reflections. Next, let α be in Δ_0, s be in S, and β be in $\Delta_{\theta,\mu}$. Then

$$p_\alpha(s\beta) = sp_{s^{-1}\alpha}\beta$$

and $s^{-1}\alpha$ is in Δ_0. By Proposition 3.10, $sp_{s^{-1}\alpha}$ is in S. Thus $p_\alpha(s\beta)$ is in $S\Delta_{e,\mu}$. Finally let r and s be in S and let α and β be in $\Delta_{e,\mu}$. Then

$$p_{r\alpha}(s\beta) = rp_\alpha r^{-1}s\beta = rs'p_\alpha\beta$$

since S is normal in $W(\mathfrak{a})$. On the right side rs' is in S and $p_\alpha\beta$ is in $\Delta_{e,\mu}$ since $p_{p_\alpha\beta} = p_\alpha p_\beta p_\alpha$ is in $W_{e,\mu}$. Thus Δ_μ is closed under its own reflections and is a root system.

We know from Proposition 3.10 that Δ_0 is reduced. Let β be in $\Delta_{e,\mu}$. Then β is even (Lemma 2.7d) and 2β is not an $\mathfrak{a}$-root, by Lemma 2.7a. Hence Δ_μ is a reduced root system.

Let $W(\Delta_\mu)$ be the Weyl group of Δ_μ. This group is generated by the p_α for α in Δ_0 and $S\Delta_{e,\mu}$. If α is in Δ_0, p_α is in S by Proposition 3.10; if

α is in $S\Delta_{e,\mu}$, say with $\alpha = s\beta$, then

$$p_\alpha = p_{s\beta} = sp_\beta s^{-1} \in sW_{e,\mu}s^{-1} \subseteq W_\mu.$$

In either case, p_α is in W_μ; thus $W(\Delta_\mu)$ is contained in W_μ. For the reverse inclusion, S is generated by reflections in members of Δ_0, by Proposition 3.10, and $W_{e,\mu}$ is the Weyl group of $\Delta_{e,\mu}$. Thus $W_\mu = W_{e,\mu}S$ is contained in $W(\Delta_\mu)$. That is, $W(\Delta_\mu) = W_\mu$. This proves the lemma.

Let $\mathfrak{g}_\mu$ be a semisimple Lie algebra split over $\mathbb{R}$ with root system Δ_μ, let $G_\mu^{\mathbb{C}}$ be a complex simply-connected group with Lie algebra $\mathfrak{g}_\mu^{\mathbb{C}}$, and let G_μ be the analytic subgroup corresponding to $\mathfrak{g}_\mu$. Since G_μ is split over $\mathbb{R}$, the group M_p for G_μ, which we call M_μ, is spanned freely over $\mathbb{Z}_2$ by the elements γ_β for β simple in Δ_μ. Define for β simple in Δ_μ

$$\sigma_\mu(\gamma_\beta) = \begin{cases} +1 & \text{if } \mu_{\xi,\beta}(0) = 0 \\ -1 & \text{if } \mu_{\xi,\beta}(0) \neq 0, \end{cases}$$

and extend σ_μ to a character of M_μ.

LEMMA 6.5: *For every α in Δ_μ*

$$\sigma_\mu(\gamma_\alpha) = \begin{cases} +1 & \text{if } \mu_{\xi,\alpha}(0) = 0 \\ -1 & \text{if } \mu_{\xi,\alpha}(0) \neq 0. \end{cases}$$

PROOF: We proceed by induction on the length $\ell(p_\alpha)$, the case $\ell(p_\alpha) = 1$ being the definition of σ_μ. We are to show that if the lemma holds for α, if β is simple in Δ_μ, and if $p_\beta\alpha \neq \alpha$, then the lemma holds for $p_\beta\alpha$. Notice by Lemma 6c of [11] that

$$\sigma_\mu(\gamma_{p_\beta\alpha}) = \sigma_\mu(\gamma_\alpha)\sigma_\mu(\gamma_\beta)^q, \tag{6.2}$$

where $q = 2\langle\alpha, \beta\rangle/|\alpha|^2$. Also when β is regarded as an $\mathfrak{a}$-root, it may not be reduced.

First suppose that β is even. Proposition 5.1a shows that $\mu_{\xi,\beta}(0) = 0$, so that $\sigma_\mu(\gamma_\beta) = +1$. Therefore (6.2) gives

$$\sigma_\mu(\gamma_{p_\beta\alpha}) = \sigma_\mu(\gamma_\alpha). \tag{6.3}$$

Proposition 5.1a shows also that $p_\beta\xi \cong \xi$, which proves the second equality in

$$\mu_{\xi,p_\beta\alpha}(0) = \mu_{p_\beta\xi,\alpha}(0) = \mu_{\xi,\alpha}(0), \tag{6.4}$$

the first equality being trivial. Equations (6.3) and (6.4) combine to say that if the lemma holds for α, then it holds for $p_\beta\alpha$.

Next suppose that $\frac{1}{2}\beta$ is an $\mathfrak{a}$-root (necessarily reduced). Then q has to be even in (6.2), and (6.3) holds. Proposition 5.1b applied to $\epsilon = \frac{1}{2}\beta$ says $p_\beta\xi \cong \xi$, and thus (6.4) holds. So again if the lemma holds for α, it holds for $p_\beta\alpha$.

Next suppose that β is an odd $\mathfrak{a}$-root, that $\frac{1}{2}\beta$ is not an $\mathfrak{a}$-root, and that $\xi(\gamma_\beta) = I$. Proposition 5.1c with $\epsilon = \beta$ gives $\mu_{\xi,\beta}(0) = 0$, whence $\sigma_\mu(\gamma_\beta) = +1$ and (6.3) holds. Proposition 5.1c shows also that $p_\beta\xi \cong \xi$, and thus (6.4) holds. Once again if the lemma holds for α, it holds for $p_\beta\alpha$.

Next suppose that β is an odd $\mathfrak{a}$-root, that $\frac{1}{2}\beta$ is not an $\mathfrak{a}$-root, that $\xi(\gamma_\beta) = -I$, and that q in (6.2) is even. Then (6.3) holds. We need therefore to prove that

$$\mu_{\xi,p_\beta\alpha}(0) = \mu_{\xi,\alpha}(0). \tag{6.5}$$

If α is even as an $\mathfrak{a}$-root, then so is $p_\beta\alpha$, and both sides of (6.5) are 0 by Proposition 5.1a; thus (6.5) holds if α is even. If α is odd as an $\mathfrak{a}$-root, then $\frac{1}{2}\alpha$ cannot be an $\mathfrak{a}$-root. [In fact, $2\langle\frac{1}{2}\alpha, \beta\rangle/|\frac{1}{2}\alpha|^2$ would have to be a nonzero multiple of 4 (since q is even) without β being a multiple of α, and this is impossible.] Thus Proposition 5.1c says

$$\mu_{\xi,p_\beta\alpha}(0) \text{ is} \begin{cases} \text{zero if } \xi(\gamma_{p_\beta\alpha}) = I \\ \text{nonzero if } \xi(\gamma_{p_\beta\alpha}) = -I \end{cases}$$

and

$$\mu_{\xi,\alpha}(0) \text{ if} \begin{cases} \text{zero if } \xi(\gamma_\alpha) = I \\ \text{nonzero if } \xi(\gamma_\alpha) = -I. \end{cases}$$

Since

$$\xi(\gamma_{p_\beta\alpha}) = \xi(\gamma_\alpha)\xi(\gamma_\beta)^q = \xi(\gamma_\alpha),$$

equation (6.5) follows.

Finally suppose that β is an odd $\mathfrak{a}$-root, that $\frac{1}{2}\beta$ is not an $\mathfrak{a}$-root, that $\xi(\gamma_\beta) = -I$, and that q in (6.2) is odd. By Proposition 5.1c, $\mu_{\xi,\beta}(0)$

is not 0 and hence $\sigma_\mu(\gamma_\beta) = -1$. Thus (6.2) gives

$$\sigma_\mu(\gamma_{p_\beta\alpha}) = -\sigma_\mu(\gamma_\alpha),$$

and the proof will be complete if we show

(6.6) $\mu_{p_\beta\xi,\alpha}(0)$ is zero if and only if $\mu_{\xi,\alpha}(0)$ is not zero.

If α is odd, then we have

$$p_\beta\xi(\gamma_\alpha) = \xi(\gamma_{p_\beta\alpha}) = \xi(\gamma_\alpha)\xi(\gamma_\beta)^q = -\xi(\gamma_\alpha). \tag{6.7}$$

Thus if α is odd and $\frac{1}{2}\alpha$ is an $\mathfrak{a}$-root, (6.6) follows from Proposition 5.1d with $\epsilon = \frac{1}{2}\alpha$ and $\delta = \beta$. If α is odd and $\frac{1}{2}\alpha$ is not an $\mathfrak{a}$-root, then we can use (6.7) and Proposition 5.1c twice with $\epsilon = \alpha$, once for ξ and once for $p_\beta\xi$, to obtain (6.6). So we conclude that (6.6) can fail only if α is even.

In this case, α is even and β is odd, so that Lemma 2.8 says that $|\alpha| \neq |\beta|$. Since q is odd, we have $|\alpha|^2 \geq 2|\beta|^2$ or else $|\beta|^2 = 3|\alpha|^2$. Let $\tilde{\alpha}$ be an extension of α to a root of $\mathfrak{a} + i\mathfrak{b}$. By Lemma 2.5, $|\tilde{\alpha}|^2 = 2|\alpha|^2$. Hence the two possibilities lead to $|\tilde{\alpha}|^2 \geq 4|\beta|^2$ or else $2|\beta|^2 = 3|\tilde{\alpha}|^2$, both of which are impossible in a reduced root system. This completes the proof of the lemma.

LEMMA 6.6:
(i) *If p is in W_μ, then $p\xi \cong \xi$ implies $p\sigma_\mu = \sigma_\mu$.*
(ii) *If p_ϵ is in W_μ, then p_ϵ is in $W'_{\xi,0}$ if and only if p_ϵ is in $W'_{\sigma_\mu,0}$.*
Consequently $R_{\xi,0}$ is isomorphic to a subgroup of $R_{\sigma_\mu,0}$, and Theorem 6.1 *reduces to the case of a minimal parabolic in a connected split semisimple Lie group of matrices.*

REMARK: As noted earlier, Theorem 6.1 follows from this lemma and Theorem 15.1 and Lemma 14.1 of [13].

PROOF: For (i), let p be in W_μ with $p\xi \cong \xi$. If α is in Δ_μ,

$$\mu_{\xi,p^{-1}\alpha}(0) = \mu_{p\xi,\alpha}(0) = \mu_{\xi,\alpha}(0). \tag{6.8}$$

Therefore

$$p\sigma_\mu(\gamma_\alpha) = \sigma_\mu(p^{-1}\gamma_\alpha p) = \sigma_\mu(\gamma_{p^{-1}a}) = \sigma_\mu(\gamma_\alpha),$$

the last equality following from Lemma 6.5 and (6.8). Since the elements γ_α generate M_μ, $p\sigma_\mu = \sigma_\mu$. This proves (i).

For (ii), suppose p_ϵ is in $W'_{\xi,0}$. By Proposition 6.2, p_ϵ is in W_μ. By definition of $W'_{\xi,0}$, $\mu_{\xi,\epsilon}(0) = 0$. Then Lemma 6.5 says that $\sigma_\mu(\gamma_\epsilon) = +1$. The equation preceding (15.1) in [13] then shows that p_ϵ is in $W'_{\sigma_\mu,0}$.

Conversely suppose p_ϵ is in $W_\mu \cap W'_{\sigma_\mu,0}$. Then $\sigma_\mu(\gamma_\epsilon) = +1$ by §15 of [13]. Lemma 6.5 then says $\mu_{\xi,\epsilon}(0) = 0$. By Lemma 19 of [3], or by [5], we have $p_\epsilon\xi \cong \xi$. Then p_ϵ is in $W'_{\xi,0}$ by definition.

To complete the proof, we note that $W'_{\xi,0}$ and $W'_{\sigma_\mu,0}$ are both in W_μ and that (ii) shows they are the same. Since $W_{\xi,0}$ is contained in W_μ (Proposition 6.2), (i) says $W_{\xi,0}$ can be regarded as a subgroup of $W_{\sigma_\mu,0}$. Finally

$$R_{\xi,0} \cong W_{\xi,0}/W'_{\xi,0} \subseteq W_{\sigma_\mu,0}/W'_{\sigma_\mu,0} \cong R_{\sigma_\mu,0},$$

and the lemma is proved.

§7. Reduction of commutativity to split case and minimal parabolic

The commutativity of the R group, as proved in §6, does not immediately imply that the commuting algebra for the representation (6.1) is commutative, only that the standard operators commute modulo scalar factors. Vogan's example, cited in the introduction, is one in which the R group is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the standard operators do not commute; however, the group in question does not satisfy the axioms of §1.

THEOREM 7.1: *If ξ is a discrete series representation of M and* $\exp \Lambda$ *is a unitary character of A, then the commuting algebra for the representation*

$$\operatorname*{ind}_{MAN \uparrow G} (\xi \otimes \exp \Lambda \otimes 1)$$

is commutative.

By Theorem 6.1, the R group is abelian. Let r and s be representatives in K of members $[r]$ and $[s]$ of $R_{\xi,\Lambda}$. Then ξ can be extended to be defined on r and on s, though not necessarily compatibly. (See Lemma 7.9 of [13].) By Lemma 14.2 of [13], the commuting algebra is commutative if and only if

$$\xi(r)\xi(s)\xi(r)^{-1}\xi(s)^{-1} = \xi(rsr^{-1}s^{-1}) \tag{7.1}$$

for each pair $[r]$ and $[s]$ in $R_{\xi,\Lambda}$; this condition is independent of the choices of representives.

LEMMA 7.2: *In order to prove Theorem* 7.1, *it is sufficient to prove* (7.1) *for each pair* $[r]$ *and* $[s]$ *in* $R_{\xi,0}$.

PROOF: Equation (7.1) for $R_{\xi,0}$, together with Lemma 14.2 of [13], shows that the commuting algebra for $\Lambda = 0$ is commutative. Let $[r]$ and $[s]$ be in $R_{\xi,\Lambda}$. Then $[r]$ and $[s]$ are in $W_{\xi,\Lambda} \subseteq W_{\xi,0}$, and so the standard intertwining operators $\xi(r)\mathscr{A}(r,\xi,0)$ and $\xi(s)\mathscr{A}(s,\xi,0)$ commute. Going over the proof that (a)$\Leftrightarrow$(b) in Lemma 14.2 of [13], we obtain (7.1) for these elements r and s. Hence (7.1) holds for $R_{\xi,\Lambda}$.

LEMMA 7.3: $R_{\xi,0}$ *is contained in S.*

PROOF: We recall the construction of §6. We have

$$R_{\xi,0} \subseteq W_{\xi,0} \subseteq W_\mu$$

by Proposition 6.2. Here $W_\mu = SW_{e,\mu}$ is the Weyl group of

$$\Delta_\mu = \Delta_0 \cup S\Delta_{e,\mu},$$

by Lemma 6.4. Each $\mathfrak{a}$-root α in $S\Delta_{e,\mu}$ is even and its reflection fixes μ. By Proposition 5.1a, $\mu_{\xi,\alpha}(0) = 0$. Since also $p_\alpha\xi \cong \xi$ (by Proposition 5.1a), we conclude that $S\Delta_{e,\mu}$ is contained in Δ'. Each member of $R_{\xi,0}$ leaves stable the positive roots of Δ'. Therefore it is enough to prove: If r in W_μ satisfies $r\alpha > 0$ for every $\alpha > 0$ in $S\Delta_{e,\mu}$, then r is in S.

We prove this statement by induction on the length $\ell(r)$ computed relative to the root system Δ_μ and the induced ordering. If $\ell(r) = 0$, then $r = 1$ and r is in S. Inductively assume the statement for length $< m$ and let $\ell(r) = m > 0$. We must have $r\epsilon < 0$ for some Δ_μ-simple root ϵ, and our assumption implies that ϵ is in Δ_0. By Proposition 3.10, p_ϵ is in S. The element rp_ϵ is in W_μ and has length $m-1$. If $\alpha > 0$ is in $S\Delta_{e,\mu}$, then $p_\epsilon\alpha$ is >0 (since α is not a multiple of ϵ) and $p_\epsilon\alpha$ is in $S\Delta_{e,\mu}$. Our assumption says that $r(p_\epsilon\alpha) > 0$. Thus $(rp_\epsilon)\alpha > 0$, and rp_ϵ satisfies $rp_\epsilon\alpha > 0$ for every $\alpha > 0$ in $S\Delta_{e,\mu}$. By inductive assumption rp_ϵ is in S. Thus r is in S, and the induction is complete.

As a consequence of Lemma 7.3 above and Lemma 63 of [12], any element r of $R_{\xi,0}$ can be decomposed as a commuting product of reflections $r = p_{\alpha_1} \dots p_{\alpha_m}$ with each α_j in Δ_0 and with m equal to the dimension of the -1 eigenspace of r. Such a decomposition we call a *nonredundant decomposition* within Δ_0. We shall examine this decomposition in detail. A set $\{\alpha_j\}$ of roots in Δ_0 will be called *superorthogonal* if the only roots of Δ_0 that are in the span of the α_j's are the $\pm\alpha_j$'s themselves.

LEMMA 7.4: *If r in $R_{\xi,0}$ decomposes as a nonredundant product $r = p_{\alpha_1} \dots p_{\alpha_m}$ of commuting reflections relative to Δ_0, then the set $\{\alpha_1, \dots, \alpha_m\}$ is superorthogonal.*

PROOF: Otherwise there would be two positive nonorthogonal, nonproportional roots α and β in the -1 eigenspace of r. Say $|\alpha| \geq |\beta|$. Since α and β are in Δ_0, they are in Δ_μ. Since $r\alpha = -\alpha$ and $r\beta = -\beta$, α and β are not in Δ'. By Lemma 19 of [3], $\mu_{\xi,\alpha}(0) = 0$ implies $p_\alpha \xi \cong \xi$ and hence $\alpha \in \Delta'$; thus we can conclude that $\mu_{\xi,\alpha}(0) \neq 0$ and $\mu_{\xi,\beta}(0) \neq 0$. By Lemma 6.5

$$\sigma_\mu(\gamma_\alpha) = \sigma_\mu(\gamma_\beta) = -1.$$

Now $\gamma_{p_\beta \alpha} = \gamma_\alpha \gamma_\beta$ since α and β are nonorthogonal with $|\alpha| \geq |\beta|$ and $\beta \neq \pm\alpha$. Hence $\sigma_\mu(\gamma_{p_\beta \alpha}) = +1$ and $\mu_{\xi, p_\beta \alpha}(0) = 0$. By Lemma 19 of [3], $p_\beta \alpha$ is in Δ'. But $r(p_\beta \alpha) = -p_\beta \alpha$ since $p_\beta \alpha$ is a linear combination of α and β, and we have a contradiction to the defining property of r.

LEMMA 7.5: *If r and s in $R_{\xi,0}$ decompose within Δ_0 as nonredundant products $r = p_{\alpha_1} \dots p_{\alpha_m}$ and $s = p_{\beta_1} \dots p_{\beta_n}$ of commuting reflections, then either $\alpha_1 = \pm\beta_j$ for some j or else α_1 is strongly orthogonal to $\beta_1, \dots, \beta_n$ within the set of $\mathfrak{a}$-roots.*

PROOF: First suppose α_1 is not orthogonal to $\beta_1, \dots, \beta_n$. We have

$$rs\alpha_1 = sr\alpha_1 = -s\alpha_1$$

since $R_{\xi,0}$ is abelian, and thus $s\alpha_1$ is in the -1 eigenspace of r. By Lemma 7.4,

$$s\alpha_1 = \pm\alpha_i \text{ for some } i. \tag{7.2}$$

The decomposition of s gives

$$s\alpha_1 = \alpha_1 - \sum_{j=1}^{n} \frac{2\langle \alpha_1, \beta_j \rangle}{|\beta_j|^2} \beta_j, \tag{7.3}$$

and our assumption is that $\Sigma(-)$ is not 0. Thus $s\alpha_1 \neq \alpha_1$. Also $s\alpha_1 = -\alpha_1$ would exhibit α_1 as in the span of $\{\beta_1, \ldots, \beta_n\}$, and we would have $\alpha_1 = \pm\beta_j$ by Lemma 7.4. Thus, arguing by contradiction, we may assume $i > 1$ in (7.2). Let us say $s\alpha_1 = \alpha_2$ for definiteness.

Forming the inner product of both sides of (7.3) with α_1, we obtain

$$\sum_{j=1}^{n} \frac{2\langle \alpha_1, \beta_j \rangle^2}{|\alpha_1|^2 |\beta_j|^2} = 1.$$

Each term on the left is a half-integer or integer ≥ 0. Thus at most two terms are nonzero. If only one term is nonzero, say the jth, then we obtain $\alpha_2 = \alpha_1 - c\beta_j$, in contradiction to Lemma 7.4. Thus exactly two terms are nonzero, say with $j = 1$ and $j = 2$, and the two terms are both 1/2. Replacing β_1 and/or β_2 by their negatives if necessary, we therefore have

$$\alpha_2 = \alpha_1 - \beta_1 - \beta_2. \tag{7.4}$$

Also

$$|\beta_1| = |\beta_2| = |\alpha_1| = |\alpha_2|,$$

the last equality holding since $\alpha_2 = s\alpha_1$.

Now the argument yielding (7.2) gives $r\beta_1 = \pm\beta_p$ and $r\beta_2 = \pm\beta_q$. Applying r to both sides of (7.4), we see that p and q are 1 and 2 in some order. In fact, the only possibilities are

$$r\beta_1 = -\beta_1 \text{ and } r\beta_2 = -\beta_2 \tag{7.5a}$$

and

$$r\beta_1 = -\beta_2 \text{ and } r\beta_2 = -\beta_1. \tag{7.5b}$$

If (7.5a) holds, then the decomposition of r gives

$$-\beta_1 = r\beta_1 = \beta_1 - \sum_{k=1}^{m} \frac{2\langle \beta_1, \alpha_k \rangle}{|\alpha_k|^2} \alpha_k.$$

By Lemma 7.4, β_1 must be $\pm\alpha_k$ for some k, and a similar result holds for β_2. This conclusion and (7.4) force either a nontrivial dependence among the α_k's or a relation $\alpha_1 = \pm\beta_j$.

Thus (7.5b) holds. Equation (7.4) and our values for inner products together give

$$p_{\beta_1}\alpha_1 = \alpha_1 - \beta_1 = \alpha_2 + \beta_2.$$

Hence

$$sr(\alpha_2 + \beta_2) = sr(\alpha_1 - \beta_1) = s(-\alpha_1 + \beta_2) = -\alpha_2 - \beta_2.$$

Consequently $p_{\beta_1}\alpha_1$ is not in Δ'.

However, we can now argue as in the proof of Lemma 7.4. We have $\sigma_\mu(\gamma_{\alpha_1}) = \sigma_\mu(\gamma_{\beta_1}) = -1$ and therefore

$$\sigma_\mu(\gamma_{p_{\beta_1}\alpha_1}) = \sigma_\mu(\gamma_{\alpha_1})\sigma_\mu(\gamma_{\beta_1}) = +1.$$

Hence $\mu_{\xi, p_{\beta_1}\alpha_1}(0) = 0$, and Lemma 19 of [3] shows that reflection in $p_{\beta_1}\alpha_1$ fixes ξ. Thus $p_{\beta_1}\alpha_1$ is in Δ', and we have a contradiction.

Hence either $\alpha_1 = \pm\beta_j$ for some j or else α_1 is orthogonal to $\beta_1, \ldots, \beta_n$. In the latter case we prove α_1 is strongly orthogonal to $\beta_1, \ldots, \beta_n$. Thus suppose $\beta_1 \pm \alpha_1$ are $\mathfrak{a}$-roots. Consideration of lengths shows $\beta_1 \pm \alpha_1$ are in Δ_0. What we have shown so far, in combination with Lemma 7.4, implies that

$$\alpha_1 \perp \{\beta_1, \ldots, \beta_n\} \text{ and } \beta_1 \perp \{\alpha_1, \ldots, \alpha_m\}.$$

Hence $r\beta_1 = \beta_1$ and $s\alpha_1 = \alpha_1$, from which we conclude that rs is -1 on $\beta_1 \pm \alpha_1$. Since rs is in $R_{\xi,0}$, we conclude that neither $\beta_1 + \alpha_1$ nor $\beta_1 - \alpha_1$ is in Δ'.

However, we can again argue as in the proof of Lemma 7.4. We have $\sigma_\mu(\gamma_{\alpha_1}) = \sigma_\mu(\gamma_{\beta_1}) = -1$ and therefore

$$\sigma_\mu(\gamma_{\beta_1-\alpha_1}) = \sigma_\mu(\gamma_{p_{\alpha_1}(\beta_1+\alpha_1)}) = \sigma_\mu(\gamma_{\beta_1+\alpha_1})\sigma_\mu(\gamma_{\alpha_1}) = -\sigma_\mu(\gamma_{\beta_1+\alpha_1}).$$

Thus $\sigma_\mu(\gamma_{\beta_1+\alpha_1}) = +1$ or $\sigma_\mu(\gamma_{\beta_1-\alpha_1}) = +1$. Thus one of $\beta_1 \pm \alpha_1$ is in Δ', and we have a contradiction.

LEMMA 7.6: *In order to prove Theorem 7.1, it is sufficient to prove that each system $\{p_{\alpha_1}, \ldots, p_{\alpha_n}\}$ of mutually strongly orthogonal root reflections in S has representatives in $N_K(\mathfrak{a})$ that commute with each other and with M_0.*

PROOF: In view of Lemma 7.2, we are to prove (7.1) for $[r]$ and $[s]$ in $R_{\xi,0}$, and the validity of (7.1) does not depend on the choice of representatives. Decompose $[r]$ and $[s]$ within Δ_0 as nonredundant products of commuting reflections; such decompositions exist by Lemma 7.3. Lemma 7.5 shows that the union of the two sets of reflections is pairwise strongly orthogonal. Choose a representative for each reflection as in the hypothesis of the present lemma, and take the obvious products of these representatives as representatives of r and s. In this way, we obtain commuting representatives for $[r]$ and $[s]$ that commute with M_0.

For these representatives the right side of (7.1) collapses to the identity, and we must show the same thing happens on the left. Since r and s commute with M_0, $\xi(r)$ and $\xi(s)$ are in the commuting algebra of $\xi|_{M_0}$, which is commutative by Lemma 4.4. Thus $\xi(r)$ and $\xi(s)$ commute, and the left side of (7.1) collapses to the identity. This proves the lemma.

We shall meet the requirement of Lemma 7.6 by giving a more constructive proof of Lemma 3.2a. We use the following notation. All roots will be relative to $\mathfrak{a} + i\mathfrak{b}$, and the members of Δ_0 are regarded as the roots that vanish on $i\mathfrak{b}$. The roots of $\mathfrak{m}$ are the roots that vanish on $\mathfrak{a}$. For each α in Δ_0 fix a root vector X_α in $\mathfrak{g}$ so that

$$[X_\alpha, \theta X_\alpha] = -2|\alpha|^{-2}H_\alpha,$$

and let $X_{-\alpha} = \theta X_\alpha$. We shall call

$$w_\alpha = \exp \frac{\pi}{2}(X_\alpha + \theta X_\alpha)$$

the *standard representative* of the reflection p_α. Note that $w_{-\alpha} = w_\alpha$ and that $w_\alpha^2 = \gamma_\alpha$.

LEMMA 7.7: *Let α be in Δ_0, let β be any root relative to $\mathfrak{a} + i\mathfrak{b}$, and let X_β be in the root space $\mathfrak{g}_\beta$ in $\mathfrak{g}^{\mathbb{C}}$.*
(a) *If α and β are strongly orthogonal, then $Ad(w_\alpha)X_\beta = X_\beta$.*
(b) *If α and β are orthogonal but not strongly orthogonal, then $Ad(w_\alpha)X_\beta = -X_\beta$.*

PROOF: (a) By strong orthogonality, $[X_\alpha, X_\beta] = [\theta X_\alpha, X_\beta] = 0$. Thus for some linear L,

$$Ad(w_\alpha)X_\beta = \exp\Big(ad\,\frac{\pi}{2}(X_\alpha + \theta X_\alpha)\Big)X_\beta = X_\beta + L\,ad(X_\alpha + \theta X_\alpha)X_\beta = X_\beta.$$

(b) Let $Y_{\beta+\alpha} = [X_\alpha, X_\beta]$ and $Y_{\beta-\alpha} = [\theta X_\alpha, X_\beta]$. From p. 143 of [6], we have

$$(ad\ \theta X_\alpha)(ad\ X_\alpha)X_\beta = (ad\ X_\alpha)(ad\ \theta X_\alpha)X_\beta = -2X_\beta.$$

Hence $(ad(X_\alpha + \theta X_\alpha))^2 X_\beta = -4X_\beta$ and

$$\begin{aligned}
Ad(w_\alpha)X_\beta &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\pi}{2}\right)^n (ad(X_\alpha + \theta X_\alpha))^n X_\beta \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{\pi}{2}\right)^{2n} (ad(X_\alpha + \theta X_\alpha))^{2n} X_\beta \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1} (ad(X_\alpha + \theta X_\alpha))(ad(X_\alpha + \theta X_\alpha))^{2n} X_\beta \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{\pi}{2}\right)^{2n} (-4)^n X_\beta \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1} (-4)^n (Y_{\beta+\alpha} + Y_{\beta-\alpha}) \\
&= (\cos \pi)X_\beta - \frac{1}{2}(\sin \pi)(Y_{\beta+\alpha} + Y_{\beta-\alpha}) = -X_\beta.
\end{aligned}$$

LEMMA 7.8: *Any system $\{p_{\alpha_1}, \ldots, p_{\alpha_n}\}$ of mutually strongly orthogonal root reflections in S has representatives in $N_K(\mathfrak{a})$ that commute with each other and with M_0.*

REMARKS: This lemma, in combination with Lemma 7.6, proves Theorem 7.1.

PROOF: Fix a root α in Δ_0 for which p_α is in the given set of reflections, and let w_α be the standard representative of p_α. Put

$$m'_\alpha = w_\alpha \zeta_\alpha$$

with ζ_α a member of $\exp \mathfrak{b}$ to be specified. If X_δ is a root vector for a root δ of $(\mathfrak{m}, i\mathfrak{b})$, then Lemma 7.7 gives

$$Ad(w_\alpha)X_\delta = \begin{cases} X_\delta & \text{if } \alpha \pm \delta \text{ are not roots} \\ -X_\delta & \text{if } \alpha \pm \delta \text{ are roots.} \end{cases} \tag{7.6}$$

We shall arrange that

$$Ad(\zeta_\alpha)X_\delta = \begin{cases} X_\delta \text{ if } \alpha \pm \delta \text{ are not roots} \\ -X_\delta \text{ if } \alpha \pm \delta \text{ are roots.} \end{cases} \tag{7.7}$$

If there are no roots δ of $\mathfrak{m}$ such that $\alpha \pm \delta$ are roots, we take $\zeta_\alpha = 1$. Otherwise let $\delta_1, \ldots, \delta_l$ be the simple roots of $\mathfrak{m}$. We claim there is exactly one simple root δ_{i_0} such that $\alpha \pm \delta_{i_0}$ are roots. There is at least one because the members of the Weyl group of $(\mathfrak{g}^{\mathbb{C}}, (\mathfrak{a} + i\mathfrak{b})^{\mathbb{C}})$ that fix $\mathfrak{a}$ send δ's for which $\mathfrak{a} \pm \mathfrak{d}$ are roots into δ's of the same type and because every root of $\mathfrak{m}$ is conjugate to a simple root. There is at most one because if δ_i and δ_j are two such, then we have

$$|\delta_i| = |\alpha| = |\delta_j|$$

and

$$\langle \alpha + \delta_i, \alpha + \delta_j \rangle = |\alpha|^2 - \langle \delta_i, \delta_j \rangle > 0 \tag{7.8}$$

by the Schwarz inequality; hence $\delta_i - \delta_j$ is a root, contradiction.

Now define H_0 in $i\mathfrak{b}$ by the condition

$$\delta_j(H_0) = \begin{cases} 1 \text{ if } j = i_0 \\ 0 \text{ otherwise,} \end{cases}$$

taking the component of H_0 in the center of $\mathfrak{m}^{\mathbb{C}}$ as 0, and define $\zeta_\alpha = \exp \pi i H_0$. Then ζ_α satisfies (7.7) on simple roots δ; for general positive δ we write $\delta = \Sigma\, n_j \delta_j$ and proceed to verify (7.7) by induction on $\Sigma\, n_j$.

Thus let $\delta > 0$ be given and choose a simple root δ_i for which $\delta - \delta_i$ is a root. Assuming that (7.7) holds for $\delta - \delta_i$ and δ_i, we are to prove it for δ. Changing notation, we see that we are to show that if δ, δ', and $\delta + \delta'$ are all roots of $\mathfrak{m}$, then

$\alpha \pm \delta$ not roots and $\alpha \pm \delta'$ not roots

$$\Rightarrow \alpha \pm (\delta + \delta') \text{ not roots} \tag{7.9a}$$

$\alpha \pm \delta$ not roots and $\alpha \pm \delta'$ roots

$$\Rightarrow \alpha \pm (\delta + \delta') \text{ roots} \tag{7.9b}$$

$\alpha \pm \delta$ roots and $\alpha \pm \delta'$ roots

$$\Rightarrow \alpha \pm (\delta + \delta') \text{ not roots.} \tag{7.9c}$$

In (7.9a) we have

$$(\alpha+\delta+\delta')-\delta=\alpha+\delta'$$
$$(\alpha+\delta+\delta')-\delta'=\alpha+\delta;$$

if $\alpha+\delta+\delta'$ turns out to be a root, one of these equations will give a contradiction unless both

$$\langle\delta+\delta',\delta\rangle\leq 0$$

and

$$\langle\delta+\delta',\delta'\rangle\leq 0.$$

The sum of these is $|\delta+\delta'|^2\leq 0$, and (7.9a) follows from this contradiction. For (7.9b), we argue by contradiction, putting $\epsilon=\delta+\delta'$ and $\epsilon'=-\delta$ and using (7.9a) to obtain the contradiction. In (7.9c) we use the argument of (7.8) to see that $\delta-\delta'$ is a root. Since α, δ, and δ' are associated with a simple component of $\mathfrak{g}$ of rank greater than 2, we are not dealing with G_2. Hence the fact that $\delta\pm\delta'$ are roots implies that $2|\delta|^2=|\delta+\delta'|^2$. Hence $|\alpha|<|\delta+\delta'|$, and $\alpha+(\delta+\delta')$ cannot be a root. This completes the verification of (7.9c) and the inductive argument for (7.7).

By (7.6) and (7.7) the elements $m'_{\alpha_1},\ldots,m'_{\alpha_n}$ commute with M_0. If α_i and α_j are two of the roots in question, then $w_{\alpha_i}w_{\alpha_j}=w_{\alpha_j}w_{\alpha_i}$ as a consequence of Lemma 7.7a, since α_i and α_j are assumed to be strongly orthogonal. Since w_{α_i} and w_{α_j} commute with $\exp\mathfrak{b}$ and since $\exp\mathfrak{b}$ is abelian, m'_{α_i} commutes with m'_{α_j}. Hence $m'_{\alpha_1},\ldots,m'_{\alpha_n}$ have the required properties.

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