INTERTWINING OPERATORS FOR SL(n, R)

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In 1947 Bargmann [1] derived the list of the irreducible unitary representations of the group $G = SL(2, R)$ of real two-by-two matrices of determinant one. For the most part, he grouped these into three series – the principal series,[1] the discrete series,[2] and the complementary series.[1] He showed that only the first two series are needed for the analysis of $L^2(G)$ and gave a number of formulas that can be interpreted [7] as the Plancherel formula for G. The principal series representations were realized in L^2 of the circle, and he observed that one representation of that form was reducible, with the others irreducible. The inner product for the complementary series was more subtle, and exhibiting it amounted to the proof of the existence of complementary series.

Similar results – the Plancherel formula, an irreducibility criterion for principal series, and conditions for existence of complementary series – are now known for a wide class of groups. We shall give in this paper a survey of these results in the context of $G = SL(n, R)$, the group of real n-by-n matrices of determinant one. Our survey is intended as an illustration of the use of an analytical tool, the theory of intertwining operators. The intertwining operators, developed in [10, 12-15, 18] give complete information about reducibility of principal series and provide an inner product

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[1]Bargmann used the term "continuous series outside the exceptional interval" for the principal series. "The exceptional interval" refers to the complementary series.

[2]Discrete series now refers to irreducible representations whose matrix coefficients are square integrable. Bargmann allowed as well two other representations in his "discrete series."

giving existence of some complementary series. It is not known whether the list of complementary series they produce is complete. For the Plancherel formula, which was proved by Harish-Chandra in a long series of papers ending with [8, 9], the intertwining operators make the Plancherel measure more explicit.

Our program will be to develop the intertwining operators for $SL(2,\mathbb{R})$, combine these operators suitably to handle a special case in $SL(n,\mathbb{R})$, and then use the special case to handle the general case. Finally, we shall turn our attention to the three problems we have mentioned, obtaining explicit results for each. A last section of the paper contains comments about a wider class of groups than $SL(n,\mathbb{R})$.

We should caution that a number of results about $SL(n,\mathbb{R})$ can be obtained by various methods that have a different scope. Typical illustrations of these approaches are the attacks on the irreducibility question by Gelfand and Graev [3], Zelobenko [22], and Wallach [20]. For the most part, we shall avoid these other methods.

1. *Operators for* $SL(2,\mathbb{R})$

For $G = SL(2,\mathbb{R})$, the principal series is indexed by a two-element set and a real parameter, so by pairs $(\pm, it)$ with t in $\mathbb{R}$. We give three realizations.

In the noncompact picture, the Hilbert space is $L^2(\mathbb{R})$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and if f is in $L^2(\mathbb{R})$, the representations $U(+, it, \cdot)$ and $U(-, it, \cdot)$ are given by

$$U(+, it, g) f(x) = |-bx+d|^{-1-it} f\left(\frac{ax-c}{-bx+d}\right)$$

$$U(-, it, g) f(x) = \operatorname{sgn}(-bx+d)\,|-bx+d|^{-1-it} f\left(\frac{ax-c}{-bx+d}\right) \cdot$$

All of these are unitary, and all are irreducible except $U(-,0,\cdot)$, which is reducible and splits into two inequivalent irreducible pieces.

For the induced picture, we introduce the subgroups

$$M = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix},\ A = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix},\ N = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix},\ V = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},\ K = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where $\varepsilon = \pm 1$ and $r > 0$. Fix $(\pm, it)$. Let the two characters of the two-element group M be defined by $\sigma_+(m) = 1$ and $\sigma_-(m) = \varepsilon$ if $m = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$. The space of the induced representation is

$$\{f : G \to \mathbb{C} \mid f(x\ man) = r^{-1-it}\sigma_\pm(m)^{-1} f(x)\}\ .$$

Since every g in G decomposes as $g = kan$, according to the Iwasawa decomposition,[3] f is determined by its restriction to K. The norm on f is taken as

$$\|f\| = \|f|_K\|_2\ ,$$

and the representation is $U(g)f(x) = f(g^{-1}x)$. Restriction from G to V provides a mapping that shows the induced picture and the noncompact picture are equivalent.[4]

In the compact picture, the induced picture is merely restricted to K. The space is the space of L^2 functions on K with $f(km) = \sigma(m)^{-1}f(k)$ and with the L^2 norm. The space is independent of t, and the formula for the group action makes sense with it replaced by a complex parameter z, except that the representations $U(\pm, z, \cdot)$ are not necessarily unitary. As z varies, the operators $U(\pm, z, g)$ act in the same space and vary analytically in z. We speak of the *nonunitary principal series.*

The other irreducible unitary representations occurring in the Plancherel formula are those of the discrete series, denoted D_n^+ and D_n^- with $n \geq 2$. We shall not write down their exact form, but give certain facts about them.

(1) $D_n^+ \oplus D_n^-$ is a subrepresentation of $U(+, n-1, \cdot)$ if n is even and of $U(-, n-1, \cdot)$ if n is odd. (This imbedding is done rigorously

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[3] Gram-Schmidt decomposition in this group.

[4] This equivalence is a scalar multiple of a unitary operator if the measure on V is taken as Lebesgue measure dx.

in the compact picture and is understood as just an imbedding of the infinitesimal action of G on those Hilbert space elements that transform under K within a finite-dimensional subspace. It is not a unitary imbedding.)

(2) When we try to extend representations from $SL(2,R)$ to $SL^{\pm}(2,R)$[5] without enlarging the space, say be defining $U\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, we can extend the principal series but not individual discrete series. However, we can extend $D_n = D_n^+ \oplus D_n^-$. The representations D_n of $SL^{\pm}(2,R)$ comprise the discrete series of $SL^{\pm}(2,R)$.

Now we can state the Plancherel formula for G. If U(g) is an irreducible representation and F is a sufficiently nice function on G, let $U(F) = \int_G F(g)U(g)\,dg$. Then

$$\|F\|_2^2 = \sum_{n=2}^{\infty} d_n(\|D_n^+(F)\|_{HS}^2 + \|D_n^-(F)\|_{HS}^2)$$

$$+ \int_{-\infty}^{\infty} \|U(+,it,F)\|_{HS}^2\, p_+(it)\,dt + \int_{-\infty}^{\infty} \|U(-,it,F)\|_{HS}^2\, p_-(it)\,dt\ ,$$

where HS denotes Hilbert-Schmidt norm and $\{d_n, p_+(it)\,dt, p_-(it)\,dt\}$ is the Plancherel measure. To give the Plancherel measure, we fix a normalization of Haar measure on G. Namely write

$$g = \begin{pmatrix}\cos\theta_1 & \sin\theta_1\\ -\sin\theta_1 & \cos\theta_1\end{pmatrix}\begin{pmatrix}e^s & 0\\ 0 & e^{-s}\end{pmatrix}\begin{pmatrix}\cos\theta_2 & \sin\theta_2\\ -\sin\theta_2 & \cos\theta_2\end{pmatrix},$$

with $0 \le \theta_1 \le 2\pi$, $0 \le \theta_2 \le 2\pi$, $0 \le s < \infty$. Then

$$dg = \frac{1}{4\pi^2}(e^{2s} - e^{-2s})\,d\theta_1\,ds\,d\theta_2$$

[5]The group of 2-by-2 real matrices of determinant ± 1.

is a Haar measure. The number d_n can be computed from the formula [6]

$$d_n^{-1} = \|f\|^{-4} \int_G |(D_n^{\pm}(g)f, f)|^2 \, dg$$

with the aid of the explicit form of $D_n^{\pm}$, and the result is that $d_n = \frac{1}{2}(n-1)$. We shall return to p_+ and p_- presently.

We come to the intertwining operators. It follows from Bargmann's classification that $U(\pm, it, \cdot)$ is equivalent with $U(\pm, -it, \cdot)$. Formally the operator given in the induced picture by

$$A(w, \pm, it)\, f(x) = \int_V f(xwv)\, dv \ ,$$

with $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and f in the space of the induced representation,[6] implements this equivalence. However, this integral is divergent, and it is necessary to proceed with care. To see the problem, one can compute $A(w, \pm, it)$ in the noncompact picture. After a change of variables, the formula is

$$A(w, +, it)\, f(x) = \int_{-\infty}^{\infty} \frac{f(x-y)\, dy}{|y|^{1-it}}$$

$$A(w, -, it)\, f(x) = \int_{-\infty}^{\infty} \frac{f(x-y)\, \mathrm{sgn}\, y \, dy}{|y|^{1-it}} \ .$$

These integrals are convergent if it is replaced by z and if $\mathrm{Re}\, z > 0$. Thus the idea is to work with the nonunitary principal series and do an analytic continuation to get the intertwining operator. In order to avoid technical problems, it is helpful to work in the compact picture, carrying over the formal operator from the induced picture.

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[6] We use the normalization that dv is Lebesgue measure. See footnote 4.

THEOREM 1. *In the compact picture,* $A(w,\pm,z)f$ *is convergent for* f *in* C^∞ *if* $\mathrm{Re}\, z > 0$ *and extends to a meromorphic function in the* z*-plane whose only singularities are at most simple poles at the nonpositive integers. Moreover,*

$$U(\pm,-z,\cdot)\,A(w,\pm,z) = A(w,\pm,z)\,U(\pm,z,\cdot)$$

as an identity of meromorphic functions. On trigonometric polynomials

$$A(w,\pm,z)^* = A(w^{-1},\pm,\overline{z}) \ . \tag{1.1}$$

Formally the operator $A(w,+,it)$ is just fractional integration of order it. In terms of Fourier transforms[7] on the line, it is well known that

$$(A(w,+,it)f)\widehat{\ }(\xi) = \gamma_+(it)|\xi|^{-it}\hat{f}(\xi) \ ,$$

where

$$\gamma_+(z) = \pi^{\frac{1}{2}-z}\,\Gamma\left(\tfrac{z}{2}\right)/\Gamma\left(\tfrac{1-z}{2}\right) .$$

See [19, p. 73]. Since $A(w^{-1},+,it) = A(w,+,it)$, we expect that $A(w^{-1},+,-it)\,A(w,+,it)$ is the multiple $\gamma_+(-it)\gamma_+(it)$ of the identity operator. Similarly we expect that $A(w^{-1},-,-it)\,A(w,-,it)$ is the multiple $\gamma_-(-it)\gamma_-(it)$ of the identity, where

$$\gamma_-(z) = \pi^{\frac{1}{2}-z}\,\Gamma\left(\tfrac{1+z}{2}\right)/\Gamma\left(\tfrac{2-z}{2}\right) .$$

Simplifying the products of gamma functions and justifying matters by using the compact picture, we arrive at the following result.

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[7] Here the Fourier transform is given by $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{2\pi i x\xi} f(x)\,dx$.

THEOREM 2. *In the compact picture*

$$A(w^{-1}, \pm, -z)\, A(w, \pm, z) = \eta_{\pm}(z)\, I$$

as identities of meromorphic functions, where

$$\eta_{+}(z) = 2\pi\, iz^{-1} \coth(-\pi iz/2)$$

and

$$\eta_{-}(z) = 2\pi\, iz^{-1} \tanh(-\pi iz/2)\ .$$

For future reference we note also that

$$\gamma_{+}(z)\gamma_{-}(-z) = \frac{2\pi}{z}\ . \tag{1.2}$$

The connection between intertwining operators and the Plancherel measure is given by the following theorem, which is of a general nature and will have an analog in SL(n, R).

THEOREM 3. $p_{\pm}(it) = \frac{\pi}{4}\eta_{\pm}(it)^{-1}$ *for real* t. *Consequently*

$$p_{+}(it) = \frac{1}{8}\, t \tanh(\pi t/2)$$

$$p_{-}(it) = \frac{1}{8}\, t \coth(\pi t/2)\ .$$

This theorem can be proved in two steps, first by relating $\eta_{\pm}(z)$ with the asymptotic behavior of certain entry functions of $U(\pm, z, \cdot)$ and second by relating the asymptotic behavior with the Plancherel measure.

These intertwining operators will be combined in various ways in our discussion of SL(n, R). We introduce normalized operators, partly as a bookkeeping device, defining

$$\mathcal{A}(w, \pm, z) = \gamma_{\pm}(z)^{-1} A(w, \pm, z)\ .$$

The identity (1.1), in combination with Theorem 2 and the fact that $\gamma_\pm(\overline{z}) = \overline{\gamma_\pm(z)}$, implies that

(i) $\mathfrak{A}(w^{-1}, \pm, -z)\mathfrak{A}(w, \pm, z) = I$

(ii) $\mathfrak{A}(w, \pm, z)^* = \mathfrak{A}(w^{-1}, \pm, \overline{z})$

(iii) $\mathfrak{A}(w, \pm, z)$ is unitary for z imaginary.

We have mentioned that only $U(-, 0, \cdot)$ is reducible among the $U(\pm, it, \cdot)$. For $z = 0$, $\mathfrak{A}(w, \pm, 0)$ is unitary and intertwines $U(\pm, 0, \cdot)$ with itself. In the case of σ_+, $\mathfrak{A}(w, +, 0)$ is scalar, whereas in the case of σ_-, $\mathfrak{A}(w, -, 0)$ is exactly the Hilbert transform and is not scalar. In short, the intertwining operators we have constructed account for the only reducibility that occurs. A similar fact will hold for $SL(n, \mathbf{R})$.

2. *Some parabolic subgroups in* $SL(n, \mathbf{R})$

Following Gelfand and Graev [3], we introduce $[n/2] + 1$ series of representations in $G = SL(n, \mathbf{R})$, with several realizations for each. The intertwining operators will exhibit the equivalence of the several realizations. Each series will consist of representations induced from a generalized upper triangular subgroup, and we begin by defining the appropriate subgroups.

The parameter that points to the appropriate series will be called k, with $0 \leq k \leq [n/2]$. Choose ℓ so that $2k + \ell = n$. Let

$$M = \left(\begin{array}{cccc|ccc} \boxed{\begin{smallmatrix}\cdot&\cdot\\ \cdot&\cdot\end{smallmatrix}} & & & & & & \\ & \boxed{\begin{smallmatrix}\cdot&\cdot\\ \cdot&\cdot\end{smallmatrix}} & & & & 0 & \\ 0 & & \ddots & & & & \\ & & & \boxed{\begin{smallmatrix}\cdot&\cdot\\ \cdot&\cdot\end{smallmatrix}} & & & \\ \hline & & & & \varepsilon_{2k+1} & & 0 \\ & 0 & & & & \ddots & \\ & & & & 0 & & \varepsilon_n \end{array}\right), \quad N = \left(\begin{array}{cccc|ccc} \boxed{\begin{smallmatrix}1&0\\ 0&1\end{smallmatrix}} & & & & & & \\ & \boxed{\begin{smallmatrix}1&0\\ 0&1\end{smallmatrix}} & * & & & * & \\ 0 & & \ddots & & & & \\ & & & \boxed{\begin{smallmatrix}1&0\\ 0&1\end{smallmatrix}} & & & \\ \hline & & & & 1 & & * \\ & 0 & & & & 1 & \\ & & & & & \ddots & \\ & & & & 0 & & 1 \end{array}\right).$$

Members of M have k two-by-two blocks, each with determinant ± 1, down the diagonal, followed by ℓ diagonal entries $\varepsilon_{2k+1}, \cdots, \varepsilon_n$ equal to ± 1. Members of N are the sum of the identity and matrices strictly upper triangular relative to M. Let $V = N^{tr}$ and let K be SO(n), the rotation subgroup.

Let $\mathfrak{a}_p$ be the vector space of diagonal matrices of trace 0, and let e_j be the linear functional on $\mathfrak{a}_p$ that picks out the j^{th} diagonal entry. For $1 \le j \le k$, let

$$f_j = \frac{1}{2}(e_{2j-1} + e_{2j}) .$$

Let $\mathfrak{a}_M$ be the subspace of $\mathfrak{a}_p$ on which all the f_j are 0, and let $\mathfrak{a}$ be the subspace for which $e_{2j-1}(H) = e_{2j}(H)$ for $1 \le j \le k$. Then $\mathfrak{a}_p = \mathfrak{a}_M \oplus \mathfrak{a}$. Define A_p, A_M, and A to be the groups of exponentials of matrices in $\mathfrak{a}_p$, $\mathfrak{a}_M$, and $\mathfrak{a}$, respectively. For example,

$$A = \begin{pmatrix} r_1 & & & & & & & \\ & r_1 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & r_k & & & & \\ & & & & r_k & & & \\ & 0 & & & & r_{2k+1} & & \\ & & & & & & \ddots & \\ & & & & & & & r_n \end{pmatrix} .$$

Then M and A normalize N and members of M and A commute with each other. Hence MAN is a group. The group MA is the group of all elements that commute with each element of A.

The groups that arise in the case $k = 0$, we shall single out by attaching subscripts p.[8] This convention defines M_p, N_p, and V_p, and it redefines A_p consistently. Put $N_M = N_p \cap M$. Note that $N_p = NN_M$ and that M is the product of M_p and the identity component M_0 of M.

[8] The subscript p is a vestige of the German $\mathfrak{p}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

The groups MAN are the basic groups we are concerned with, but we also want some variants of them, and we shall use the linear functionals f_i, $1 \leq i \leq k$, and e_j, $2k+1 \leq j \leq n$, to define these variants. To begin with, each nonzero difference of two of these linear functionals is called a *root*, or root of $\mathfrak{a}$, and we associate a subgroup of N or V to each. Let E_{ij} be the matrix that is 1 in the i-j$^{\text{th}}$ entry and 0 elsewhere. Define subgroups by

$$N_{f_i-f_j} = I + \mathbb{R}E_{2i-1,2j-1} + \mathbb{R}E_{2i,2j-1} + \mathbb{R}E_{2i-1,2j} + \mathbb{R}E_{2i,2j}$$

$$N_{f_i-e_j} = I + \mathbb{R}E_{2i-1,j} + \mathbb{R}E_{2i,j}$$

$$N_{e_i-f_j} = I + \mathbb{R}E_{i,2j-1} + \mathbb{R}E_{i,2j}$$

$$N_{e_i-e_j} = I + \mathbb{R}E_{ij} .$$

If L is a root, the dimension of N_L is called the *multiplicity* of L. We associate a variant of N to each of the $(k+\ell)!$ enumerations of the $k+\ell$ linear functionals f_i, $1 \leq i \leq k$, and e_j, $2k+1 \leq j \leq n$. Fix such an enumeration, and adopt the convention that a functional minus a functional farther along in the list is a *positive root*. The remaining roots are the negative roots. With N_L defined above when L is a root, let

$$N_0 = \prod_{\substack{L=\text{root} \\ L>0}} N_L .$$

(If the functionals are enumerated in the original order, N_0 is N. If they are in reverse order, N_0 is V. There are $(k+\ell)! - 2$ other possibilities.) We shall be concerned with all the groups MAN_0 constructed this way.[9] Write V_0 for N_0^{tr}.

[9] These are some of the subgroups of G that are called parabolic in the literature. In fact, these are exactly all the parabolic subgroups with reductive part MA.

3. *Formal Intertwining Operators*

Fix one of the groups $P_0 = MAN_0$ constructed in the previous section. The series of representations of G that goes with P_0 is parametrized by (ξ, λ), where

ξ = irreducible discrete series representation of M on Hilbert space H^{ξ}

$\lambda = e^{\Lambda}$ = character of A, not necessarily unitary.

Here we can regard Λ as a complex-valued real-linear functional on $\mathfrak{a}$. To describe ξ more explicitly, let m in M be written as

$$m = (m_1, \cdots, m_k, \varepsilon_{2k+1}, \cdots, \varepsilon_n)$$

where the m_j's are the $SL^{\pm}(2, R)$ blocks comprising the top part of M. Then[10]

$$\xi(m) = \left(\prod_{\text{certain } j} \varepsilon_j \right) D_{N_1}(m_1) \oplus \cdots \oplus D_{N_k}(m_k) . \qquad (3.1)$$

Let $\mu(a)$ be the positive number by which Lebesgue measure on N_0 is multiplied when N_0 is conjugated by A, so that μ is a certain positive character of A. (If $\{L_j\}$ is the set of positive roots defining N_0 and n_j is the multiplicity (1, 2 or 4) of L_j, then $\mu = \exp(\Sigma n_j L_j)$.)

In the induced picture the space for the representation $U_{P_0}(\xi, \lambda, \cdot)$ is

$$\{f : G \to H^{\xi} \mid f(x\, man_0) = \mu(a)^{-\frac{1}{2}} \lambda(a)^{-1} \xi(m)^{-1} f(x)\}$$

with norm

$$\|f\| = \|f|_K\|_2 ,$$

where Haar measure for K has total mass one. The group action is

$$U_{P_0}(\xi, \lambda, g) f(x) = f(g^{-1} x) .$$

The representation is unitary if λ is unitary.

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[10]Equation (3.1) gives an irreducible ξ if $n > 2k$. However, if $n = 2k$, ξ is the sum of two inequivalent discrete series. An exact parametrization of the irreducible ξ's will be given in Section 6.

The compact picture is the restriction of the induced picture to K, and the noncompact picture is the restriction of the induced picture to V_0. Gelfand and Graev [3] describe the noncompact picture more explicitly.

Let $P_1 = MAN_1$ and $P_2 = MAN_2$ with the same MA. For f in the space of $U_{P_1}(\xi, \lambda, \cdot)$ in the induced picture, we consider

$$A(P_2 : P_1 : \xi : \lambda) f(x) = \int_{V_1 \cap N_2} f(xv)\, dv \ ,$$

where dv is Lebesgue measure (Haar measure) in the coordinates on $V_1 \cap N_2$. As with $SL(2, \mathbb{R})$, there are convergence problems, but at least on a formal level we have

$$U_{P_2}(\xi, \lambda, \cdot) A(P_2 : P_1 : \xi : \lambda) = A(P_2 : P_1 : \xi : \lambda) U_{P_1}(\xi, \lambda, \cdot) \ .$$

4. *Special case*, $k = 0$

In this section we assume that $k = 0$, and we drop the subscripts p to simplify notation. M is now finite abelian of order 2^{n-1}. The irreducible representations of M are one-dimensional, and we use σ for a typical one (instead of ξ).

One special feature of this situation is that the n! possible choices for N_0 are all conjugate within G. In fact, let M' be the normalizer of A in K. Members of M' have one ± 1 in each row and column, and 0's elsewhere. A member of M' conjugates an N_0 to an N_1, leaving N_0 stable if and only if it is in M. The group M'/M is the full symmetric group on n letters and permutes the N_0's simply transitively. It acts also by permuting the e_i's, and the permutation that maps N_0 to N is exactly the one that restores the ordering of the e_i's to the natural one.

Thus let $P = MAN$ and let $P_0 = MAN_0 = w^{-1}Pw$ with w in M'. We investigate $A(P_0 : P : \sigma : \lambda)$. This operator is not given by a convolution integral in the noncompact picture as it stands. However, let us introduce

$R(w)f(x) = f(xw)$; $R(w)$ intertwines $U_{P_0}(\sigma, \lambda, \cdot)$ with $U_P(w\sigma, w\lambda, \cdot)$, where $w\sigma(m) = \sigma(w^{-1}mw)$, $w\lambda(a) = \lambda(w^{-1}aw)$. Thus we expect

$$A_P(w, \sigma, \lambda) = R(w)A(P_0 : P : \sigma : \lambda)$$

to intertwine $U_P(\sigma, \lambda, \cdot)$ with $U_P(w\sigma, w\lambda, \cdot)$, and this operator is given by a convolution integral in the noncompact picture.

For example, in $SL(3, \mathbb{R})$ with σ trivial and $w = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, we pass to the noncompact picture and make a change of variables to find f on V maps to the function of $v_0 \in V$ given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |z|^{-1+s_1}|xy-z|^{-1+s_2} f\left(v_0\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}\right) dx\, dy\, dz\ .$$

This is essentially a convolution, but the singularity of the kernel is not limited to a single point. On the face of it, the problem of analytic continuation of this integral would seem to be much harder than the problem in $SL(2, \mathbb{R})$.

However, even in $SL(n, \mathbb{R})$, the problem reduces to the case of $SL(2, \mathbb{R})$. First, if w as a permutation is a consecutive transposition $(i\ i{+}1)$, the operator $A_P(w, \sigma, \lambda)$ is an $SL(2, \mathbb{R})$ operator in disguise. To understand matters, we write out $A_P(w, \sigma, \lambda) f(v_0)$ in the noncompact picture,[11] decomposing v_0 as a product $v_0 = v_0'' v_0'$, where v_0'' is 0 in the $(i{+}1, i)^{th}$ entry and v_0' is 0 in all off-diagonal entries but the $(i{+}1, i)^{th}$. If we regard v_0'' as fixed, then the operator is an $SL(2, \mathbb{R})$ operator for the imbedded subgroup in the i^{th} and $(i{+}1)^{st}$ rows and columns. The σ and λ for the subgroup are obtained by restriction. Consequently the operators corresponding to consecutive transpositions admit analytic continuations. For general w, decompose the permutation

[11] To pursue matters rigorously, one uses also the compact picture.

as a product of consecutive transpositions in as short a fashion as possible and take the composition of the corresponding operators.[12] For normalizing factors γ we can use the corresponding product of $SL(2, \mathbf{R})$ factors. Put $\mathcal{A}_P = \gamma^{-1} A_P$. The result is that A is given unambiguously by a convergent integral for certain $\lambda = e^{\Lambda}$'s and extends to a meromorphic function for Λ in $\mathbf{C}^{n-1}$. Also $\mathcal{A}$ is unambiguously defined, is meromorphic, and has the following properties.

THEOREM 4.

(i) $U_P(w\sigma, w\lambda, \cdot)\mathcal{A}_P(w, \sigma, \lambda) = \mathcal{A}_P(w, \sigma, \lambda) U_P(\sigma, \lambda, \cdot)$

(ii) $\mathcal{A}_P(w_1 w_2, \sigma, \lambda) = \mathcal{A}_P(w_1, w_2\sigma, w_2\lambda)\mathcal{A}_P(w_2, \sigma, \lambda)$

(iii) $\mathcal{A}_P(w, \sigma, \lambda)^* = \mathcal{A}_P(w^{-1}, w\sigma, w\bar{\lambda}^{-1})$

(iv) $\mathcal{A}_P(w, \sigma, \lambda)$ *is unitary if* λ *is unitary.*

5. *General Case,* k *Arbitrary*

Return to the case of general k and to the notation of Sections 2-3. Fix k and MA. It can happen that two choices of N_1 are not conjugate within G, and consequently the intertwining operator $A(P_2 : P_1 : \xi : \lambda)$ cannot be transformed into a convolution operator (in the noncompact picture) in any evident way. A typical example occurs with $SL(3, \mathbf{R})$, $k = 1$, when $P_1 = MAN$ and $P_2 = MAV$. The H^{ξ}-valued function f on $V \cong \mathbf{R}^2$ is transformed as follows: If the image is evaluated at $\begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix}$ with x in $\mathbf{R}^2$, the value is

$$\int_{v \in \mathbf{R}^2} |1+x\cdot v|^{-\frac{3}{2}+z} \xi\begin{pmatrix} |1+x\cdot v|^{-\frac{1}{2}}(I+xv^{tr}) & 0 \\ 0 & \operatorname{sgn}(1+x\cdot v) \end{pmatrix} f\begin{pmatrix} 1 & 0 \\ v^{tr} & 1 \end{pmatrix} dv .$$

This is not a convolution; in addition, the singularity of the kernel is one-dimensional, and for any $v \neq 0$, the behavior of the integrand in x depends on the asymptotic behavior of $\xi(m)$ as $m \to \infty$.

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[12] This scheme for reducing the problem has a long history, beginning with Gelfand and Neumark [4, Chapter III]. It was developed further by Kunze and Stein [15] and completed by Schiffmann [18].

Despite these complications, there is a simple trick by which we can handle such integrals. The two key facts are (1) the imbedding of discrete series of $SL^{\pm}(2,R)$ in the nonunitary principal series and (2) the double induction formula for representations (induction in stages).

Recall from (3.1) that $\xi(m)$ is built from various discrete series $D_{N_j}(m_j)$ of $SL^{\pm}(2,R)$. Now M is essentially a direct sum of copies of $SL^{\pm}(2,R)$ and it follows from the imbedding of discrete series for $SL(2,R)$ and $SL^{\pm}(2,R)$ that ξ imbeds in

$$\omega = \operatorname*{ind}_{M_pA_MN_M \uparrow M} (\sigma \otimes \lambda_M \otimes 1) ,$$

where $A_M = A_p \cap M$, $N_M = N_p \cap M$, and

$$\lambda_M = \frac{1}{2} \sum_{j=1}^{k} (N_j - 1)(e_{2j-1} - e_{2j}) .$$

Moreover, if m_{ij} denotes the diagonal matrix with -1 in the i^{th} and j^{th} diagonal entries and $+1$ in the other diagonal entries, then

$$\sigma(m_{2j-1,2j}) = (-1)^{N_j}, \qquad 1 \leq j \leq k$$

$$\sigma(m_{ij})I = \xi(m_{ij}), \qquad 2k < i < j \leq n .$$

Since $\xi \subseteq \omega$, the double induction formula says that

$$\operatorname*{ind}_{MAN_0 \uparrow G} (\xi \otimes \lambda \otimes 1) \subseteq \operatorname*{ind}_{MAN_0 \uparrow G} \left(\operatorname*{ind}_{M_pA_MN_M \uparrow M} (\sigma \otimes \lambda_M \otimes 1) \otimes \lambda \otimes 1 \right)$$

$$= \operatorname*{ind}_{MAN_0 \uparrow G} \left(\operatorname*{ind}_{M_p(A_MA)(N_MN_0) \uparrow MAN_0} (\sigma \otimes (\lambda_M \otimes \lambda) \otimes 1) \right)$$

$$= \operatorname*{ind}_{M_pA_p(N_0)_p \uparrow G} (\sigma \otimes (\lambda_M \otimes \lambda) \otimes 1) ,$$

where $(N_0)_p = N_M N_0$. Therefore $U_{P_0}(\xi,\lambda,\cdot)$ is imbedded in the representation $U_{M_pA_p(N_0)_p}(\sigma,\lambda_M\otimes\lambda,\cdot)$, which is one of the representations of the special case $k=0$ considered in Section 4.

Correspondingly the intertwining operator $A(P_2:P_1:\xi:\lambda)$, which is given formally by an integral over $V_1 \cap N_2$, can be identified with a restriction of the special case intertwining operator

$$A(M_pA_p(N_2)_p : M_pA_p(N_1)_p : \sigma : \lambda_M \otimes \lambda) ,$$

which is given formally by an integral over

$$(N_1)_p^{tr} \cap (N_2)_p = V_M V_1 \cap N_M N_2 = V_1 \cap N_2 .$$

Thus the convergence and analytic continuation of the formal intertwining operator in the case of general k is reduced to the case $k = 0$.[13]

It is a consequence of formula (1.2) that we can normalize $A(P_2:P_1:\xi:\lambda)$ by the same factor as the corresponding operator with $k=0$ and arrive at the following conclusion, in which $\mathfrak{A} = \gamma^{-1}A$.

THEOREM 5.

(i) $U_{P_2}(\xi,\lambda,\cdot)\mathfrak{A}(P_2:P_1:\xi:\lambda) = \mathfrak{A}(P_2:P_1:\xi:\lambda)U_{P_1}(\xi,\lambda,\cdot)$

(ii) $\mathfrak{A}(P_3:P_1:\xi:\lambda) = \mathfrak{A}(P_3:P_2:\xi:\lambda)\mathfrak{A}(P_2:P_1:\xi:\lambda)$

(iii) $\mathfrak{A}(P_2:P_1:\xi:\lambda)^* = \mathfrak{A}(P_1:P_2:\xi:\bar{\lambda}^{-1})$.

Sometimes it happens that N_1 and N_2 are conjugate. Let M' be M times the normalizer of A in K, and let w be in $M' \cap K$. Suppose $P = MAN$ and $P_0 = MAN_0 = w^{-1}Pw$. Then we can define

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[13] This reduction works if f transforms within a finite-dimensional space under K. To make sense of part (i) of Theorem 5, we should multiply U_{P_1} and U_{P_2} on each side by the projection on such a subspace.

$$A_P(w,\xi,\lambda) = R(w)\,A(P_0 : P : \xi : \lambda)$$

$$\mathfrak{A}_P(w,\xi,\lambda) = R(w)\,\mathfrak{A}(P_0 : P : \xi : \lambda)$$

as in Section 4. Then $\mathfrak{A}_P(w,\xi,\lambda)$ intertwines $U_P(\xi,\lambda,\cdot)$ and $U_P(w\xi, w\lambda, \cdot)$, and we obtain a result similar in form to Theorem 4, but with σ replaced by ξ.

6. *Plancherel Formula*

From the work of Romm [17] and Harish-Chandra [9, Theorem 11], the Plancherel formula for $G = SL(n, R)$ takes the following general form:

$$\|F\|_2^2 = \sum_{k=0}^{[n/2]} \sum_{\xi \text{ of } M} d(\xi) \int_{\mathfrak{a}'} \|U(\xi, i\Lambda, F)\|_{HS}^2 \,\mu_\xi(i\Lambda)\, d\Lambda\,, \tag{6.1}$$

where the outside sum index is the number of two-by-two blocks in M, ξ is an irreducible discrete series representation of M with formal degree $d(\xi)$, μ_ξ is a function on the complexification of the dual $\mathfrak{a}'$ of $\mathfrak{a}$, and Lebesgue measure $d\Lambda$ on $\mathfrak{a}'$ is suitably normalized. We deal with the problem of making this formula totally explicit. The formulas of [9] would reduce this problem for $SL(n, R)$ to the cases $n = 2$, 3, and 4. The theory of intertwining operators, coupled with the results of [9], reduces the problem for $SL(n, R)$ immediately to the case $n = 2$.

Fix k. The first step is to parametrize the discrete series of M. Let M_0 be the identity component of M, and let $M^\#$ be the product of M_0 and the center of M. If ξ is an irreducible discrete series of M, then the restriction of ξ to $M^\#$ is a sum $\xi_1 \oplus \cdots \oplus \xi_j$ of irreducible representations, and $M/M^\#$ permutes the classes of the ξ_i's without fixed points. Therefore ξ is an induced representation, in fact is induced from any of the ξ_i's. Now ξ_i is still irreducible on M_0, which is the direct sum of k copies of $SL(2, R)$, and so is determined by a tuple $(N_1, \cdots, N_k)$ of

$SL(2,R)$ parameters[14] (with each $|N_j| \geq 2$), together with the restriction of ξ_i to the center of M. Thus ξ_i (and hence ξ) is determined by the data

$$(N_1, \cdots, N_k) \quad \text{and} \quad \xi(m_{2k+1,j}) \quad \text{for} \quad 2k+2 \leq j \leq n\,.$$

For $2k+2 \leq j \leq n$, define

$$s_j = \begin{cases} +1 & \text{if } \xi(m_{2k+1,j}) = I \\ -1 & \text{if } \xi(m_{2k+1,j}) = -I\,. \end{cases}$$

The quotient $M/M^{\#}$ determines whether two sets of data lead to equivalent ξ's, and we arrive at the following criteria: If $k < n/2$, two sets of data lead to equivalent ξ's if and only if the tuples of s_j's are identical and the tuples of N_j's differ only by sign changes. If $k = n/2$, two sets of data lead to equivalent ξ's if and only if the tuples of N_j's differ by an even number of sign changes.

Apart from a normalization that we shall consider later, the numbers $d(\xi)$ are the products of the corresponding numbers for the $SL(2,R)$'s. If ξ has data $(N_1, \cdots, N_k, s_{2k+2}, \cdots, s_n)$, then

$$d(\xi) = \text{Const} \times \prod_{j=1}^{k} (|N_j| - 1)\,.$$

To get at μ_ξ, we combine (1) Harish-Chandra's theory that relates asymptotics with the Plancherel measure and (2) identities that relate the intertwining operators with asymptotics. The result is Theorem 6 below. By Theorem 5

$$A(P : P^{tr} : \xi : e^{\Lambda})\, A(P^{tr} : P : \xi : e^{\Lambda}) = \eta_{P,\xi}(\Lambda) I$$

for a complex-valued meromorphic function $\eta_{P,\xi}$.

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[14] Let us agree to associate the parameter N_j to $D^+_{N_j}$ and the parameter $-N_j$ to $D^-_{N_j}$.

THEOREM 6. *If* M *has* k *two-by-two blocks, then*

$$\mu_\xi(\Lambda) = \mathrm{Const}(k)\,\eta_{P,\xi}(\Lambda)^{-1}$$

for an explicitly given constant depending on k *and the normalization of Haar measure.*

The bookkeeping necessary to compute $\eta_{P,\xi}$ has been done by the normalizing factors for the intertwining operators. Let ξ be imbedded in

$$\operatorname*{ind}_{M_pA_MN_M \uparrow M} (\sigma \otimes e^{\Lambda_M} \otimes 1) .$$

The normalizing factors for the operators in the definition of $\eta_{P,\xi}$ are the same as for suitable operators when $k = 0$, and these in turn are products of $SL(2, \mathbb{R})$ factors. The product expansion for the normalizing factors yields a product expansion for $\eta_{P,\xi}$ in terms of η_+ and η_-, and the result is

$$\eta_{P,\xi}(\Lambda) = \prod_{i<j}{}' \,\eta_{\sigma(m_{ij})}\left(\frac{2(\Lambda + \Lambda_M, e_i - e_j)}{|e_i - e_j|^2}\right), \tag{6.2}$$

where $\prod'$ means that the factors corresponding to $(i, j) = (2i' - 1, 2i')$ with $i' \le k$ are omitted. The factors on the right side are of the form $\eta_+(z)$ and $\eta_-(z)$ and are given in Theorem 2.

We can be more explicit. We know that

$$\Lambda_M = \frac{1}{2} \sum_{j=1}^{k} (|N_j| - 1)(e_{2j-1} - e_{2j}) .$$

Let

$$N_j' = |N_j| - 1$$

and

$$\Lambda = \sum_{i=1}^{k} \Lambda_i(e_{2i-1} + e_{2i}) + \sum_{j=2k+1}^{n} \Lambda_j e_j .$$

We do not know σ completely but know that $\xi(m) = \sigma(m)I$ for m in the center of M. Direct computation, even with this incomplete knowledge, shows that all the tanh and coth factors arising from the η_+'s and η_-'s cancel unless $2k < i < j$.[15]

We can now combine these computations with Theorems 6 and 2, obtaining the Plancherel measure except for a factor depending on k. Let $s_{2k+1} = +1$. Then

$$d(\xi)\mu_\xi(i\Lambda) = c_k\left[\prod_{j=1}^{k} N_j'\right]\left[\prod_{i<j\leq k}\left\{\left(\tfrac{1}{4}(N_i'-N_j')^2 + (\Lambda_i-\Lambda_j)^2\right)\left(\tfrac{1}{4}(N_i'+N_j')^2 + (\Lambda_i-\Lambda_j)^2\right)\right\}\right]$$
$$\times\left[\prod_{\substack{i\leq k\\ j>2k}}\left(\tfrac{1}{4}N_i'^2 + (\Lambda_i-\Lambda_j)^2\right)\right]\left[\prod_{2k<i<j}(\Lambda_i-\Lambda_j)\,{\tanh \atop \coth}(\pi(\Lambda_i-\Lambda_j)/2)\right] \tag{6.3}$$

with tanh if $s_is_j = +1$ and coth if $s_is_j = -1$.

To write down c_k, we have to specify normalizations for Haar measures. On $G = K(\exp \mathfrak{a}_p)N_p$, we use as a Haar measure

$$dg = e^{2\rho_p(H)}\,dk\,dH\,dn\ ,$$

where

- dk = on K has total mass one
- dH on $\mathfrak{a}_p$ is Lebesgue measure when $\mathfrak{a}_p$ has norm the square root of the sum of the squares of the entries
- dn on N_p is Lebesgue measure in the coordinates
- g in G is decomposed as $g = k(\exp H)n$ in KA_pN_p
- ρ_p on $\mathfrak{a}_p$ is half the sum of the positive roots defining N_p.

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[15] The cancellation occurs root-by-root. That is, it occurs in lots of 4 when $i < j \leq 2k$ and in lots of 2 when $i \leq 2k < j$.

To fix $d\Lambda$ in the Plancherel formula, we normalize dH on $\mathfrak{a}$ to be Lebesgue measure when $\mathfrak{a}$ is equipped with norm the square root of the sum of the squares of the entries. Then we normalize $d\Lambda$ on the dual $\mathfrak{a}'$ so that

$$f(0) = \int_{\mathfrak{a}'} \left(\int_{\mathfrak{a}} e^{i\Lambda(H)} f(H)\, dH \right) d\Lambda \quad \text{for} \quad f \quad \text{in} \quad C^{\infty}_{\text{com}}(\mathfrak{a}) .$$

The constants contributing to c_k are the constant of Theorem 6, the constant contributing to $d(\xi)$, and the coefficients 2π that were dropped each time the factor η_+ or η_- appears in $\eta_{P,\xi}$. From Theorem 11 of [9] relating the Plancherel measure to "c-functions" and from Theorem 3 of [13] relating "c-functions" to intertwining operators, we find that the constant of Theorem 6 is

$$\text{Const}(k) = (k!\,\ell!)^{-1} \int_V e^{-2\rho H(v)}\, dv ,$$

where ρ is half the sum of the positive roots defining N. (This is different from ρ_p if $k \neq 0$.) This integral is computed in [2] and [5]. If

$$c(z) = \pi^{-\frac{1}{2}} \Gamma\left(\frac{z}{2}\right) \Big/ \Gamma\left(\frac{z+1}{2}\right) ,$$

then

$$\int_V e^{-2\rho H(v)}\, dv = \pi^{\frac{1}{2}(n^2-n)-k} \prod_{i<j}{}' c\left(\frac{2\langle 2\rho-\rho_p, e_i-e_j \rangle}{|e_i-e_j|^2} \right)$$

with $\prod'$ as in formula (6.2).

In the expression for $d(\xi)$, Haar measure for $SL^{\pm}(2, R)$ is to be normalized in the same fashion as Haar measure for $SL(n, R)$, and this is different from the normalization in Section 1. The representation ξ of M is induced from a representation ξ_0 of the subgroup $M^{\#}$, and

$$d(\xi_0) = (2\pi\sqrt{2})^{-k} \prod_{j=1}^{k} N'_j \ .$$

To pass to M, we write $M = M^\# F$ as a semidirect product with $M^\#$ normal. Each element of F has order 2. The normalized Haar measure of M, restricted to $M^\#$, is $|F|^{-1}$ times the normalized Haar measure of $M^\#$. We find that

$$d(\xi) = |F|\, d(\xi_0) \ .$$

Here

$$|F| = \begin{cases} 2^k & \text{if} \quad k < n/2 \\ 2^{k-1} & \text{if} \quad k = n/2 \ . \end{cases}$$

Thus the contribution to c_k from $d(\xi)$ is

$$|F|(2\pi\sqrt{2})^{-k} = \begin{cases} (\pi\sqrt{2})^{-k} & \text{if} \quad k < n/2 \\ \frac{1}{2}\,(\pi\sqrt{2})^{-k} & \text{if} \quad k = n/2 \ . \end{cases}$$

Finally, Theorem 2 says that a factor $(2\pi)^{-1}$ must be included in c_k for each η_+ or η_- that appears in $\eta_{P,\xi}$, and there are $\frac{1}{2}(n^2-n)-k$ such factors.

We conclude that

$$c_k = 2^{-(n^2-n)/2} \times \begin{Bmatrix} 1 & \text{if} \quad k < n/2 \\ \frac{1}{2} & \text{if} \quad k = n/2 \end{Bmatrix} \times (2\pi^{-2})^{k/2}(k!(n-2k)!)^{-1}$$

$$\times \prod_{i<j}{}' \, c\left(\frac{2<2\rho-\rho_p, e_i-e_j>}{|e_i-e_j|^2}\right). \tag{6.4}$$

7. *Reducibility Criterion*

For each k with $0 \leq k \leq [n/2]$, we shall use intertwining operators to decide which of the representations $U_P(\xi, \lambda, \cdot)$ of $G = SL(n, R)$ are reducible when λ is unitary. From the work of Gelfand and Graev [3], it follows that $U_P(\xi, \lambda, \cdot)$ splits into at most two irreducible pieces, necessarily inequivalent. With a little extra work, one can decide when this splitting actually occurs. However, we shall not follow this approach, but shall use the general framework of intertwining operators.

Fix k. As in Section 5, let M' be M times the normalizer of A in K. The quotient $W(\mathfrak{a}) = M'/M$ is called the Weyl group of $\mathfrak{a}$. It operates on the class of ξ and on λ. Namely if w is in $M' \cap K$, put

$$w\xi(m) = \xi(w^{-1}mw) \quad \text{and} \quad w\lambda(a) = \lambda(w^{-1}aw) .$$

The $w\lambda$ and the class $[w\xi]$ of $w\xi$ depend only on the coset of w in M'/M. The group $W(\mathfrak{a})$ is easily computed and can be regarded as all permutations of the f_i, $1 \leq i \leq k$, times all permutations of the e_j, $2k < j \leq n$.

Suppose $[w\xi] = [\xi]$. Then it is possible to extend ξ to a representation of the smallest group containing M and w, with the extended representation acting on the same space. That is, we can define $\xi(w)$. This definition is unique up to a scalar equal to a j^{th} root of unity if j is the least positive integer such that w^j is in M.

In this case the operator $\xi(w)\mathfrak{A}_P(w, \xi, \lambda)$ satisfies

$$U_P(\xi, w\lambda, \cdot)\xi(w)\mathfrak{A}_P(w, \xi, \lambda) = \xi(w)\mathfrak{A}_P(w, \xi, \lambda)U_P(\xi, \lambda, \cdot) .$$

If also $w\lambda = \lambda$, then the operator $\xi(w)\mathfrak{A}_P(w, \xi, \lambda)$ commutes with $U_P(\xi, \lambda, \cdot)$ and will exhibit $U_P(\xi, \lambda, \cdot)$ as reducible if the operator is not scalar and if λ is unitary. The operator $\xi(w)\mathfrak{A}_P(w, \xi, \lambda)$ depends only on the coset $[w]$ of w in M'/M.

It is easy to determine whether $[w\xi] = [\xi]$. Let us enlarge the parameter set of $[\xi]$ to include $s_{2k+1} = 1$, writing it as

$$(N_1,\cdots,N_k, s_{2k+1},\cdots,s_n) \ ,$$

and let us admit

$$(N_1,\cdots,N_k, -s_{2k+1},\cdots,-s_n)$$

as a further equivalence. Then w acts as a permutation of the N_i's and of the s_j's, and $[w\xi] = [\xi]$ if and only if the final parameter set is equivalent with the original one.

For each $[w]$ in $W(\mathfrak{a})$, one can show that the normalizing factor for $A_P(w,\xi,\lambda)$ is nowhere vanishing for λ unitary. Let W_{ξ,λ_0} be the subgroup of $W(\mathfrak{a})$ of elements $[w]$ such that $[w\xi] = [\xi]$ and $w\lambda_0 = \lambda_0$. Then we have the following result [14].

THEOREM 7. *Let λ_0 be unitary. The operators $\xi(w)\mathcal{A}_P(w,\xi,\lambda_0)$ with w in W_{ξ,λ_0} such that the normalizing factor for $A_P(w,\xi,\lambda)$ is regular at $\lambda = \lambda_0$ form a basis for the vector space of bounded linear operators commuting with $U_P(\xi,\lambda_0,\cdot)$.*

Theorem 7 indicates a computation that will decide the reducibility question, since the normalizing factors ultimately are products of $SL(2,R)$ normalizing factors. If $[w]$ is W_{ξ,λ_0} and w does not act as the identity permutation on the indices $1,\cdots,k$, it is not hard to see that the normalizing factor fails to be regular. Thus we may assume w acts only on the s_j's. For simplicity, assume $\lambda = 1$. If $[w]$ is written as a product of consecutive transpositions in as short a fashion as possible, each factor in the corresponding decomposition of $\mathcal{A}_P(w,\xi,1)$ will be the identity or a Hilbert transform, and the condition of Theorem 7 is that all the factors be Hilbert transforms. One can then work out that w must map the parameter set

$$(N_1,\cdots,N_k, s_{2k+1},\cdots,s_n)$$

into

$$(N_1,\cdots,N_k, -s_{2k+1},\cdots,-s_n) \ .$$

THEOREM 8. *Let* ξ *have parameter set*

$$(N_1, \cdots, N_k, s_{2k+1}, \cdots, s_n)$$

and let λ_0 *be unitary. If there exists a permutation* $p \neq 1$ *of indices* $2k+1, \cdots, n$ *such that* $p(s_{2k+1}, \cdots, s_n) = (-s_{2k+1}, \cdots, -s_n)$ *and* $p\lambda_0 = \lambda_0$, *then* $U_P(\xi, \lambda_0, \cdot)$ *is reducible and splits into two inequivalent irreducible pieces. Otherwise* $U_P(\xi, \lambda_0, \cdot)$ *is irreducible. In particular, reducibility can occur only if* $\ell = n - 2k$ *is even and does occur when* ℓ *is even and positive if* ξ *and* λ_0 *are suitably chosen.*

8. *Complementary series*

The K-finite vectors for $U_P(\xi, \lambda, \cdot)$ are the members f of the representation space such that the span of $U_P(\xi, \lambda, K)f$ is finite-dimensional. Such vectors are dense in the representation space.

Informally $U_P(\xi, e^\Lambda, \cdot)$ is in the complementary series if Λ is not purely imaginary and if there exists an inner product on the space of K-finite vectors that makes $U_P(\xi, e^\Lambda, \cdot)$ unitary. The difficulty with this definition is that $U_P(\xi, e^\Lambda, x)$ need not leave stable the space of K-finite vectors. We can repair the difficulty by using the infinitesimal representation of $U_P(\xi, e^\Lambda, \cdot)$, i.e., the corresponding representation of the Lie algebra. Thus we say $U_P(\xi, e^\Lambda, \cdot)$ is in the *complementary series* if there exists an inner product $<\cdot, \cdot>$ on the K-finite vectors with respect to which the infinitesimal representation is skew-Hermitian, i.e.,

$$<U_P(\xi, e^\Lambda, X)f, g> = -<f, U_P(\xi, e^\Lambda, X)g>$$

for X in the Lie algebra.

If we assume that $<f, g> = (Lf, g)$ for an operator L and the usual L^2 inner product $(\cdot, \cdot)$ and if we take into account the identity $U_P(\xi, e^\Lambda, -X)^* = U_P(\xi, e^{-\bar{\Lambda}}, X)$, then we find the condition is that L be positive definite Hermitian and satisfy

$$U_P(\xi, e^{-\bar{\Lambda}}, X) L = L U_P(\xi, e^{\Lambda}, X) .$$

This equation is satisfied if L is a suitable intertwining operator and Λ is related suitably to $-\bar{\Lambda}$. Namely, if w is in $K \cap M'$, if $[w\xi] = [\xi]$, and if $w\Lambda = -\bar{\Lambda}$, this equation holds with

$$L = \xi(w)\mathfrak{A}_P(w, \xi, e^{\Lambda}) .$$

Moreover, this L is Hermitian if w, as a permutation, has order two. So a sufficient condition for complementary series is that this L be positive definite.

A technique for showing that L is positive definite is described in detail in [11] and [12]. The idea is that a continuous family of nonsingular Hermitian operators on a finite-dimensional space is everywhere positive definite if it is somewhere positive definite. We shall introduce assumptions that make L equal to the identity (which is positive definite) at $\Lambda = 0$. Because of the relations that intertwining operators satisfy, nonsingularity must persist until $\mathfrak{A}_P$ has a pole at Λ or $-\bar{\Lambda}$. Information about the poles of $\mathfrak{A}_P$ ultimately is largely a question about $SL(2, \mathbb{R})$.

In order to make maximum use of this technique, we must find all permutations $[w]$ of order two such that $[w\xi] = [\xi]$ and $\xi(w)\mathfrak{A}_P(w, \xi, 1)$ is scalar. From [14] one knows each such permutation is a product of transpositions with the same property. For such a transposition $[w]$, one can prove that $\xi(w)\mathfrak{A}_P(w, \xi, 1)$ is scalar if and only if $s_{w(j)} = s_j$ for $2k < j \leq n$ if ξ has parameters $(N_1, \cdots, N_k, s_{2k+1}, \cdots, s_n)$. Putting these facts together and making the necessary computations, we arrive at the following result.

THEOREM 9. *Let ξ have parameter set $(N_1, \cdots, N_k, s_{2k+1}, \cdots, s_n)$. Suppose $[w]$ is a permutation of order two such that $[w\xi] = [\xi]$ and $s_{w(j)} = s_j$ for $2k < j \leq n$. Then every complex Λ that is not purely imaginary and satisfies*

(i) $w\Lambda = -\overline{\Lambda}$

(ii) $|\mathrm{Re}\,\frac{<\Lambda,a>}{<a,a>}| < 1$ *for every root* a *of* $\mathfrak{a}$ *of the form* $a = f_i - f_j$

(iii) $|\mathrm{Re}\,\frac{<\Lambda,a>}{<a,a>}| < \frac{1}{2}$ *for every root* a *of* $\mathfrak{a}$ *of the form* $a = e_i - e_j$

is such that $U_P(\xi, e^{\Lambda}, \cdot)$ *is in the complementary series.*

9. *Wider Class of Groups*

Most of the results mentioned in this paper for SL(n, R) have generalizations to connected real semisimple Lie groups of matrices. The role of SL(2, R) in Section 1 is played by groups of real-rank one. Convergence for the intertwining integrals was handled by [15], analytic continuation was obtained independently in [18] and [12], and the normalization was done in [12]. The group that generalizes V is not always abelian, and the Fourier transform is not an appropriate tool; instead, the normalizing factors are constructed by means of Weierstrass canonical products.

For the general group, the representations that appear in the Plancherel formula are induced from parabolic subgroups MAN. The representations of MAN are assumed to be discrete series on M and unitary characters on A. In particular, MAN plays no role unless M has discrete series representations. The Plancherel formula is announced in [8] and [9].

The case $k = 0$ in SL(n, R) corresponds to the case of a minimal parabolic subgroup, in which M is compact. Schiffmann [18] realized that the intertwining operators in this case satisfied some relations even before normalization and exhibited them as compositions of real-rank-one operators. The normalization is done in [12].

The theory for the nonminimal parabolics is in [13] and [14]. The reduction to the case of minimal parabolics is in [13]. See also [21]. In the general case, theorems about reducibility appear in [14] and [10], and a theorem about complementary series appears in [14]. In the general case, the problem of deciding which intertwining operators are scalar and which are linearly independent is less transparent, but is solved by a detailed study of the group W_{ξ,λ_0}.

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