Anneli Lax's Research Mathematics

The mid 1950s was a period of intense interest in a basic existence-uniqueness condition for linear partial differential equations (PDEs) known as the "Cauchy problem". Lars Gårding had proved a fundamental result for constant-coefficient linear PDEs, and Jean Leray was just beginning his study of global solutions for linear PDEs with holomorphic coefficients.

The general *Cauchy problem* concerns a linear partial differential operator L of order m in n variables. The equation under study is $L(u) = 0$. Some submanifold S of dimension $n - 1$ is given, and initial values of the unknown function and its first $m - 1$ outgoing normal derivatives are specified on S. Some hypotheses are imposed on L, S, and the initial values. The question is whether there exists locally a unique solution of the equation on one side of the surface so that the initial conditions are satisfied.

For the situation of interest, Anneli Lax, with Richard Courant, had already proved a theorem that reduced one direction for the question in n variables to the question in 2 variables. It involved a parametrized family of 2-dimensional problems and gave a sufficient condition for the n-dimensional existence-uniqueness in terms of the sufficiency in 2-variables; the sufficiency in 2 variables had already been proved by E. E. Levi in 1909.

Lax's thesis dealt with the necessity in the 2-variable case. Let us state the result precisely when L has constant coefficients. Take the variables to be (x, t), and let $a_{i,j}$ be the coefficient of $\partial^{i+j}u/\partial^i x\partial^j t$. By a linear change of variables if necessary, we may assume that $a_{0,m} \neq 0$. Group the terms according to their order, and define

$$p_k(z) = a_{0,k}z^k + a_{1,k-1}z^{k-1} + \cdots + a_{k,0}.$$

If the roots of $p_m(z)$ are $\lambda_1, \ldots, \lambda_m$, then the top-order terms of L may be written

$$a_{0,m}\left(\frac{\partial}{\partial t} - \lambda_1\frac{\partial}{\partial x}\right)\cdots\left(\frac{\partial}{\partial t} - \lambda_m\frac{\partial}{\partial x}\right).$$

We assume that the λ_i are real but not necessarily distinct.

The lines $x = -\lambda_i t + c$ for each i and c are called *characteristics*. These have long been known to play a special role. This role can already be seen for the special case $(\partial/\partial t - \lambda(\partial/\partial x))u = 0$, whose general solution is $f(x + \lambda t)$ for any function f of one variable. Specifying initial data on a noncharacteristic line determines f everywhere, but specifying data on a characteristic line $x = -\lambda t + c$ determines f only at the one point c.

A curve S in the (x, t) plane is called *noncharacteristic* for L if it is nowhere tangent to a characteristic. The equation $L(u) = 0$ is said to be *properly solvable* relative to S if, for some k, all sets of k times differentiable initial data determine a unique solution of the Cauchy problem in a one-sided neighborhood of S.

It was known that the Cauchy problem is properly solvable for any noncharacteristic curve if the real numbers λ_i are distinct. Lax's theorem allows repetitions among the λ_i:

Theorem. *The Cauchy problem for the constant-coefficient equation $L(u) = 0$ in 2 variables and a noncharacteristic curve is properly solvable if and only if the greatest common divisor of the polynomials*

$$p_m(z), \frac{dp_m(z)}{dz}, \ldots, \frac{d^k p_m(z)}{dz^k}$$

divides $p_{m-k}(z)$ for $k = 1, \ldots, m - 1$.

In the previously known case in which the λ_i are distinct, the greatest common divisor in the theorem is 1 and therefore divides all $p_{m-k}(z)$.

Gårding's earlier result gave a different necessary and sufficient condition in the 2-variable constant-coefficient case, but it did not permit verification of the condition by inspection of the coefficients. Lax's thesis went on to consider the 2-variable variable-coefficient case. She proved that a certain condition generalizing the one in the above theorem was necessary and sufficient for proper solvability when the curve is noncharacteristic. The condition is now sometimes called the Levi-Lax condition. L. Svensson extended the theorem to n variables in 1968.

—Anthony W. Knapp

> Anneli would become interested in the food stamp budget. She would get around to the textbook, but only after understanding the kid's view.

Lax the Friend

Anneli Lax's accomplishments in mathematics, in writing, and in teaching are perhaps the easiest to document. Less concrete but perhaps more lasting are her contributions to the support of others in the field. For Anneli Lax was a steadfast and valued friend to many, offering support in countless tangible and intangible ways. One beneficiary of this support, Louise Raphael, writes:

> During my sabbatical year at Courant, I had the deep pleasure of living with the Lax family. Anneli had a genius for friendship. She was most loyal to all of her friends and accepted and loved them as they are.
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> Anneli was a "mathematical egalitarian". She respected brilliance and could hold her own. She was mathematically